Holism as the significance of gauge symmetries[*]

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October 1, 2019

Abstract

The thought experiment known as Galileo's ship exemplifies a relational empirical significance of subsystem boosts. The inertial state of motion of the ship is immaterial to how events unfold in the cabin, but is registered in the values of relational quantities such as the distance and velocity of the ship relative to the shore.

Can gauge transformations in Yang-Mills theory—taken as mere descriptive redundancy—exhibit a similar empirical significance? This question has been debated in the last fifteen years in philosophy of physics.

I will argue that the answer is 'yes'. To clearly pose the problem, I focus on physical, i.e.: gauge-invariant information. Recent results prove that given two subsystems' physical information, the universal state obtained by the composition of the subsystems is not always uniquely determined (Gomes & Riello, 2019). Moreover the state's residual variety is encoded in the action of a symmetry group on a subsystem. While for Galileo's ship the variety is encoded in boosts and translations, in gauge theory this variety has the structure of the finite-dimensional Lie group of the theory.

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1 Introduction

In its broadest terms, a symmetry is a transformation of a system which preserves the values of a relevant (usually large) set of physical quantities. Of course, this broad idea is made precise in various different ways: for example as a map in the space of states, or on the set of quantities; and as a map that must respect the system's dynamics, e.g. by mapping solutions to solutions or even by preserving the value of the Lagrangian functional on the states.

The broad idea is also associated with various debates.[1] For example, should we say that a symmetry transformation applied to the whole universe cannot yield a different physical state of affairs? And relatedly: should we prefer a reduced i.e. quotiented formalism, so that if presented with a state space $\mathcal{S}$ partitioned into the orbits of a group of symmetries acting on $\mathcal{S}$, we prefer the reduced state space whose elements are the orbits, i.e. $[s] \in \mathcal{S}/\sim$? (where $s \sim s$ $(s, s' \in \mathcal{S})$ means that s and s' are related by a symmetry transformation and square brackets, '$[\cdot]$', denote an entire equivalence class).

My focus in this paper is to put these "defining features" to work on just one recent philosophical debate about the empirical significance of symmetries. More specifically, the debate is about whether gauge symmetries can have a *direct* empirical significance. Of course, all hands agree that symmetries have various important empirical implications. The obvious examples come from the Noether theorems: the restrictions on the equations of motion entailed by Noether's second theorem, and the (approximately) conserved charges given by Noether's first theorem. Accordingly, in this debate, such familiar implications are usually labeled 'indirect'.

But some familiar symmetries of the whole Universe, such as velocity boosts in classical or relativistic mechanics (Galilean or Lorentz transformations), have a *direct* empirical significance when applied solely to subsystems. Thus Galileo's famous thought-experiment about the ship—that a process involving some set of relevant physical quantities in the cabin below decks proceeds in exactly the same way whether or not the ship is moving uniformly relative to the shore—shows that sub-system boosts have a direct, albeit relational, empirical significance. For though the inertial state of motion of the ship is undetectable to experimenters confined to the cabin, yet the entire system, composed of ship and sea[2] registers the difference between two such motions, namely in the different relative velocities of the ship to the water. Such examples rely on what are called 'external symmetries', i.e. symmetries which shift spacetime points around.

So the question arises: *Can other symmetries—especially gauge symmetries—have a similar direct empirical significance when applied to subsystems?*

Gauge symmetries are normally taken to encode descriptive redundancy: a view I will endorse. That is, they arise in a formalism that uses more variables then there are physical degrees of freedom in the dynamical system described. (They are also internal: unlike a boost or spatial translation, they do *not* shift spacetime points around).

This descriptive redundancy means that the natural answer to our question is 'No'. For surely, a "freedom to redescribe" could not have the content needed for a direct empirical significance, like the one illustrated by Galileo's ship. This

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[1] See the essays in (Brading & Castellani, 2003) and the references therein.

[2] From now on, I will prefer "sea" to "shore"; this restriction eliminates the need to discuss translations in addition to boosts (Maudlin, 1993), and places the two subsystems in direct contact, as in the case we will explore.

'No' answer was developed in detail by Brading and Brown (Brading & Brown, 2004) (henceforth BB) in response to various discussions such as Kosso (Kosso, 2000). They take themselves—I think rightly, in this respect—to be articulating the traditional or orthodox answer.

But in fact, the answer is 'Yes'. Indeed, the 'Yes' answer has been argued for by Greaves and Wallace (Wallace & Greaves, 2014) (henceforth GW), building on (Healey, 2009). And I will agree with some aspects of GW's analysis of symmetries. But, unlike either GW or BB, I will recast the topic to focus on gauge-*invariant* information about—i.e. states of—regions.[3]

Thus, including my own view, the shape of the debate can be summarized in three different definitions, which I will now informally sketch (exact definitions will appear in later sections). First, the broad notion of 'direct empirical significance' is a matter of existence of transformations of the universe possessing the following two properties (articulated in this way by BB):

(i) the transformation should lead to an empirically different scenario, and

(ii) the transformation should be a symmetry of the subsystem in question (e.g. Galileo's ship).

If such transformations exist, the symmetries of the theory in question are said to exhibit *direct empirical significance* (henceforth DES). BB argue that gauge symmetries cannot exhibit DES in this sense.

GW articulate DES for gauge theory in a manner that fosters their 'Yes' answer to the above. They focus on subsystems as given by regions, and thereby identify transformations possessing properties (i) and (ii) by first formulating the putative effects of such transformations on the gauge fields in these regions. The emerging transformations with DES are in 1-1 correspondence with a group which is a quotient of two infinite-dimensional groups.

In particular, they focus their attention on the group of *relational* DES. In this case, for a given subsystem state s, the relational DES transformations are in 1-1 correspondence with the following quotient:[4]

$$\mathcal{G}_{\mathrm{DES}}^{\mathrm{GW}}(s) \simeq \mathcal{G}(s_{|\partial})/\mathcal{G}_{\mathrm{Id}}.$$

where $\mathcal{G}(s_{|\partial})$ are the gauge transformations of the region which preserve the state s *at the boundary* of the region, and $\mathcal{G}_{\mathrm{Id}}$ are the gauge transformations of the region which are the identity at the boundary (these terms and statements will be made precise in Section 4.2 and appendix A).

I myself will argue for a 'Yes' answer, but will approach the question from an entirely different perspective, treating DES in terms of gauge-*invariant* information. Namely, I will identify DES through the occurrence of *subsystem-holism*.[5]

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[3]In this paper I will focus on just BB's and GW's work, postponing to (Gomes, n.d.) discussion of other authors, especially (Friederich, 2014) and (Teh, 2016), and of related work on the interpretation of gauge theory in general, such as (Myrvold, 2010); cf. footnote 5

[4]Although GW allow for the larger, non-strictly relational quotient group, of all subsystem symmetries quotiented by the interior ones, they do not investigate this larger (infinite dimensional) group, whose physical meaning—if any—is unclear (Wallace & Greaves, 2014, p.86,87).

[5]The term 'holism' has connotations I would rather avoid. Hence in this first appearance, I included the qualifier 'subsystem'. But I will often use the single word 'holism', to mean my 'subsystem-holism'. In the context of gauge systems under study here, my *subsystem-holism* is identical in spirit to Myrvold's *global patchy non-separability* (Myrvold, 2010), which he articulated/developed for the holonomy approach to gauge theories. But I refrain from adopting this nomenclature because (i) I do not focus on holonomies, and (ii) it does not apply to finite-dimensional systems like Galileo's ship. I will briefly comment again on this relation in footnote 66 in section 6.1, leaving the full analysis to the upcoming (Gomes, n.d.).

Subsystem-holism is said to occur when the physical state of those subsystems composing the whole fail to determine the physical state of the whole. In other words, there can be many physical states of the whole which are formed from the same physical states of the individual subsystems. Schematically, if the subsystems are 'sea' and 'ship', and there are equivalence relations, $\sim$, applicable to states of subsystems and the whole, and given the physical states (i.e. "gauge-invariant") $[s_{\text{sea}}], [s_{\text{ship}}]$, there is a many-to-one relation, encoded by the set I:

$$[s_{\text{sea and ship}}]_{(i)} = [s_{\text{sea}}] \cup_{(i)} [s_{\text{ship}}], \qquad i \in I,$$

where $i \in I$. One can thus see that the variety of states of the whole is *not* encoded in either subsystem: it is encoded in the *relations* between the two subsystems, as denoted by $\cup_{(i)}$.

In certain situations, such as in Galileo's ship, there is remarkable order to this variety of physical states of the whole, an order also encoded in the structure of I. Namely, each element of this variety can be transformed into another by a subsystem symmetry which does not extend beyond the boundary of the subsystem. In other words, I carries the structure of finite-dimensional symmetry group: boosts and translations for Galileo's ship and, for the gauge theory, (sub)groups of the Lie group characterizing said gauge theory. Such transformations indeed have a 1-1 relation to those emerging from GW's relational characterization of DES. In this manner, DES becomes a matter of *subsystem-holism*, and holism is enabled by the non-local character of gauge-invariant information which will be investigated here.

1.1 Relevance to other debates

To develop my position, we must revisit one other main debate in the philosophy of gauge, already mentioned at the start of the previous section. Namely, given a theory whose set of universe-descriptions—'states'—is partitioned by a group of symmetries, we can take one of two attitudes:

(a) **Reduction**:– try to write down a quotiented, *reduced*, theory whose states correspond to the cells of the partition; or

(b) **Sophistication of symmetries**[6]:– resist quotienting the given theory, but take its symmetry-related states to be isomorphic.

I'll advocate a third position (c), which applies only in the presence of subsystems.

Thus, for the entire universe, I unequivocally endorse reduction. In the same vein, I will also assume that the theory in question *empirically* discerns two different states s_1 and s_2 of the universe *if and only if* $[s_1] \neq [s_2]$.

For subsystems, the question is more subtle, for there are two perspectives we can take: one from the inside, or intrinsic; and one from the outside, or extrinsic. If we are interested in discriminating between (intrinsically) distinct physical possibilities, then surely those states which cannot be (intrinsically) discriminated are to be counted as one, and so 'reduction' still applies.

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[6]The 'sophistication' position has long been advocated for diffeomorphisms and spacetime metrics (see (Pooley, 2013) and references therein). The nomenclature was originally used for sophisticated substantivalism: points of spacetime may have identity, but this identity comes only through the complex web of inter-relations between different fields of the theory, and is in this way entirely contingent on the state, i.e. not absolute. Some general features of this position have more recently also been suggested for gauge theories (Caulton, 2015; Dewar, 2017; Gomes, 2019). Dewar describes it thus: "Whereas a reduced theory converts a class of symmetry-related models into a single model, sophistication converts a class of symmetry-related models into a class of isomorphic models."

But, when we combine the subsystem with the rest of the world, we are required to exploit subsystem symmetries in a real physical sense: as emphasized first by Rovelli (Rovelli, 2014), reduced representations of subsystems cannot be straightforwardly coupled to each other. For coupling, we need to keep gauge-variant elements in the theory. In (Gomes, 2019), it was similarly argued that reduction should only be endorsed for the entire universe: coupling regional states may require a readjustment of the states as particular gauge representatives of the physical states (cf. appendix B.1),[7] and thus gauge-information should not be entirely eliminated for regions.

Therefore we should delineate a third attitude towards gauge-related subsystems:

(c) **External sophistication and internal reduction of subsystem symmetries**:–fix unique representations of the intrinsic physical states of the subsystems (i.e. from the internal perspective), but then allow these representations flexibility from an external perspective, as e.g. required for the smooth coupling of subsystems. In practical terms, (c) will be implemented by "covariant gauge-fixings". For example, by fixing an isolated subsystem's coordinates to be given by its own center of mass coordinates. The center of mass is still embedded in Euclidean space, and thus still allowed some flexibility to move around from an outside perspective, whenever it comes in contact with another subsystem. This example is very much in the spirit of Rovelli's "squadrons" (see the appendix B.1).

In the context of my approach to DES through holism, the flexibility allowed by option (c), 'External sophistication' for short, is employed for melding the subsystems' physical content into the physical content of the joint state.

In sum, technicalities apart, my main claim is that both Galilean boost symmetry for particle systems and gauge symmetry for certain field theories carry direct empirical significance through subsystem-holism. This holism is *empirically significant*, since it registers physical—i.e. gauge-invariant—differences in the entire system and we take such differences to be also empirically distinguishable. Moreover, the implied underdetermination of the physical state of the whole universe by the physical state of its subsystems is encoded in a *subsystem symmetry*, but only as seen from the 'outside perspective'.

In the following, I will clarify two outstanding previous construals of DES (Brading & Brown, 2004; Wallace & Greaves, 2014), and then identify precisely where I believe these previous papers fall short. In (Brading & Brown, 2004), BB claim gauge theories can have no DES, while in (Wallace & Greaves, 2014), GW articulate their disagreement with BB, advocating the presence of DES also in gauge theories. But neither of these papers proceed in terms of physical content or holism, and neither's construal of "symmetries with DES" can be instantiated in terms of symmetry groups acting on subsystems. Thus, by the criteria of this paper, neither properly characterizes the direct empirical significance of symmetries. I will lay out my alternative and describe how it resolves the defects of BB's and GW's characterization of DES. The plan of the paper is described in more detail in the next section.

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[7]Rovelli focused on the coupling between different types of fields, or, in the finite-dimensional case, on the coupling of two different particle systems (Rovelli, 2014). In (Gomes, 2019) I extended that requirement to the coupling of fields in regions. See also (Dougherty, 2017) for a 'stack-theoretic' argument supporting the problems of reduction for the coupling of regions: the defined notion of separability there requires the preservation of gauge-related representations, to be kept as isomorphic but not identified. That is, I construe Dougherty's view as a defense of position (b) motivated by the composition of subsystems (but using stack-theory).

1.2   Roadmap

In section 2, I first describe the terms of the standard debate on DES. Before moving on to the particular construals of the debate by GW and BB in the context of gauge theory, I need to address one loose pillar on which their work is founded.

Both GW and BB take gauge potentials as a starting point. But, I claim, they both also neglect the main feature of gauge potentials: the non-locality of its gauge-invariant content. It is this feature which licenses holism in gauge theories, and it thus deserves scrutiny and motivation.[8] In section 3 I will thus defend the use of gauge potentials from various perspectives. The goal is identify a particular type of non-locality as the motivation for the use of gauge potentials.

Having established that a particular type of holism lies at the root of theoretical gauge symmetries, I then proceed to a more careful analysis of the debates on DES. In section 4 I will formalize the GW argument for DES in Yang-Mills and describe its shortcomings.

Ignoring all the caveats described within that section, we indeed find that the DES symmetries should be isomorphic to a finite-dimensional quotient of groups,[9] still in line with my main claim that subsystem-holism is encoded in a 'regional rigid' group.

But in terms of the gauge-invariant data (cf. Definition 2 in section 2.4.1) this amounts to only a pyrrhic victory. Since the group of DES symmetries can only be characterized as an isomorph of a quotient group, it is not precisely a 'regional rigid' group of symmetries: as I show in appendix A, there is in fact no precise way to characterize the action of the quotient group over the region. This is in line with GW's view that since global symmetries are members of the group of local symmetries, one cannot have DES for the smaller group of global symmetries and none for the larger group of local symmetries. But this argument ignores that in certain circumstances there is a natural way to single out, or "pluck" out, or sieve, the global symmetries from the local ones. Moreover, this "plucking" can be done in a gauge-invariant way, consistently with my proposal.

I will base DES on the residual conjoining variety in states of the universe ("whole") arising from "gluing" physical data: made precise in Definition 2. For this we must deal appropriately with boundaries in gauge theories, which we do in section 5. The main obstacle we need to overcome is to distill regional gauge-invariant data whilst retaining the flexibility required to glue such data into a universal gauge-invariant description.[10] But the obstacle was thoroughly described in (Gomes, 2019), and already overcome in (Gomes, Hopfmüller, & Riello, 2019;

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[8] Unfortunately, the choice of restricting investigations to electromagnetism is standard in the literature of philosophy of physics. Non-Abelian theories are oftern portrayed as solely important for the physicist, and as not qualitatively novel enough to merit study by philosophers. But electromagnetism is, also in a *qualitative* sense, a degenerate case of Yang-Mills. The action of the group U(1) is independent of the state A, and this independence effaces many interesting structures that can only be fully discerned in the more elaborate non-Abelian case. I only introduce the bare minimum notation for discussing the more general Yang-Mills case, and, at the cost of more content, relegate the more complete treatment of Yang-Mills to appendix C.

[9] In the spacetime asymptotic limit, this quotient is essentially the same as that described by Teh in (Teh, 2016) to embody DES. There is one obvious problem here: as recent work illustrates (Strominger, 2018) there are many lingering intricacies involved in taking the asymptotic limit (see also (Riello, 2019; Barnich & Brandt, 2002) for descriptions of spacetime asymptotics which are more in line with the tools of the present paper). Second, my main interest is in a relational construal of DES, and asymptotics are quite problematic in that regard, since they establish a reference system which has no dynamics of its own. I will thus restrict the discussion to finite regions.

[10] I should stress that the physical states will be extracted solely from smooth gauge fields. I am not allowing any physical state which possesses no smooth representative into the picture.

Gomes & Riello, 2019); so I briefly summarize the obstacle in appendix B.1 and how it is overcome in a second appendix B.2.

Generically,[11] the gluing *is* uniquely fixed by the regional gauge-invariant data. But, in certain cases, there *is* a subsystem-holism, giving way to a variety of universes. This variety is in 1-1 correspondence with a rigid group of transformations (and also with the occurrence of conserved charges) which do possess gauge-invariant significance. This correspondence thus establishes my main claim.[12] Section 5 also completely characterizes the occurrence of subsystem-holism for electromagnetism as coupled to a Klein-Gordon scalar field, vindicating my main claims in that specific context.

In section 6, I conclude.

2 Direct empirical significance

I start, in section 2.1 by reporting the construal of DES in terms of properties of certain transformations of the Universe. This description of DES immediately runs into some cumbersome notation when applied to gauge theories. Therefore, in section 2.2 I introduce a new terminology which better distinguishes the relevant categories of transformations. Then, having got the right nomenclature for addressing DES in the context of gauge theories, in section 2.3 I apply it to re-express the debate in these better terms. In section 2.4, I then proceed to offer an appetizer of my criticism to BB and GW's construal of DES, and supplant those construals with my own. Thus here I describe the relation between *regional (or subsystem)-holism and DES*.

2.1 DES as a transformation of the universe

BB frame the definition of DES in terms of two conditions. First, a transformation cannot be a symmetry of the entire universe, otherwise it would not have any direct empirical significance. But second, it needs to act as a symmetry for subsystems, otherwise the transformation in question could hardly be called a symmetry. Thus Brading and Brown define:

Definition 1 (Direct Empirical Significance (DES) as a transformation)
To bear direct empirical significance a transformation must satisfy the following conditions:[13]

1. **Transformation Condition**: *the transformation of a subsystem of the universe with respect to a reference system must yield an empirically different scenario. In our words: the transformation in question is not a symmetry of the world as a whole.*

2. **Subsystem Symmetry Condition**: *The interior evolution of the untransformed and transformed **subsystems** must be empirically indistinguishable.*

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[11]This is true for simply-connected manifolds; a necessary qualification of this statement, since *non*-simply-connected topologies should be, in some sense, generic.

[12]All the mathematical developments for these results were originally obtained in (Gomes & Riello, 2019) (see section 4 in particular).

[13]Both Teh and GW add a condition of dynamical isolation between the two subsystems. Teh takes this to justify an asymptotic treatment for the subsystem in question. We won't need to make this isolation condition explicit: it emerges from the criteria. More on this in section 5.2.

> *In our words: the transformation should count as a symmetry when applied to the subsystems composing the entire system.*

Thus in the example of Galileo's ship, the entire system—both ship and sea—is in different states if the ship is heading through calm waters towards the North-west at 50km/h or towards the South at 20km/h. The entire system thus satisfies the first condition (*Transformation*). Nonetheless, inside the cabin, you would not be able to distinguish the two scenarios: so the subsystem satisfies the second condition.[14]

In Definition 1, I made explicit reference to subsystems (plural) of the universe. This assumption is usually implicit in the behavior of the reference subsystem ("the sea"), whose state (evolution, etc.) is assumed to be untouched by the transformation. It is an idealization contingent on some approximate dynamical isolation of the subsystems under study. (Kosso, 2000) recognizes the importance of an untransformed reference system for the symmetry transformation to be observable, saying (p. 91): "An internal, global transformation is, in principle, observable. As with an external transformation, the change requires some unchanged system of reference." Similarly, (p. 87): "To observe the transformation is to observe both the unchanged reference and the changed system." . This condition is agreed upon by Brading and Brown (p. 646) (Brading & Brown, 2004). As we will see in section 5, I won't need to have all subsystems of the universe be invariant, but the reference system—standing like a collar or buffer between the given subsystem and the rest of the world—will be assumed to be invariant as a physical system. GW also only investigate cases in which such a reference is in place (see e.g. p. 86 (Wallace & Greaves, 2014)).

We can now restate BB's intuition as follows. According to section 1.1 symmetry transformations of the entire universe cannot have empirical siginificance; they can sit comfortably with the 'gauge as redescription', or 'reduction' view. Therefore we must, in a certain sense, 'localize' the transformation to observe it. But if we make gauge transformations locally specifiable, can't we always match a subsystem gauge transformation, taken as a redescription, to the identity on the surrounding environment, so as to give a mere redescription of the universe as a whole, thereby violating the first condition's (*Transformation*) demand for an empirically distinguishable scenario?

2.2 Two distinctions

At this point in the discussion, standard terminology gets in the way of clarity. When used in conjunction with subsystem-Universe distinctions, the words 'local' and 'global' acquire other possible meanings, and and may pull intuition in different directions (see p. 648 of (Brading & Brown, 2004)). To continue the discussion, it is useful to introduce a nomenclature that distinguishes these meanings.

- *Universal*: A universal transformation is one that applies to the world as a whole. In the field-theoretic case considered in this paper, a universal transformation corresponds to a smooth gauge transformation supported on the entire space(time) manifold. The set of universal transformations may depend on an infinite or finite number of parameters.

- *Regional*: A regional transformation is one that applies only to a subsystem

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[14] Again: these symmetries also entail, by Noether's first theorem, associated conserved charges, such as the linear momentum for the velocity boost.

of the world.[15] In our examples, the subsystem will consist of a (sub)region of the world, i.e. a proper bounded submanifold of the spacetime manifold. The set of regional symmetry transformations may also depend on an infinite or finite number of paramaters.

- *Malleable (aka 'local')*: A malleable transformation depends on an infinite number of parameters: e.g. it is of the form of an arbitrary smooth function over a given manifold. Here, the usual label is 'local'. But using 'local' invites confusion with the above category, 'regional'. I will therefore prefer the term 'malleable'. A malleable symmetry transformation can be either regional or universal.

- *Rigid (aka 'global')*: A 'rigid' symmetry transformation depends only on a finite number of parameters. This is to be contrasted with *malleable*. The usual label is 'global'. But again, this term invites confusion, namely with the above category, 'universal'. So I will therefore prefer the term 'rigid'. A rigid symmetry transformation can be either regional or universal.

Therefore a symmetry transformation may lie in any combination of the above categories: regional and rigid, regional and malleable, universal and malleable, or universal and rigid.[16]

Regional transformations are under-studied in the physics literature, but are known to hide many surprises: see e.g. (Regge & Teitelboim, 1974; Balachandran, Chandar, & Momen, 1996; Donnelly & Freidel, 2016; Gomes et al., 2019; Gomes, 2019). As to universal transformations, the rigid ones are familiar; they are associated with the standard treatments of Noether's first theorem, and thus correspond to conserved charges (Noether, 1971; Kosmann-Schwarzbach & Schwarzbach, 2011; Olver, 1986; Brading & Brown, 2000). The malleable universal transformations are associated to constraints, or relations between the equations of motion (such as the Hamiltonian constraints or the Bianchi identities of general relativity and the Gauss constraint in electromagnetism).[17]

2.3 The debate reexpressed

Using this nomenclature, we can re-express Definition 1 and better address the subtleties of applying it to gauge symmetry. Thus Definition 1 says that a DES-bearing transformation is one which is not a universal symmetry and yet one whose restrictions are regional symmetries.

Finite-dimensional theories, i.e. ones which do not involve fields, generally only have rigid symmetries (such as translations, etc.). In those cases, the strictly regional ones are agreed by all to be *directly* empirically distinguished from the universal ones—by giving different values for appropriate physical quantities, viz. relational quantities relating the transformed subsystem to the rest of the universe.

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[15]This is often called 'subsystem' symmetry (Wallace & Greaves, 2014; Teh, 2016), but here I employ this alternative nomenclature because my interest will be solely in subsystems formed by spacetime restriction.

[16]Teh (Teh, 2016) labels the transformations with underlining: as <u>local</u> (meaning regional), and *local* (meaning malleable), and <u>global</u> (meaning universal), and *global* (meaning rigid); but I feel this also invites confusion.

[17] In the Hamiltonian treatment, the symmetries are represented as flows in the constrained phase space, with orbits being the integral manifold to which the flows are tangent (see (Earman, 2003) for a celebration of the virtues of the Hamiltonian treatment). In the Lagrangian treatment, symmetries are represented as orbits in configuration space (Lee & Wald, 1990). A powerful formalism which lies in between the Hamiltonian and Lagrangian is the *covariant symplectic formalism* (Lee & Wald, 1990; Crnkovic & Witten, 1987). It is most useful in discussing canonical (or Hamiltonian) features of a system while retaining easy access to spacetime covariance.

This is of course what Galileo's ship illustrates. In this case, the clear distinction between universal and regional rigid symmetries is illustrated in an uncontroversial case of DES.

But the situation for malleable symmetries seems different. These are the symmetries which act on spacetime fields, and can be independently specified by an action/transformation at each spacetime point. In certain examples, their generators are spacetime vector fields; they are in others Lie-algebra-valued scalar fields, acting on an internal space over each spacetime point.

Indeed it is easy to imagine a malleable symmetry acting on a region of spacetime and not on another, which can thereby serve as the reference system. Our intuition, which will be spelled out in section 4.2, is that to keep the reference system intact, the malleable transformation should smoothly tend to the identity at the boundary between the regions, lest it create discontinuities in the fields. But then, it seems (as in the end of section 2.1) we could suitably extend any regional malleable symmetry to the rest of the Universe simply with the identity transformation. The conjunction of the two regional transformations—one that tends to the identity at the boundary and the other the identity on the rest of the universe—would be a universal malleable symmetry, which, according to the usual view of gauge as redescription, cannot have empirical significance.

According to BB (Brading & Brown, 2004), this is precisely the case: the malleability of gauge symmetries prevents them from relating different states of affairs. In other words, due to their malleability, gauge symmetries would fail to satisfy Definition 1's first condition (*Transformation*), i.e. the requirement of an empirically distinguishable scenario, and so would have no empirical significance in the way that regional rigid symmetries do. As BB write:

> "Thus, a transformation applied to one subsystem will involve the other subsystem, even if only because the transformation of the gauge field goes smoothly to the identity. In conclusion, there can be no analogue of the Galilean ship experiment for local gauge transformations, and therefore local gauge symmetry has only indirect empirical significance (being a property of the equations of motion)." (p. 657)

GW oppose this conclusion (Wallace & Greaves, 2014). They believe there is a set of symmetry transformations, called by them 'non-interior, boundary-preserving', which do have direct empirical significance.[18] Their position will be fully clarified in section 4; it is based on attempts to directly instantiate the content of Definition 1. I also will oppose BB's conclusion for roughly the same reasons as GW.

2.4   My own position in the debate: rigid variety

Overall, I will argue for a position not considered by either the GW or the BB camp: an appropriate selection of rigid regional symmetries—but not all the malleable ones!—can retain direct empirical significance (DES) in both the finite-dimensional case and in the field-theoretic case. In very specific circumstances, and according to a precise method, the rigid symmetries will be plucked from the malleable ones and will give rise to a transformation with DES as per Definition 1.

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[18]GW allow in principle for non-boundary-preserving symmetries, but, by their own admission, do not investigate them. The reasons we will neglect them is because they don't conserve a reference system (see (Kosso, 2000), p. 87).

I will be led to this analysis by showing how, in the Yang-Mills mathematical formalism, the GW characterization of DES stumbles. This failure is due to complications that the presence of boundaries impose on gauge theories.[19] Intuitively, these problems arise because gauge-invariant observables are to some degree non-local, whereas boundaries are localized objects.

But once the correct tools are in place, we can bypass this stumbling block, proposing an alternative characterization of direct empirical significance in terms of the difference between regional and universal *gauge-invariant* states; a difference between the whole and the sum of the parts in gauge theory. Indeed, I believe such a path to DES is more cogent than the phrasing used in Definition 1; and yet, as we will see, it is in a sense equivalent to it in the case of Yang-Mills. It is moreover entirely compatible with the characterization of gauge as descriptive redundancy.

In the first subsection here, 2.4.1, I will give a brief summary of the definition I will employ and results obtained through it. In the next subsection 2.4.2, I will provide an appetizer regarding the differences between my approach and that of BB and GW.

2.4.1 Rigid variety and relational DES

In the following, to make matters concrete, the field-theories I will focus on are general classical Yang-Mills theories in the presence of matter. The spacetime fields in question will be the standard, smooth gauge potentials, A, and charged scalars, ψ, valued in the appropriate vector spaces, which I discuss in more detail below, in section 3.3. I denote the doublet of these two fields by: $\varphi = (A, \psi)$, and the space of such doublets by $\Phi \ni \varphi$. The subsystems will consist of regions in the manifold, to which the fields are restricted. This initial set-up is standard in the debate about the direct empirical significance of gauge symmetries, and is applicable to all of the approaches considered here. Within this context, I define:[20]

Definition 2 (Regional holism) *Given a manifold Σ, that is decomposed as $\Sigma = \Sigma_+ \cup \Sigma_-$, along the boundary $\partial\Sigma_\pm = \Sigma_+ \cap \Sigma_- =: S$; given the regional fields, $\varphi^\pm$ supported on $\Sigma_\pm$, **regional-holism** occurs if the joint gauge-invariant contents of $\varphi^\pm$ are compatible with more than one gauge-invariant content of φ, i.e. of the field over the entire manifold Σ.*

More formally, we label each legitimate/physically possible composition of the two given regional states to form a physically possible universal state by i, with i belonging to some index set I, which can depend on the constituent states. So *regional-holism* corresponds to the set I having more than one element. Thus, denoting the gauge-equivalence class by square brackets, indicating the ith composition of states by $\cup_{(i)}$ and the resulting universal state as $\varphi^{(i)}$, we write

$$[\varphi_{(i)}] = [\varphi^+] \cup_i^S [\varphi^-], \quad i \in I \quad \text{with} \quad [\varphi_{(i)}] \neq [\varphi_{(i')}] \quad \text{iff} \quad i \neq i' \tag{2.1}$$

or, in terms of the standard Yang-Mills potential and matter fields:

$$[A^{(i)}, \psi^{(i)}] = [A^+, \psi^+] \cup_i^S [A^-, \psi^-], \qquad i \in I, \tag{2.2}$$

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[19]Boundaries in gauge theories are a topic on their own, and bring a suite of new and interesting problems; see e.g.: (Regge & Teitelboim, 1974; Balachandran et al., 1996; Casini, Huerta, & Rosabal, 2014; Donnelly & Wall, 2015; Donnelly & Freidel, 2016; Gomes & Riello, 2017; Geiller, 2017; Speranza, 2018; Gomes & Riello, 2018; Gomes et al., 2019; Camps, 2019; Gomes & Riello, 2019; Gomes, 2019).

[20]For the relation to Myrvold's "patchy separability" see footnotes 5 and 66.

So here i is neither a spacetime index nor necessarily related to a Lie-algebra index: it is just an element of an index set $I([\varphi^{\pm}])$—whose dependence on the given pair $[\varphi^{\pm}]$ will be henceforth omitted—and $\cup_i^S$ represents a valid *gluing*, i.e. composition, of the two gauge-invariant data $[\varphi^{\pm}]$ along S. The global fields φ are in the same class of differentiability as the regional ones (albeit the latter will lie on manifolds with boundary). If I is empty there is *no* possible gluing, i.e. the regional gauge-invariant states are incompatible and cannot conjoin into a universal physical state (*regional incompatibility*). If I has a single element, the gluing is unique, and then we say there is *regional determination*. If otherwise, i.e. if I has more than one element, the universal physical state is underdetermined by the regional physical states and we say there is *regional-holism*.[21] In this case, we will also say the universal state possesses *residual variety*.

Note that as it stands Definition 2 is in line with both the Galileo's ship analogy and with the idea of gauge transformations as mere re-description. The analogy states that, in general, the physical states $[\varphi_{\text{sea}}]$ and $[\varphi_{\text{ship}}]$ can be glued in a variety of ways. Definition 2 is *essentially* relational: any variety will be solely a variety of relations between the subsystems. In the ship case, this variety is classified by boosts, i.e. I has a 1-1 correspondence with the group of Galilean boosts. This example illustrates how a transformation taking $[\varphi^{(i)}] \to [\varphi^{(i')}]$ could recover DES as prescribed by Definition 1.

In the following, we will see that there are circumstances in which Yang-Mills subsystems indeed admit transformations with DES through *regional-holism* in this full sense. Namely, for certain regional gauge-invariant data which can be glued together i.e. composed to give a physically possible universal state, there remains a *residual variety* of universal gauge-invariant data obtained from this gluing. This variety is parametrized by regional *rigid* transformations, not regional malleable transformations, and is encoded by the external action of a finite-dimensional Lie group on a subsystem.

In other words, in some cases there is a 'regional rigid symmetries'-worth of universal states which are *regionally* gauge-equivalent to each other but have a *relational* physical distinctness, i.e. are physically distinct (not universally gauge-equivalent) due to relational differences. The particular structure of I will depend on the regional field content in each case. It is the field content which identifies and "plucks out" the rigid subgroups of the full infinite-dimensional malleable group that are to be endowed with DES.

In Yang-Mills theory, regional-holism will occur only in conjunction with those conditions which are necessary for the existence of conserved global charges, as related to the rigid subgroup I (namely, only for reducible configurations, in which case I is the group of reducibility parameters (Barnich & Brandt, 2002)). This procedure thus establishes a link between what is known as an indirect consequence of gauge—the conservation of charges—and a 'direct' one (DES).

2.4.2 The differences to GW and BB: an appetizer

I start this appetizer by noting that Definition 2's characterization of DES in terms of physical i.e. gauge-invariant, states was unavailable to both (Wallace & Greaves, 2014; Brading & Brown, 2004). GW are more explicit in taking the subsystem states to uniquely determine the universal state:

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[21]In the nomenclature of (Myrvold, 2010), *regional determination* would be 'global separability', but *regional incompatibility* is not there countenanced (cf. footnotes 5 and 66).

> "Firstly, in doing so we make the assumption that knowing the state of the subsystem and its environment suffices to specify the state of the total system. [...] For example, it is true for Yang-Mills gauge theories in the connection formalism but not in the holonomy formalism." (p. 67)

While their assumption holds for the gauge and matter fields, it does *not* hold for their physical content, i.e. for their equivalence classes.[22] And as we have seen, it also would not hold for finite-dimensional systems, such as Galileo's ship. Of course, the inverse relation *does* hold for all of these systems: given the state of the whole, we can uniquely find the state of the subsystems. But what is being studied here is not a decomposition of the whole into the parts, but a *subsystem-first* viewpoint of composition of the whole: given the subsystems how do we put them together?

Although I have focused on GW's statements above, BB's error has similar roots (also according to GW). Echoing the above GW quote, we would have been unsurprised to find in (Brading & Brown, 2004) a similar one: "we make the assumption that knowing the *gauge transformation* of the subsystem and its environment suffices to specify a *gauge transformation* of the total system." Indeed, the message of this fictitious sentence is essentially endorsed in (Brading & Brown, 2004) and forms the core of GW's criticism, as we will see. Again the inverse relation is unproblematic: a universal gauge transformation *does* uniquely yield the regional gauge transformations.

I will call GW's quoted assumption, that the subsystem states uniquely determine the universal state, '*Antiholism*'.[23] As will be explained below in section 3 (section 3.2 in particular), the main lesson of gauge theory, indeed the main lesson of the Aharonov-Bohm effect (Belot, 1998; Earman, 2019), is that the use of gauge fields belies a type of non-locality (of type L9 in the categorization of (Earman, 1987)). 'Antiholism' at least partly rejects this lesson.

3   Gauge theory

This section will run as follows: First, in section 3.1 I note that admitting many different *local* descriptions of the same underlying *non-local* physical content is an interesting fact about the system itself. It is an often understated *indirect* significance of gauge, and it belies a particular brand of non-locality that results from spacetime covariant, generalized Gauss's laws. Even if the Aharonov-Bohm effect evinces non-locality more directly, non-locality is already implicit in the Gauss laws and in the lack of magnetic monopoles. Using variables that naturally incorporate these laws—i.e. the gauge potentials—*incidentally* gives us a more palatable explanation of the Aharonov-Bohm effect. This perspective is defended in section 3.2, in which I demonstrate the necessity of employing this particular sort of gauge redundancy for such theories. This argument shows why equivalence classes of gauge-potentials parametrize the physical content of the theory; it is an argument

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[22]In electromagnetism, given the space of loops (smooth embeddings $\gamma : S^1 \to \Sigma$), one can form a basis of gauge-invariant observables by the holonomies, $\exp\left(i \int_\gamma A\right)$ (this can be accomplished more generally for non-Abelian theories using Wilson loops, (Barrett, 1991)). This is why the GW criteria (cf. section 4.1) does not apply for formulations of theories using holonomies.

[23]Antiholism bears more than a little family resemblance with 'regional *determination*'—defined after Definition 2 as those cases in which I consists of a single element. The differences are the following: the first, and obvious one is that antiholism is an assumption. But even if we were to phrase 'regional determination' as an assumption, we wouldn't capture the fact that one is stated in terms of the physical content, $[\varphi^\pm]$, and the other in terms of the states $\varphi^\pm$. I therefore have decided to keep the terminology separate.

for holism as the meaning of theoretical gauge redundancy. Finally, in section 3.3 I fix some notation for the gauge theories I will consider in this paper.

3.1 Gauge redundancy as physically significant

Apart from this paper's main topic of DES, it is important to note another more general way in which gauge has a physical significance. This is compatible with the idea that gauge is a matter of descriptive redundancy; and it is not contentious, though often it is not articulated. Namely, it is a physically significant fact about a system that it *can* be described by a gauge theory: i.e. that it admits different descriptions related by gauge transformations (of whatever kind is appropriate to the system).

Such an understanding of gauge theories was in (Gomes, 2019) put at the fore-front: gauge theories were characterized as systems admitting many (infinitely many!) local *perspectives* on the same non-local state of affairs.[24]

Indeed, such a broad characterization yields an analogy with usual (i.e. non-gauge) symmetries, such as symmetries of spacetime, e.g. the Galilean group, or even just of space e.g. the Euclidean group. The fact that these symmetries fix, i.e. preserve, the relevant geometric structures (i.e. an element of the Galilean group fixes the spatial and temporal metrics, and the flat connection, of neo-Newtonian spacetime M, and an element of the euclidean group fixes the spatial metric of Euclidean space E = the affine space of $\mathbb{R}^3$) implies that a specific choice of one of the coordinate charts adapted to the structure (a specific inertial coordinate system on M, a specific cartesian coordinate system on E) introduces items that one might call unnecessary or even redundant. For it is arbitrary—a "breaking of the symmetry"—to make *this* choice of origin, *this* choice of spatial axes etc, when others are equally good. But of course, all agree that it is physically significant, indeed very informative, that the system supports such a description: i.e. that it can be described with such a coordinate chart, that is related to other similar charts, in the way parametrized by the symmetry group.

Similarly, one should surely say, about a gauge theory. Thus in the analogy, the analogue of a specific choice of a coordinate chart is a gauge-fixing. And just as it is physically significant that a system supports descriptions by inertial coordinate systems related by the Galilean group, or descriptions by cartesian coordinates related by the euclidean group: so also it is a physically significant fact about a system that it *can* be described by the various gauge-fixings, and that these are related in such and such a manner (given by the group $\mathcal{G}$ of gauge transformations).[25]

To sum up:— The fact that a system supports a variety of equivalent procedures for describing it is a physically significant fact about the system. And it is a fact one is liable to forget, if one focuses only on one such procedure, or if one focuses only on the equivalence (and thus the arbitrariness, or *redundancy*, associated with any one procedure). And this moral applies equally to e.g. space and spacetime symmetries, and to gauge symmetries. As we will see it is this notable fact about gauge systems that will crucially enter my constructions.

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[24] As noted in section 1.1 Dewar has proposed a similar view (Dewar, 2017), which he dubbed "sophistication', but without the focus on subsystems, which is where I believe the view reaps rewards.

[25] As a notable exception of the literature, in (Brading & Brown, 2000), p. 99, this fact is highlighted, but only for the group of global gauge transformations.

3.2 Why use gauge potentials?

We will take the simple example of electromagnetism to illustrate the non-trivial fact that there are *non-local* physical facts that *can* (and should) be expressed in a manner that admits different *local* descriptions.

In other words, here I will show why we must we use A_μ (where μ, ν, etc. stand in for abstract spacetime indices) as opposed to the good old electric and magnetic fields, E and B. Without getting into all of the intricacies of the debate, I will provide two simple inter-related arguments which can appropriately be labeled 'mathematical' and 'empirical'.

3.2.1 The mathematical argument

The standard line of argument for introducing the gauge potential starts from considerations of global symmetry: namely by localizing the global symmetry of the electric potential. A transformation of the electric field which is the gradient of an arbitrary function of space and time can still be a symmetry because the magnetic potential could always adjust to compensate for the putative dynamical effects of the transformation. Of course, not all dynamics are compatible with such symmetries. Indeed, in hindsight, many general properties of Maxwellian dynamics are straitjacketed by the use of the gauge potential as a fundamental variable. Namely, the use of this variable implies the validity of the Bianchi identities, $\partial_{[\rho} F_{\mu\nu]} = 0$, and, from an effective field theory point of view, the first order contribution to the equations of motion must be the sourced Maxwell equations, $\partial^\mu F_{\mu\nu} = j_\nu$ (see (Maldacena, 2016) for a quick derivation). With a retrodictive lens, we could say that

> Gauge symmetry works for electrodynamics in the sense of accurately predicting the nature of the interaction, and it is this success that motivated applying the gauge principle in other contexts, in particular to the nuclear interactions. (p. 94) (Kosso, 2000)

To many, this sounds like reason enough to introduce gauge fields, but I will try to further buttress those reasons: we can mathematically motivate the introduction of the gauge fields in several alternative ways, and in the following we will look at simplified versions of different arguments than the above.

Since the standard line above starts with the electric field, let us instead focus on another one of Maxwell's equations, $\nabla \cdot \vec{B} = 0$. This equation says that magnetic force lines must always close in space. So we cannot build any magnetic field we would like in the lab, even for an instant. In other words, the equation constrains the magnetic initial data of a dynamical system. This constraint is non-local, as is the Gauss law $\nabla \cdot \vec{E} = J$: to find possible initial data, one must solve a strictly spatial partial differential equation, which therefore involves an integral over its spatial domain.[26]

But Helmholtz and Poincaré have taught us that, at least in simply connected domains, we can write any divergenceless $\vec{B}$ in terms of an *unconstrained* potential

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[26] To be concrete: we mostly ignore the influence of electric charges behind Andromeda because they largely cancel out; if there were immense charges there now, they would affect the possible electric fields in my room. This sort of non-locality can be called a *quasi-locality*, as we can replace the knowledge of charges behind Andromeda for knowledge of boundary conditions around the system of interest. But these boundaries are not themselves freely specifiable, and so on. See (Earman, 1987) for a discussion of this type of non-locality, which he calls 'type L9' and (Gomes & Riello, 2019) for a categorization of which degrees of freedom embody which types of non-locality.

$\vec{A}$, namely as: $\vec{B} := \mathrm{curl}\vec{A}$. The guiding principle for introducing the gauge potential in this manner is that we want to use only local variables, i.e. variables that describe possible physical situations without further constraints. If we want to maintain this principle—while keeping with the origin of $\vec{A}$ as a local explanation for the lack of magnetic monopoles—then it is clear that the relevant symmetry of our variables should be $\vec{A} \to \vec{A} + \mathrm{grad} f$ for a smooth function f, and *not* $\vec{A} \to \vec{A} + \vec{A}'$ such that $\mathrm{curl}\vec{A}' = 0$, which would again be a non-local constraint on $\vec{A}'$.[27] In other words, if we accept the guiding principle above—i.e. of local description of possibly non-local facts—on its own terms, as the sole reason for employing redundant variables in the first place, it is clear that, independently of topology, the relevant symmetry should be the more restricted $\vec{A} \to \vec{A} + \mathrm{grad} f$.

If we now also take into account relativity, or, more elementarily, the 'moving magnet and conductor problem' (as in the opening of Einstein's 1905 paper), we know that boosts of frame will convert (parts of) the magnetic field into parts of the electric field. The only way to accommodate both a description of $\vec{B}$ in terms of $\vec{A}$ and the above conversions under boosts is to have $\vec{A}$ form the spatial components of a single covariant 4-dimensional object: which requires us to see the electrostatic potential as a time component for $\vec{A}$, forming A. Indeed, this derivation stands in for a more general fact: if the electric and magnetic forces are to form facets of the same 4-dimensional object, one *must* employ the gauge fields.[28] And once we have introduced A as the fundamental variable, we are back to the general arguments above straitjacketing the dynamics.

Conversely, to write a Lagrangian or Hamiltonian formulation yielding the Maxwell equations one must employ the gauge potentials. That is, there is *no* local action yielding Maxwell equations with sources which can be written without employing (the equivalent of) gauge potentials.[29] The same point can be made in the Hamiltonian formulation.[30]

Still in the 'converse' direction, once one has such an action, there is necessarily a joint transformation of variables (to be fully specified in section 3.3), $A \mapsto A^g, \psi \mapsto \psi^g$, which leaves the value of the action (and the equations derived from it) invariant. And this gauge *variance* of the fundamental variables is deeply significant for a much wider choice of theories: by employing A as our basic variable, as opposed to gauge invariant fields (such as $\vec{E}, \vec{B}$), we are assured to write a spacetime Lorentz-invariant Lagrangian *which will automatically lead to the validity* of some local charge conservation law, such as the one coming from the (covariant) Gauss laws,[31]

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[27] This latter symmetry is erroneously taken to describe the relevant gauge orbits of electromagnetism by some of the literature, see e.g. (Earman, 2019, Sec. 3.3) and (Belot, 1998, P. 543) (cf. footnote 39). I will postpone a full discussion of this point to (Gomes, n.d.).

[28] This general fact is reflected in the equations of motion: an explicitly Lorentz covariant formulation of the Maxwell equations will display the electromagnetic curvature tensor $F_{\mu\nu}$, which could be taken as a fundamental gauge-invariant variable, without recourse to A_μ. But it is important to note that $F_{\mu\nu}$ is not just any anti-symmetric two-tensor; it must also satisfy the Bianchi identities, $\partial_{[\rho} F_{\mu\nu]} = 0$ (where the subscript square brackets denote anti-symmetrization). Therefore, at least in topologically simple domains, $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. The equations of motion of non-Abelian Yang-Mills theory employ an even more overt dependence on the gauge potential variables, A.

[29] See theorem 3.3.5 in (Bleecker, 1981) for a closely related argument which leads to this conclusion.

[30] Indeed, the properties of the Poisson bracket already evinces non-locality: in 3 space dimensions (with i, j, k, etc standing in for spatial indices), $\{E^i(x), B^j(y)\} = \epsilon^{ijk} \partial_k \delta(x, y)$ is not a canonical Poisson bracket and the presence of the derivative on the right-hand-side is the signature of a nonlocal behavior.

[31] Satisfied by those currents which generate the global gauge transformation (eg. the global phase shifts of the wavefunction for charged scalars). That is, using the gauge fields as variables of the action functional has a great advantage: we automatically ensure conservation of the associated Noether current in a spacetime

without having to add non-local constraints on the initial data.

In sum, the Gauss laws and the lack of magnetic monopoles indicate a more natural choice of fields to represent electromagnetic phenomena; a choice which naturally leads to these laws without having to invoke non-local constraints on physical facts.

3.2.2 The empirical argument

The action functional should carry all the physical content of the theory, both classically and quantum mechanically.[32] Due to the gauge-symmetry of the action, which employs A as a fundamental field, it is only the equivalence class $[A]$, defined by $A \sim A^g$, which matters for physics.

I will define the state space of the gauge fields as $\mathcal{A} \ni A$, with the respective space of equivalence classes being given by $[\mathcal{A}]$. So, is the empirical content of the equivalence classes $[A]$ identical to that of $\vec{E}$ and $\vec{B}$? Or more restrictedly, is the empirical content of the spatial components $\vec{A}$ identical to the empirical content of the magnetic fields? No, it is not.

The fact that they are not empirically identical is usually illustrated by the Aharonov-Bohm effect (see e.g. (Belot, 1998)). But I will not need a full description of that famous experiment. Instead, I will resort to the attitude towards empirical difference already stated in section 1.1: if a theory employs the state space Φ which is partitioned into equivalence classes, then for $\varphi, \varphi' \in \Phi$, a difference in $[\varphi] \neq [\varphi']$ is *in principle* an empirical difference. So the question is: is there a difference between $[\mathcal{A}]$ and the space of magnetic fields?

We know the relation between the space of $\vec{A}$s and of $\vec{B}$s: for each $\vec{A}$ there is a $\vec{B}$, and two gauge-related $\vec{A}$s correspond to the same $\vec{B}$. But, we want to ask: can two different equivalence classes, $[\vec{A}] \neq [\vec{A}']$, correspond to the same $\vec{B}$? If they can, equivalence classes of gauge potentials will distinguish between physical situations that the magnetic fields will not.[33]

Indeed they can, and these examples are easy to find: Let $\vec{A}' = \vec{A} + \kappa$, for κ a one-form such that $\mathrm{d}\kappa = 0$, (where d is the exterior derivative) but also such that there is no smooth potential function for κ, i.e. no f such that $\mathrm{d}f = \kappa$. Such a κ is closed ($\mathrm{d}\kappa = 0$) but not exact ($\kappa \neq \mathrm{d}f$), and, once we allow the spatial manifold to be non-simply connected, such κ's are in 1-1 correspondence with a topological

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region. In the covariant symplectic formalism this is extremely easy to see: the variation of a Lagrangian density dependent on the field φ gives: $\delta L = \mathrm{EL}\delta\varphi + \mathrm{d}\theta$, where θ is the covariant symplectic potential (a 1-form on field space Φ), EL are the Euler-Lagrange densities, $\delta\varphi$ is the variation of φ and d is the spacetime exterior derivative. If L is gauge invariant, on-shell of the Euler-Lagrange equations we obtain $\mathrm{d}(\theta(\delta_\xi\varphi)) = \mathrm{d}j_\xi = 0$, where ξ is an infinitesimal gauge transformation, $\delta_\xi\varphi$ is the effect it has on the field φ, and $j_\xi = \theta(\delta_\xi\varphi)$ is the Noether current density. The fact that the total derivative of the Noether current density is zero expresses conservation over a spacetime region. Conversely, the Noether current density is only conserved for all ξ if the sourced equations of motion are satisfied, e.g. for Yang-Mills theories, $D^\mu F_{\mu\nu} = J_\nu$ (where J_μ is the matter current density and D the gauge covariant exterior derivative). See (Lee & Wald, 1990; Crnkovic & Witten, 1987; Gomes et al., 2019) for more details.

[32]With the possible addition of boundary conditions and a path integral measure—whose choice is equivalent to fixing the factor ordering ambiguities in canonical quantization. For a renormalizable theory, a lack of local gauge-invariance of the measure can have catastrophic consequences; namely, the presence of local anomalies spoiling renormalization properties of the theory. From a broader, effective point of view, we can characterize the theory even more sparsely: the field content and the symmetries of the effective action functional suffice.

[33]Note that this possibility—that equivalence classes of gauge potentials will distinguish between physical situations that the magnetic fields will not—is precluded if we take not the more restricted equivalence relation $\vec{A}' \sim \vec{A} + \mathrm{d}f$, but the broader $\vec{A}' \sim \vec{A} + \vec{A}''$ such that $\mathrm{curl}\vec{A}'' = 0$, as (Earman, 2019; Belot, 1998) do, being led to erroneous conclusions. (cf. footnotes 27 and 39).

feature of the manifold Σ, viz. the first homology of Σ. In sum, it is clear that there exist $\vec{A}, \vec{A}'$ such that $\vec{B} = \vec{B}'$ (seen as functions of $\vec{A}$) and yet $[\vec{A}] \neq [\vec{A}']$.

The setting of the Aharanov-Bohm experiment—which I will not go into in detail—experimentally explores these two possible physical states, albeit by introducing the additional theoretical baggage of particle quantum mechanics.[34] The effect registers only the holonomy of the gauge potential around a completely isolated solenoid. Whether the solenoid contains or not an active magnetic flux will affect an interference pattern for electrons shot around the solenoid. In other words, given a curve $\gamma : S^1 \to \Sigma$ interpolating between the source and taget of electrons going around the isolated solenoid, since the phase shift observed in the interference pattern on the target screen depends solely on $\exp\left(i \oint_\gamma A\right)$, one can change the values of the gauge-fields by any gauge transformation—which can radically change the values of $\vec{A}$ at any given point along γ— while the physical observable, being non-local in the right way,[35] does not change.

The lesson of the Aharonov-Bohm experimental confirmation is that observables are given by $[A]$, and are thus non-local.[36] As emphasized by Earman in (Earman, 2019), a certain type of non-locality of electromagnetic phenomena *is* the lesson of the experiment. And this lesson is not contingent on a manifold being non-simply connected. But the lesson has not been widely absorbed ((Belot, 1998) excepted, but see footnote 39). In Earman's words:

> Thus, despite the fact that non-simple connectedness (of the electron configuration space) is essential, by definition, to the AB effect, it is not essential to some of the key issues to which it served to call attention. The philosophical literature seems incapable of absorbing this fact, as if it were under the thrall of the patently invalid inference that goes: 'The AB effect uses non-simple connectedness; the AB effect reveals a [certain] kind of nonlocality; ergo the nonlocality derives from non-simple connectedness.' (p.196)[37]

Underneath Lagrangians, Lorentz covariance, etc., lies the true message of gauge fields: Nature is, to some extent, even classically non-local (Earman, 2019). Although we can more freely specify possibilities using the gauge potential—i.e. without non-local constraints—the extraction of the purely physical, i.e. gauge-invariant content now is also a non-local affair. There is no free-lunch. The notable fact about the theory is that different local perspectives (or gauge-fixings) on the same non-local physical state are available (Gomes, 2019).[38] That is, the no-signalling type of

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[34]In this respect, it is reminiscent of other "bastardizations" of theories—such as the introduction of eddies and splashes under Galileo's ship, or a GPS in the cabin, or light coming in from the beach—which are necessary to experimentally reflect a difference of states which the theory only registers universally, i.e. only in the relations between subsystems.

[35]Namely, under $A \mapsto A + \mathrm{d}f$, Stoke's theorem guarantees that the closed integral does not change.

[36](Belot, 1998) packs a similar message: "Until the discovery of the Aharonov-Bohm effect, we misunderstood what electromagnetism was telling us about our world" (p. 532). He also emphasizes that the choice of interpretation of the theory is one between non-locality and indeterminism: "In particular, [...] we are forced to accept that electromagnetism is either indeterministic or nonlocal." (p.533). I will disregard the possible indeterministic interpretation here.

[37]Earman goes on to say: "To be sure, if actual physical space or the configuration space of an actual electron were non-simply connected, then some form of nonlocality would be in the offing. But this form of nonlocality can tell us nothing about the local vs. nonlocal nature of observables for actual systems since actual physical space and the configuration space of an actual electron are simply connected. Nevertheless, as just noted, gauge-invariant observables for the quantum electrodynamics of actual systems do have a nonlocal character." I would only add that the same could have been said without invoking 'quantum' properties.

[38]They are not available for scalar fields, for instance, and wouldn't be necessary for a purely classical

non-locality at play here is special: it allows the non-local physical state to accept multiple local descriptions, and it is intimately associated to the (covariant) local Gauss law (see footnote 31, or (Strocchi, 2015)). The particular sort of holism that it entails is the meaning of theoretical gauge symmetries.

In sum, we were led to a perspective-independent picture of electromagnetic phenomena in terms of the equivalence classes $[A]$.[39] It is a picture both mathematically and empirically preferred to one in terms of electric and magnetic fields. But deploying A's to replace $\vec{E}$ and $\vec{B}$ in every corner gives rise to a new challenge: we must now formulate all of our criteria—including those for DES—in terms of gauge-independent data.

However, the elements of $[\mathcal{A}]$, being expressed as quotients, are not directly parametrizable. We overcome this problem by exploiting the notable fact that gauge theories allows multiple local representation of non-local physical states. That is, we overcome it with a gauge-fixing, by determining/selecting a class of elements of $\mathcal{A}$ which uniquely represent elements of $[\mathcal{A}]$ (cf. (Gomes, 2019, Sec. 2)). A selective class fixes further conditions which the representatives must satisfy. Nonetheless, it is not the A, but the $[A]$s—or the appropriately gauge-fixed versions, uniquely representing elements of $[\mathcal{A}]$ as elements of $\mathcal{A}$—which should be taken as the objects of physical law. This is true both for classical and quantum physical statements about the theory, which is why worries about a possible failure of determinism in gauge theories are misguided (Strocchi, 2015): the laws as expressed in the quotient configuration space $[\mathcal{A}]$ are completely deterministic (as can be shown by using a gauge-fixing).

3.3   General Notation

This section can be skipped without much loss for those accustomed to standard notation of Yang-Mills theory.

We are given a manifold Σ, which for our illustrative purposes in this subsection will represent a space(time) endowed with an Euclidean metric.[40] We also suppose a finite-dimensional *charge group*, e.g. $G = \text{SU}(N)$, with Lie algebra $\mathfrak{g} := \text{Lie}(G)$, e.g. $\mathfrak{g} = \mathfrak{su}(N)$. Given the charge group we define the group of gauge transformations $\mathcal{G} = C^\infty(\Sigma, G)$, with composition given by pointwise action of G, i.e. $(gg')(x) := g(x)g'(x)$, and the respective infinitesimal version, $\text{Lie}(\mathcal{G}) = C^\infty(\Sigma, \mathfrak{g})$ (with pointwise Lie algebra commutator). An element of $\mathcal{G}$ is a map: $g(\cdot) : \Sigma \ni x \mapsto g(x) \in G$. The gauge fields and its gauge-transformed version A^g are given by Lie-algebra valued space(time) 1-forms. For most of the paper, I won't need more detail than this, but I reserve the more detailed definitions for the appendix.

In the main text of the paper, I will only require the simpler case of Maxwell

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description of the classical equations of motion of electromagnetism in a simple, fixed spacetime background. It is also easy to come up with a non-local theory which does not accept local perspectives.

[39]In the Hamiltonian formalism, the equivalence class becomes of the symplectic pair, $(\vec{E}, \vec{A}) \sim (\vec{E}, \vec{A}+\text{grad}f)$. Note that this is not the same as Belot's interpretation 2 in (Belot, 1998): there he takes $\vec{A} \sim \vec{A}'$ iff they are associated to the same magnetic field. Of course, the Aharonov-Bohm effect would not respect *that* equivalence, leading Belot to—erronously—declare the "the coarse-grained, gauge-invariant [reduced] interpretation is sunk." (p. 548).

[40]The Euclidean metric provides a simpler interpretation of the results on gluing, in the next section. See (Gomes & Riello, 2019) for more on this topic, and for how one can import the relevant results to the Lorentzian signature, "3+1" context. In any case, the philosophy of physics literature on DES ignores the contrast between Euclidean and Lorentzian signatures.
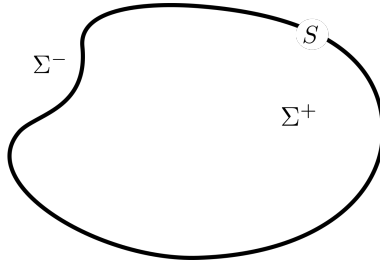
Figure 1: The two spacetime regions $\Sigma^{\pm}$, separated by a manifold with codimension one, S.

electrodynamics coupled to scalar Klein-Gordon theory. Thus, for the Abelian case, $G = U(1)$, we would write

$$A^g = A + \partial \ln g \quad \text{and} \quad \psi^g = g\psi, \tag{3.1}$$

which are indeed the transformations taken by GW, with $g = \exp(-i\chi)$ (Wallace & Greaves, 2014) (p. 66, eq 3).

I will also assume that the manifold Σ is endowed with a Riemannian metric, g_{ij}, and that it is decomposed into two regions (cf. figure 1): $\Sigma_{\pm}$, with boundary $\partial \Sigma_{\pm} =: S = \Sigma_{+} \cap \Sigma_{-}$. For now, each of $\Sigma, \Sigma_{\pm}$ is assumed topologically trivial; and, if any of these manifolds is not compact, then all the fields on them will be restricted to have some suitably fast fall-off rate.[41]

I will denote the regional, unquotiented configuration spaces of each field sector (gauge field, matter field, and doublet (gauge and matter fields) respectively) as $\mathcal{A}_{\pm}$, $\Psi_{\pm}$ and $\Phi_{\pm}$. I will omit the subscript $\pm$ for the corresponding universal configuration spaces (i.e. $\mathcal{A}, \Psi, \Phi$, respectively). The restricted groups of gauge transformations will be denoted in analogous fashion: $\mathcal{G}_{\pm} = C^{\infty}(\Sigma_{\pm}, G)$, and all abstract quotient spaces are denoted by square brackets, as in $[\Phi^{\pm}] := \Phi^{\pm}/\mathcal{G}_{\pm}$, and $[\Phi] := \Phi/\mathcal{G}$.

If the fields compose smoothly, the left hand side below is a smooth field:

$$A = A_{+}\Theta_{+} + A_{-}\Theta_{-}, \qquad \text{and} \qquad \psi = \psi_{+}\Theta_{+} + \psi_{-}\Theta_{-}. \tag{3.2}$$

Here Θ_{+} and Θ_{-} are the characteristic functions of the regions Σ_{+} and Σ_{-} i.e. they are distributions: unity in the region, and zero outside, with some conventional value at the boundary which is immaterial for our purposes. Smoothness requires equality of the following quantities at S: $(\partial^n A_{+})_{|S} = (\partial^n A_{-})_{|S}$ and mutatis mutandis for ψ in place of A; where the superscript n denotes all derivatives: first order, second order, i.e. $n = 1, 2$, etc; and for any combination of independent directions, i.e. x, y, z parametrizing Σ.

4   DES according to GW

I start in section 4.1 by briefly setting the stage and reviewing the GW description of DES symmetries in gauge theories for those subsystems which are demarcated by

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[41]In figure 1, one could think of Σ_{-} as a collar around Σ_{+}; this would not block our treatment. The only complication would be to then consider further boundary conditions on Σ_{-}, and so on. We therefore restrict our attention to the case where Σ_{-} encompasses the "rest of the universe", i.e. the entire "environment" in the language of GW.

spacetime bounded regions.[42] I will then implement their description more directly in the setting of Yang-Mills theories in section 4.2. I will partially realize GW's vision, obtaining a quotient between infinite-dimensional Lie groups. So up to the end of section 4.2.2, my conclusion largely matches theirs, including a similar criticism of BB.

In section 4.2.2 I exhibit how GW's characterization of DES goes beyond what is allowed by BB's characterization. Then, in section 4.3 I will further elaborate on the significance of the result, and what I see as its problems: roughly, that the quotient group does not have a natural action on either subsystem, and, relatedly, that it is not constructed from gauge-invariant ingredients. This, and a summary in section 4.4, set the stage for my "non-naive" treatment of DES, in section 5.

4.1 The GW setup

For our purposes here, I will focus on GW's definition of a relational directly empirically significant symmetry (a relational DES), as one that satisfies the two conditions—*Transformation* and *Symmetry*— given by BB in Definition 1. To recap, in this context GW seek a transformation that: (i) is not a symmetry of the entire system (*Transformation*), and yet (ii) is a symmetry of the subsystem and of its environment (*Symmetry*). If that can be achieved, then the only thing that has changed, since the transformation is not a symmetry and yet leaves the two subsystems individually unchanged, is the *relation* between the two subsystems.[43]

To precisely state the main claims of GW in the context of Yang-Mills theory in regions of spacetime, I have to first provide a glossary, translating their terms and notation to those of Yang-Mills classical field theory. In field-theoretic language, here is the glossary, with (GW) indicating their terms (I have also added some comments on each item's relevance for DES):

- **System and environment:** The 'system', $\mathcal{S}$ (GW, p.67), will be characterized by the regional field-space (i.e. the unquotiented configuration space) Φ_+ over the Σ_+ region, and the environment $\mathcal{E}$ (GW), will be characterized by the regional field-space Φ_-. As is the case for GW's environment, Σ_- is the (closure of the) complement of Σ_+ within Σ (see figure 1), where Σ is the universe, $\mathcal{U}$ (GW, p.67).

- **Composition of system and environment:** The states of the subsystems (or of the subsystem and environment), must be able to compose nicely, so that they together correspond to a possible universal state of affairs. The condition of 'conjunction', or composition, viz. called $s * e$ (GW, p.68), for $s = \varphi_+ \in \Phi_+$ and $e = \varphi_- \in \Phi_-$ will be implemented in the field-theoretical setting as the requirement that the two fields join smoothly (a requirement endorsed by GW). Thus $\varphi_{\pm}$ compose to yield a universe state, labelled $\varphi := \varphi_+\Theta_+ + \varphi_-\Theta_-$, iff: $\partial^n \varphi^+_{|S} = \partial^n \varphi^-_{|S} \quad \forall n \in \mathbb{N}$.

- **Interior symmetry:** An 'interior symmetry' (GW) of the subsystem is simply a regional gauge transformation which goes smoothly to the identity at the boundary, i.e. $g_+ \in \mathcal{G}_+$ such that $g_{+|S} = \mathrm{Id}$ and $\partial^n g_{+|S} = \partial^n \mathrm{Id}_{|S} = 0$. The

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[42]GW aim their characterization to extend beyond the domain of gauge theories and subsystems demarcated by spacetime regions. I will have no such ambitions.

[43]Although GW allow for the possibility that the environment is not kept fixed, i.e. that DES is not purely relational, they do not investigate this possibility, and neither will I (and neither do Kosso or BB). It seems that non purely relational DES have little to say about regularities of nature. We will get back to this once we have gone over the necessary basic definition of GW's terms.

idea is that such regional transformations may always be extended by the identity on the environment, $g_- = \text{Id}$; and are therefore composable to a valid universal gauge transformation, namely $g = g_+\Theta_+ + g_-\Theta_-$, independently of the environment state, φ_-. That is, for initially composable $\varphi_\pm$, i.e. any valid $\varphi = \varphi_+\Theta_+ + \varphi_-\Theta_-$, moving φ_+ by an 'interior symmetry' will still lead to a valid (i.e. smooth) configuration $\varphi' := \varphi_+^{g_+}\Theta_+ + \varphi_-\Theta_-$. With pointwise group multiplication of G, the 'interior symmetries' $\mathcal{G}_{\text{Id}_S}^+$ forms a group; in fact a normal subgroup of $\mathcal{G}_+$.[44]

- **Boundary-preserving symmetry of** $\tilde{\Phi}_+ \subset \Phi_+$**:** For a subset of the regional configurations $\tilde{\Phi}_+$, the 'boundary-preserving symmetry group' (GW) is the subset of the regional gauge transformations $\tilde{\mathcal{G}}_+ \subset \mathcal{G}_+$ that preserve those field-configurations on the boundary:[45]

$$\tilde{\mathcal{G}}_+(\tilde{\Phi}_+) := \{g \in \mathcal{G}^+ \,|\, \forall \varphi_+ \in \tilde{\Phi}_+, \partial^n \varphi_{+|S}^g = \partial^n \varphi_{+|S} \;\forall n \in \mathbb{N}\}$$

When the subset $\tilde{\Phi}_+$ to which $\tilde{\mathcal{G}}_+$ refers is not clear from context, I will make it explicit, denoting the boundary-preserving symmetries of a configuration φ_+ by $\tilde{\mathcal{G}}_+(\varphi_{+|S})$ (and of a subset $\tilde{\Phi}_+$ by extension, $\tilde{\mathcal{G}}_+(\tilde{\Phi}_+)$); otherwise I will abbreviate to $\tilde{\mathcal{G}}_+$. Note also that since $(\varphi^g)^{g'} = \varphi^{gg'}$, the boundary state is preserved, and $\tilde{\Phi}_+$ forms a group. Moreover, $\mathcal{G}_{\text{Id}_S}^+$ is again a normal subgroup of $\tilde{\mathcal{G}}_+$ (the proof follows as in footnote 44).

- **State equivalence, '$\sim$':** Here there is not much difference between GW's construal and ours. Two subsystem states $\varphi_\pm, \varphi'_\pm$ are theoretically equivalent, i.e. $\varphi_\pm \sim \varphi'_\pm$ if and only if they are related by a subsystem symmetry, $g_\pm \in \mathcal{G}^\pm$, and mutatis mutandis for the universal state.

Thus GW's general tactic for identifying symmetries with DES will be to look for *non-interior, boundary-preserving symmetries*: discussed at the start of section 4.2.1. But first we need to echo the role of BB and Kosso's 'reference systems', discussed at the end of section 2.1. There we saw that both Kosso and BB recognized the importance of leaving a reference system unchanged. What about GW? At first, they allow for changes of the reference subsystem: "Non-interior symmetries that are not boundary-preserving on the subsystem states of interest can have direct empirical significance, but to realize this significance, the environment state must be altered." (p. 73). However, they quickly realize that without a reference system, there will be too many 'spurious', non-nomological cases of such symmetries, cases which tell us nothing about regularities of nature. That is because one could simply take *any* universal symmetry—boundary-preserving or not—and then "make some small and irrelevant, but intrinsic, further change to the state of the environment." (p. 73).

GW thus conclude that

> "The task of demarcating transformations that appear to correspond in some interesting way to the theoretical symmetry of interest, on the one hand [...] and implementations of this [...] cheap trick, on the other, lies beyond the scope of this article." (p.74)

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[44]I.e. for $g \in \mathcal{G}^+$ and $h \in \mathcal{G}_{\text{Id}_S}^+$ we have $g^{-1}hg(x) = x$ for $x \in S$ and therefore $g^{-1}hg \in \mathcal{G}_{\text{Id}_S}^+$.

[45] Although GW only require that the value of the field at the boundary be preserved, according to the smoothness criterion (which GW also endorse), continuity should hold for an arbitrary number of derivatives of the field at the boundary also.

And later, confirming a preference for the relational construal of DES, they state (translated into our language):

> "If φ_-, φ'_- are intrinsically different states of the environment,[46] then the transformation $\varphi_+\Theta_+ + \varphi_-\Theta_- \to \varphi_+^{g_+}\Theta_+ + \varphi'_-\Theta_-$ is a change in physical state of the universe, and one that is not purely relational. The empirical symmetry will nevertheless 'correspond to' the theoretical subsystem symmetry $\varphi_+^{g_+}$, insofar as (and only insofar as) there is a principled connection between the subsystem symmetry transformation g_+ and the environment transformation $\varphi_- \to \varphi'_-$. We have not investigated the possibilities for such 'principled connections'."(p. 86)

Therefore, in line with Kosso, BB and (partly) with GW, we endorse the assumption of a physical reference system whose physical state is to remain unmodified.[47] With this assumption, we adopt a purely relational interpretation of DES.

Once translated to our nomenclature and setting, we can state the tactic for identifying those symmetries that have relational DES: find two physical states, one for each of two regions ('the ship' and 'the sea'), that can be related ('glued together') in different ways. Phrased in terms of states of the universe ('universal states'), and for subsystems rather than regions:—

1. There are two universal states φ and φ' that are physically different, i.e. not related by a universal symmetry, i.e. $\varphi' \nsim \varphi$; and

2. The universe is composed of two subsystems, for each of which, its physical state in φ: $\varphi_\pm$, is physically identical to its state in φ', i.e. $\varphi_\pm \sim \varphi'_\pm$.

3. Since, $\varphi_\pm \sim \varphi'_\pm$ but $\varphi' \nsim \varphi$: the difference between φ and φ' thus lies in the relations ('gluings') between the subsystems.

Clearly, 1 to 3 would indeed apply to Galileo's ship. That is, if boundaries were of no concern: $\Phi_+ \ni \varphi_+$ would be the possible ship states, $\Phi_- \ni \varphi_-$, the possible sea states, $\varphi = \varphi_+\Theta_+ + \varphi_-\Theta_-$ a particular state of ship and sea at relative rest; and finally $\varphi' := \varphi_+^{g_+}\Theta_+ + \varphi_-\Theta_-$ some joint state of ship and sea at constant relative velocity for which the sea remains untransformed.[48]

4.2 Expression of Brown-Brading DES conditions in the GW set-up for Yang-Mills

The first aim of this section, elaborated in subsection 4.2.1 is to give, in Definition 3 and equation (4.10) below, the GW expression of DES in terms of *non-interior, boundary-preserving symmetries*. Then, in section 4.2.2, I articulate the root of the differing conclusions of BB and GW regarding DES.

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[46]I take it that by "intrinsically different" they mean not only $\varphi_- \neq \varphi'_-$ but also $[\varphi_-] \neq [\varphi'_-]$, but this goes unstated there, partly because of a difference in focus.

[47]In principle, this reference may be a small collar around the subsystem of interest. The limitation of our analysis to two subsystems is of an entirely pragmatic nature: We strictly need only two subsystems—one to act as the subsystem displaying DES symmetries and the other as reference—but we could have included more. In other words, using this minimal setting allows us to ignore the (unimportant, at least for theory-building) further relations between reference system and the rest of the universe (see also footnote 41).

[48] But as will shortly become apparent, the GW characterization of DES does not deal directly with physical, i.e. gauge-invariant, properties of the regions.

4.2.1 Construction of GW's relational DES

Before attempting to implement this notion of DES in the gauge theoretic case, I should address a preliminary concern: whether '*interior subsystem symmetries*' precisely identify the subgroup that may *not* exhibit relational DES according to 1-3.

GW justify the special status of interior symmetries by appealing to the fact that these are the symmetries which preserve any boundary state, and therefore always yield a composable subsystem state before and after a transformation. But any gauge transformation, even if it is *not* the identity at S, may be suitably extended into Σ_-. So even if we agree to keep the *physical* state of the environment as a fixed reference, as per the discussion in section 4.1 (see also footnote 41 and the penultimate paragraph of section 2.1), what justifies keeping the *representation* of the environment, the state φ_-, fixed as well?

In other words, given *composable* regional states, $\varphi_\pm$, should GW's relational DES be characterized by the existence of g_+, g_- such that:

$$\varphi \not\sim \varphi' \quad \text{with} \quad \varphi = \varphi_+ \Theta_+ + \varphi_- \Theta_- \quad \text{and} \quad \varphi' := \varphi_+^{g_+} \Theta_+ + \varphi_-^{g_-} \Theta_- \ ? \qquad (4.1)$$

Or are they justified in restricting attention to gauge transformations that leave the 'sea' unmodified? I.e. can we restrict our search to solely those g_+ satisfying

$$\varphi \not\sim \varphi' \quad \text{with} \quad \varphi = \varphi_+ \Theta_+ + \varphi_- \Theta_- \quad \text{and} \quad \varphi' := \varphi_+^{g_+} \Theta_+ + \varphi_- \Theta_- ? \qquad (4.2)$$

To answer this concern, from (4.1), note that $\varphi' \not\sim \varphi$ implies $(\varphi')^g \not\sim \varphi, \forall g \in \mathcal{G}$. We are justified in investigating DES for $\varphi'' := (\varphi')^g$ instead. Thus, we can employ the universal gauge transformations to bring one of the subsystems back to the original state. That is *given* any candidates $\varphi_\pm$ for (4.1) and any doublet g_+, g_- yielding $\varphi \not\sim \varphi'$ in (4.1), we choose any $g' \in \mathcal{G}$ such that $g'_{|\Sigma_-} = g_-^{-1}$. We then have an equivalent condition to (4.1):

$$\varphi \not\sim \varphi'' \quad \text{with} \quad \varphi = \varphi_+ \Theta_+ + \varphi_- \Theta_- \quad \text{and} \quad \varphi'' = \varphi_+^{(g'_{|\Sigma_+})g_+} \Theta_+ + \varphi_- \Theta_- \qquad (4.3)$$

Since we are quantifying over all g_+, i.e. all we are concerned with is that $\varphi_\pm \sim \varphi''_\pm$, we conclude that we can indeed restrict our investigations to (4.2).

From a relational standpoint, this restriction makes intuitive sense: if all we are concerned with are relational differences, we can use the environment states as a fixed reference. In sum: freely applying universal symmetries—as they are taken to leave the physical system unchanged according to section 1.1—guarantees that we can use the particular representation of one of the subsystems as a fixed reference.

It is also clear that *interior symmetries* do not have relational DES. That is because the two regional gauge transformations *cannot* smoothly join:

$$\text{there is no } g \in \mathcal{G} \quad : \quad g_{|\Sigma_+} = g_+, \quad g_{|\Sigma_-} = g_-, \qquad (4.4)$$

for otherwise $\varphi' = \varphi^g \sim \varphi$. For interior symmetries, (4.4) clearly fails—the following is a valid (smooth) group element:

$$g \in \mathcal{G} \quad : \quad g_{|\Sigma_+} = g_+ \in \mathcal{G}_{\text{Id}_S}^+, \quad g_{|\Sigma_-} = \text{Id}; \qquad (4.5)$$

and therefore, since $\varphi \sim \varphi'$, we have a contradiction with (4.2), i.e.

$$\varphi \sim \varphi' \quad \text{with} \quad \varphi = \varphi_+ \Theta_+ + \varphi_- \Theta_- \quad \text{and} \quad \varphi' := \varphi_+^{g_+} \Theta_+ + \varphi_- \Theta_-. \qquad (4.6)$$

Accordingly, GW's attempt to express BB's two conditions for DES (cf. see the start of section 4.1), is summed up in the following definition:

Definition 3 (Relationally non-trivial transformations of the fields) *Given configurations $\varphi_\pm$ which are the composable, the group $\mathcal{G}_{DES}^{GW}(\varphi_+) \subset \mathcal{G}^+$ bearing relational DES according to GW,*[49] *are the* **relationally non-trivial transformations**, *defined by:*

Writing $\qquad \varphi'' = \varphi_+ \Theta_+ + \varphi_- \Theta_-$, *and* $\varphi' := \varphi_+^{g^+} \Theta_+ + \varphi_- \Theta_-$

$$\text{we say} \quad g^+ \text{ has DES}, \quad g^+ \in \mathcal{G}_{DES}^{GW}(\varphi_+) \subset \mathcal{G}^+ \quad \text{iff} \quad \varphi \not\sim \varphi'. \quad (4.7)$$

We now spell out how this definition forces us to *non-internal, boundary-preserving* symmetries. According to the composibility criterion, the first line of (4.7) implies:

$$\begin{cases} \partial^n \varphi^+{}_{|S} = \partial^n \varphi^-{}_{|S} \\ \partial^n \varphi_+^{g^+}{}_{|S} = \partial^n \varphi^-{}_{|S} \end{cases} \qquad \forall n \in \mathbb{N} \qquad (4.8)$$

which together yield:

$$\partial^n \varphi^+{}_{|S} = \partial^n \varphi_+^{g^+}{}_{|S} \qquad (4.9)$$

We find therefore, that as a *necessary but not sufficient condition*, g_+ must preserve the value of φ^+ and all its derivatives, at S, i.e. it must be a *near-boundary stabilizer* of φ^+. In other words, such a g_+, if it exists, must be an element of the *boundary-preserving symmetries* $\tilde{\mathcal{G}}_+(\varphi_{+|S})$.

But the other condition we obtained above, according to (4.5) and (4.6), is that g^+ cannot be an interior symmetry, $g_+ \notin \mathcal{G}_{\mathrm{Id}_S}^+$. That is, if any $g^+ \in \mathcal{G}_{\mathrm{DES}}^{\mathrm{GW}}(\varphi_+)$ exists, it can be composed with any $g_+ \in \mathcal{G}_{\mathrm{Id}_S}^+$ to yield precisely the same subsystem and universal physical states.

In other words, we must take the quotient of two infinite-dimensional groups, finding, according to Definition 3:[50]

$$\mathcal{G}_{\mathrm{DES}}^{\mathrm{GW}}(\varphi_+) \simeq \tilde{\mathcal{G}}_+(\varphi_{+|S})/\mathcal{G}_{\mathrm{Id}_S}^+. \qquad (4.10)$$

But take note: so far this is only a necessary condition, and we have only found a (quotient) group which is isomorphic to the group of (relational) DES symmetries according to GW: we have found no subgroup of $\mathcal{G}^+$. As we can see from appendix A, the information we can extract from this partial result is that, if it exists, the group $\mathcal{G}_{\mathrm{DES}}^{\mathrm{GW}}$ must be rigid (and in 1-1 relation to a quotient, see footnote 50).

After a brief intermezzo in comparing these results with the conclusions of BB in section 4.2.2 below, I will take this matter up again in section 4.3.

4.2.2 The fault lines between GW and BB

We can now see how (4.10) and the argument and discussion leading to it encodes the disagreement between (Wallace & Greaves, 2014) and (Brading & Brown, 2004).

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[49]As we will see, it turns out this group only depends on φ_+ (including its boundary values), which is why we write $\mathcal{G}_{\mathrm{DES}}^+(\varphi_+)$ as opposed to $\mathcal{G}_{\mathrm{DES}}^+(\varphi_\pm)$.

[50]This is not quite the same conclusion of GW. They write: (p. 75) "Empirical symmetries correspond 1-1, not to non-interior (theoretical) symmetries themselves, but to equivalence classes of non-interior symmetries under this equivalence relation: that is, to elements of the quotient group $\mathcal{G}_+/\mathcal{G}_{\mathrm{Id}_S}^+$" (where I have substituted my notation for theirs). I.e. in this conclusion they allow for non-relational DES. But, as I explained at the end of section 4.1, on the discussion of reference systems, they do not explore this extended notion in their paper; rightfully so, in my opinion. Indeed, the more rigorous results presented here in the derivation of DES can only be implemented with the assumption of a fixed physical reference system; otherwise one cannot even justify the special character of the interior gauge transformations.

First, it is important to note that for generic configurations φ_+, there are *no* non-trivial transformations with (relational) DES, since for most φ_+ it is true that $\tilde{\mathcal{G}}_+(\varphi_{+|S}) = \mathcal{G}^+_{\mathrm{Id}_S}$.

For example, in the case of electromagnetism with coupled Klein-Gordon scalar fields, if $\psi^+_{|S} \neq 0$, no non-trivial stabilizers of $\psi^+_{|S}$ exist, since $g^+\psi^+_{|S} = \psi^+_{|S}$ if and only if $g^+_{|S} = \mathrm{Id}$. But by considering vanishing matter fields at the boundary (essentially an isolation condition) we could indeed have a non-trivial quiotient in (4.10): transformations that tend to constants different than the identity at the boundary are not quotiented out and yet are boundary-preserving on any such state (see appendix A).

Thus BB's opposite conclusion is based on disregarding the requirements of composition of states of subsystems. For they take any two regional gauge transformations which fail to coincide at the boundary to be disallowed.

(They also consider electromagnetism with a scalar field; more specifically, interferometry, viz. a beam-splitter discussed by 't Hooft ('t Hooft, 1980). But we will not need details.) They write:

> "The only remaining option is to consider a region where the wavefunction can be decomposed into two spatially separated components, and then to apply local gauge transformation to one region (i.e. to the component of wavefunction in that region, along with the electromagnetic potential in the region) and not to the other. But then either the transformation of the electromagnetic potential results in the potential being discontinuous at the boundary between the 'two subsystems', in which case the relative phase relations of the two components are undefined (it is meaningless to ask what the relative phase relations are), or the electromagnetic potential remains continuous, in which case what we have is a special case of a local gauge transformation on the entire system." (p. 656)

BB are right in one respect: once you have a universal configuration one *cannot* apply gauge transformations which are discontinuous, i.e. produce a delta function at the interface (see p.79 and 82 (Wallace & Greaves, 2014) for their endorsement of continuity of universal gauge transformations). But it is also true that one could have regional configurations being acted upon by regional gauge transformations which don't match at the boundary (see p. 83 of GW for their endorsement of *this* point). In this second instance, one aims to take the regional subsystem states first and have different gauge transformations applied to them before gluing. This is a *subsystems*-first viewpoint. It poses questions like the ones I am posing in this paper: given regional subsystem states, which universal states are compatible with their composition?

Summarizing, in their quote, BB mix the *universe-first* questions with subsystem-first ones. From a universe-first perspective, all hands agree that one cannot have a gauge transformation which is discontinuous at the boundary. From this perspective it is true that $g_+ = g_{|\Sigma_+}$ and $g_- = g_{|\Sigma_-}$; i.e. that the regional gauge transformations are mere restrictions of a universal gauge transformation. Of course, such an assumption would preempt any search for an empirically significant symmetry, according to (4.4). But from a subsystem viewpoint, these conclusions are unwarranted: one *starts from the regional states* and then glues them. From this perspective, it is the effect of the regional gauge transformations that matter: one does not try to glue the 'disembodied' gauge transformations either. Following this

prescription, the differing constant phases arriving S from the two regions are only disallowed under the extra assumption that the matter fields ψ are non-vanishing at S. If they *do* vanish at the boundary, one need not worry about different constant phase shifts.

GW spot this error, and assert that what should be fundamental is only the continuity of the glued gauge and matter fields, A, ψ, not of phase shifts. In their words (but my notation) (p. 83):

> The key to seeing why this argument fails is noting that what is given, when we are given the pre and post-transformed states of the universe, is not a function from spacetime to the gauge group, but merely the effect of whatever transformation is being performed on the particular pre-transformation (universe) state (ψ, A). And if this particular ψ happens to vanish on the overlap region S, then nothing about the corresponding gauge transformations g_+, g_- can be 'read off' from their effects on the wavefunction in that region [assuming they are constant near S]. It is therefore possible that the universe transformation being performed might correspond to the effect of (say) *some constant gauge transformation g_+ in Σ_+, and a different constant gauge transformation g_- in Σ_-, so that there is no way of patching g_+, g_- together* to obtain a single smooth function from the whole of spacetime to the gauge group. [my italics]

I whole-heartedly agree with this verdict. But it means one must endorse the subsystems-first approach to relational DES. But I believe GW are not entirely consistent in applying this approach, as we will shortly see.

Nonetheless, GW are one step closer to an analysis based solely on physical entities than are BB: they take the representations of the subsystem fields to be important, not the gauge transformations themselves. But, as mentioned in the introduction, they are not yet at exactly the right level' and in fact they can't be, due to the non-local nature of gauge observables and their '*Antiholism*' assumption, particularly as endorsed in conjunction to a *subsystem-first* methodology.

4.3   Problematic issues of the GW result

In this subsection, I will explore two related problematic issues I see with GW's results. First, I will discuss the drawbacks of the quotient nature of their result in section 4.3.1. Second, in section 4.3.2, I will criticize their unwitting abandonment of a "subsystem-first" viewpoint when considering the composition of subsystems. Lastly, in section 4.3.3, I will defend *the extraction* of rigid subgroups from their surrounding malleable group from the charges of GW. I will do this by imbuing such extractions with physical significance.

4.3.1   The 1-1 relation to a quotient group

In my view, one issue with GW's final position, as expressed in equation (4.10), is that we only have a group isomorphism between $\mathcal{G}_{\text{DES}}^{\text{GW}}$, whatever it may be, and a *quotient* group, $\tilde{\mathcal{G}}_+(\varphi_{+|S})/\mathcal{G}_{\text{Id}_S}^+$. The existence of this isomorphism is only a *necessary* condition that the DES group $\mathcal{G}_{\text{DES}}^{\text{GW}}$ would have to satisfy, "empirical symmetries correspond 1-1 [...] to elements of a quotient group" (p. 75. See also footnote 50).

But otherwise GW do not see the quotient nature of their result as problematic. In fact, they see it as exonerating their notion of DES from the charges they make

against the orthodox view on DES:

> in any theory that has a malleable symmetry group, the rigid symmetries remain as a subgroup of that malleable symmetry group. (For example, in general relativity, the rigid translations and boosts form a subgroup of the group of all diffeomorphisms.) It is therefore logically impossible that all rigid symmetries, but no malleable symmetries, can have direct empirical significance; (p. 61)

(again using my nomenclature). That is, their charge on the orthodox view is that it attributes DES only to certain rigid subgroups of malleable groups. Recognizing the quotient group as being isomorphic (in 1-1 relation) to the symmetries exhibiting DES, GW argue, assuages this concern:

> The reason lies in a structural difference between the original problematic claims and our replacements: rather than holding that all elements of a (malleable) symmetry group have one property while elements of some subgroup thereof have a contradictory property, we hold that (for any given subsystem) there is a subgroup of subsystem symmetry transformations (the 'interior' ones) that cannot have empirical significance, and that it is elements of the quotient of the larger group by this subgroup that are *candidates for correspondence to physical operations*. There is thus no object of which we assert both that it does, and that it does not, have some given property. (p. 87) [my italics]

But how can a quotient group represent physical operations, which *do presumably act* on a subsystem? So one asks: can we extend GW's constructions so as to have this quotient group define some group action on the regional configuration? Such an action is necessary if we are to recognize (from the 'outside') two regional configurations which are related by a DES-transformation, as we can do with Galileo's ship.

In this respect, in appendix A, I analyze the quotient group and attempt to complete GW's characterization of DES in the above way. I find a *negative* answer to the question above: there is *no* natural choice of $\mathcal{G}_{\text{DES}}^{\text{GW}} \subset \mathcal{G}^+$. My best efforts culminate in equation (A.7). It is true that (A.7) represents a possible action of the quotient (4.10) on the regional configurations. If we were to trust it, it would be proof that GW's construal of DES yields a rigid regional symmetry group (but not a malleable one!). Therefore, the conclusion here is: if one insists only on the 1-1 relation between the quotient group and the DES group—i.e. the one which *could* act as physical operations on a subsystem—one is left with a badly underdetermined regional group: there is no unique way of realizing the said physical operations (as there is with Galileo's ship); even if the quotient is finite-dimensional, there is a continuous infinity of equivalent representations of the same finite-dimensional group of DES on the region.

4.3.2 Subsystems first

To attain (A.7), I have to be extremely cavalier about important facts: which, I will now argue, largely undermine its significance, and even that of the quotient (4.10). (I also criticize (A.7) at the end of the appendix A). The most important defect concerns the way I (and GW: see section 8 of (Wallace & Greaves, 2014)) here implemented '*Composition of system and environment*' (see section 4.1). Namely, according to a smoothness criterion of configurations at the boundary (also explicit

in equation (4.8)). GW argue that composition of states, $*$, is a partial operation, by writing:

> "Finally, and of most direct importance for this paper, note that one should not assume that just any old pair [of regional states] is a possible state of the total system. In general, there are boundary conditions which must be satisfied in order for a given subsystem-environment pair to be a solution of the theory's dynamical equations. In the case where the total system is a field theory, these will be boundary conditions in the differential-equation sense: requirements that the fields and their derivatives match up on the boundary." (p. 68)

As with GW's criticism of BB, smoothness is a perfectly valid criterion once you have the universal state and want to *de*compose it: the two regional configurations *should* run smoothly into each other. But that does not directly imply anything about the existence of relational DES, which is about the composition of regional subsystems.

The defect in this passage by GW, as already presaged in section 2.4.2, is: from the subsystem-first viewpoint, why should we be concerned with whether the field in a given arbitrary representation is smoothly glued? Shouldn't we focus instead on whether the physical, i.e. gauge-invariant contents of the two regional configurations can be composed? This focus would still allow a pure gauge discontinuity at the boundary. In other words, from a subsystem-first viewpoint, we are only given the intrinsic regional physical states. Shouldn't we somehow describe gluing for $[\varphi_+]$ and $[\varphi_-]$, as opposed to finding conditions by which φ_+ and φ_- are each a restriction of one universal state? This criticism clearly echoes GW's criticism of BB, and, as we will see, the solution of one qualitatively also extends to the other.

In fact, we can adapt the paragraph of section 4.2.2 summarizing GW's criticism of BB to also represent my criticism of GW.[51] Leaving the respective GW criticism of BB in parenthesis, it reads:

> "From a universe-first viewpoint, all hands agree that one cannot have a state (respectively gauge transformation) which is discontinuous at the boundary. This leads GW (resp. BB) to the unwarranted assumption that $\varphi^+ := \varphi_{|\Sigma_+}$ and $\varphi^- := \varphi_{|\Sigma_-}$ (resp. $g_+ = g_{|\Sigma_+}$ and $g_- = g_{|\Sigma_-}$); i.e. that the regional field configurations (resp. regional gauge transformations) are mere restrictions of a universal configuration (resp. universal gauge transformation). But from a subsystem viewpoint, one need not worry about states differing by a gauge transformation at the boundary (resp. one need not worry about different constant phase shifts at the boundary)[52]; one glues the physical content, not their arbitrary representations as configurations (resp. one glues the states, not the gauge transformations)."

4.3.3 Well-defined rigid subgroups

As described in section 4.3.1, one of GW's main criticisms of the orthodox view that gauge symmetry cannot have DES is, essentially, that one cannot endow significance solely to a subgroup of a group.

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[51] Such a side-by-side comparison of the two views mimicks the fictitious BB quote in section 2.4.2, representing the focus of GW's attack.

[52] For vanishing matter at the boundary, otherwise one does need to worry. Similarly for my criticism of GW, if the states don't differ by a gauge transformation at the boundary, one does need to worry.

But is this really true? What are *the* rigid translations and boosts of a generic spacetime metric? Poincaré transformations are not well-defined (i.e. geometrically defined) in a generic background metric. They are defined by Killing fields in a Minkowski metric. In that sense, that subgroup is *physically distinguished*, but only in particular circumstances; it is only meaningful over a Minlowski metric.[53]

In specific backgrounds (e.g. Minkowski), there will be a physically well-defined rigid subgroup of the malleable transformations, but this subgroup will be effaced once one moves to generic backgrounds. As can be gleaned in the quotes from section 4.3.1, GW's criticism of the orthodox view rightly attacks a "disembodied" notion of a gauge group. That is, if one focuses just on the group itself, and not on its action on states, one cannot single out a rigid subgroup in any meaningful way. In fact, as the last GW quote of section 4.2.2 illustrates, they criticize BB's dismissal of DES as based on the same oversight: GW's criticism points out that the transformations themselves may be discontinuous, but their effect on the regional states does not entail a discontinuity on the glued state. In other words, GW agree: it is the *effect* of the transformation that matters.

Contra GW, it seems therefore entirely possible to associate DES only to certain physically meaningful rigid symmetries. That means that either in Minkowski or in more general backgrounds, malleable transformations that don't belong to these subgroups would lack DES. That is, rigid symmetries *may* have DES in a given special background and yet lose that significance for a generic background. Such normal subgroups are "plucked out" from the surrounding malleable group by satisfying certain equations, e.g. the Killing equations, which are themselves physically significant.

Such subgroups of rigid symmetries are usually called *stabilizers*, and the configurations they stabilize are called *reducible*. And the same concepts apply to gauge theories. In the Abelian case all $A \in \mathcal{A}$ are reducible, and they all possess the same stabilizers, namely, the constant gauge transformation. The non-Abelian case is much more similar to the spacetime case (see C.1 for a brief description): generic A are *not* reducible, but some are. Being reducible is a physically significant fact: in the quotient space $[\mathcal{A}]$, the orbits of reducible configurations are qualitatively different than the generic orbits.[54]

Indeed, as shown in full non-Abelian generality in (Gomes & Riello, 2019) and in section 5 for the Abelian case, it is precisely such stabilizer subgroups which are plucked out from all the malleable gauge transformations and endowed with DES.

4.4 Summary

In this Section, I have partially vindicated the position of GW, in that I argued that a gauge transformation in a gauge theory can have direct empirical significance

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[53]If one is thinking not in terms of active diffeomorphisms, but of coordinate transformations, then indeed, one can single out translations and boosts, but only with respect to that coordinate system. More broadly, there are generically no constant gauge-transformations: they usually require a global section to be defined. One should also note that although for principal fiber bundles one has a natural action of the charge group, G, this is not the case for associated bundles. One cannot define a "constant" action of the gauge group: it can only be constant with respect to a given section (see (Kobayashi & Nomizu, 1963)).

[54]Note that the notion of reducibility covaries with the notion of stabilizers. Namely, if a configuration φ is reducible, with a given stabilizer $f \in \mathcal{G}$, then for any given $g \in \mathcal{G}$, φ^g will also be stabilized by gfg^{-1}. This qualitative difference between the orbits renders the quotient space into a *stratified* manifold: i.e. a space formed by a concatenation of boundaries (see (Kondracki & Rogulski, 1983; Fischer, 1970) and, for a philosopher-friendly description of stabilizers, and their relation to conserved charges, (Gomes, 2019, Secs. 3.3.5-7))

that is displayed/revealed/encoded in the values of relational fields—in a manner analogous to, although also different in important respects from—a rigid regional transformation of Galileo's ship.

In other words: at least naively, there are cases in which, given two smoothly joined regional configurations, we can regionally—but *not* universally—gauge-transform. That is: the universal state resulting from gluing is not a gauge-transform, a "mere re-description", of the original universal state.

On the other hand, this vindication was limited. The analogy with Galileo's ship is misleading: there, the difference of states is manifest in different values of relational physical quantities. There, we could relate the different universal states by a group acting on a subsystem (albeit from an external point of view). For the GW constructions, no regional symmetry group was naturally identified as representing the quotient group (4.10). We were led to this limited answer by our cavalier attitude towards the role of bounded regions with regard to gauge-equivalence classes in theories with non-local constraints. In other words, this section's attempt to characterize DES symmetries was misled by GW's assumption of '*Antiholism*': that the universal state is uniquely determined by the regional states. The 'antiholism' assumption is also associated with a rejection of a "subsystems-first" viewpoint on gluing: a rejection which, according to GW, was responsible for BB's dismissal of DES for gauge theories is strikingly similar (see section 4.3.2).

In my view, these defects at least partially vitiate the development of their essential argument for relational DES.

As we will see in the next section 5, we will also adopt essentially the same criteria as 1-3 in section 4.2.1. The main difference is that we start from the regional gauge-invariant content and *then* build different universal physical states. Given two states $\varphi_\pm$, smoothness at the interface S is not a good criterion to characterize gauge-independent facts about the universal configuration. I will rectify these issues by using Definition 2 instead of Definition 3 to characterize DES symmetries in gauge theories. Our "option (c)", representing 'external sophistication' towards subsystem symmetries (cf section 1.1) will be the key to overcoming 'Antiholism' in a tractable manner. We will see that for Definition 2, we can indeed realize DES as a group of rigid regional symmetries plucked out "from under the nose" of the malleable symmetries.

5 DES as subsystem-holism

Having broadly argued that holism lies at the origin of theoretical gauge redundancy in section 3.2, I will now show that holism is the source of symmetries with direct empirical significance.

In this section I will explore a definition of DES through an underdetermination of universal *gauge-invariant* data from regional *gauge-invariant* data. That is, I will explore DES according to *regional-holism* described in Definition 2. Section 5.1 introduces the obstacles to gluing the physical content of regions and briefly reports on the appendices relevant to this issue. The following section, 5.2, shows how gluing employs a mixture of 'reduction' and 'external sophistication' (option (c) in section 1.1).

In Section 5.2.1 I describe the positive and negative aspects of using matter fields to parametrize the gauge-invariant content of the theory. I conclude that, unlike gauge fields, matter fields are generically ill-suited for that role. I then

describe to what extent we can employ the gauge-fixing of gauge fields to uniquely parametrize the space of physical states over a bounded region. This is done in section 5.2.2. In section 5.2.3 I describe precisely how 'external sophistication' is deployed to facilitate the gluing of the regional physical states. Finally, in section 5.2.4, I vindicate my main claims about holism and its relation to DES: viz that there is a rigid group of regional symmetries parametrized by the residual variety of universal physical states as composed by identical regional physical states.

5.1 Introduction to the problem of gluing

According to our account, both GW and BB ignored *the composition* of gauge-invariant data of the *subsystems*, ultimately undermining our faith in the results of the last section (Section 4.3). Both works ignored, at some level, the point that DES uses a 'subsystem-first' viewpoint: one is given subsystems and then inquires about their composition.

In effect, the formulation of the issue in the present Section can be seen as the natural culmination of the attempts to characterize DES, first by BB and then by GW. BB assumed the U(1) phase had to be smoothly composed between two regions. GW assumed the field-configurations had to be smoothly composed between the two regions. And I will only assume that the gauge-independent content of the two regions has to be smoothly composed between the two regions.

At first sight, the question we face in this Section is much simpler than all the previous construals of DES: how do we make sense of the equation (cf. (2.1)):

$$[\varphi]_{(i)} = [\varphi^+] \cup_i^S [\varphi^-], \qquad i \in I \quad ? \tag{5.1}$$

But, looking closely, there are obstacles: I will separate some of the technical ones from the underlying conceptual ones. The technical ones have been sorted out elsewhere (Gomes, 2019; Gomes et al., 2019; Gomes & Riello, 2019), and are thus relegated to appendix B. Since the content of the appendices is still important to our main questions, I summarize it here.

In appendix B.1, I will describe some of the issues with gluing—i.e. with composing gauge systems—which block a naive resolution of the problem announced in equation (5.1). The obstruction arises if we try to directly compose the gauge-invariant content of the given regions. Resorting to a gauge-fixing seems like the natural resolution, but *naive* gauge-fixing also brings complications. I address these complications in this first subsection of the appendix.

In appendix B.2, I will summarize a resolution of these obstructions.[55] The resolution involves first a parametrization of the regional and universal physical content. In the Abelian case (but not the non-Abelian one, which requires more technology, see appendices B.2 and C), the parametrization is obtained by complete regional and universal gauge-fixings. By 'complete' I mean: they do not limit the type of gauge transformations at the boundary, a limitation that is usually imposed for gauge theories in bounded regions (cf. (Regge & Teitelboim, 1974; Balachandran et al., 1996; Giulini, 1995; Strominger, 2018) and also appendix B.1), and is reflected in GW's special treatment of "interior symmetries" (see section 4).

Thus I endorse 'reduction' (cf. Section 1.1) for determining the regional and universal physical content. But, to describe gluing, I endorse 'external sophistication':

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[55]This resolution was developed in the above technical references (especially (Gomes & Riello, 2019)), using natural geometric structures, such as connection-form, in the field space of gauge theories.

i.e. allowing all the different representations of the same regional physical content to be counted as isomorphic but *not* identified from the outside view. That is: this external flexibility of representation is employed for composing the intrinsically unique representations of the regional physical contents into unique representations of each possible universal physical content. This broad point of view on the necessity of gauge degrees of freedom was defended in (Rovelli, 2014) ("gauge as coupling") and deployed to resolve the coupling of regions inYang-Mills theory in (Gomes, 2019; Gomes & Riello, 2019).

Thus, what remains to be done is to characterize the circumstances under which regional-holism arises. In section 5.2 I will illustrate this for Abelian gauge theories (i.e. electromagnetism with a scalar field). For this simple case, I will explicitly show that the whole state is underdetermined by the regional states, and that the ensuing variety of universal states is equivalent to a copy of the charge group (U(1)). I will thus prove my main claim that there is a regional (or subsystem) rigid group of symmetries with (relational) DES, emerging from residual variety, as per Definition 2.

I will analyze the non-Abelian case in section C.1. There, a concession must be made: due to the non-linear character of the theory, gluing takes place at the perturbative level, and as such it requires the specification of which underlying configuration is to suffer the perturbations. Nonetheless, the formalism transforms covariantly with respect to gauge transformations of the perturbed configuration,[56] and one is able to retain, in the non-Abelian context, all the interesting results obtained in section 5.2. In appendix C, I will briefly review the interesting structure of gluing and regional-holism in Yang-Mills that is effaced in the Abelian case.

5.2   DES as a regional-holism

This is the section where we prove the main claims of this paper, I will organize it as follows: in section 5.2.1 I will explain why we use the gauge fields A to find unique gauge-invariant representations of the *doublet* $\varphi = (A, \varphi)$. To put it more dramatically, I answer the following natural question: non-locality creeps into $[\Phi]$ through $[\mathcal{A}]$, therefore wouldn't it be a simpler endeavor, not to mention a mathematically local one, to just parametrize the orbits of $[\Phi]$ using the matter fields? I will present the obstacles in this route.

Naturally, after answering this pressing question, we proceed in Section 5.2.2 to show precisely how we fix a unique representation for the regional physical content of $[\Phi_\pm]$ through the gauge fields, and an almost unique one for the universal physical states in $[\Phi]$. This completes the 'reduce' stage of our account. In section 5.2.3 we proceed to the actual gluing of these representatives. As advocated (for regions) in (Gomes, 2019) and (for coupling subsystems) in (Rovelli, 2014), we need to employ some gauge flexibility in order to compose the physical content of the individual regions into a universal physical state. This completes the 'external sophistication' step of our account. Finally, in Section 5.2.4 we put it all together, proving our main claim of rigid variety as per equation (2.1) and Definition 2.

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[56]This sounds like the BRST treatment of gauge theories (see (Henneaux & Teitelboim, 1992) for a review), by which one retains global transformations even if one eliminates the degeneracy in the propagator of the theory through a perturbative gauge-fixing. And indeed, the tools used in this work in the non-Abelian context (cf. appendix B.2) recover the properties of BRST ghosts; see (Gomes & Riello, 2017) and (Gomes et al., 2019), Section 3.1, for more on this.

5.2.1 Matter field parametrization of the reduced spaces $[\Phi], [\Phi_\pm]$

Before we begin in earnest the analysis of regional holism for the gauge fields, we need to adjudicate against the use of matter fields for the parametrization of the gauge orbits. Since the gauge transformations act on ψ linearly and 'ultralocally' (i.e. with no spacetime derivatives and with linear dependence on ψ, $\psi \mapsto \exp(i\xi)\psi$), they only act non-trivially on ψ where ψ does not vanish; (in contrast, gauge transformations may act on the gauge-fields everywhere, even where A vanishes). This implies that matter fields themselves can only be used to single out unique physical representatives (i.e. 1-1 parametrizations of the reduced, or quotient spaces $[\Phi], [\Phi_\pm]$, cf. Section 3.3 for the notation), if those matter fields are non-vanishing *everywhere* on spacetime.[57] It turns out that if they are non-vanishing everywhere, then they *do* parametrize gauge-invariant observables pointwise in spacetime, and so there is no "gap" between the two sides of (5.1). In other words, in that case there is *regional determination* according to Definition 2.

But generically, points in Φ have matter fields which *do* vanish somewhere.[58] Let us thus turn to the case in which ψ does vanish somewhere, so that we are obliged to parametrize $[\varphi]$ through the gauge-field.

5.2.2 Gauge field parametrization of $[\Phi], [\Phi_\pm]$ through $\mathcal{A}, \mathcal{A}_\pm$.

Given the regions $\Sigma_\pm$, we consider two states, $\varphi^\pm = (A^\pm, \psi^\pm)$. We need to uniquely characterize the physical content of such states.

In the literature concerning gauge theory in bounded regions (cf. (Regge & Teitelboim, 1974; Balachandran et al., 1996; Giulini, 1995; Strominger, 2018)), gauge transformations are usually restricted, or 'fixed by hand'—and not through a restriction on the states—at the boundary (see (Gomes, 2019) for a more thorough criticism and explanation of the standard procedure). But, to ensure that we retain the full physical content of the regional states and the capacity to relate arbitrary configurations, there must be no *prior* restrictions on the gauge transformations at the boundary: there should be no breaking or fixing of the gauge at the boundary. Similarly, even the interior gauge transformations of the previous of section 4 should *not* be singled out. This is crucial, both conceptually—why should the "redundant descriptive fluff" at the boundary be any different than in the bulk?—as well as technically.

Appendix B does the heavy-lifting. What it shows is that, *without loss of generality*, i.e. if we do not restrict the gauge transformations at the boundary, we can restrict our attention to those "fixed" $A_\pm^{\mathrm{f}}$ such that

$$\begin{cases} \mathrm{div}(A_\pm^{\mathrm{f}}) & = 0 \\ s \cdot A_\pm^{\mathrm{f}} & = 0 \end{cases} \tag{5.2}$$

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[57]If that is the case, we can build a field-space connection-form, as in appendix B.2, from the matter fields. Such a connection-form, described in section 7 of (Gomes et al., 2019), was called "the Higgs connection". It becomes important in considerations of spontaneous symmetry breaking as explained in the following footnote. It can also be used to "deflate" the Aharonov-Bohm effect (agreeing, in this setting, with the conclusions of (Wallace, 2014)).

[58]This consideration precedes the classical/quantum distinction: it applies to configuration space. An exception can be made for the Higgs field, which can eliminate part of the gauge redundancy (only of the Lie subgroup that is non-trivially acting on it, i.e. of the subgroup of SU(2) which does not stabilize the Higgs). This is essentially how spontaneous symmetry breaking appears in the description given by (François, 2019; Attard, François, Lazzarini, & Masson, 2018), and also in the context of the (functional) Higgs connection-form in (Gomes et al., 2019), section 7.

where s is the normal to the boundary S, and $\cdot$ is induced by the inner product on Σ. It might seem surprising that we can restrict our attention to such constrained boundary conditions on $A_\pm$ and yet still encompass the entire gamut of possible regional physical states. The reason for this is that any other state—including ones with other boundary behavior—differs from such a $A_\pm^{\mathrm{f}}$ by a unique regional gauge transformation; again, a regional gauge transformation *unrestricted at the boundary*. Such regional physical states will therefore also identify representations of the same state which are *not* related by an interior symmetry (see also section 4).[59]

More precisely, given any $A_\pm$ (even such that $s \cdot A_\pm \neq 0$), there exists a *unique* $A_\pm^{\mathrm{f}}$ and a *unique* (A-dependent) U(1) gauge transformations $\xi_\pm^{\mathrm{f}} \equiv \xi_\pm^{\mathrm{f}}(A_\pm)$ such that:

$$A_\pm^{\mathrm{f}} = A_\pm + \mathrm{d}\xi_\pm^{\mathrm{f}}(A_\pm); \tag{5.3}$$

(cf. footnote 61 for the uniqueness claim). The $\xi_\pm^{\mathrm{f}}(A_\pm)$ are defined by inserting (5.3) into (5.2) and solving for $A_\pm$.[60] The solution is non-local, but only within the region.

Under a gauge transformation $A \mapsto A^g$ the covariance properties of $\xi_\pm^{\mathrm{f}}(A_\pm)$ guarantee that

$$A_\pm + \mathrm{d}\xi_\pm^{\mathrm{f}}(A_\pm) = A_\pm^g + \mathrm{d}\xi_\pm^{\mathrm{f}}(A_\pm^g),$$

therefore, as a functional of $A_\pm$, the gauge-fixed $A_\pm^{\mathrm{f}}$ is indeed physical, or gauge-invariant one-form-valued functional, i.e.

$$A_\pm^{\mathrm{f}}(A_\pm^g) = A_\pm^{\mathrm{f}}(A_\pm) \tag{5.4}$$

(from now on I will omit the A-dependence of ξ^{f} and A^{f}).

Moreover, since matter fields 'co-rotate' with $A_\pm$, they just get "taken for the ride" in this gauge-fixing. That is, since the $\xi_\pm^{\mathrm{f}}$ in (5.3) are unique, a given doublet, $\varphi_\pm = (A_\pm, \psi_\pm)$ is also projected to a unique $\varphi_\pm^{\mathrm{f}}$, as in:

$$\varphi_\pm^{\mathrm{f}} := (A_\pm^{\mathrm{f}}, \psi_\pm^{\mathrm{f}}) = (A_\pm + \mathrm{d}\xi_\pm^{\mathrm{f}}, \exp\left(i\xi_\pm^{\mathrm{f}}\right)\psi_\pm) . \tag{5.5}$$

In the asymptotic flat case, $\exp\left(i\xi^{\mathrm{f}}\right)\psi$ is known as the "Dirac dressed electron" (Dirac, 1955). It is an electron that is "dressed" by an appropriate Coulombic tail, rendering the electron also gauge-invariant (as can be easily checked from the gauge-covariant transformation properties of ξ^{f}). The function $A_\pm^{\mathrm{f}}$ is known as the "radiative" projection of the photon. Here we have extended both notions to finite bounded regions (see (Gomes et al., 2019; Gomes & Riello, 2019)).

In sum, we find *unique* representatives for the full regional physical field content, and therefore we can identify (using the notaion '$\equiv$') the content with the representative:

$$[\varphi_\pm] \equiv \varphi_\pm^{\mathrm{f}} = (A_\pm^{\mathrm{f}}, \psi_\pm^{\mathrm{f}}). \tag{5.6}$$

To count/discern the possible universal physical states, we apply the corresponding gauge fixing for unbounded, finite regions. In the absence of boundaries, i.e. for the universal state A, the significance of (5.2) reduces to the familiar statement that the condition

$$\mathrm{div}(A^{\mathrm{f}}) = 0; \tag{5.7}$$

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[59] Although GW are careful to distinguish *theoretical* from *empirical* symmetries, they do single out interior symmetries, as we saw in section 4; this is denied by the present approach.

[60] That is, by: $\xi_\pm^{\mathrm{f}} = \nabla^{-2}(\mathrm{div}(A_\pm))$ where ∇^2 is the Laplacian with the following Neumann boundary conditions $s \cdot \xi_\pm^{\mathrm{f}} = s \cdot A_\pm$. This gives rise to a transversal photon with particular boundary conditions.

is a bona-fide (partial) gauge condition, called the (Euclidean) Lorentz gauge, or, in the non-Abelian setting, the (perturbative) covariant Landau gauge. It is 'partial' because a different choice, related to A^{f} by a constant shift, would still satisfy (5.7).[61] It implies that the representative of the universal physical state is only determined up to a global phase shift. This will be important in what is to come.

This discussion summarized, in the Abelian case, what is reported in the entirety of appendix B: where, for the sake of generality of the presentation, results are given for the non-Abelian Yang-Mills theory in the perturbative setting.

5.2.3 Gluing the regional physical states

But how can gluing the physical content of two states differ from just naively composing the two states themselves, as advocated by GW and presented in section 4.2? The answer is: because the obstruction to smooth composition may be one of mere representation; or, in the nomenclature of (Gomes, 2019), it may be a mere 'difference in perspective'. In other words: because two states may not match at the boundary, and yet still be conjoinedinto the same universal physical content.

In the Abelian setting that we focus on here, since we are assuming there is no matter at the boundary S, states that differ at the boundary can have their physical contents joined into a universal viable state if and only if:

$$(A^{\mathrm{f}}_+ - A^{\mathrm{f}}_-)_{|S} = \mathrm{d}\xi_S \tag{5.8}$$

where $\xi_S \in \mathbb{C}^\infty(S)$ is interpretable as a gauge transformation intrinsic to the boundary. Indeed, condition (5.8) is met if and only if there are no boundary charges at S, in which case the boundary is called 'fiducial' (Gomes & Riello, 2019). Thus we are able to glue A^{f}_+ and A^{f}_- forming some A^{f}, and therefore glue φ^{f}_+ and φ^{f}_- forming some φ^{f}, if and only if the boundary is fiducial.[62]

So far, we thus have (5.7) at the universal level (determining $[\varphi]$ up to a global phase), and (5.2) at the regional level (determining $[\varphi_\pm]$). Now, according to 'external sophistication of symmetries', advocated in section 1.1, from the 'external', or universal perspective, we should still distinguish the various different states $A^{\mathrm{f}}_\pm + \mathrm{d}\xi_\pm$. That is, from the outside perspective we refrain from imposing restrictions on $\xi_\pm$. We will need to employ this flexibility, since gauge 'readjustments' will be required to glue the (purely physical content of the) regional states $\varphi^{\mathrm{f}}_\pm$ given in (5.6), forming a (purely physical description of the) universal state, φ^{f}.

In other words, given $A^{\mathrm{f}}_\pm$ satisfying (5.2), we seek gauge-readjustments for gluing, $\xi^{\mathrm{gl}}_\pm$, such that the glued state:

$$A^{\mathrm{f}} := (A^{\mathrm{f}}_+ + \mathrm{d}\xi^{\mathrm{gl}}_+)\Theta_+ + (A^{\mathrm{f}}_- + \mathrm{d}\xi^{\mathrm{gl}}_-)\Theta_- \tag{5.9}$$

is *smooth* and satisfies (5.7) (we remind the reader that $\Theta_\pm$ are the characteristic function of the regions $\Sigma_\pm$, cf Section 3.3).

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[61] In the Lorentzian setting, one would still have a left-over freedom of choosing a universal harmonic function, $\xi^{\mathrm{f}} \to \xi^{\mathrm{f}} + \chi$ where $\nabla^2\chi = 0$. But for Σ simply-connected and of Euclidean signature, the only harmonic function is $\chi = c$, a constant. In the regional case, the residual freedom would be $\xi^{\mathrm{f}}_\pm \to \xi^{\mathrm{f}}_\pm + \chi_\pm$ where $\nabla^2\chi_\pm = 0$ *and* $s \cdot \mathrm{d}\chi = 0$, but the only solution to these two conditions are $\chi_\pm = 0$.

[62] If the boundary is not fiducial, we need to include boundary information separately, and, as stated above in Section 5.2.1, there will be no underdetermination in the gluing process. Translated, this agrees with the conclusion GW arrive at for the "Faraday cage", as being disanalogous to Galileo's ship: see item (3) in page 74 of (Wallace & Greaves, 2014) and foregoing discussion. Different charges at the boundary would change the physical state of the reference system, and therefore the significance of DES would not be purely relational, and thus beyond the scope of this paper.

This is the point where we wriggle away from the 'reduction position'. Although $A^{\mathrm{f}}_{\pm}$ are indeed gauge-invariant as functionals of $A_{\pm}$ (as per (5.4)), we can still apply gauge transformations "from the outside": these do not act on the regional representation $A_{\pm}$, but directly on the intrinsic physical state itself. Naively, these transformations simply add a gradient to a one-form-valued functional, $A^{\mathrm{f}}_{\pm}$.

A closed formula for solutions $\xi^{\mathrm{gl}}_{\pm}$ of (5.9), *as functions of the boundary difference* $(A^{\mathrm{f}}_{+} - A^{\mathrm{f}}_{-})_{|S}$, was found in (Gomes & Riello, 2019). For illustration purposes, I display the solution here:

$$\xi^{\mathrm{gl}}_{\pm} = \zeta^{\pm\Pi}_{(\pm)} \quad \text{with} \quad \Pi = \left(\mathcal{R}^{-1}_{+} + \mathcal{R}^{-1}_{-} \right)^{-1} \left((\nabla^2_S)^{-1} \mathrm{div}_S (A^{+} - A^{-})_{|S} \right),$$

where the subscript S denotes operators and quantities intrinsic to the interface surface S, $\zeta^{u}_{(\pm)}$ is a harmonic function on (resp.) $\Sigma_{\pm}$ with Neumann boundary condition $s^i \partial_i \zeta^{u}_{(\pm)} = u$, and $\mathcal{R}$ is the Dirichlet-to-Neumann operator (for the meaning of these operators, see the solution for the general non-Abelian Yang-Mills gauge theories, equation (C.4) in the appendix).

Unlike the $\xi^{\mathrm{f}}_{\pm}$ used for unique regional representation (which are solely functions of the $A_{\pm}$ being reduced), the $\xi^{\mathrm{gl}}_{\pm}$ used for gluing—which are functions of boundary differences—are underdetermined: there is a freedom associated to constant regional shifts: $\xi^{\mathrm{gl}}_{\pm} \to \xi^{\mathrm{gl}}_{\pm} + c_{\pm}$, for $c_{\pm} \in \mathbb{R}$, which was to be expected from the form of (5.9) (and the lack of boundaries for the joint region).

Note that this procedure relieves us of the difficult task which eventually defeated our attempt to physically realize DES as per Definition 3 (see section 4.3 and equation (A.7) in appendix A). That is, we don't need to invent a 'regional expression' of elements of a quotient group. Here, the regional elements which can potentially exhibit DES have already been "plucked out" from the malleable group of regional gauge transformations.[63] It is also important to note that the existence of a stabilizer is a gauge-invariant notion (and for non-Abelian Yang-Mills, the stabilizer itself covaries with choice of base point, see footnote 54). The underdetermination of $\xi^{\mathrm{gl}}_{\pm}$ in (5.9) is in fact independent of the gauge-fixing, and would be of precisely the same form had we found another complete regional gauge-fixing.[64]

5.2.4 Establishing the main claims

Therefore, given regional physical states $\varphi^{\mathrm{f}}_{\pm}$, here a conjunction of e.g. a transverse projection of the photons and a Dirac dressing of the charges, we can glue them to form a two-parameter collection of universal states:

$$\varphi^{\mathrm{f}}_{(c^{+},c^{-})} = (A^{\mathrm{f}}, \psi^{\mathrm{f}}_{(c^{+},c^{-})}) :=$$
$$((A^{\mathrm{f}}_{+} + \mathrm{d}\xi^{\mathrm{gl}}_{+}), \exp\left(i(\xi^{\mathrm{gl}}_{+} + c_{+})\right)\psi^{\mathrm{f}}_{+})\Theta_{+} + ((A^{\mathrm{f}}_{-} + \mathrm{d}\xi^{\mathrm{gl}}_{-}), \exp\left(i(\xi^{\mathrm{gl}}_{-} + c_{-})\right)\psi^{\mathrm{f}}_{-})\Theta_{-}$$
$$(5.10)$$

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[63] Crucially, the same occurs in the non-Abelian case. The difference is that there the regional rigid symmetries that are "plucked out" are state-dependent and not just generated by constants; i.e. they are Lie algebra valued fields which stabilize the perturbed states. In precisely the same way Killing directions of Riemannian metrics depend on the underlying metric which they stabilize, so to these stabilizer fields depend on the perturbed state they are required to stabilize, but there is always at most a finite-dimensional (i.e. rigid) space of such stabilizers (cf. Appendix C).

[64] However, finding an alternative is not an easy task: the gauge must be fixed entirely by constraints on A. See section 1.3.2 in (Gomes, 2019). That it is independent of the gauge-fixing can be traced back to the fact that stabilizers are gauge-invariant notions.

This would give us *two* copies of U(1) parametrizing the universal physical states compatible with the regional physical contents. But of course, if $c_+ \equiv c_-$, we have a global constant phase shift, which precisely matches the left-over underdetermination of the universal physical state $[\varphi]$ by (5.7) (cf. footnote 61 and comments at end of Section 5.2.2). Therefore, we are left with a physical variety parametrized by the difference, $\bar{c} := c_+ - c_-$, which is insensitive to any global phase shift and gives a single copy of U(1). We can always set $c_- = 0$ if we want to fix the subsystem Σ_- as the reference, and then regional-holism is encoded by a regional action of a rigid symmetry (ie. U(1)):

$$((A_+^{\mathrm{f}} + \mathrm{d}\xi_+^{\mathrm{gl}}), \exp(i(\xi_+^{\mathrm{gl}}))\psi_+^{\mathrm{f}}) \mapsto ((A_+^{\mathrm{f}} + \mathrm{d}\xi_+^{\mathrm{gl}}), \exp(i(\xi_+^{\mathrm{gl}} + \bar{c}))\psi_+^{\mathrm{f}}).$$

Finally, as a corollary of these constructions, we are able to state our main result of this section for gluing physical states:

Corollary 1 (Rigid variety for U(1)**)** *For electromagnetism as coupled to a Klein-Gordon scalar field in a simply-connected universe: given the physical content of two regions, $[\varphi_\pm]$, for matter vanishing at the boundary but not in the bulk of the regions, the universal state is underdetermined, resulting in a variety parametrized by an element of* U(1). *In the notation of* (2.1) *and Definition 2:*

$$[\varphi]_{(i)} = [\varphi_+] \cup_{(i)} [\varphi_-], \quad i \in I \simeq \mathrm{U}(1) \tag{5.11}$$

Thus we have found, in the case of electromagnetism coupled to a Klein-Gordon scalar field, a *rigid variety* through *regional-holism*, as defined in Definition 2 in Section 2.4. Moreover, by fixing a reference subsystem (Σ_-), we can construe this U(1) as acting as a group of (rigid), regional symmetries over Σ_+. Thus we have positively answered our question (5.1), posed in the beginning of Section 5, and fully vindicated our main claims: namely, that regional-holism is a source of empirically significant subsystem symmetries.

Note also the direct parallel with Galileo's ship: as in (5.11), we there also recognize subsystem-holism as encoded by the (external) action of a finite-dimensional group (in suggestive notation):

$$[\text{sea and ship}]_{(i)} = [\text{sea}] \cup_{(i)} [\text{ship}], \qquad i \in I = \text{Boosts} \ltimes \text{Euclidean}$$

and, once again, $[\text{sea and ship}]_{(i)} = [\text{sea and ship}]_{(i')}$ if and only if $i = i'$.

Agreed, the non-Abelian Yang-Mills case is more complicated: non-linearities render the corresponding I of the equation corresponding to (5.11) dependent on the physical state. Nonetheless, our constructions *are* valid at a perturbative level, i.e. one needs to first fix a ground state and then perturb it. If the perturbed state is the 'vacuum', i.e. $[A^*]$ for some $A^* = 0$, we recover the full Lie algebra of the gauge group through the analogue of (5.11). These questions are explored in appendix C.

6 Conclusions

6.1 Summary

Broadly, in this paper I have explored the role of subsystem-holism in the context of gauge theories, where I dissected the meaning and occurrence of "empirically significant subsystem symmetries"—whose existence and characteristics are still matters

of debate in the philosophy of physics. To make matters concrete, I have focused on Yang-Mills theory and restricted subsystems to be demarcated by spacetime regions.

In order to better adjudicate the debate between the opposing sides—represented by Greaves and Wallace (GW) on one side, and Brading and Brown (BB) on the other— it was first necessary to clear the ground by introducing new nomenclature. The standard nomenclature of local and global gauge symmetries is perfectly adequate if the system under study is the entire Universe, as is usually the case. But if one wants to discuss subsystems, and needs to distinguish between symmetries acting also at these different levels, the standard nomenclature becomes awkward. The awkwardness is apparent when one refers, e.g.: to 'a global subsystem symmetry'. The new nomenclature uses four labels so as to express two logically independent distinctions: namely whether symmetries act only regionally or universally, and whether they depend on an infinite or a finite set of parameters, i.e. whether they are rigid or malleable when acting throughout a spacetime region or throughout the universe. It also disentangles possible confusions with 'non-local and local' functions.

With these definitions in place, and in the specific setting of Yang-Mills theory in bounded regions, I found myself in agreement with GW in their criticism of BB. Namely, BB prematurely dismissed the possibility of DES tranformations by assuming that such a transformation would always create discontinuities in the boundary between the regions. The difference between GW and BB can be shortly summarized thus: one should be concerned with smooth composition of regional gauge and matter fields (GW), as opposed to a smooth composition of gauge transformations (BB).

But then I examined the GW interpretation of DES and also found it wanting. The interpretation, in terms of "boundary-preserving non-interior subsystem symmetries", can only be construed as an isomorph of a quotient of infinite-dimensional groups. Arguments for how this quotient could be instantiated by the regional action of a finite-dimensional group were given, and then discarded as non-unique at best, and ill-defined at worst. Without stipulating any action for this quotient group, its interpretation is left hazy.

The root of this unpleasant feature vitiating GW's relational construal of DES is the lack of regard for regional physical information. Namely, GW consider directly the gluing of the gauge and matter fields, as described in section 4. But the appropriate formulation of DES needs to traffic in regional gauge invariant variables, as it does for Galileo's ship.

This assumption led GW to '*Antiholism*': regional states should uniquely define the universal state. GW admit this condition fails for gauge theories in certain gauge-invariant bases (holonomies for the Abelian case), but it even clearly fails for Galileo's ship. It seems inconsistent with an appropriate definition of DES for gauge theories; gauge invariant quantities are to some extent non-local (L9 in Earman's classificatory scheme (Earman, 1987)), which means there is in principle room to explore between the whole and the sum of the parts. That is, there should be room for *holism*.

In different ways, both BB and GW implement criteria inherited from a universal perspective, not a subsystem-centric one. Namely, it is true that given the smooth universal state, one cannot apply a transformation which is discontinuous on the subsystem interface (vindicating BB's dismissal of DES). Similarly, given the universal state, its restrictions to regions do always smoothly join up at the

boundary (vindicating GW's construal of DES). But the question we want to ask—including GW—is: given first the subsystems, how can we join them? It is only from the subsystem-centric perspective that the relational construal of DES makes sense; if one deviates from this perspective, one becomes immediately vulnerable to BB's blocking of DES. In other words, if one is concerned first and foremost with the *decomposition* of universal quantities into regional ones, then all hands agree that both gauge transformations and states should smoothly glue without discontinuity. But if one is concerned with the *composition* of regional quantities, such discontinuities are not necessarily damning, as they may lack physical significance.

Having noticed these issues, we could cover the difference between the present results to GW and from there to BB as follows: one should be concerned with the smooth composition of the physical data, as given by equivalence classes, and not with smooth composition of gauge and matter fields in a given representation (GW), and much less with the smooth composition of gauge transformations (BB).

Finding the criteria for smooth composition of the information contained in the equivalence classes $[A]$ still presents a challenge. The reason is that, at least for gauge-fields (as opposed to matter fields), observables, or the quotient/reduced field configurations $[A]$ are, to "flog a dead horse", non-local, while the boundary is local. It doesn't seem enough to say two regional states don't match at a boundary. The question is: can their physical content be composed into a universal physically viable state?

The way we got around the problem of gluing physical content was to use an 'externally sophisticated view of symmetries', as originally advocated by Rovelli (Rovelli, 2014) (see also (Gomes, 2019) for the same issue in a context of regional subsystems). Namely, here I employed 'reduction' as a means to identify the regional and universal physical contents—each non-locally determined *within* its corresponding domain,—but then I employed sophistication for gluing.

We found that for electromagnetism as coupled to a scalar Klein-Gordon field—the Klein Gordon fields were taken to vanish at the boundary in between but not in the bulk within the regions—there can be multiple universal physical states formed by gluing the same regional physical states. That is, a failure of uniqueness in the gluing creates the gap which fits a variety of universal physical states. In other words, a variety of universal physical states can be built from the same regional physical states. These regional and universal states fulfill Definition 2 for *regional-holism*, and thereby also correspond to purely relational DES-bearing universal transformations, as described in Definition 1.

I want to highlight an important fact: even if I have used 'external sophistication', Corollary 1, including its significance for DES, is completely compatible with viewing gauge degrees of freedom as "descriptive fluff". The fact that the theory admits such a particular sort of redundancy is related to a the particular sort of non-locality of its gauge-independent degrees of freedom. It is this non-locality that is responsible for the gap between the regional gauge-invariant information and the universal gauge-invariant information; it is the gap from which DES emerges.

Incidentally, the work of (Gomes & Riello, 2019) reported here precisely delineates the sort of non-locality involved. For generic regional states in Yang-Mills theory, one does *not* have regional-holism: in most circumstances we can describe the physical state of the whole by describing the physical state of its composing regions.[65] When we cannot, subsystem-holism ensues, and with it, DES as I have

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[65]But the specification of each intrinsic regional state is (regionally) non-local. Cf. footnotes 5 and 66 for the relation to Myrvold's "global patchy separability".

defined it. Transformations with DES as per definition 1, seen as rigid symmetries externally applied to subsystems, are perfectly compatible, and indeed ultimately arise from, *a variety of universal* gauge-invariant states as expressed purely in terms of the *regional* gauge-invariant content, $\varphi_{\pm}^{\mathrm{f}}$.

Finally, for non-Abelian theories, in appendix C I reported on further results of (Gomes & Riello, 2019) showing that for non-simply connected manifolds, such a variety of universal physical states can be parametrized by sub-algebras of $\mathfrak{g}$: the Lie algebra of the charge group G. For both Abelian and non-Abelian Yang-Mills, for non-simply connected manifolds we get something beyond G: namely, we also get the first homology group, whose dimension is finite and given by the so-called first (equivariant) Betti number. Such a contribution from non-trivial topology was to be expected from the source of our results: holism.[66]

In the debate landscape, we thus locate ourselves somewhere in between the two opposing views. On the side of orthodoxy,[67] we find that indeed only rigid, but no malleable symmetries may have direct empirical significance in gauge theory. In fact, surprisingly, the rigid ones correspond to 'the global' gauge transformations, G (for non-Abelian: only for very particular perturbed configurations). On the side of GW, we find that gauge theories may indeed harbor symmetries that have direct empirical significance. But to force my conclusion into either pigeonhole, or even a combination, would be to shave off some of the important subtleties of this situation.

In a few words: for the physical, i.e. gauge-invariant, content of the gauge fields, the whole is more than the sum of the parts. Their difference manifests itself in a direct empirical significance of the charge group. The same can be said for Galileo's ship scenario, where the difference manifests itself in the direct empirical significance of boosts and translations.

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[66]Here I comment on the relation between regional-holism and Myrvold's 'global patchy-non-separability' (Myrvold, 2010, p.427), studied for electromagnetism in holonomy variables (cf. footnotes 22 and 5). Holonomies (or Wilson loops), in the absence of charged matter, give a (vastly overcomplete) basis of gauge-invariant variables (Barrett, 1991) and therefore we should be able to reproduce Myrvold's conclusions. For simply connected regions Σ like ours, by composing regional loops $\gamma_{\pm} \in \Sigma_{\pm}$ going in opposite directions at the boundary S it is true, as Myrvold argues, that we recover the gauge-invariant holonomy corresponding to a larger loop γ not contained in either region. Therefore, according to Myrvold, separability fails only for non-simply connected manifolds, where the holonomies of $\gamma_{\pm}$ cannot recover the universal holonomy of γ. I have two comments to make on the relation to the present work: (i) in the absence of matter, indeed non-trivial topology is the only source of regional-holism, and we recover Myrvold's conclusions. But with matter, we can close off curves which are not loops by putting charges at the ends of the curves, thereby obtaining gauge-invariant holonomies (cf. (Gomes et al., 2019, Sec. 4.3.2)). These sourced holonomies correspond to the variety we found for electromagnetism in corollary 1, which occurs even for simply-connected Σ when charged matter is present in the regions $\Sigma_{\pm}$. (ii) unfortunately, Myrvold's loop composition doesn't work in the same way for the non-Abelian theory: although the appropriate regional loops themselves will compose as curves in the manifold, Wilson loops—giving the gauge-invariant content of the holonomies—involve traces, and the traces ruin the composition properties: the corresponding regional gauge-invariant quantities do not compose. Our construction (see appendix C) gets around that. The conclusions, in Myrvold's nomenclature, are then: global patchy separability fails for non-Abelian theory if and only if: the manifold is non-simply connected, or charged matter is present inside the regions *and* furthermore this charged matter obeys (perturbative) regional conservation laws (i.e. and the perturbed gauge-field has stabilizers). These relations will be explained in mode detail in (Gomes, n.d.).

[67]In fact, in this point, we are largely in the side of Kosso (Kosso, 2000), and not so much on that of Brown and Brading, who want to claim that even rigid internal symmetries have no DES.

6.2   A bonus, a subtlety, and an issue

I will end with one bonus fact, not hitherto mentioned, one subtlety and one issue, both to be addressed in the future.

The bonus: Many issues arise within the symplectic treatments of gauge theories in bounded regions. Two of the most pressing were the fact that the generator of the gauge transformation was not symplectic, and the fact that the symplectic charges were spurious (Donnelly & Freidel, 2016; Gomes et al., 2019; Gomes & Riello, 2019). By using charges which only traffic in (representations of) [Φ] we resolve these two issues. But more importantly for this work, the non-trivial symplectic charges appearing in this framework are precisely related to the group I, obtained through Definition 2. They are related to the total charge of a region and to the stabilizers of the gauge field in that region. This also means that the charge is obtained, in a certain sense, by one region with respect to another.[68] It also exposes an enticing connection between DES and what is usually considered an *indirect* significance of gauge: namely, the occurrence of conserved charges (and their relations to the physical fields). In other words, we find a 1-1 relation between DES-symmetry generators and non-trivial conserved charges in the Yang-Mills setting.

The subtlety: If we had matter fields which were non-vanishing everywhere,[69] we could have used the matter fields as references, obtaining the gauge-independent information of the theory point by point. In this case, there would have been no *regional-holism*. This seems in line with both the fact that non-trivial topology can also give rise to DES, and that the Aharonov-Bohm effect can be deflated using non-vanishing matter fields (Wallace, 2014).

The issue: For some of the constructions on gluing, I employed an underlying assumption that the spacetime metric was Euclidean. For a Lorentzian, 3+1 framework, some subtleties need to be taken into account (Gomes & Riello, 2019), but I believe essentially the same result should follow. This point is important for the actual observability of these symmetries.

Acknowledgements

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[68]As Strocchi emphasizes in his brief review for philosophers (Strocchi, 2015):

"A local Gauss law implies that states carrying a (corresponding) global gauge charge cannot be localized; this means that the presence of a charge in the space time region O can be detected by measuring observables localized in the (spacelike) causal complement O' ; this represents a very strong departure from standard QFT, where "charges" in O are not seen by the observables localized in O'." (p. 12)

[69]And in the fundamental representation of the full group, G, not some subgroup, and without stabilizer subgroups. Also, matter fields will generically vanish somewhere, and so this supposition fails (but see footnote 58).

APPENDIX

Notation for Yang-Mills theory coupled to matter

We define, for $\mathrm{d}x^\mu$ a basis of 1-forms over Σ and τ_α a basis of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$

$$A = A_\mu^\alpha \mathrm{d}x^\mu \tau_\alpha \in \Lambda^1(\Sigma, \mathfrak{g}) \quad \text{and} \quad A^g := g^{-1}Ag + g^{-1}\partial g \tag{.1}$$

Infinitesimally, i.e. for an infinitesimal gauge-transformation $\xi \in \mathrm{Lie}(\mathcal{G})$, the gauge field transforms as

$$A \mapsto A + \mathrm{D}\xi \quad \text{where} \quad \mathrm{D}\xi := \partial\xi + [A, \xi]. \tag{.2}$$

For full generality we can introduce charged fermions in a fundamental representation: so for some vector space W, the 4-component Dirac spinor field (i.e. in $\mathbb{C}^4$) as:

$$\psi \in C^\infty(\Sigma, \mathbb{C}^4 \otimes W) \quad \text{and} \quad \psi^g = g\psi.$$

And I will write the joint configuration as

$$\varphi = (A, \psi)$$

I will denote the regional, unquotiented configuration spaces of each field sector (gauge field and matter, respectively) as $\mathcal{A}_\pm = \{A_\pm \in \Lambda^1(\Sigma_\pm, \mathrm{Lie}(G))\}$, $\Psi_\pm = \{\psi_\pm \in C^\infty(\Sigma_\pm, \mathbb{C}^4 \otimes W)\}$. And, for the joint configuration spaces, I write $\Phi_\pm = \{\varphi_\pm := (A_\pm, \psi_\pm) \in \mathcal{A}_\pm \times \Psi_\pm\}$, writing $\Phi, \Psi, \mathcal{A}$. I will omit the subscript $\pm$ for the corresponding universal configuration spaces. The restricted groups of gauge transformations will be denoted in analogous fashion: $\mathcal{G}_\pm = C^\infty(\Sigma_\pm, G)$, and all abstract quotient spaces are denoted by the square brackets, as in $[\Phi^\pm] := \Phi^\pm/\mathcal{G}_\pm$, and $[\Phi] := \Phi/\mathcal{G}$.

A   Realizing the GW-DES-quotient group

A.1   Extracting a regional group action

It is very difficult to describe the nature of the quotient $\tilde{\mathcal{G}}_+(\varphi_{+|S})/\mathcal{G}_{\mathrm{Id}_S}^+$—which represents DES according to GW—in general, but more can be said if we focus our analysis in two ways: i) we consider only the gauge transformations which are connected to the identity, i.e. of the type $g = \mathrm{Pexp}(t\xi)$ for $\xi \in \mathrm{Lie}(\mathcal{G})$ and the (path-ordered) exponential. ii) we will consider a 'best-case' scenario, i.e. consider the sort of boundary configuration that would admit the largest group of relationally non-trivial transformations.

For any given gauge-field A, the infinitesimal version of a stabilizing gauge transformation is given by ξ such that:

$$\mathrm{D}\xi = \partial\xi + [A, \xi] = 0 \tag{A.1}$$

This can be seen as the analogue to the Killing equation, which also takes an infinitesimal diffeomorphism transformation of the metric to vanish. In both gravity and Yang-Mills, there are at most a finite number of independent gauge directions which won't change the given configuration, i.e. which stabilize it. In other words, infinitesimally, any stabilizer sub-algebra is generated by $\mathcal{K} = \mathrm{Span}\{\xi_r\}_{r=1\cdots n}$, for

$\xi_r \in \mathrm{Lie}(\mathcal{G})$, where and n is bounded by the dimension of $\mathfrak{g}$. For gravity, the configuration with the largest dimension of $\mathcal{K}$ is Minkowski, for non-Abelian gauge fields, the universal configuration with the largest stabilizer possible is $\tilde{A} = 0$. One can see $\tilde{A}$ as vacuum, and uniform and so fixed by many transformations. We thus characterize $\mathcal{K}_{\mathrm{max}}$ by:

$$\mathrm{D}\xi = 0 = \partial(\xi^\alpha)\tau_\alpha = 0 \qquad (A.2)$$

where we have reintroduced the Lie algebra basis on the right hand side. Equation (A.2) therefore gives us one independent spacetime constant, c, per dimension of $\mathfrak{g}$. Therefore, this maximal Killing (or infinitesimal stabilizer) algebra is given by:[70]

$$\mathcal{K}_{\mathrm{max}} := \mathrm{Span}\{\xi_\alpha(x) = c_\alpha\}_{\alpha=1\cdots\dim(\mathfrak{g})} \qquad (A.3)$$

where I have made the a priori possible spacetime dependence of ξ explicit. Note that for any other configuration admitting a stablizer near the boundary, the stabilizer algebra will be of a dimension lower than that of $\mathfrak{g}$. Therefore, for our purposes of characterizing the largest possible algebra giving rise to (4.7), there is no loss in assuming we are in this 'best-case scenario'.

In accordance with our 'best-case' scenario analysis, we also suppose that $\psi_+ = 0$ in a neighborhood of the boundary, i.e. that $\partial^n \psi_{+|S} = 0$. With this assumption, any ξ_+ stabilize also the matter field near the boundary.

Following our 'best-case' scenario, we assume that the given configuration A_+ is such that $\partial^n A_{+|S} = 0$. Therefore the *boundary-preserving* gauge transformations will be given by $\xi_+ \in \mathrm{Lie}(\mathcal{G}_+)$ such that:

$$\tilde{\mathcal{K}}_+(\varphi^+{}_{|S}) := \mathrm{Span}\{\xi_+ \in \mathrm{Lie}(\mathcal{G}_+) \ : \ \partial^n \xi_{+|S} = \partial^n c_\alpha = \begin{cases} 0 & \text{if} \quad n \neq 0 \\ c_\alpha & \text{if} \quad n = 0 \end{cases} \} \qquad (A.4)$$

In other words, $\tilde{\mathcal{K}}_+$ is given by vector fields which approach a constant near the boundary. In the class of smooth functions, this boundary behavior does not fix the behavior of the elements of $\tilde{\mathcal{K}}_+$ in the interior of the regions.[71] In the smooth case, there is still an infinite-dimensional space of generators for $\tilde{\mathcal{K}}_+$, including the entirety of the interior symmetryes, generated by $c_\alpha \to 0$. We call the set of Lie algebra valued scalar fields

$$\mathcal{I}_+ := \{\xi_+ \in \mathrm{Lie}(\mathcal{G}_+) \ : \ \partial^n \xi_{+|S} = 0\} \qquad (A.5)$$

the interior symmetry generators. Such interior symmetries generate a normal subgroup of the boundary preserving symmetries which we would like to quotient out.

From (A.4) and (A.5), it seems $\mathcal{I}_+$ generates all the elements of $\tilde{\mathcal{K}}$ when $n \neq 0$, and therefore it intuitively seems that there are at most $\dim(\mathfrak{g})$ elements of $\tilde{\mathcal{K}}_+$ which are independent of elements of $\mathcal{I}_+$. But it is very difficult to formalize this intuition: it could easily depend on the differentiability class we are focusing. Nonetheless, in an attempt to formalize this intuition, we decompose general boundary stabilizing elements of the gauge transformation algebra by employing

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[70] It is left as an exercise to the reader to show that it really does form a Lie algebra (see (Kobayashi & Nomizu, 1963)).

[71] But were we to increase the class of differentiability from smooth to analytic, , determining all the derivatives of the function at the boundary would determine its value everywhere (in this case, this would amount to setting $\xi^+ = c_\alpha$).

the non-zero constants c_α throughout the manifold Σ_+, i.e. for $\xi_+ \in \tilde{\mathcal{K}}_+$ we can then uniquely write it as a sum:

$$\xi_+ = (\xi^o_{+\alpha}(x) + c_\alpha)\tau^\alpha \qquad \text{where} \qquad \xi^o_+ \in \mathcal{I}_+ \, , \, c^\alpha \in \mathbb{R}, \, \alpha = 1 \ldots \dim(\mathfrak{g}) \qquad \text{(A.6)}$$

For the purposes of finding a *relationally non-trivial transformation*, according to Definition 3, we saw that we could ignore, or, since the map $\xi \mapsto D\xi$ is linear, subtract away, any such ξ^o_+ as being interior. We then find that, at most, the regional transformations that can have direct empirical significance are generated by:

$$\xi^{\text{DES}} = c_\alpha \tau^\alpha \qquad \text{(A.7)}$$

In other words, the regional transformations allowed to have DES according to the GW criterion made explicit in Definition 3 are generated by at most $\dim(\mathfrak{g})$ elements, and are therefore also *rigid*.

A.2 Problem with this resolution

The major issue with this result concerns my attempt to decompose regional transformations into a part that would have DES, and a part that wouldn't (an interior symmetry), given in (A.6). Although this decomposition seems natural, nothing in the theory is singling it out. Even if another choice would have equally had a $\dim(\mathfrak{g})$ number of generators of '*relationally non-trivial transformations*', such generators could drastically differ from c_α away from the boundary S. Therefore, there is no unique action of these transformations on the bulk of the region, and we cannot single out those field-configurations φ_+, φ_- which are related by such supposedly directly empirically significant transformations.[72] In other words, there is no unique embedding of $\mathcal{G}^+_{\text{DES}}(\varphi_+)$ into $\mathcal{G}^+$.

A second issue arises in moving away from the 'best-case' scenario. For if A_+ does not vanish close to S, and yet still has near-boundary stabilizers, such stabilizers will not be constants. And therefore it becomes unclear how they should be continued away from S into the bulk of the region Σ_+.[73]

B The functional connection-form as a resolution to the issues of gauge-fixing in bounded regions

B.1 The issues with gluing gauge systems

This section will roughly summarize the arguments given by Rovelli in (Rovelli, 2014) and in section 1.3 in (Gomes, 2019), which point to the difficulties in making complete sense of equation (5.1).

In gauge theories, one is usually concerned mostly with the gauge-invariant/reduced/quotient space of observables. Gauge—as mere descriptive fluff—should be shaved off. I will here grant that reality should be fundamentally about gauge-invariant objects, *as*

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[72]I thank Aldo Riello for pointing this out to me.

[73]One could in principle have a unique continuation if the differentiability class of the gauge transformation was chosen as analytic, as opposed to C^∞. But another issue might arise from using distinct differentiability classes. In the analytic class, $\mathcal{G}^+_{\text{Id}_S}$ has a single element. Since $\tilde{\mathcal{G}}_+(\varphi_{+|S})$ is also finite-dimensional in that case, the quotient $\tilde{\mathcal{G}}_+(\varphi_{+|S})/\mathcal{G}^+_{\text{Id}_S}$ could be still be well-defined and of the right dimension, $\dim(\mathfrak{g})$. However, the dimension of the non-relational group of DES, $\mathcal{G}_+/\mathcal{G}^+_{\text{Id}_S}$ could crucially depend on the differentiability class. I have not investigated these issues here.

long as we have acess to the entire universe. For trading solely in gauge-invariant language we are unable to relate merely regional gauge-invariant data. This is true whether we employ standard notions of gauge-fixing, or find a way to describe regional systems solely in (regionally) gauge-invariant terms. In the first part of this subsection, I will briefly recount Rovelli's obstruction to gluing gauge-invariant data, and in the second I will deal more properly with the field-theoretic version and gauge-fixing.

B.1.1   Rovelli's spaceships –

For Rovelli, gauge-variant[74] objects are necessary to couple certain types of subsystems. To illustrate the idea, he describes two squadrons, each made up of N spaceships, which are just coming into contact with each other. Given q_j^1 as the position of the j-th ship in the first squadron, the Lagrangian for the first squadron is given by

$$L_1 = \frac{1}{2} \sum_{i=1}^{N-1} (\dot{q}_{i+1}^1 - \dot{q}_i^1)^2 \tag{B.1}$$

which has a time-dependent displacement symmetry acting as $\delta_1 q_i^1 := q_i^1 + f_1(t)$ (mutatis mutandis for the second squadron, with '2' in place of '1'). For each squadron one can find gauge-invariant variables by taking the difference in position of two ships $\bar{q}_i^1 = q_{i+1}^1 - q_i^1$, $i = 1, \cdots N - 1$ where the barred variables, $\bar{q}_i^1$ are gauge-invariant. Although there are N spaceships in each squadron, I can describe the subsystems individually in a gauge-invariant manner by using the gauge-fixed $N - 1$ variables; of course, I could have chosen many different parametrizations of the gauge-invariant variables.

Now the second squadron appears and fighting begins. Interaction terms between the two squadrons might no longer be independently gauge-invariant under δ_1, δ_2. They still could be, if expressed in terms of the barred variables, but for the example Rovelli gives, $L_{\text{int}} = \frac{1}{2}(\dot{q}_1^1 - \dot{q}_N^2)^2$, they are not.

To couple the subsystems, we must retain information about the origin of each subsystem. Note that one could do this by adding an extra variable which is regionally physically determined, say the center of mass of the squadron. One then expresses all physical quantities in terms of differences to the center of mass, while still leaving the center of mass itself as an arbitrary position variable in Euclidean space, e.g $q_{\text{cm}}^1(q_1^1, \cdots q_N^1)$. I.e. in each squadron there is a unique, relational representation of physical processes. When the two squadrons come together, there will be a natural readjustment of the two center of mass position variables, so that they coincide within Euclidean space and with the total center of mass; the readjustment is just

$$\delta q_{\text{cm}}^1 := q_{\text{cm}}(q_1^1, \cdots q_N^1, q_1^2, \cdots q_N^2) - q_{\text{cm}}^1(q_1^1, \cdots q_N^1) = \frac{1}{2}(q_{\text{cm}}^2 - q_{\text{cm}}^1). \tag{B.2}$$

In section B.2 we will perform a similar split with gauge theories: there will be a single description of physical processes, but that description will be flexible enough to accommodate merging/gluing/composition of subsystems.

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[74]'Gauge-variant' is prefearble to the double-negative 'gauge non-invariant', as Rovelli and others refer to the property.
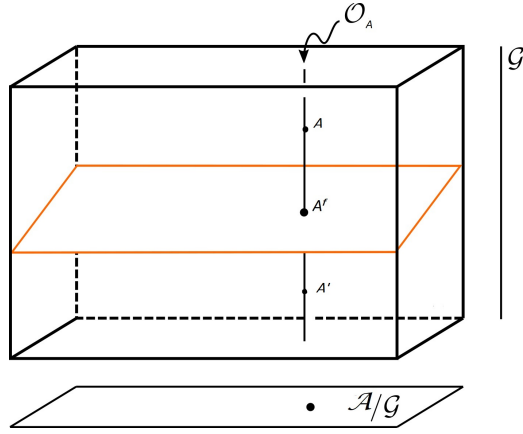
Figure 2: A gauge-fixing over the configuration space $\mathcal{A}$. $\mathcal{O}_A$ stands in for the *orbit* of A: all the configurations gauge related to A.

B.1.2   Problems with gauge-fixing –

To see how the same gap between regional and universal gauge-invariance pointed out by Rovelli appears in gauge field theories, we cannot resort to a description of the reduced/quotient/gauge-invariant space of observables, $\Phi/\mathcal{G}$, for there usually is no parametrization of this space. Instead, we must employ gauge-fixings. In terms of configuration space Φ, a gauge-fixing is a section, i.e. a graph over the quotient base space (or a 1-1 map between the quotient base space and its image), as in figure B.1.2.

Many obstacles arise in the treatment of gauge theories within bounded regions (see (Gomes et al., 2019; Donnelly & Freidel, 2016; Gomes & Riello, 2019; Gomes, 2019) for a review), but I will focus on only two:[75] *i)* the lack of flexibility of gauge-fixings, and *ii)* the unphysical boundary conditions for the gauge transformations.

We start with *i)*. This issue arises because we are considering more than one subsystem, or region, of the world.

We will fix the gauge by imposing conditions on the shape of the representation of the gauge field itself (ignoring matter for now). In other words, what is sought is a function $f(A) = 0$ which completely fixes the form of ξ in (.2). But, as (.2) evinces, the solution for ξ should be in some sense non-local, because in one sense or another, one must resolve the spacetime derivatives appearing in these equations.

Let us take the Abelian case (i.e. electromagnetism) for illustrative purposes. A common choice of gauge-fixing is given by $f(A) := \text{div}(A) = 0$. To verify whether choosing such a form of the gauge field fully fixes the representation ambiguity, we compute

$$\text{div}(A + \partial \ln g) = 0 \Rightarrow \nabla^2(\ln g) = \text{div} A \tag{B.3}$$

In the Yang-Mills case, the analogous to equation (B.3) is known as the Landau gauge (or the Lorenz gauge in Lorentzian signature), and its solutions have similar properties as in the Abelian case. Up to a constant gauge transformation, given an initial A, there is a unique solution $g(A)$, which fixes the representation of A as the package

$$A^{\text{f}} := A + \partial \ln g(A) \tag{B.4}$$

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[75] A large fraction of these obstacles appear within the most popular setting for gauge theories: the symplectic geometric setting. There, issues with symplectic charges and symplectic generators for gauge theories in bounded regions become very visible. I will not concern myself with these issues here.

Note that if we start with any other $A' = A^g$, i.e. another element $A' \in \mathcal{O}_A$ as per figure B.1.2, we will still obtain the same gauge-fixed configuration

$$A^{\mathrm{f}} = A' + \partial \ln g'(A') = A'^{\mathrm{f}} \tag{B.5}$$

where $g'(A')$ is the unique solution to $\nabla^2(\ln g') = \operatorname{div} A'$. When there are no boundaries, such a gauge-fixing procedure can eliminate all redundancy; we have a 1-1 relation between $[A]$ and A^{f}. Since we know precisely the form of A^{f}, given in (B.4), perhaps we can find out how the gluing in equation (5.1) works?

But alas, there are problems. A gauge-fixed description of the theory should traffic solely in the variables of the sort A^{f}. Of course, if this is taken to be the case, there is no room for any flexibility.[76] Given two regions, as in figure 3.3, given two $A^{\mathrm{f}}_{\pm}$'s which don't match at the boundary, there is no manner by which we can find a universal physical description, $A^{\mathrm{f}}_{\pm}$; there are no knobs to turn to readjust the two regional descriptions so that together they form a unified whole.

What we would like instead of a gauge-fixing is to leave the possibility of such a knob—in Rovelli's language, a 'handle'. In other words, to have regional unique descriptions for the physical content, but to somehow still allow transformations to shift it around. In the next section, we will see how this can be achieved.

The second issue arising in the presence of regions, is due to the boundary between these regions. The problem here is that, in the presence of boundaries, an equation of the form (B.3) does not fix a unique representative of the physics: it must be conjoined by a boundary condition. But the problem is not in the boundary condition itself, it is that one must stipulate a boundary condition *for the gauge parameter g*, without reference to the field-configuration itself. Say we have chosen $g^+_{|S} = 1$. Given an A_+, we solve (B.3) with this condition, obtaining A^{f}_+. But now suppose, instead of A_+, we decide to solve (B.3) for $A'_+ = A^{g'_+}_+$ which is such that $g'_{+|S} \neq 1$. Then we will find a different gauge-fixed configuration, i.e. unlike (B.5) we find $A^{\mathrm{f}} \neq A'^{\mathrm{f}}$. We could of course just forbid any gauge transformations at the boundary, but then a suite of questions follow: what if the boundary is just fiducial? If we are allowed to just fix the gauge there "by hand" as it were (as opposed to by stipulating a condition on the gauge field itself, as in f$(A) = 0$), why bother leaving the possibility of gauge redundancy elsewhere? Isn't gauge just descriptive fluff? Why should it acquire special status on (even imaginary) boundaries? What we need is to find a way of representing physical processes without giving the boundaries special treatment.

B.2   The field-space connection form

Intuitively, we want a "covariantized" version of figure B.1.2. We want to be able to strike some balance between a unique physical representation and some flexibility to match between different physical representations. We want, something like B.2, which provides a foliation of configuration space, and not just a section.[77]

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[76]Perhaps this is best seen in the Hamiltonian formalism. There gauge transformations come from (symplectic) flows in phase space, and these flows are generated by constraints: phase space functions of the form $f(p,q) = 0$. Upon gauge-fixing, one finds a dual constraint, $h(p,q) = 0$. For instance, suppose $f(p,q) = p_1$, the momentum coordinate of the first particle. Then a natural gauge-fixing is to fix also the position of that particle, by $h(p,q) = q_1$. Then one can in effect set these two variables to zero together, erasing them from the description of the system. They are no longer included in the (reduced) phase space.

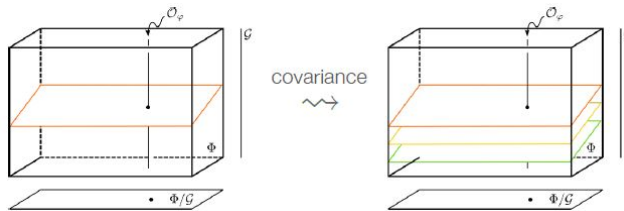[77]Moving between leaves is essentially equal to performing BRST transformations.

Figure 3: A covariant foliation over the configuration space $\mathcal{A}$.

It turns out such a balance exists. To reach it, we must build specific tools that work at the infinitesimal level, for the small gauge transformations (i.e. those generated by $\mathrm{Lie}(\mathcal{G})$). These geometric tools have been described elsewhere, here I give merely a summarized version of their construction.

At the infinitesimal/perturbative level, such a covariant treatment is equivalent to the following: instead of gauge-fixing we apply a decomposition: we decompose the perturbation δA into a unique representative of the physical content—as in δA^{f}—and a "pure gauge component of δA. This pure gauge component functions like a handle, or a knob, we can readjust to match with other physical processes.

In effect, we perform a type of Helmholtz decomposition for the perturbative gauge fields (Gomes et al., 2019): for 3-dimensional vector fields X, Helmholtz proved that one can uniquely decompose $X = X_{\mathrm{c}} + X_{\mathrm{d}}$, with $\mathrm{curl}(X_{\mathrm{d}}) = 0 = \mathrm{div}(X_{\mathrm{c}})$. For topologically simple regions, we can write $X_{\mathrm{d}} = \mathrm{grad} f$, for some $f \in C^{\infty}(M)$. We can extend the decomposition to higher dimensions, in the presence of boundaries, and for vector fields valued in a non-Abelian Lie-algebra. We will decompose an infinitesimal process, i.e. a perturbation of the fields, into an element which is pure gauge, $X_{\mathrm{d}} = \mathrm{grad} f$, and an element which uniquely represents the physics—an element corrsponding to a gauge-fixed version of the field—X_{c}.[78]

The geometric insight that allows us to generalize the Helmholtz decomposition is the following: the standard Helmholtz decomposition is just an orthogonal decomposition in configuration space, with respect to a particular inner product in configuration space. I.e. for differential forms $\lambda, \lambda_{\perp} \in \Lambda^n(\Sigma), \kappa \in \Lambda^{n-1}(\Sigma)$ (Σ is topologically simple):

$$\lambda = \mathrm{d}\kappa + \lambda_{\perp}, \qquad \lambda_{\perp} \quad \text{unique and such that} \quad \mathrm{d}^*\lambda_{\perp} = 0$$

Coming from $\langle \mathrm{d}\kappa, \lambda_{\perp} \rangle = 0$ where, e.g. for $\lambda \in \Lambda^1(\Sigma)$:

$$\langle \lambda_1, \lambda_2 \rangle := \int_{\Sigma} \mathrm{d}^d x \sqrt{g}\, g_{ij}\, \lambda_1^i\, \lambda_2^j \tag{B.6}$$

and the adjoint d^* is similarly defined by the adjoint: $\langle \gamma', \mathrm{d}\kappa' \rangle = \langle \mathrm{d}^*\gamma', \kappa' \rangle$. In sum, given an inner product in the space of fields, Helmholtz (or Hodge) establishes orthogonal decomposition.

This decomposition can be precisely formulated in terms of a projector onto the gauge orbits. Or, more precisely, a projector onto the tangent space of the orbits, which is called a *vertical space*. Since the tangent space to the orbits is isomorphic to $\mathrm{Lie}(\mathcal{G})$ (which generates it by acting on the configurations),[79] we can represent

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[78] For a more complete analysis regarding the relevant extensions of the Helmholtz decomposition in the presence of boundaries, and for non-Abelian theories, see section 5 of (Gomes & Riello, 2019).

[79] Barring some important obstructions which we will discuss in section 5.2.
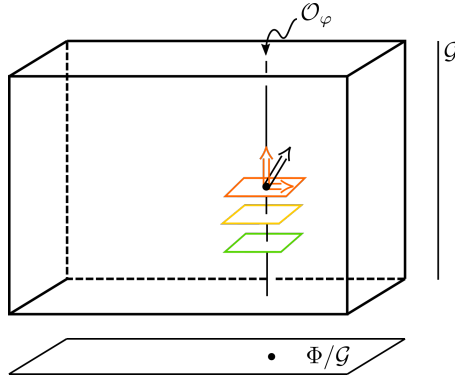
Figure 4: An orthogonal decomposition of a perturbation to a given field configuration. The upwards pointing component is vertical, or pure gauge, component, and the the right-pointing is a horizontal, or physical component of the perturbation. The different colors at different levels of the orbit represent the covariance properties of the decomposition.

such projectors by what is called a functional connection form, $\varpi \in \Lambda^1(\Phi, \text{Lie}(\mathcal{G}))$, i.e. a connection-form in the infinite-dimensional configuration space, seen as a principal fiber bundle (for more on this, see (Gomes & Riello, 2017; Gomes et al., 2019) and references therein).

And we can now decompose a general element of the tangent space to A, namely, $\delta A \in T_A \mathcal{A}$, into a pure gauge part, and a unique physical representative part:

$$\delta A = (\delta A - \text{D}(\varpi(\delta A))) + \text{D}(\varpi(\delta A)) \tag{B.7}$$

The term inside the parenthese is purely 'horizontal', i.e. orthogonal to the orbits, or as per the Helmholtz decomposition, called δA_c, and the second term is just the effect of an infinitesimal gauge transformation, $\xi = \varpi(\delta A) \in \text{Lie}(\mathcal{G})$, as applied to the base configuration A. If δA is given by the original black double-arrow in figure B.2, then $\text{D}(\varpi(\delta A))$ is the upwards pointing red component, and $(\delta A - \text{D}(\varpi(\delta A)))$ is the right-pointing red component.

To write down the precise form of ϖ, we use the natural generalization of (B.6) to the non-Abelian case, which requires the use of a trace, $\text{d} \to \text{D}$, given in (.2), and the consideration of a bounded Σ. Nonetheless, it gives precise equations for the vertical projector (as encoded in ϖ) which comes endowed with covariant boundary conditions. Namely, the defining equations for ϖ turn out to be:

$$\begin{cases} \text{D}^2(\varpi(\delta A)) & = \text{D}^i \delta A_i \quad \text{(bulk)} \\ s_i \text{D}^i(\varpi(\delta A)) & = s^i \delta A_i \quad \text{(boundary)} \quad \text{field-dependent, covariant} \end{cases} \tag{B.8}$$

where s^i is the normal to the boundary S, and $\text{D}^2 = \text{D}^\mu \text{D}_\mu$. Such field-dependent boundary conditions allow us to bypass the limitations of the second objection above, namely, that we had to fix all gauge transformations even at fiducial boundaries.

By suitably adjusting the leftover pure gauge component of the regional perturbations, one one is able to overcome the obstructions involved in gluing the gauge-invariant data across regions. The question then is: given two unique regional representations of the physics as per the generalized Helmholtz decomposition, $X_\text{c}^\pm$, can we exploit the regional gauge transformations to yield a unique representation of the universal physics, X_c? Yes, we can, and I briefly report a proof of this in the next section.

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C Gluing for non-Abelian Yang-Mills theory

In the non-Abelian case we have the equations analogous to (5.2) and (5.7), namely:

$$\begin{cases} D^i \delta A_i^{\pm} & = 0 \\ s^i \delta A_i^{\pm}{}_{|S} & = 0 \end{cases} \tag{C.1}$$

$$D^i \delta A_i = 0 \tag{C.2}$$

And we use apply them to establish gluing through the analogous to (5.9)

$$\delta A := (\delta A^+ + D\xi^+)\Theta_+ + (\delta A^- + D\xi^-) \tag{C.3}$$

We obtain the following conditions on $\xi_{\pm}$:

$$\begin{cases} D^2 \xi_{\pm} = 0 & \text{on} & \Sigma_{\pm} \\ s_i D^i (\xi_+ - \xi_-) = 0 & \text{on} & S \\ (\xi_+ - \xi_-)_{|S} = (D_S^2)^{-1}(D_S^c(\delta A_c^+ - \delta A_c^-)_{|S}) & \text{on} & S \end{cases}$$

(subscript S means intrinsic to the surface).

We can solve these emerging conditions (see section 4 in (Gomes & Riello, 2019)), obtaining:

$$\xi^{\pm} = \zeta_{(\pm)}^{\pm \Pi} \quad \text{with} \quad \Pi = \left(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1} \right)^{-1} \left((D_S^2)^{-1} D_S^c (\delta A_c^+ - \delta A_c^-)_{|S} \right) \tag{C.4}$$

where, in each region, ζ^{ϕ} stands for a covariantly harmonic function, satisfying $D^2 \zeta = 0$, with Neumann boundary conditions at S given by $s^i D_i \zeta_{|S} = \phi$, and where the subscript S denotes "intrinsic to S", and the intrinsic coordinates to S are given by c, and where $\mathcal{R}$ is the so-called Dirichlet-to-Neumman operator. Briefly, $\mathcal{R}$ functions as follows: a given harmonic function with Dirichlet conditions—these conditions are the input of $\mathcal{R}$—will possess a certain normal derivative at the boundary; i.e. will induce certain Neumann conditions there—these conditions are the output of $\mathcal{R}$. That is, let ζ^u be a harmonic function with Neumann boundary condition, then for ζ_u a covariantly harmonic function with Dirichlet boundary condition, $(\zeta_u)_{|S} = u$, the Dirichlet to Neumann operator $\mathcal{R}$ is defined as $\zeta_u = \zeta^{\mathcal{R}(u)}$ (i.e. it finds the harmonic function with Neumann condition corresponding to one with a Dirichlet condition).

C.1 Non-Abelian regional-holism

There are two sources of underdetermination of solutions to (C.3): one topological and one from homogeneous fields. The first arises from the possibly non-trivial first (equivariant) homology group of Σ, we look at the first in section C.1.1 and at the second in section C.1.2.

C.1.1 Topological holism

We may have a universal field δA satifying (C.2) which, when restricted to each $\Sigma_{\pm}$ is pure gauge, i.e. of the form $D\xi_{\pm}$. In other words, the room for discrepancy is equivalent to broken homology cycles. This is easier to see in the Abelian case, where δA would be a one form which is exact in the simply connected patches,

but only closed in the entire manifold (see (Gomes & Riello, 2019), section 4.7 for an example). By the Poincaré lemma, this occurs if and only if the topology of the manifold is non-trivial. In other words, universal physical processes in this case may not come from regional physical processes; some physical processes are universal/global in nature.

But of course for topologically non-trivial manifolds we should expect the whole to contain more information than the sum of the parts, foiling *Antiholism*! The arising regional-holism would contribute with the suitable homology group to the variety set I in (2.1). Such a contribution has a finite number of generators (given by the first Betti number, i.e. the rank of the first homology group).[80]

C.1.2   Stabilizer ambiguity

Barring non-trivial topology, is there another source of underdetermination of $\xi^{\pm}$ also in the non-Abelian case? To arrive at an answer, note that we are using the gauge-fields as a reference. It is through them that we fix the ξ and parametrize the reduced/quotient configuration space $[\Phi] = \Phi/\mathcal{G}$. Therefore one source of regional-holism would be a possible underdetermination of $\xi_{\pm}$ by (5.9). And indeed, although (in topologically simple spacetimes) the solution to (5.9) uniquely determines the glued field δA, equation (5.9) does not *uniquely* determine $\xi^{\pm}$. As described in (Gomes, 2019), the gauge fields may have some transformations in their 'blind-spots'.

This underdetermination occurs if and only if there exist $\chi_{\pm} \in \mathrm{Lie}(\mathcal{G})$ such that

$$\mathrm{D}\chi_{\pm} = \partial\chi_{\pm} + [A_{\pm}, \chi_{\pm}] = 0 \tag{C.5}$$

for $A = A_{+}\Theta_{+} + A_{-}\Theta_{-}$ being the smooth field configuration around which we are considering perturbations.[81] This is, again, the "Killing" (or infinitesimal stabilizer) equation for gauge transformations (see appendix A and equation (A.1)). Namely, underdetermination of $\xi_{\pm}$ occurs if and only if there exist $\chi_{\pm}$ which *stabilize* the entire $A_{\pm}$, in which case we could use either $\xi_{\pm}$ or $\xi_{\pm} + \chi_{\pm}$ for gluing.

Such stabilizers are generalizations of the constant potential shifts in electromagnetism. But here, unlike what we found in section 4.2, in equation (4.10), it does not suffice to leave the states at the boundary unchanged; here the entire regional field must be stabilized by $\chi_{\pm}$! Thankfully, this regional property relieves us of the difficult task which eventually defeated us in our attempt to physically realize DES as per Definition 3: we don't need to build a regional continuation of elements of a quotient group. Here, the regional elements which can potentially exhibit DES have already been plucked out from $\mathcal{G}^{+}$.

As with Killing directions for spacetime geometries, such elements are hard to come by: they're generically trivial (for non-Abelian charge groups G), and, when non-trivial, they are generated by a finite-dimensional basis.

In sum: there may still be some small variety in $\xi^{\pm}$, which is innocuous as far as the gauge field is concerned. However, as we will shortly see, this variety can be relationally felt by regional matter fields.

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[80]This topological fact should be consequential to the labeling of the inequivalent representations of the θ vacua (see (Strocchi, 2015)).

[81]This assumption is not crucial, since the way we implement the perturbative gauge fixing is covariant with respect to gauge transformations in $\mathcal{A}$.

C.1.3 Obtaining variety from the stabilizer ambiguity

Let us write the matter field counterpart to (5.9):

$$\delta\psi = (\delta\psi^+ + \xi^+\psi_+)\Theta_+ + (\delta\psi^- + \xi^-\psi) \tag{C.6}$$

Since we are not using the matter fields to parametrize the quotient/moduli/reduced configuration spaces, there are no analogues of equations (5.2) and (5.7) to be implemented for it. But smoothness of the field does impose certain conditions. In fact, different choices of stabilizer, adding $\chi_+ \neq \chi_-$ to ξ_+, ξ_-, respectively, will render the matter fields incompatible at the boundary.[82]

Therefore, if the matter fields do *not* vanish at the boundary, we can assume there is again regional determinacy as per Definition 2. So we assume they vanish there; a condition postulated by GW as "dynamical isolation" of the two regions. And also, if the matter field vanishes not only at the boundary, but everywhere, we are back to (5.9), and no regional-holism arises, yet again. Thus, I assume: *the matter fields vanish at the boundary, but not in the bulk* of the regions.

To illustrate the precise emergence of the Lie algebra $\mathfrak{g}$, we resort to a 'best-case' scenario, as in appendix A. That is, apart from the conditions stipulated above for the matter fields, we will take the gauge field configuration around which we are perturbing to be the 'vacuum', $A = 0$. In that case, the gauge covariant differential just becomes the standard differential (as in the Abelian, electromagnetic case), i.e. $\mathrm{D} \to \partial$, and the stabilizer equation becomes $(\partial\chi^\alpha(x))\tau_\alpha = 0$. Then not only is the space of stabilizers χ a finite-dimensional vector space, closed under commutation, but for this case it forms a Lie algebra homomorphic to $\mathfrak{g}$.

Then we have a parametrized solution

$$\delta\psi_{(\chi^+,\chi^-)} = (\delta\psi^+ + (\xi^+ + \chi^+)\psi_+)\Theta_+ + (\delta\psi^- + (\xi^- + \chi^-)\psi) \tag{C.7}$$

But we are still not done. For $\chi^+ = \chi^-$, $\delta\psi$ is still universally gauge equivalent to $\delta\psi_{(\chi^+,\chi^-)}$, i.e. they differ by a universal, rigid gauge transformation. Therefore, what really matters for universal variety is the *difference*: $\chi_+ - \chi_-$. This difference is generated by just one copy of $\mathfrak{g}$. That is, given the $\delta A_\pm, \delta\psi_\pm$, we have:

$$(\delta A, \delta\psi_{(i)}) = (\delta A^+ + \mathrm{D}\xi^+_{(i)}, \delta\psi^+ + \xi^+_{(i)}\psi^+)\Theta_+ + (\delta A^- + \mathrm{D}\xi^-, \delta\psi^- + \xi^-\psi^-)\Theta_- \tag{C.8}$$

where $\xi^+_{(i)}(x) := (\xi^+_\alpha(x) + i_\alpha)\tau^\alpha$, for spacetime constant coefficients i^α, with τ_α a basis of $\mathfrak{g}$ (which also implies $\mathrm{D}\xi^+ = \mathrm{D}\xi^+_{(i)}$), i.e. the i parametrize the Lie algebra of the charge group G.

Finally, we have arrived at our destination: I have shown that the variety set I, at least infinitesimally and in the best-case scenario, can in some sense recover the charge group G, i.e. I have characterized the infinitesimal version of (5.1), with:

$$[\delta\varphi^{(i)}] = [\delta\varphi^+] \cup^S_{(i)} [\delta\varphi^-], \quad i \in \mathfrak{g}, \quad \text{with} \quad [\delta\varphi^{(i)}] \neq [\delta\varphi^{(i')}] \quad \text{iff} \quad i \neq i' \tag{C.9}$$

For non-Abelian groups, it is not clear how to obtain the finite version of this equation, replacing $\delta\varphi$ with φ, i.e. with

$$[\varphi^{(i)}] = [\varphi^+] \cup^S_{(i)} [\varphi^-], \qquad i \in I \simeq G \tag{C.10}$$

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[82] That is, unless $\chi^\pm$ also stabilize $\psi^\pm$. For non-Abelian fields, there may still be internal stabilized directions, i.e. for particular configurations $\tilde\psi_\pm \neq 0$, there may be $\tilde\chi_\pm \neq 0$ such that $\tilde\chi_\pm\psi_\pm = 0$. This is not true for U(1), there $\tilde\chi_\pm\psi_\pm \neq 0$ for any non-vanishing gauge transformation and matter field. But I do not know of any non-trivial, shared stabilizers for both fields in such a situation.

But in the Abelian case, the tools utilized here *do* allow an integration to the finite setting.[83]

Moving away from the best-case scenario of $A = 0$: to obtain *regional-holism* the base configuration A must still be somewhat homogenous; it must admit internal directions which leave it unchanged. But as mentioned above, a general theorem about Killing fields (and stabilizers), shows that Killing directions are always generated by a finite dimensional basis; in the case of internal gauge transformations, this basis consists of at most $\dim(\mathfrak{g})$ elements, closed under commutation, etc. Thus even away from the best-case, we obtain I's isomorphic to sub-algebras of $\mathfrak{g}$ (we would also have to replace τ_α in the definition of $\delta\psi_{(i)}$ for a choice of basis $\{\chi_\alpha\}$ of the appropriate Killing, or stabilizing fields).

Note that, as had to be the case, according to Definition 2, there exists a *regional variety* implicit in equation (C.8), and this variety recovers also the definition of DES through transformations, if the transformations referred to in Definition 1 are taken to be e.g.: $(\delta A, \delta\psi) \to (\delta A, \delta\psi_{(i)})$.

References

Attard, J., François, J., Lazzarini, S., & Masson, T. (2018). The dressing field method of gauge symmetry reduction, a review with examples. In J. Kouneiher (Ed.), *Foundations of mathematics and physics one century after hilbert: New perspectives* (pp. 377–415). Cham: Springer International Publishing. Retrieved from https://doi.org/10.1007/978-3-319-64813-2_13 doi: 10.1007/978-3-319-64813-2_13

Balachandran, A. P., Chandar, L., & Momen, A. (1996). Edge states in gravity and black hole physics. *Nucl. Phys.*, *B461*, 581-596. doi: 10.1016/0550-3213(95)00622-2

Barnich, G., & Brandt, F. (2002). Covariant theory of asymptotic symmetries, conservation laws and central charges. *Nucl. Phys.*, *B633*, 3-82. doi: 10.1016/S0550-3213(02)00251-1

Barrett, J. W. (1991, Sep 01). Holonomy and path structures in general relativity and yang-mills theory. *International Journal of Theoretical Physics*, *30*(9), 1171–1215. Retrieved from https://doi.org/10.1007/BF00671007 doi: 10.1007/BF00671007

Belot, G. (1998, 12). Understanding Electromagnetism. *The British Journal for the Philosophy of Science*, *49*(4), 531-555. Retrieved from https://doi.org/10.1093/bjps/49.4.531 doi: 10.1093/bjps/49.4.531

Bleecker, D. (1981). *Gauge theory and variational principles.* Dover Publications.

Brading, K., & Brown, H. R. (2000). Noether's theorems and gauge symmetries.

Brading, K., & Brown, H. R. (2004). Are gauge symmetry transformations observable? *The British Journal for the Philosophy of Science*, *55*(4), 645–665. Retrieved from http://www.jstor.org/stable/3541620

Brading, K., & Castellani, E. (Eds.). (2003). *Symmetries in physics: Philosophical reflections.* Cambridge University Press.

Camps, J. (2019). Superselection Sectors of Gravitational Subregions. *JHEP*, *01*, 182. doi: 10.1007/JHEP01(2019)182

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[83]This is related to the 'dressing formalism", see section 9 of (Gomes et al., 2019) and (François, 2019; Attard et al., 2018).

Casini, H., Huerta, M., & Rosabal, J. A. (2014, Apr). Remarks on entanglement entropy for gauge fields. *Phys. Rev. D*, *89*, 085012. Retrieved from `https://link.aps.org/doi/10.1103/PhysRevD.89.085012` doi: 10.1103/PhysRevD .89.085012

Caulton, A. (2015). The role of symmetry in the interpretation of physical theories. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, *52*, 153 - 162. Retrieved from `http://www.sciencedirect.com/science/article/pii/S1355219815000635` doi: https://doi.org/10.1016/j.shpsb.2015.08.002

Crnkovic, C., & Witten, E. (1987). Covariant description of canonical formalism in geometrical theories. in "three hundred years of gravitation.". In S. W. Hawking & W. Israel (Eds.), *Three hundred years of gravitation* (p. 676-684). Cambridge.

Dewar, N. (2017, 09). Sophistication about Symmetries. *The British Journal for the Philosophy of Science*, *70*(2), 485-521. Retrieved from `https://doi.org/10.1093/bjps/axx021` doi: 10.1093/bjps/axx021

Dirac, P. A. M. (1955). Gauge invariant formulation of quantum electrodynamics. *Can. J. Phys.*, *33*, 650. doi: 10.1139/p55-081

Donnelly, W., & Freidel, L. (2016). Local subsystems in gauge theory and gravity. *JHEP*, *09*, 102. doi: 10.1007/JHEP09(2016)102

Donnelly, W., & Wall, A. C. (2015). Entanglement entropy of electromagnetic edge modes. *Phys. Rev. Lett.*, *114*(11), 111603. doi: 10.1103/PhysRevLett.114 .111603

Dougherty, J. (2017). Sameness and separability in gauge theories. *Philosophy of Science*, *84*(5), 1189-1201. Retrieved from `https://doi.org/10.1086/694083` doi: 10.1086/694083

Earman, J. (1987, September). Locality, nonlocality and action at a distance: A skeptical review of some philosophical dogmas. In R. Kargon, P. Achinstein, & W. T. Kelvin (Eds.), *Kelvin's baltimore lectures and modern theoretical physics : historical and philosophical perspectives* (pp. 449 – 490). Cambridge: MIT Press. Retrieved from `http://d-scholarship.pitt.edu/12972/`

Earman, J. (2003). Tracking down gauge: An ode to the constrained hamiltonian formalism. In K. Brading & E. Castellani (Eds.), *Symmetries in physics: Philosophical reflections* (pp. 140–62). Cambridge University Press.

Earman, J. (2019, May 01). The role of idealizations in the aharonov–bohm effect. *Synthese*, *196*(5), 1991–2019. Retrieved from `https://doi.org/10.1007/s11229-017-1522-9` doi: 10.1007/s11229-017-1522-9

Fischer, A. (1970). The theory of superspace. In *Proceedings of the relativity conference held 2-6 june, 1969 in cincinnati, oh. edited by moshe carmeli, stuart i. fickler, and louis witten. new york: Plenum press, 1970., p.303.*

François, J. (2019). Artificial versus substantial gauge symmetries: A criterion and an application to the electroweak model. *Philosophy of Science*, *86*(3), 472-496. Retrieved from `https://doi.org/10.1086/703571` doi: 10.1086/703571

Friederich, S. (2014, 04). Symmetry, Empirical Equivalence, and Identity. *The British Journal for the Philosophy of Science*, *66*(3), 537-559. Retrieved from `https://doi.org/10.1093/bjps/axt046` doi: 10.1093/bjps/axt046

Geiller, M. (2017). Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity. *Nucl. Phys.*, *B924*, 312-365. doi: 10.1016/j.nuclphysb.2017.09 .010

Giulini, D. (1995). Asymptotic symmetry groups of long-ranged gauge configurations. *Modern Physics Letters A*, *10*(28), 2059-2070. Retrieved from `https://doi.org/10.1142/S0217732395002210` doi: 10.1142/S0217732395002210

Gomes, H. (n.d.). More on gauge theory and holism. *(to appear)*.

Gomes, H. (2019). Gauging the boundary in field-space. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*. Retrieved from `http://www.sciencedirect.com/science/article/pii/S1355219818302144` doi: https://doi.org/10.1016/j.shpsb.2019.04.002

Gomes, H., Hopfmüller, F., & Riello, A. (2019). A unified geometric framework for boundary charges and dressings: Non-abelian theory and matter. *Nuclear Physics B*, *941*, 249 - 315. Retrieved from `http://www.sciencedirect.com/science/article/pii/S0550321319300483` doi: https://doi.org/10.1016/j.nuclphysb.2019.02.020

Gomes, H., & Riello, A. (2017). The observer's ghost: notes on a field space connection. *JHEP*, *05*, 017. doi: 10.1007/JHEP05(2017)017

Gomes, H., & Riello, A. (2018, Jul). Unified geometric framework for boundary charges and particle dressings. *Phys. Rev. D*, *98*, 025013. Retrieved from `https://link.aps.org/doi/10.1103/PhysRevD.98.025013` doi: 10.1103/PhysRevD.98.025013

Gomes, H., & Riello, A. (2019). Quasilocal degrees of freedom in Yang-Mills theory.

Healey, R. (2009, 08). Perfect Symmetries. *The British Journal for the Philosophy of Science*, *60*(4), 697-720. Retrieved from `https://doi.org/10.1093/bjps/axp033` doi: 10.1093/bjps/axp033

Henneaux, M., & Teitelboim, C. (1992). *Quantization of gauge systems*. Princeton University Press.

Kobayashi, S., & Nomizu, K. (1963). *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on.

Kondracki, W., & Rogulski, J. (1983). *On the stratification of the orbit space for the action of automorphisms on connections. on conjugacy classes of closed subgroups. on the notion of stratification.* Inst., Acad. Retrieved from `https://books.google.co.uk/books?id=LK0JrgEACAAJ`

Kosmann-Schwarzbach, Y., & Schwarzbach, B. E. (2011). *The noether theorems: Invariance and conservation laws in the twentieth century.* Springer, New York, NY.

Kosso, P. (2000). The empirical status of symmetries in physics. *The British Journal for the Philosophy of Science*, *51*(1), 81–98. Retrieved from `http://www.jstor.org/stable/3541749`

Lee, J., & Wald, R. M. (1990). Local symmetries and constraints. *J. Math. Phys.*, *31*, 725-743. doi: 10.1063/1.528801

Maldacena, J. (2016). The symmetry and simplicity of the laws of physics and the Higgs boson. *Eur. J. Phys.*, *37*(1), 015802. doi: 10.1088/0143-0807/37/1/015802

Maudlin, T. (1993). Buckets of water and waves of space: Why spacetime is probably a substance. *Philosophy of Science*, *60*(2), 183–203. Retrieved from `http://www.jstor.org/stable/188350`

Myrvold, W. C. (2010, 03). Nonseparability, Classical, and Quantum. *The British Journal for the Philosophy of Science*, *62*(2), 417-432. Retrieved from `https://doi.org/10.1093/bjps/axq036` doi: 10.1093/bjps/axq036

Noether, E. (1971). Invariant variation problems. *Transport Theory and Sta-*

tistical Physics, *1*(3), 186-207. Retrieved from https://doi.org/10.1080/00411457108231446 doi: 10.1080/00411457108231446

Olver, P. (1986). *Applications of lie groups to differential equations.* Springer-Verlag New York.

Pooley, O. (2013). 'substantivalist and relationalist approaches to spacetime'. in the oxford handbook of philosophy of physics. In R. Batterman (Ed.), (chap. 15). Oxford University Press.

Regge, T., & Teitelboim, C. (1974). Role of Surface Integrals in the Hamiltonian Formulation of General Relativity. *Annals Phys.*, *88*, 286. doi: 10.1016/0003-4916(74)90404-7

Riello, A. (2019). Soft charges from the geometry of field space.

Rovelli, C. (2014). Why Gauge? *Found. Phys.*, *44*(1), 91-104. doi: 10.1007/s10701-013-9768-7

Speranza, A. J. (2018). Local phase space and edge modes for diffeomorphism-invariant theories. *JHEP*, *02*, 021. doi: 10.1007/JHEP02(2018)021

Strocchi, F. (2015). Symmetries, Symmetry Breaking, Gauge Symmetries.

Strominger, A. (2018). *Lectures on the infrared structure of gravity and gauge theory.* Princeton University Press.

Teh, N. J. (2016). Galileo's gauge: Understanding the empirical significance of gauge symmetry. *Philosophy of Science*, *83*(1), 93-118. Retrieved from https://doi.org/10.1086/684196 doi: 10.1086/684196

't Hooft, G. (1980). Gauge theories and the forces between elementary particles. *Scientific American, 242, pp. 90-166*.

Wallace, D. (2014). Deflating the Aharonov-Bohm Effect. *arxiv: 1407.5073*.

Wallace, D., & Greaves, H. (2014). Empirical consequences of symmetries. *British Journal for the Philosophy of Science*, *65*(1), 59–89.