DEMYSTIFYING LANFORD'S THEOREM

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Abstract. The issue of identify if/where an irreversible ingredient enters in the proof of the Lanfords theorem has been discussed recently in the literature. Sometimes, in focusing on the details, we can loose the overall picture. In this brief commentary I express few reflection about the general place of Lanfords theorem at the foundation the problem of the origin of irreversibility in particular. I argue that the importance of this theorem is more mathematical/technical than conceptual or explanatory.

1. On explaining irreversibility

The problem of reconciling a reversible micro-dynamics with the thermodynamic behaviour has been ind it continues to be a debated topic in foundation of physics. Recently some articles appeared on this Journal has focus the on the Lanford's Theorem (henceforth **LT**) and in particular on what in its proof is responsible for the emergence of irreversibility[9, 1]. I am not going to enter directly in this debate here. What I intend to do is to develop some reflection on the *conceptual* relevance of **LT** inside the so called problem of the origin of irreversibility. This requires first a clarification on what I mean here by the *problem of irreversibility*. For *thermodynamic behavior* I have in mind the classical image of the gas that, initially confined in a corner of a container, uniformly spreads over the entire available space as soon as the confining partition is removed. Explaining irreversibility in this case means to account, on the base of a reversible micro-dynamics, for the existence and uniqueness of a special macro-state ($\mathcal{M}_{equilibrium}$). This special state acts as an *attractor state* i. e. the system converges to it and remains there:

$$\mathcal{M} \longrightarrow \mathcal{M}_{equilibrium}, \tag{1.1}$$

no matter what the initial state $\mathcal{M}$ is [3].

In modern terms, Boltzmann's solution of the problem so defined rely on the concept of **typicality**[8, 6]. In a closed system, the volume of the phase-space region corresponding to a macrostate is a measure of how commonly we can expect to observe it in that state. Combinatorial arguments show that Hamiltonian mechanical systems have energy hyper-surfaces dominated by one macrostate, the one that we identify as equilibrium state [2]. $\mathcal{M}_{equilibrium}$ *dominates* in the sense that its measure, calculated as a fraction of the whole phase space, is close to 1. The explanation of the the observed irreversibility based on Boltzmann's original ideas (henceforth for sake of syntheses I will call it **Boltzmann statistical program** (**BSP**)) is based on the fundamental ingredients [2]:

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a): many degrees of freedom in a macroscopic system;
b): separation of observational scales and;
c): the initial conditions: the system is supposed to start in a atypical (i.e with low probability) state in the past;

to this list we need to add the following:

d): *isotropy* and *independence* of velocities of the particles involved.

On the other hand efforts have been made in the understanding in how/where irreversibility emerges in kinetic equations that model the behaviour of systems that begin away from equilibrium. Boltzmann's transport equation is the focus here:

$$\partial_t f + v \cdot \Delta_x f = Q(f, f) \tag{1.2}$$

This equation describes the evolution of $f(x, v, t)$ due to diffusion and collisions with respectively the second therm on the left and the therm on the right.It is the well known *molecular chaos hypothesis* (Stosszahlansatz), i.e. that colliding particles can be considered uncorrelated, is considered the ingredient that accounts for the time asymmetry. Boltzmann was nor able to justify rigorously this hypotheses and here is where **LT** enters into the picture. In the derivation of the Boltzmann equation, **LT** needs to enter in the details of the model, the geometry of hard-sphere collisions, etc.. As a by-product the theorem shows that if a kind of *Stosszahlansatz* is present at $t = 0$, it will be present even later. In **LT** the fact that a version of *Stosszahlansatz* works for different times is not an assumption but it is derived in a rigorous way. *Rigorous* here means that expression like *almost* or *approximately true* are substituted with something like "$Prob(something) = 1$ *if some counter* $\longrightarrow \infty$". To do this some highly idealized assumption must be considered like the Grad-Boltzmann limit and a very particular choice of the initial conditions. About the **LT** it has been said that (p. 86 from [4], my italics):

> We already mentioned that the Lanford result that we formulated and proved in this section is unsatisfactory, because its validity time is unsatisfactorily short on physically relevant scales. On the other hand, the *conceptual impact* of the result was remarkable and persists; we have proved that a rigorous transition from reversible to irreversible dynamics is possible, and this is significant even if the time interval in question is extremely short.

Here I disagree with statements like this that stress the conceptual role of the **LT** because they give the false impression that either the **BSP** is not enough solid for explaining the behavior of the gas or that **BSP** is in need of a rigorous result to be complemented with. This is not the case and I will illustrate my argument with an example.

This framework in its most general form considers non-interacting particles. Indeed it must be stressed that observed irreversibility does not origin from interactions of the constituents and there is no need of any other particular dynamical property. Assumption d) is seldom stressed in literature but it is fundamental ingredient of the **BSP**. In particular *isotropy* is a strong ontological statement about a fact of Nature: in a collection of particles, velocities are uniformly distribute in

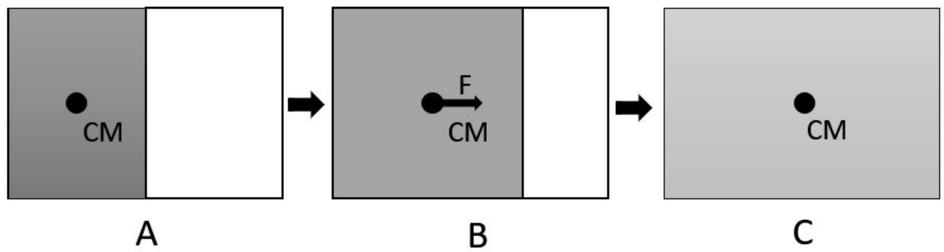

Figure 1. If isolated gas evolution is described by its center of mass (CM) irreversibility is introduced.

every spatial direction. Now, Assuming[1] that the **BSP** works in explaining also the existence and properties of $\mathcal{M}_{equilibrium}$ for a *dilute gas* (interaction due to binary collisions), the question is: what is the place of Lanford's theorem in this overall picture? Does its mathematical rigor add something to the statistical argument or not? And if yes what? **LB** is usually presented as rigorous derivation of the (not invariant under time reversal) Boltzmann equation for the mechanical model of the hard-sphere system.

To begin with, I propose a provocative look at Boltzmann ideas.

The **BSP** is a general framework that does not take into account the details of the constituents of the system and the dynamics. The consequence is that the origin of irreversibility does not lie in those details. Even in the Boltzmann gas, interactions (i. e. binary collisions) are *not* the driving force to account for the approaches to equilibrium. We properly should say that the gas approaches equilibrium *despite* collisions (see the discussion in p. 93 of [7]). Interactions in general play a role in *the way* the system approaches equilibrium and ultimately in the structure of $\mathcal{M}_{equilibrium}$. The irreversibility of the very dilute gas (the existence of $\mathcal{M}_{equilibrium}$ with its features) is explained by the **BSP** because (binary) collisions do not disrupt the main logic of the argument: if *isotropy* and *independence* are true at the beginning we can assume the they will be approximately true even later. In particular the isotropic symmetry at the initial time is not broken by collisions. *Stosszahlansatz* works because it capture a *typical* characteristic in the dilute case, if two particles are initially uncorrelated, it is unlikely that they have collided before a given time t. In this sense the *Stosszahlansat* is not the ultimate source of irreversibility of the dilute gas.

2. A simple model

As an illustrative here i will briefly discuss a different look at the problem of the emergence of irreversibility developed in detail in [5]. The idea is to describe the evolution of an isolated gas as a system of particles and focusing on the dynamics of its center of mass (**CM**). Describing the dynamics of a gas by its center of mass is an extreme form of *dimensional projection* that shows easily how irreversibility emerges in a classical, deterministic, multi-particle system. With reference to figure 1 if the gas starts in the configuration A with the corresponding **CM**, spontaneous evolution

[1]There are different opinions about if and how far this framework works but a discussion of them all is beyond the bounds of this brief commentary.

will lead to a displacement of **CM** as shown. This is a dynamical description where we observe irreversibility since the reverse evolution $C \longrightarrow B \longrightarrow A$ is prevented. The dynamics of **CM** can be explained by simple mechanical arguments. When the partition is removed the gas is confined at A. The net external force due to the momentum transferred to the particles by the walls during collisions is now different from zero and the **CM** moves to B and C and then stops. In C the net forces acting on the **CM** is again zero. The **CM** evolution stops in B since the net external force due to walls is equal to zero. This is easily justified provide we accept a usual assumption about independence and isotropy of velocities of particles[2]. This model is interesting because: a) Irreversibility is introduced in the *reduced* mechanical description; b) interactions between the particles are irrelevant because they do not affect the dynamics of **CM**; c) independence and isotropy are again fundamental basic assumption; d) the model can be made rigorous in an appropriate limit regime (number of particles $n \longrightarrow \infty$, etc.).

3. Irreversibility: a dialectical account

Reductionism is the scientific approach for which higher-level domain phenomena can be explained by reference to the properties of the lower-level entities that make them up. From a bird's-eye view, Boltzmann arguments, culminating in its famous formula:

$$\mathcal{S} = k \log \mathcal{W}, \tag{3.1}$$

do not eventually end up in a reductionist project, they represents instead a spectacular example of *dialectics*: in a process of synthesis, two apparently irreconcilable domains, *thesis* and *antithesis* (in this case mechanics and thermodynamics) are resolved in a middle ground theory that is what we now call *statistical mechanics.* Indeed to solve the problem of irreversibility this is what we need. I summarize it in the following:

Statement 1. *To account for $\mathcal{M}_{equilibrium}$ for the expanding ideal gas, the **BSP** does not try to extract irreversibility out of a deterministic, time-reversible Hamiltonian dynamics in a continuous chain of logical steps. What **BSP** accomplish is to show that irreversibility and time-reversible Hamiltonian dynamics can coexist under an appropriate conceptual framework.*

BSP and **LT** operate at different levels. **BSP** is a master framework that can be applied to account for the irreversibility observed in a dilute gas. It is so to speak a top-down logic of explanation: from the general to the particular. **LT** works bottom-up, it aims to derive irreversibility from first principles for a particular model and under severe assumptions. I claim that the importance of **LT** is more technical-mathematical than conceptual or physical.

On the other end there is a misconception that a the final word on the problem of the origin of irreversibility can be fulfilled ultimately by a pure reductionist project where irreversibility is extract out of a deterministic, time-reversible Hamiltonian underlying dynamics, being it classical or quantum. As I explained what is needed

[2]A similar argument, with slight modification, can be done even in the case that the gas stars at the center of the vessel. For details the reader can refer to [5].

is a framework in which we can accommodate both: irreversibility *and* a deterministic, time-reversible Hamiltonian dynamics and this is what **BSP** *conceptually* accomplishes the task.

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