Open problems that concern computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in ZFC as they refer to current knowledge about $\mathcal{X}$

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Conditions (1)-(8) below concern sets $\mathcal{X} \subseteq \mathbb{N}$. (1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. (2) No known algorithm decides the finiteness of $\mathcal{X}$. (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$. (5) $\mathcal{X}$ is widely known in number theory. (6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$. (7) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\mathrm{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$. (8) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies $\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$. We do not know any set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(4) and (5). The same is true, if condition (5) is replaced by condition (6) or (7) or (8). For every explicitly known integer n, some simply defined set $\mathcal{X} \subseteq \mathbb{N}$ includes the set $(-\infty, n] \cap \mathbb{N}$ and satisfies conditions (1)-(4). Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form n^2+1. The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(3) and (5)-(8). The set $\mathcal{X} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions (1)-(3) and (6)-(8). Some hypothetical statement implies that these sets $\mathcal{X}$ satisfy condition (4).

Key words and phrases: computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge about $\mathcal{X}$, explicitly known integer n, finiteness (infiniteness) of $\mathcal{X}$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of $\mathcal{X}$, n bounds $\mathcal{X}$ if $\mathcal{X}$ is finite, no known algorithm decides the finiteness of $\mathcal{X}$, open mathematical problem that cannot be formalized in ZFC.

1. Basic definitions and lemmas

Definition 1. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Lemma 2. $((7!)!)! \approx 10^{10^{16477.87280582041}}$.

Proof. We ask Wolfram Alpha about $0.0 + ((7!)!)!$. □

Definition 2. *We say that an integer* $n \geqslant -1$ *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$*, if* $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$*, cf.* [10] *and* [11].

Definition 3. *We say that a non-negative integer n is a weak threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$*, if* $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathrm{card}(\mathcal{X}) \leqslant n$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $n \geqslant -1$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer n is a weak threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all weak threshold numbers of $\mathcal{X}$ form the set $[\mathrm{card}(\mathcal{X}), \infty) \cap \mathbb{N}$.

Theorem 1. *For every set* $\mathcal{X} \subseteq \mathbb{N}$*, if an integer* $n \geqslant -1$ *is a threshold number of* $\mathcal{X}$*, then* $n+1$ *is a weak threshold number of* $\mathcal{X}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$ and for every integer $n \geqslant -1$, the inclusion $\mathcal{X} \subseteq (-\infty, n]$ implies that $\mathrm{card}(\mathcal{X}) \leqslant n + 1$. □

Let $\mathcal{P}_{n^2+1}$ denote the set of primes of the form $n^2 + 1$. We do not know any weak threshold number of $\mathcal{P}_{n^2+1}$. The same is true for the sets

$$\left\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

and

$$\{n \in \mathbb{N} : n! + 1 \text{ is a square}\}$$

Lemma 3. *For every positive integers x and y,* $x! \cdot y = y!$ *if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. *(Wilson's theorem,* [1, p. 89]*). For every integer* $x \geqslant 2$*, x is prime if and only if x divides* $(x-1)! + 1$.

The conditions below concern sets $\mathcal{X} \subseteq \mathbb{N}$.

(1) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite.

(2) No known algorithm decides the finiteness of $\mathcal{X}$.

(3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$.

(4) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) $\mathcal{X}$ is widely known in number theory.

(6) There is no known equality $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined simpler than $\mathcal{X}$.

(7) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies
$(\mathcal{Y} \subseteq \mathcal{X}) \wedge (\mathrm{card}(\mathcal{X} \setminus \mathcal{Y}) < \omega)$.

(8) No known set $\mathcal{Y}$ is defined simpler than $\mathcal{X}$ and satisfies
$\mathrm{card}((\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})) < \omega$.

(4•) There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathrm{card}(\mathcal{X}) \leqslant n$.

(1◇) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X} = \mathbb{N}$.

(2◇) No known algorithm decides the equality $\mathcal{X} = \mathbb{N}$.

(1*) There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is finite.

2. Open Problems 1 and 2

The following two open problems cannot be formalized in *ZFC* as they refer to current knowledge about $\mathcal{X}$.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(3), (4•), *and* (5)*?*

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* (1)-(5)*?*

Theorem 2. *Open Problem 2 claims more than Open Problem 1.*

Proof. Condition (4) implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition (4) implies condition (4•). □

Open Problems 1 and 2 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

3. Partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ is infinite, see [4, pp. 37–38] and [7]. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Theorem 3. *The set $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{M}$ satisfies conditions* (1)-(4).

Proof. Condition (1) holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β. Thus condition (2) holds. Condition (3) holds trivially. Since the set $\mathcal{M}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function.

Lemma 5. *For every non-negative integer n, $\left[\dfrac{3n - 3\beta + 3}{3n - 3\beta + 2}\right]$ equals* 0 *or* 1. *The first case holds when $n \leqslant \beta - 1$. The second case holds when $n \geqslant \beta$.*

Lemma 6. *The function*

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - \left[\sqrt{n}\right]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geqslant \beta$ infinitely many times.

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{(t+0)^2 + t,\ (t+1)^2 + t,\ (t+2)^2 + t,\ \ldots\right\} \cap [\beta, \infty)$$

□

Theorem 4. *The set*

$$\mathcal{X} = \left\{n \in \mathbb{N} : 2 + \left[\frac{3n - 3\beta + 3}{3n - 3\beta + 2}\right] \cdot \left(\left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 - 1\right) \text{ is prime}\right\}$$

satisfies conditions `(1)`-`(4)`.

Proof. Condition `(3)` holds trivially. By Lemma 5, $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 + 1 \text{ is prime}\right\}$$

By Lemma 6, the set $\mathcal{H}$ is empty or infinite. The second case holds when

$$\exists\, k \in \mathbb{N} \cap [\beta, \infty)\ k^2 + 1 \text{ is prime} \qquad \texttt{(G)}$$

The equality $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of $\mathcal{X}$ and conditions `(1)` and `(4)` hold. Condition `(2)` holds as due to known physics we are not able to confirm the statement `(G)` by a direct computation. □

4. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

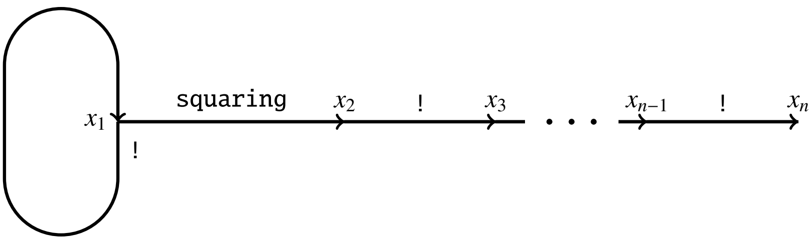

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 7. *For every positive integer n, the system* $\mathcal{U}_n$ *has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(f(1), \ldots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k :\ i, k \in \{1, \ldots, n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\Big\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations* $\mathcal{S} \subseteq B_n$ *has at most finitely many solutions in positive integers* $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant f(n)$. The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. An elementary reasoning shows that the statements Ψ_1 and Ψ_2 are true.

Theorem 5. *For every statement* Ψ_n*, the bound* $f(n)$ *cannot be decreased.*

Proof. It follows from Lemma 7 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 6. *For every integer* $n \geqslant 2$*, the statement* Ψ_{n+1} *implies the statement* Ψ_n.

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 7. *Every statement* Ψ_n *is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

5. A conjectural solution to Open Problem 2

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &=& x_3 \\ x_3! &=& x_4 \\ x_5! &=& x_6 \\ x_8! &=& x_9 \\ x_1 \cdot x_1 &=& x_2 \\ x_3 \cdot x_5 &=& x_6 \\ x_4 \cdot x_8 &=& x_9 \\ x_5 \cdot x_7 &=& x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

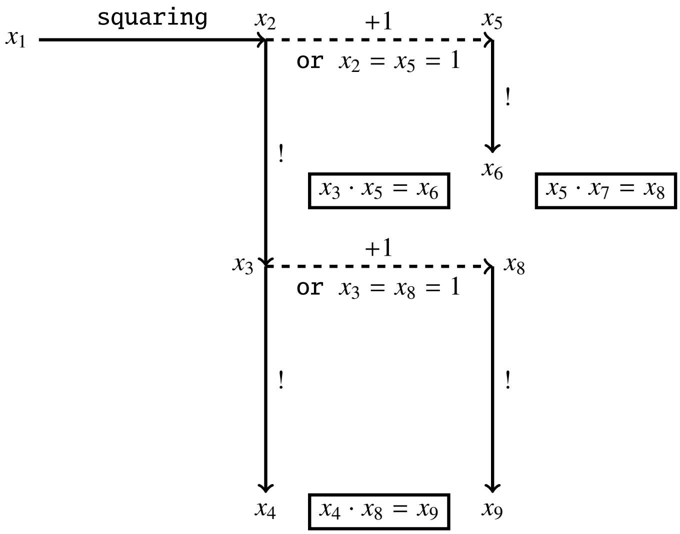

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 8. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{array}{rcl}
x_2 & = & x_1^2 \\
x_3 & = & (x_1^2)! \\
x_4 & = & ((x_1^2)!)! \\
x_5 & = & x_1^2 + 1 \\
x_6 & = & (x_1^2 + 1)! \\
x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 & = & (x_1^2)! + 1 \\
x_9 & = & ((x_1^2)! + 1)!
\end{array}
$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 8 follows from Lemma 4. □

Lemma 9. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{A}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Let Φ_9 denote the statement Ψ_9 restricted to the system $\mathcal{A}$. Apoloniusz Tyszka believes that the statement Φ_9 is true.

Theorem 8. *The statement Φ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 8, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

The statement Φ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 8 and 9, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$.

Lemma 10. $\mathrm{card}(\mathcal{K}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}} \approx 10^{10^{450.6930560314272}}$.

Proof. The following `PARI/GP` ([6]) command

```
isprime(1+9*4^747,{flag=2})
```

returns $\%1 = 1$. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([9, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10\hat{}(9 * 4\hat{}747))$. □

Theorem 9. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions* `(1)-(3)` *and* `(5)-(8)`. *The set $\mathcal{X} = \mathcal{K}$ satisfies conditions* `(1)-(3)` *and* `(6)-(8)`. *The statement Φ_9 implies that these sets $\mathcal{X}$ satisfy condition* `(4)`.

Proof. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 10 implies condition `(1)` for both sets $\mathcal{X}$. Conditions `(3)` and `(6)-(8)` hold trivially for both sets $\mathcal{X}$. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$. Thus condition `(2)` holds for both sets $\mathcal{X}$. Suppose that the statement Φ_9 is true. By Theorem 8, $f(7)$ is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. By Theorem 8, $\underbrace{9 \ldots 9}_{f(7) \text{ digits}}$ is a threshold number of $\mathcal{X} = \mathcal{K}$. Thus condition `(4)` holds for both sets $\mathcal{X}$. □

6. Open Problems 3 and 4

The following two open problems cannot be formalized in *ZFC* as they refer to current knowledge about $\mathcal{X}$.

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1⋄)-(2⋄)`, `(2)-(3)`, `(4•)`, *and* `(5)`*?*

Open Problem 3 claims more than Open Problem 1 as condition `(1⋄)` implies condition `(1)`.

Open Problem 4. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1⋄)-(2⋄)` *and* `(2)-(5)`*?*

Open Problem 4 claims more than Open Problem 2 as condition (1◇) implies condition (1).

Theorem 10. *Open Problem 4 claims more than Open Problem 3.*

Proof. Condition (4) implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition (4) implies condition (4•). □

Open Problems 3 and 4 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

7. A partial solution to Open Problem 4

Let $\mathcal{V}$ denote the set of all positive multiples of elements of the set

$$\left\{n \in [\beta+1, \infty) \cap \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

Theorem 11. *The set* $\mathcal{X} = \{0, \dots, \beta\} \cup \mathcal{V}$ *satisfies conditions* (1◇)-(2◇) *and* (2)-(4).

Proof. The inequality $\mathrm{card}(\mathcal{X}) > \beta$ holds trivially. Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [2, p. 23]. These two facts imply conditions (1◇) and (2◇). Condition (3) holds trivially. Since the set $\mathcal{V}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition (4) holds. The question of finiteness of the set

$$\left\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

remains open, see [3, p. 159]. By this and Lemma 1, the question of emptiness of the set

$$\left\{n \in [\beta+1, \infty) \cap \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\right\}$$

remains open. Therefore, the question of finiteness of the set $\mathcal{V}$ remains open. Consequently, the question of finiteness of the set $\mathcal{X}$ remains open and condition (2) holds. □

8. Open Problems 5 and 6

The following two open problems cannot be formalized in *ZFC* as they refer to current knowledge about $\mathcal{X}$.

Open Problem 5. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1*), (2)-(3), (4•), *and* (5)*?*

Open Problem 6. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1*) *and* (2)-(5)*?*

Theorem 12. *Open Problem 6 claims more than Open Problem 5.*

Proof. Condition (4) implies that $\mathrm{card}(\mathcal{X}) < \omega \Longrightarrow \mathcal{X} \subseteq (-\infty, |n|]$. Since $|n| \geqslant -1$, Theorem 1 guarantees that condition (4) implies condition (4•). □

Open Problems 5 and 6 remain open, if condition (5) is replaced by condition (6) or (7) or (8).

9. Partial solutions to Open Problem 6

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [5].

Lemma 11. *([8, p. 297]). It is conjectured that* $x! + 1$ *is a square only for* $x \in \{4, 5, 7\}$.

Let $\mathcal{W}$ denote the set of all integers x greater than β such that $x! + 1$ is a square.

Theorem 13. *The set*

$$\mathcal{X} = \{0, \ldots, \beta\} \cup \{k \cdot x : \ (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions `(1*)` *and* `(2)-(4)`.

Proof. Condition `(1*)` holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{W}$ is conjecturally empty by Lemma 11. Condition `(3)` holds trivially. We do not know any algorithm that decides the emptiness of $\mathcal{W}$ and the set

$$\mathcal{Y} = \{k \cdot x : \ (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition `(2)` holds. Since the set $\mathcal{Y}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

Let C denote the following system of equations:

$$\begin{cases} x_1! & = & x_2 \\ x_2! & = & x_3 \\ x_5! & = & x_6 \\ x_4 \cdot x_4 & = & x_5 \\ x_3 \cdot x_5 & = & x_6 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system C.

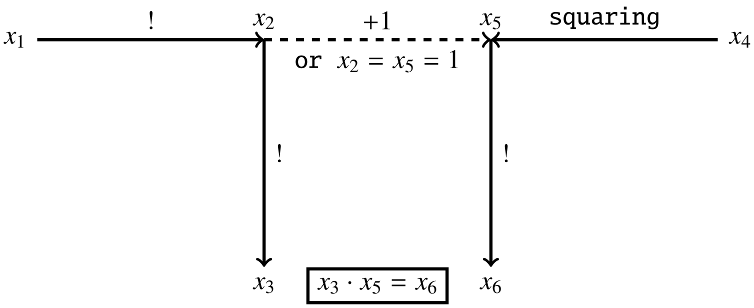

Fig. 3 Construction of the system C

Lemma 12. *For every* $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, *the system* C *is solvable in positive integers* x_2, x_3, x_5, x_6 *if and only if* $x_1! + 1 = x_4^2$. *In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{array}{lcl} x_2 & = & x_1! \\ x_3 & = & (x_1!)! \\ x_5 & = & x_1! + 1 \\ x_6 & = & (x_1! + 1)! \end{array}$$

Proof. It follows from Lemma 3. □

Let Φ_6 denote the statement Ψ_6 restricted to the system $\mathcal{C}$. Apoloniusz Tyszka believes that the statement Φ_6 is true.

Theorem 14. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Φ_6 guarantees that each such solution (x_1, x_4) satisfies $x_1 < 24!$.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 12, the system $\mathcal{C}$ is solvable in positive integers x_2, x_3, x_5, x_6. The statement Φ_6 implies that $x_6 = (x_1! + 1)! \leqslant f(6) = f(5)!$. Hence, $x_1! + 1 \leqslant f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24!$. □

Theorem 15. *Let $\mathcal{X}$ denote the set of all non-negative integers n which have $((k!)!)!$ digits for some $k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\}$. We claim that $\mathcal{X}$ satisfies conditions* (1*), (2)-(3), *and* (6)-(8). *The statement Φ_6 implies that $\mathcal{X}$ satisfies condition* (4).

Proof. Let $d = ((7!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^{d-1}, \ldots, \underbrace{9\ldots9}_{d \text{ digits}}\} \subseteq \mathcal{X}$. Hence, $\mathrm{card}(\mathcal{X}) \geqslant 9 \cdot 10^{d-1}$. By this and Lemmas 2 and 11, condition (1*) holds. Conditions (2)-(3) and (6)-(8) hold trivially. By Theorem 14, the statement Φ_6 implies that $\underbrace{9\ldots9}_{\beta \text{ digits}}$ is a threshold number of $\mathcal{X}$. Thus condition (4) holds. □

Acknowledgement. Sławomir Kurpaska prepared three diagrams in *TikZ*. Apoloniusz Tyszka wrote the article.

References

[1] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[2] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers,* American Mathematical Society, Providence, RI, 2012.

[3] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry,* Springer, New York, 2001.

[4] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT,* Springer, London, 2012.

[5] M. Overholt, *The Diophantine equation $n! + 1 = m^2$,* Bull. London Math. Soc. 25 (1993), no. 2, p. 104.

[6] `PARI/GP` *online documentation,* `http://pari.math.u-bordeaux.fr/dochtml/html/Arithmetic_functions.html`.

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form $n^2 + 1$,* `http://oeis.org/A002496`.

[8] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[9] S. Y. Yan, *Number theory for computing,* 2nd ed., Springer, Berlin, 2002.

[10] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity,* Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm.

[11] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer,* in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

Sławomir Kurpaska
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl
Website: http://www.cyf-kr.edu.pl/~rttyszka