Antiunitary Equivalence

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Abstract

In algebraic approaches to quantum mechanics and quantum field theory, debates about physical equivalence revolve around the concept of *unitary equivalence*. Virtually all of these discussions suppress a closely related concept, *antiunitary equivalence*. The goal of this paper is to begin the project of disentangling the relationship between these two concepts. I provide necessary and sufficient conditions for the existence of antiunitary intertwiners between representations, clarify that antiunitary equivalence does not entail unitary equivalence, and argue that there are interpretational subtleties that are both physically important and unique to the antiunitary case.

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1 Introduction

In algebraic approaches to quantum mechanics and quantum field theory (QFT), debates about physical equivalence revolve around the concept of *unitary equivalence* (or more generally *quasiequivalence*). Many philosophers have defended unitary equivalence as a sufficient condition for the physical equivalence of two concrete algebraic representations (e.g., Halvorson and Müger 2006; Ruetsche 2011; Baker et al. 2015). A few have gone further and defended its necessity (e.g., Arageorgis 1995; Halvorson and Clifton 2001), while others have countered that physically equivalent representations can be related by more general symmetry relations (e.g., Baker 2011).

Virtually all of these discussions suppress a closely related concept, *antiunitary equivalence*. Ruetsche (2011, p. 28) briefly acknowledges its possible significance, although her ensuing analysis of physical equivalence restricts attention to the unitary case without further discussion. Similarly, Strocchi's influential account of spontaneous symmetry breaking briefly notes the antiunitary case before excluding it by decree (2008, p. 116). Baker and Halvorson (2013) intend their distinction between represented and implemented algebraic symmetries to encompass the antiunitary case, although they note that "for purposes of this paper, we ignore the difference between unitary and antiunitary" (p. 468). Earman (2003) is a rare exception, explicitly endorsing a picture of spontaneous symmetry breaking that covers both unitary and antiunitary symmetries, although he does not go into any detail about the similarities or differences between the two cases.

Overall, the attitude towards antiunitary equivalence in the philosophy literature has been to relegate it to footnotes (if it is mentioned at all). But this indifference has the potential to generate significant confusion. Do arguments linking unitary equivalence to physical equivalence and spontaneous symmetry breaking extend to antiunitary equivalence with minor mathematical adjustment (as sometimes implied)? Or do antiunitary symmetries generate a distinct set of interpretive problems that must be handled separately from the unitary case?

The goal of this paper is to begin the project of disentangling the relationship between unitary and antiunitary equivalence. In §2, I distinguish three clusters of concepts centered around *unitary*, *antiunitary*, and *Jordan equivalence*. In §3, I give necessary and sufficient conditions for the existence of antiunitary intertwiners between representations, clarifying that antiunitary equivalence does not entail unitary equivalence. In §4 I argue that these cases are physically significant — in relativistic QFT, conjugate charge representations are not unitarily equivalent in general, but they are always antiunitarily equivalent. In §5 I consider the ramifications these observations have for debates surrounding physical equivalence. (Spontaneous symmetry breaking will be explored in a separate paper.) I contend that there are important interpretational subtleties unique to the antiunitary case. As interpreters, we cannot continue to gloss over the distinction between unitary and antiunitary.

2 Three Notions of Equivalence

Let $\mathfrak{A}$ be a unital C^*-algebra. A *representation* of $\mathfrak{A}$, $(\pi, \mathcal{H})$, consists of a Hilbert space, $\mathcal{H}$, and a *-homomorphism, π, mapping $\mathfrak{A}$ into the set of bounded linear operators on $\mathcal{H}$. Given two Hilbert spaces, $\mathcal{H}$, $\mathcal{K}$, an *isometry* is a linear operator $T : \mathcal{H} \to \mathcal{K}$, such that $TT^* = T^*T$. (This ensures that T identifies $\mathcal{H}$ with a closed subspace of $\mathcal{K}$, preserving the Hilbert space norm.) If $(\phi, \mathcal{K})$ is also a representation of $\mathfrak{A}$, $(\pi, \mathcal{H})$ is a *subrepresentation* of $(\phi, \mathcal{K})$ iff there exists an isometry such that $T\pi(A)T^* = \phi(A)$ for all $A \in \mathfrak{A}$. Finally, a *unitary operator* is an isometry, U, such that $UU^* = U^*U = I$. (Thus U identifies $\mathcal{H}$ with $\mathcal{K}$.)

With these preliminaries in place, we have the following trio of concepts, familiar from both the mathematics and the philosophy literature. Given two representations $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$, of $\mathfrak{A}$, the representations are

- *unitarily equivalent* iff there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, such that $U\pi_1(A)U^* = \pi_2(A)$ for all $A \in \mathfrak{A}$,
- *quasiequivalent* iff π_1 and π_2 have unitarily equivalent subrepresentations,
- *disjoint* iff no (nonzero) subrepresentation of π_1 is unitarily equivalent to a subrepresentation of π_2.

Quasiequivalence is a well-defined equivalence relation on representations since the product of two unitary operators is unitary. Note that a common alternative definition of quasiequivalence (extensionally equivalent to the one given here), requires that the von Neumann algebras $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$ are *-isomorphic. For irreducible representations, quasiequivalence reduces to unitary equivalence. Therefore quasiequivalence generalizes the notion of unitary equivalence to reducible representations, which can be decomposed into direct sums of irreducible subrepresentations. For clarity, we will amend standard usage and refer to quasiequivalent and disjoint representations as *unitarily quasiequivalent* and *unitarily disjoint*.

Motivated by the possibility of symmetries represented by antiunitary operators, we can define a parallel trio of concepts. A linear operator, A, acts linearly on Hilbert space vectors, i.e., $A(cx + dy) = cAx + dAy$ for all $c, d \in \mathbb{C}$, $x, y \in \mathcal{H}$. An *antilinear operator*, B, acts antilinearly on vectors, i.e., $B(cx + dy) = \bar{c}Bx + \bar{d}By$, where the overline denotes complex conjugation. An *antiunitary operator*, $V : \mathcal{H} \to \mathcal{K}$, is an antilinear operator such that $VV^* = V^*V = I$. Two representations are

- *antiunitarily equivalent* iff there exists an antiunitary operator $V : \mathcal{H}_1 \to \mathcal{H}_2$, such that $V\pi_1(A)V^* = \pi_2(A)$ for all $A \in \mathfrak{A}$,
- *antiunitarily quasiequivalent* iff π_1 and π_2 have antiunitarily equivalent subrepresentations,
- *antiunitarily disjoint* iff no (nonzero) subrepresentation of π_1 is antiunitarily equivalent to a subrepresentation of π_2.

These definitions almost perfectly mirror the cluster surrounding unitary equivalence. Anitunitary quasiequivalence reduces to antiunitary equivalence for reducible representations, and two representations are antiunitarily quasiequivalent iff $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$ are *-conjugate-isomorphic.[1] There is one important formal difference — antiunitary quasiequivalence is not an equivalence relation. The product of two antiunitary operators is unitary, so transitivity can fail. Moreover, there are examples of C^*-algebras that are not conjugate-isomorphic to themselves (Connes, 1975), so reflexivity can fail too.

[1]There is a bit of subtly here. We could have defined antiunitary equivalence by setting $V\pi_1(A^*)V^* = \pi_2(A)$ for all $A \in \mathfrak{A}$, as is more common in parts of the mathematics literature. In this case $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$ are *-anti-isomorphic rather than a *-conjugate-isomorphic. Anti-isomorphisms reverse products, $AB \mapsto BA$, but preserve the complex unit, $i \mapsto i$, while conjugate-isomorphisms preserve products, $AB \mapsto BA$, but conjugate the complex unit, $i \mapsto -i$. Ultimately, the difference does not matter, two C^*-algebras are conjugate-isomorphic iff they are anti-isomorphic. The definition we have chosen here maps more cleanly onto the discussion of conjugate group representations in §3. Conjugate group representations are equivalent to dual group representations, a notion that fits better with talk of *-anti-isomorphisms. See Swanson (2019, Lem. 1-2) for more details.

This problem is easily fixed. The product of a unitary and antiunitary operator is antiunitary, so the disjunction of unitary and antiunitary quasiequivalence is a well-defined equivalence relation on representations. This generates a third trio of concepts. Two representations are

- *Jordan equivalent* iff there exists a unitary or antiunitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, such that $U\pi_1(A)U^* = \pi_2(A)$ for all $A \in \mathfrak{A}$,
- *Jordan quasiequivalent* iff π_1 and π_2 have Jordan equivalent subrepresentations,
- *Jordan disjoint* iff no (nonzero) subrepresentation of π_1 is Jordan equivalent to a subrepresentation of π_2.

Some mathematical physicists (e.g., Varadarajan 1968; Emch 1972) have proposed Jordan quasiequivalence as the right notion of physical equivalence in quantum theory. The motivation here is twofold. First, Wigner's theorem ensures that any symmetry that preserves transition probabilities and superselection structure can be represented by either a unitary or antiunitary operator. Second, it is typically assumed that only the self-adjoint elements of C^*-algebras directly represent physical quantities. The spectral properties of these operators are completely captured by the canonical *Jordan product* on $\mathfrak{A}_{SA} \subset \mathfrak{A}$, defined by $A \bullet B := \frac{1}{2}(AB + BA)$. On this basis, it is tempting to view the rest of the C^*-algebra as superfluous mathematical structure. Physical equivalence need only preserve the structure of $\mathfrak{A}_{SA}$ as a (real) Jordan algebra. This is exactly the notion captured by Jordan quasiequivalence (Emch, 1972, p. 149–55).

For now we will leave open whether Jordan quasiequivalence is sufficient for physical equivalence. (We will encounter some reasons for skepticism in §5). Before diving into this question, we need to investigate the relationship between unitary and antiunitary equivalence in more detail.

3 The Existence of Antiunitary Intertwiners

To see how inattention to the subtleties surrounding unitary and antiunitary equivalence can lead us astray, consider the following bad chain of reasoning:

> If two irreducible representations are (unitarily) disjoint, then their set of intertwiners is empty. Thus there are no operators

— unitary, antiunitary, or otherwise — intertwining them. So antiunitary equivalence always entails unitary equivalence, and we can ignore it for all practical purposes.

The first inference is fine if we define the set of intertwiners (as commonly done in the mathematics literature) as the set of bounded *linear* operators $T : \mathcal{H}_1 \to \mathcal{H}_2$, such that $T\pi_1(A)T^* = \pi_2(A)$ for all $A \in \mathfrak{A}$. But then the second inference does not follow. Alternatively, if we define the set of intertwiners more permissively as the set of bounded *linear or antilinear* operators $T : \mathcal{H}_1 \to \mathcal{H}_2$, such that $T\pi_1(A)T^* = \pi_2(A)$ for all $A \in \mathfrak{A}$, then the first inference is bad.[2]

Does antiunitary equivalence entail unitary equivalence? In general, no. To find an example that clearly demonstrates this, though, we have to do a bit of digging. To get started, let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathfrak{A}$. The conjugate Hilbert space, $\bar{\mathcal{H}}$, is obtained by changing the action of $\mathbb{C}$ on $\mathcal{H}$, $cx \mapsto \bar{c}x$, for all $c \in \mathbb{C}$, $x \in \mathcal{H}$. Define the *conjugate representation*, $(\bar{\pi}, \bar{\mathcal{H}})$, by setting $\bar{\pi}(A) = \pi(A)$.

Proposition 1. *Let $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ be two irreducible representations of $\mathfrak{A}$. There exists an antiunitary intertwiner, $V\pi_1(A)V^* = \pi_2(A)$, for all $A \in \mathfrak{A}$, iff $(\pi_1, \mathcal{H}_1)$ is unitarily equivalent to $(\bar{\pi}_2, \bar{\mathcal{H}}_2)$.*[3]

Proof. Define the antilinear map, $J : \mathcal{H}_2 \to \bar{\mathcal{H}}_2$, by setting

$$J(cx + dy) = \bar{c}x + \bar{d}y \ , \tag{1}$$

for all $c, d \in \mathbb{C}$ and $x, y \in \mathcal{H}_2$. It follows that $J^2 = I$ and $\langle Jx, Jy \rangle = \overline{\langle x, y \rangle} = \langle y, x \rangle$, so J is an antiunitary involution. Since $\pi_2(A)$ is linear,

$$\begin{aligned} J\pi_2(A)cx &= Jc\pi_2(A)x \\ &= \bar{c}\pi_2(A)x \\ &= \pi_2(A)\bar{c}x \\ &= \bar{\pi}_2(A)cx \ , \end{aligned} \tag{2}$$

[2] Note that if T is a linear intertwiner, then the existence of polar decompositions ensures that $T = U|T|$, where U is unitary, and thus there exists a unitary intertwiner. Similarly, if T is antilinear, then $T = V|T|$ where V is antiunitary, and thus there exists an antiunitary intertwiner.

[3] This is a natural generalization of Varadarajan (1968, Lem. 3.8) which only applies to unitary group representations.

and so $J\pi_2(A)J = \bar{\pi}_2(A)$, for all $A \in \mathfrak{A}$.

Suppose that π_1 is unitarily equivalent to $\bar{\pi}_2$, so there exists a unitary $U : \mathcal{H}_1 \to \bar{\mathcal{H}}_2$, such that $U\pi_1(A)U^* = \bar{\pi}_2(A)$. Using J, we can construct an antiunitary intertwiner between π_1 and π_2:

$$V\pi_1(A)V^* := JU\pi_1(A)U^*J = \pi_2(A) \ . \tag{3}$$

To prove the converse, suppose that there exists and antiunitary intertwiner, $V\pi_1(A)V^* = \pi_2(A)$. The existence of polar decompositions in the universal enveloping von Neumann algebra, $\mathfrak{A}^{**}$, ensures that any antiunitary operator $V = JU$, where U is unitary and J is the antiunitary conjugation operator defined previously (Blackadar, 2006, I.5.2.6). This entails that,

$$U\pi_1(A)U^* = J\pi_2(A)J = \bar{\pi}_2(A) \ , \tag{4}$$

so π_1 is unitarily equivalent to the conjugate representation of π_2. □

Note that our definition of conjugate representations for arbitrary C^*-algebras is a straightforward generalization of the standard definition of conjugate group representations. If G is a locally compact group, then irreducible strongly continuous unitary representations of G are also irreducible representations of the (full) group C^*-algebra, and vice versa (Blackadar, 2006, II.10.2.1-4). This connection leads to the following result:

Corollary. *There exist antiunitarily equivalent C^*-algebra representations that are not unitarily equivalent.*

Proof. If G is a locally compact group, then each irreducible strongly continuous representation of G is also an irreducible representation of the group C^*-algebra. In general, conjugate representations of G are not unitarily equivalent, but by proposition 1 they are always antiunitarily equivalent. For the important special case of compact simply connected Lie groups, each pair of irreducible conjugate representations is unitarily equivalent iff the associated Weyl group contains -1 (Simon, 1996, Lem. IX.10.1). This is the case for $SU(2)$, $SO(2n+1)$, $SO(4n)$, and $Sp(n)$, but it is not true for $SU(n)$, $U(n)$, $SO(n)$, or $O(n)$ in general. □

Group representation theory gives us a nice stock of examples where antiunitary equivalence does not entail unitary equivalence. The $SU(2)$,

$SO(2n+1)$, $SO(4n)$, and $Sp(n)$ cases also provide several examples of antiunitarily equivalent representations that are also unitarily equivalent. Are there any cases of unitarily equivalent representations that are not antiunitarily equivalent? The examples of C^*-algebras which are not conjugate-isomorphic to themselves, noted briefly in the last section, are probably the easiest to point to, although they are otherwise rather thorny entities. Taking direct sums of irreducible representations from these various camps generates a zoo of examples falling under the various notions of equivalence introduced in §2.

Our focus on group theory is not entirely accidental. Conjugate group representations play an important role in QFT, where they are used to model field configurations with conjugate charge. In general, these representations will be unitarily disjoint but antiunitarily quasiequivalent, providing a physically significant example of the mathematical phenomena just unearthed.

4 Charges and CPT Symmetry

In algebraic QFT, we have three mathematically equivalent ways to describe superselected charges. The first is via the category of *localized transportable morphisms*. Such morphisms map the net of local observable algebras into the algebra of bounded operators on the vacuum Hilbert space, $\varrho : \mathfrak{A} \to B(\mathcal{H}_\omega)$. They are localized in some doublecone or spacelike cone, acting as the identity in the causal complement, and they can be transported to any similarly shaped region by the adjoint action of certain unitary operators. If ω is the vacuum state, $\omega \circ \varrho$ describes a state with charge Q localized in the relevant region. The category of localized transportable morphisms has a rich mathematical structure which can be used to describe tensor products of charged states as well as conjugate charges. The conjugate of ϱ is the unique (up to unitary equivalence) morphism $\bar{\varrho}$ such that $\omega \circ \varrho \circ \bar{\varrho}$ contains a component in the vacuum sector.

The second way to characterize charge structure, revealed by the pioneering analysis initiated by Doplicher, Haag, and Roberts (1969a,b) and later generalized by Buchholz and Fredenhagen (1982), is via the category of representations of the net of local observable algebras that satisfy certain special boundary conditions. A representation of the net satisfies the *DHR/BF selection criterion* iff it is quasiequivalent to the vacuum representation in the causal complement of some doublecone or spacelike cone. Such representations have a folium of states that look like the vacuum everywhere except within the relevant cone. The key to the DHR/BF analysis lies in showing that the category of representations satisfying this selection criterion is equivalent to the category of localized transportable morphisms. This allows us to pull back the tensor product and conjugate structure from the category of charge morphisms to the category of DHR/BF representations. Although more abstract, this second viewpoint is seen as more explanatorily fundamental. (In practice, though, it is usually much easier to work with charge morphisms.)

There is a third way to describe charges that has more in common with the approach taken in Lagrangian and constructive QFT. Rather than an observable algebra, we can start with a net of field algebras, $\mathfrak{F}$, acting faithfully on a Hilbert space, $\mathcal{H}$, and a compact global gauge group, G. The observable algebra, $\mathfrak{A}$, is identified with the G-invariant subalgebra of $\mathfrak{F}$. Under the action of G, $\mathcal{H}$ decomposes into a direct sum of superselection sectors,

$$\mathcal{H} = \bigoplus_{\xi \in \hat{G}} \mathcal{H}_\xi \ , \tag{5}$$

where each sector is a direct sum of irreducible strongly continuous representations of G labeled by the same group character $\xi \in \hat{G}$. Amazingly, each of these irreducible representations is also an irreducible representation of $\mathfrak{A}$ satisfying the DHR/BF selection criteria. Even more amazingly, Doplicher and Roberts (1990) prove that we can naturally reconstruct the field algebra and gauge group starting from only the DHR/BF category. (The DHR/BF category is dual to the category of representations of G in a certain well-defined sense.) The punchline is that everything that we want to say about global gauge charges in algebraic QFT can ultimately be cached out in terms of boundary conditions on the net of observable algebras in a completely gauge-free manner.[4]

Returning to the theme of antiunitary equivalence, for each pair of conjugate charge morphisms, ϱ, $\bar{\varrho}$, there are associated conjugate DHR/BF representations, π, $\bar{\pi}$. These representations are direct sums of irreducible DHR/BF representations, each of which is also an irreducible strongly continuous representation of the dual gauge group G. Proposition 1 then entails that irreducible, conjugate DHR/BF representations are always antiunitarily

[4]It should be noted that the DHR/BF analysis has not yet been extended to cover charges in theories with local gauge symmetry or massless particles.

equivalent, and conjugate DHR/BF representations are always antiunitarily quasiequivalent. Moreover, the corollary entails that except in certain special cases where the representations are self-conjugate, DHR/BF conjugate representations are not unitarily quasiequivalent.

We can prove more. In any causal, Lorentz-invariant, thermodynamically well-behaved QFT, there is a global reflection symmetry that reverses the direction of time, flips spatial parity, and conjugates charge. Swanson (2019) gives a detailed analysis of the logic of the so-called *CPT theorem* in algebraic QFT. At the algebraic level, a CPT transformation can be viewed as an involutive Jordan-automorphism of the net of observable algebras, i.e., a bijection $\theta : \mathfrak{A} \to \mathfrak{A}$ such that $\theta^2 = 1$ and which preserves the canonical Jordan product. In addition, θ must act as a full spacetime inversion on local algebras, $\theta\mathfrak{A}(O) = \mathfrak{A}(-O)$, conjugate charge morphisms, $\theta \circ \varrho = \bar{\varrho} \circ \theta$, and have the correct commutation relations with the Poincaré transformations, $\theta \circ \alpha_{a,\Lambda} = \alpha_{-a,\Lambda} \circ \theta$. (The Haag-Kastler axioms posits a privileged representation of the Poincaré transformation acting as net automorphisms.) We expect that vacuum states in many (if not all) models of QFT will be CPT-invariant. It follows that θ will be implemented by an antiunitary operator, Θ, in each vacuum representation.[5] If this is the case, then Θ intertwines conjugate charge representation:

Proposition 2. *If CPT symmetry is implemented in vacuum representations, then the antiunitary CPT operator intertwines conjugate DHR/BF charge representations.*

Proof. If CPT symmetry is implemented in each vacuum representation, $(\pi_\omega, \mathcal{H}_\omega)$, then $\Theta\pi_\omega(A)\Theta = \pi_\omega(\theta(A))$, for all $A \in \mathfrak{A}$. Conjugate charge representations, π and $\overline{\pi}$, are unitarily equivalent to the representations $(\pi_\omega \circ \varrho, \mathcal{H}_\omega)$ and $(\pi_\omega \circ \bar{\varrho}, \mathcal{H}_\omega)$, where ϱ and $\bar{\varrho}$ are the associated localized transportable

[5] Baker and Halvorson (2013) draw an important distinction between *implemented* and *represented* algebraic symmetries that can be naturally generalized to include the antiunitary case. If $\alpha : \mathfrak{A} \to \mathfrak{A}$ is a Jordan automorphism, and π is a representation of $\mathfrak{A}$, then so is $\pi \circ \alpha(\mathfrak{A}) := \pi(\alpha(\mathfrak{A}))$. Wigner's theorem entails that α can always be *represented* by some unitary or antiunitary operator, W, such that $W\pi(\mathfrak{A})W^* = \pi \circ \alpha(\mathfrak{A})$. This does not mean that W implements a Jordan equivalence between π and $\pi \circ \alpha$, however, only that the adjoint action of W is a bijection between the two representations. Jordan equivalence requires in addition that the adjoint action of W intertwines π and $\pi \circ \alpha$ pointwise, i.e., $W\pi(A)W^* = \pi \circ \alpha(A)$ for all $A \in \mathfrak{A}$. In this case we say that α is *implemented* by W.

DHR/BF morphisms. We then have:

$$\begin{aligned}\Theta\pi_\omega \circ \varrho(A)\Theta &= \Theta\pi_\omega(\varrho(A))\Theta \\ &= \pi_\omega(\theta(\varrho(A))) \\ &= \pi_\omega(\theta \circ \varrho(A)) \\ &= \pi_\omega(\bar{\varrho} \circ \theta(A)) \\ &= \pi_\omega \circ \bar{\varrho}(\theta(A)) \ ,\end{aligned}$$

for all $A \in \mathfrak{A}$. The first line follows from the definition of $\pi_\omega \circ \varrho$. The second follows from the hypothesis that CPT symmetry is implementable and the third from the definition of DHR/BF morphism composition.[6] The fourth line follows from the commutation relations between θ and any charge morphism and the fifth again from the definition of morphism composition. Therefore, Θ intertwines conjugate charge representations, $\Theta\pi(A)\Theta = \bar{\pi}(\theta(A))$, for all $A \in \mathfrak{A}$. □

The fact that the CPT operator implements an antiunitary equivalence between charge representations will be an important interpretational data point that we need to consider in the following section.

5 Physical Equivalence?

We return now to the question left hanging in §2, does Jordan quasiequivalence entail physical equivalence? Since only self-adjoint elements of $\mathfrak{A}$ represent physical quantities, and all of their spectral properties are preserved by Jordan equivalence, it is tempting to answer yes. But there are reasons to resist this temptation.

If the non-self-adjoint portion of a C^*-algebra is really physically irrelevant, then it should be possible to formulate quantum mechanics entirely in terms of Jordan algebras. There was a historical program that attempted to do just this, but it eventually ran out of steam for non-trivial mathematical reasons (McCrimmon, 2004, Ch. 1). There is no natural notion of a tensor product between two Jordan algebras, and as Hanche-Olsen (2006) eventually proved, any Jordan algebra with a tensor product is really just

[6] This is especially simple in the DHR case where, as a consequence of Haag duality, the charge morphisms are endomorphisms of $\mathfrak{A}$. In the more general BF case, this is no longer true, but wedge duality allows us to prove a similar morphism composition rule.

a C^*-algebra in disguise. Thus, insofar as we view tensor product structure as physically significant, we should be wary of ignoring the non-self-adjoint part of the C^*-algebra.

There is a second reason for doubt. As Alfsen and Shultz (1998) emphasize, in both classical and quantum theories, observables play dual roles — they represent physical quantities and they generate symmetries. Each self-adjoint element of $\mathfrak{A}$ acts as the infinitesimal generator of a unique 1-parameter group of statespace automorphisms. The non-self-adjoint part of $\mathfrak{A}$ encodes this generating relationship between observables and symmetries using a canonical Lie product, $A \star B := \frac{i}{2}(AB - BA)$. (Alfsen and Shultz prove that this is equivalent to the choice of a certain kind of generalized orientation structure on state space.) Insofar as we think that this generating relationship carries physical significance, we should proceed with caution.

Evidently, more care is needed. Following Ruetsche (2011, Ch. 2), we bracket interpretational issues related to the measurement problem and look for a presumptive notion of physical equivalence for partially interpreted QFTs. Ruetsche's favored notion takes the form of a translation scheme between the kinematics and dynamics of two models, consisting of three bijections:

- $i_s : \mathcal{S} \to \mathcal{S}'$, between the physically possible states of each model, that preserves transition probabilities,

- $i_q : \mathcal{Q} \to \mathcal{Q}'$, between the physically possible quantities of each model, that preserves algebraic relations between quantities and is such that for all $\phi \in \mathcal{S}$ and $A \in \mathcal{Q}$, $\phi(A) = \phi'(A')$ where $\phi' = i_s(\phi)$ and $A' = i_q(A)$,

- $i_d : \mathcal{D} \to \mathcal{D}'$, between the dynamics of each model (i.e., $\mathbb{R}$-valued 1-parameter flows on state space, d_t), such that $d_t(\phi)(A) = d'_t(\phi')(A')$, where $d'_t = i_d(d_t)$.

Ruetsche goes on to argue that unitary equivalence yields such a translation scheme, and so unitarily equivalent (and by extension unitarily quasiequivalent) representations should be interpreted as presumptively physically equivalent. Essentially the same argument can be applied to antiunitary equivalence. An antiunitary mapping, $V : \mathcal{H} \to \mathcal{H}'$ induces a bijection on states that preserves transition probabilities (by Wigner's theorem), and since it intertwines the corresponding representations, it induces a Jordan isomorphism that preserves all Jordan-algebraic relations between self-adjoint observables (including all spectral properties). Moreover it identifies the flow generated by the Hamiltonian, H, with the flow generated by $H' = VHV^*$, and consequently, in the Heisenberg time-evolution picture:

$$Ve^{itH}Ae^{-itH}V^* = e^{-itH'}A'e^{itH'} \tag{6}$$

for all $A \in \mathfrak{A}$. Therefore, $d_t(\phi)(A) = d'_t(\phi')(A')$.

There are two important subtleties that distinguish the antiunitary case from the unitary one. First, even though an arbitrary Jordan automorphism is guaranteed to map pure states to pure states, it may not map physical pure states satisfying some relevant selection criteria to physical pure states. A unitarily implemented Jordan automorphism is guaranteed to map pure states to pure states in the same superselection sector. So if the original state satisfies the relevant selection criteria, the symmetry-transformed state does as well. In contrast, an antiunitarily implemented Jordan automorphism can map pure states in one superselection sector to those in another sector. Prima facie, there is no guarantee that this new sector will satisfy the relevant selection criteria.

Fortunately, in the DHR/BF case, we avoid this problem. By proposition 1, any antiunitary intertwiner will map an irreducible DHR/BF representation onto a representation unitarily equivalent to its conjugate and therefore in the conjugate superselection sector. But this is a quirk related to the existence of conjugate charges. There is no guarantee that it will hold in general for other physical selection criteria. Absent a general argument ruling this possibility out, it would be a mistake to automatically identify antiunitarily equivalent representations as physically equivalent in an arbitrary quantum theory. If some generalization of the DHR/BF criterion turns out to be a necessary constraint on physical states in any well-behaved relativistic QFT, then the problem might be avoided, but only in this restricted context.

There is a second subtly, however, that raises problems even for well-behaved relativistic QFTs. Although antiunitary equivalence preserves all Jordan-algebraic relations, it does not preserve all algebraic relations. In particular, it does not preserve the Lie product, encoding the generating relationship between observables and 1-parameter groups of state space symmetries. Unitary equivalence, in contrast, preserves this generating relationship. But antiunitary equivalence does not completely make hash out of it. Rather, antiunitary intertwiners systematically reverse the Lie product. If π carries the Lie product $A \star B$, any antiunitarily equivalent representation, π', is unitarily equivalent to $\bar{\pi}$, which carries the Lie product $(A \star B)^{op} := B \star A = -(A \star B)$. In principle, both Lie products can encode the same physical information, corresponding to opposite choices of orientation structure on state space.[7] What possible physical significance could such a choice have?

One obvious physical consequence, indicated by (6), is that the dynamical bijection, i_d, reverses temporal orientation. But even though the dynamics d'_t flow in the opposite temporal direction, they are not necessarily the time-reverse of d_t. This requires further that V commutes with H, leaving the form of the dynamical law unchanged, and that V acts uniformly on spatial degrees of freedom. Proving that the CPT operator, Θ, does both of these things is a non-trivial part of the CPT theorem, justifying why Θ can be viewed as a generalized time-reversal operator. There is a long tradition of identifying time-reversed states as physically equivalent in theories where the laws are time-reversal invariant. Following this tradition, we may choose to interpret Θ-transformed states as physically equivalent, but the same reasoning will not apply to arbitrary cases of antiunitary equivalence.

Nonetheless, I think that a decent case for interpreting at least some antiunitarily equivalent representations as physically equivalent can be made along these lines. For philosophers sympathetic to relationalist metaphysics, this is where the action is. Any such relationalist argument, however, faces an unexpected challenge. As we have seen in §4, if CPT symmetry is implemented in vacuum representations, Θ intertwines conjugate DHR/BF representations. In these cases, the relationalist is forced to interpret conjugate charge sectors as physically equivalent.

Such an identification threatens to undermine the explanatory success of the DHR/BF picture of antimatter. It is the existence of conjugate objects in the category of localized transportable morphisms that allows for the description of conjugate charges. In order for this notion to be captured by the specification of boundary conditions on the observable net, the category of DHR/BF representations must likewise possess conjugate objects. It is not obvious that "quotienting" the DHR/BF category by identifying conjugate objects yields a mathematical structure capable of discharging the explanatory jobs handled by the usual DHR/BF analysis. To be clear, the relationalist proposal would not eliminate conjugate charges entirely. It would

[7] The statespace of a C^*-algebra is a compact convex set, whose minimal, norm-exposed faces are affinely isomorphic to either a line or a Euclidean 3-ball. The Lie product determines which self-adjoint operator, A or $-A$, generates clockwise rotations of these facial 3-balls.

still make sense to distinguish between conjugate charges within a given representation, but there would no longer be a difference between a representation with two Q-charges and one $\overline{Q}$-charge, and a representation with two $\overline{Q}$-charges and one Q-charge. Rather than two possible boundary conditions in this case, net global charge Q and net global charge $\overline{Q}$, there is only one boundary condition, one unit of net global charge. But if we take the lesson of the DHR/BF analysis to heart, the explanatorily fundamental way to describe charges in algebraic QFT is via possible boundary conditions on the net of gauge-invariant observables. The relational difference between conjugate charges within a representation is then explained by the different boundary conditions captured by the conjugate representations π and $\bar{\pi}$. If we identify these boundary conditions as physically equivalent, we lose the ability to tell this type of gauge-invariant story.

These worries pose a distinct challenge for interpreting antiunitary equivalence as sufficient for physical equivalence in the context of relativistic QFT. How should we view antiunitary intertwiners that do not implement generalized time-reversal symmetries? If conjugate charge representations are physically equivalent, how do we recover the explanatory power of the DHR/BF picture? Interpreters of QFT should take note.

References

Alfsen, E. and Shultz, F. (1998). Orientation in operator algebras. *Proceedings of the National Academy of Sciences, USA*, 95:6596–6601.

Arageorgis, A. (1995). *Fields, Particles and Curvature.* Ph.D. dissertation, University of Pittsburgh.

Baker, D. J. (2011). Broken symmetry and spacetime. *Philosophy of Science*, 78:128–148.

Baker, D. J. and Halvorson, H. (2013). How is quantum spontaneous symmetry breaking possible? *Studies in History and Philosophy of Modern Physics*, 44(4):464–69.

Baker, D. J., Halvorson, H., and Swanson, N. (2015). On the conventionality of parastatistics. *British Journal for the Philosophy of Science*, 66(4):929–76.

Blackadar, B. (2006). *Operator Algebras.* Springer.

Buchholz, D. and Fredenhagen, K. (1982). Locality and the structure of particle states. *Communications in Mathematical Physics*, 84(1):1–54.

Connes, A. (1975). A factor not anti-isomorphic to itself. *Bulletin of the London Mathematical Society*, 7:171–174.

Doplicher, S., Haag, R., and Roberts, J. E. (1969a). Fields, observables and gauge transformations. I. *Communications in Mathematical Physics*, 13:1–23.

Doplicher, S., Haag, R., and Roberts, J. E. (1969b). Fields, observables and gauge transformations. II. *Communications in Mathematical Physics*, 15:173–200.

Doplicher, S. and Roberts, J. E. (1990). Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Communications in Mathematical Physics*, 131:51–107.

Earman, J. (2003). Rough guide to spontaneous symmetry breaking. In Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pages 335–346. Cambridge UP, Cambridge.

Emch, G. (1972). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, volume XXVI of *Monographs and Texts in Physics and Astronomy.* Wiley.

Halvorson, H. and Clifton, R. (2001). Are Rindler quanta real? Inequivalent particle concepts in quantum field theory. *British Journal for the Philosophy of Science*, 52:417–470.

Halvorson, H. and Müger, M. (2006). Algebraic quantum field theory. In Butterfield, J. and Earman, J., editors, *Philosophy of Physics*, pages 731–922. Elsevier.

Hanche-Olsen, H. (2006). JB-algebras with tensor products are C^*-algebras. In *Operator Algebras and their Connections with Topology and Ergodic Theory: Proceedings of the OATE Conference held in Busteni, Romania, Aug. 29–Sept. 9, 1983*, volume 1132 of *Lecture Notes in Mathematics*, pages 223–9. Springer.

McCrimmon, K. (2004). *A Taste of Jordan Algebras*. Springer.

Ruetsche, L. (2011). *Interpreting Quantum Theories*. Oxford UP, Oxford.

Simon, B. (1996). *Representations of Finite and Compact Groups*, volume 10 of *Graduate Studies in Mathematics*. American Mathematical Society.

Strocchi, F. (2008). *Symmetry Breaking*. Springer, Berlin/Heidelberg.

Swanson, N. (2019). Deciphering the algebraic CPT theorem. *Studies in History and Philosophy of Modern Physics*, 68:106–25.

Varadarajan, V. (1968). *Geometry of Quantum Theory*. Springer.