Indeterminism, Gravitation, and Spacetime Theory

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1 Introduction: Varieties of Indeterminism

Contemporary discussions of physical determinism that engage with modern spacetime and gravitational theory (e.g., Earman (1986, 2007)) have typically focused on the question of the *global* uniqueness of solutions for two sorts of initial-value problems: that of the evolution of a field or a number of massive point particles in a fixed spacetime, or of four-dimensional spacetime itself from a three-dimensional "slice" within the context of general relativity. (I set the "indeterminism" sometimes associated with the Hole Argument aside. Cf. Brighouse (1994).) Paradigm examples of indeterminism in these contexts involve, respectively, the non-collision singularities of the "space invaders" scenario, where the worldliness of gravitating massive particles appear from spatial infinity after an arbitrary time, and the existence of non-unique maximal extensions, where a given "slice" can be evolved in one of many incompatible ways.

Here I shall investigate another sort of indeterminism, which, though studied in the literature on determinism in classical physics, is not typically considered in light of spacetime theory. In this sort of indeterminism, a localized, point-like physical system with equations of motion described by a set of differential equations has an initial-value problem with many solutions—typically uncountably infinitely many. This is a failure of *local* uniqueness of solutions. While systems exhibiting this property have been known since at least the 19th century (Fletcher, 2012; van Strien, 2014), a simple, concrete example due to Norton (2003, 2008) has recently captured philosophers' attention. In this example, a massive point particle begins at rest on the surface of a peculiarly shaped cylindrically symmetric rigid dome. The difference in height of a point on the dome from

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its apex, h, can be given as a function of the radial distance to that point from the apex along the surface of the dome, r:

$$h(r) = (2b^2/3g)r^{3/2}, \tag{1}$$

where g is the (constant) acceleration due to gravity at the surface of the Earth and b is a dimensional constant. If the dome is fixed rigidly to a flat surface in a uniform gravitational field yielding acceleration g, then the net force on the massive particle is the component of the total gravitational force tangential to the surface:

$$F_{\parallel} = mg\sin\theta = mg(dh/dr) = mb^2\sqrt{r}, \tag{2}$$

where θ is the angle between the horizontal and the radial tangent to the dome's surface at the particle's location.[1] Now, if the particle's initial position at $t = 0$ is the apex of the dome, then its initial value problem has as a solution

$$r(t) = \begin{cases} 0, & \text{if } t \leq T, \\ \frac{1}{144}(b[t-T])^4, & \text{if } t > T, \end{cases} \tag{3}$$

for *every* positive constant T. In other words, it is compatible with the Newtonian dynamics for the particle to fall down an arbitrary side of the dome after an arbitrary time resting at the top.

Mathematically, the local uniqueness to an initial value problem can be guaranteed only when further conditions hold of the equations of motion (Agarwal and Lakshmikantham, 1993). In the case of Newtonian point particles, one such condition is that the force on the particle be *locally Lipschitz continuous* at the initial conditions (Arnol'd, 1992, 36–38, 104–105). In general, a function $F(x) : D \to \mathbb{R}^m$, with $D \subseteq \mathbb{R}^n$, is *Lipschitz continuous on D* just when there is a constant $K > 0$ such that for all $x, y \in D$, $|F(x) - F(y)| \leq K|x - y|$. F is locally Lipschitz continuous at $x \in D$ when there is some neighborhood U of x on which it is Lipschitz continuous. It is easy to show that $F_{\parallel}$ in equation 2 is not Lipschitz continuous at $r = 0$: $|F_{\parallel}|/|r|$ is not bounded above as $r \to 0$. Physically, this corresponds to an initial condition for the particle in which its net force (hence net acceleration) vanishes, but the derivative of that force in any spatial direction is infinite. This infinite derivative is necessary, but not sufficient, for nonuniqueness of the initial value problem, as the example of a ball rolling off a table illustrates.

The ensuing discussion of Norton's dome has mostly concentrated on furnishing various arguments against the dome system's legitimacy, either as an incomplete or incorrect application of Newtonian theory, or as an unphysical or otherwise improper idealization in that theory. (See Fletcher (2012) for an overview of many of these objections.) By contrast, Malament (2008) and Wilson (2009) have rejected the question as being too simplistic, for it assumes that there is a mathematically univocal, common conception of Newtonian mechanics. Elaborating on this idea, Fletcher (2012) argues for a plurality of closely related theories of Newtonian mechanics, some of which may have indeterministic models, partly on the grounds that indeterminism arises from mathematical features of the equations of motion for a model rather than from any identifiably common physical feature thereof.

The goal of this paper is to expand on this thesis by suggesting that the sort of indeterminism given by multiple solutions to a test particle's initial value problem depends neither on the physics

[1]Note that since $0 \leq \theta \leq 90°$, $0 \leq dh/dr \leq 1$, hence $0 \leq r \leq g^2/b^4$ and $0 \leq h \leq 2b^2/3g^4$: the dome must have a finite height.

in question being non-relativistic, nor on the specification of a particular form of forced motion. Its aspiration, in other words, is the construction of a certain relativistic spacetime, some of whose timelike geodesics passing through a point are not uniquely determined by the specification of their tangent vector there. In such a spacetime, the woldlines of free massive test particles modeled by these geodesics exhibit the sort of indeterminism displayed by Norton's dome. This indeterminism also combines in a novel way some of the features of both of the two sorts of indeterminism described at the beginning of this section: on the one hand, it applies to particular sets of worldliness of massive test particles without varying spacetime structure, and on the other, it does so without introducing new matter entering the universe from spatial infinity.

To arrive at this example, I will attempt to make two successive modifications to an electrostatic example of Fletcher (2012), described in §2.1, which displays the same sort of indeterminism as the dome system. First, in §2.2, I adapt it from Newtonian to special relativistic physics. The second modification, in §2.3, attempts to replicate this sort of indeterminism with the unforced, or geodesic, motion of a test particle in a general relativistic spacetime.

These examples raise the question about the sorts of spacetimes, relativistic or otherwise, which admit of non-unique solutions to the geodesic equation for some initial conditions. I suggest an answer some of these question in §3 to the effect that any spacetime exhibiting this sort of indeterminism must at some point violate the strong energy condition (SEC), which heuristically can be understood as the statement that the effects of gravity are locally attractive (i.e., causal geodesic tend to converge). In other words, repulsion, whether through forced or natural motion, is necessary for indeterminism. This immediately implies that indeterminism cannot manifest in either relativistic vacuum spacetimes or through pure gravity in Newtonian spacetimes.

Other sorts of questions raised by these examples, discussed in the concluding §4, are of an interpretive nature. In addition to them providing more evidence for a pluralistic understanding of relativity theory, they also unsettle the usual, though tacit, assumption that the specification of a relativistic spacetime is a complete determination of all the events of a model world. One option is to augment spacetime structure to fix the actual worldlines of test particles; another, more ambitious but less developed option is to take the concept of the test particle and the worldline more seriously as idealizations, delimiting their range of fruitful application more precisely. I provide some, though not entirely conclusive, reasons that the latter option is to be preferred.

2 Indeterminism through Forced and Unforced Motion

In the sections below concerning relativistic spacetimes, I use the abstract index notation (Malament, 2012, §1.4; Wald, 1984, §2.4) according to which lowercase superscript (resp. subscript) roman letters $(a, b, c, \dots)$ on a symbol representing a tensor denote the label of the vector (resp. covector) space(s) in which the tensor lives. As an example of this notation: those sections will consider relativistic spacetimes (M, g_{ab}), where M denotes a smooth, four-dimensional, paracompact and smooth real manifold and g_{ab} denotes a Lorentz metric of signature $(1, 3)$ on this manifold.

Note as well that section 2.1 is based on Fletcher (2012, §3.4).

2.1 Indeterminism of Forced Motion in Newtonian Spacetime

Consider the following spherically symmetric electric charge distribution in a Newtonian spacetime:

$$\rho(r) = \begin{cases} 5C\epsilon_0/2\sqrt{r}, & \text{if } 0 < r \leq R, \\ 0, & \text{if } r = 0 \text{ or } r > R, \end{cases} \tag{4}$$

where r is the radial distance from the center of symmetry, ϵ_0 is the permittivity of free space, and C is a dimensional constant. Even though $\lim_{r\to 0}\rho(r) = \infty$, the total charge Q is finite:

$$Q = \int \rho dV = 4\pi C\epsilon_0 R^{5/2}. \tag{5}$$

By Gauss's Law, the radial component of the electric field for $r \leq R$ is

$$E_r(r) = \frac{1}{4\pi\epsilon_0 r^2}\int_{B_r} \rho dV = C\sqrt{r}, \tag{6}$$

where B_r is the ball of radius r. A test particle with charge q starting from rest at a radial distance $r \leq R$ from the center of the charge distribution experiences a Coulomb force

$$F_r(r) = qE_r(r) = qC\sqrt{r}. \tag{7}$$

Thus, if the particle is initially at rest at the origin, it has uncountably many solutions $r(t)$ to its equation of motion exactly in analogy with equation 3.

2.2 Indeterminism of Forced Motion in Special Relativity

For convenience, choose units so that the numerical value of the speed of light c is 1, as is the permittivity of free space ϵ_0. Consider Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ along with a constant, unit timelike vector field t^a whose geodesic congruence represents the worldlines of a family of inertial observers. In order to reproduce an analog of the example in §2.1, pick (the image of) one such worldline γ as the axis of symmetry for a charge distribution

$$\rho_{|p} = \begin{cases} 5C/2\sqrt{r_{|p}}, & \text{if } 0 < r_{|p} \leq R, \\ 0, & \text{if } r_{|p} = 0 \text{ or } r_{|p} > R, \end{cases} \tag{8}$$

where, much as before, C is a dimensional constant and $r_{|p}$ is the distance from $p \in M$ to (the image of) γ along the spacelike geodesic orthogonal to it. Now, given any observer with four-velocity ξ^a at a point, they can reconstruct the Faraday tensor F_{ab} and the charge-current density J^a according to

$$F_{ab} = E_a\xi_b - \xi_a E_b + \varepsilon_{abcd}\xi^c B^d, \tag{9}$$
$$J^a = \rho\xi^a + j^a, \tag{10}$$

where E^a is their observed electric field, B^a is their observed magnetic field, j^a is their observed current density, and ε_{abcd} is a volume form. Thus we may assign, for the family of inertial observers determined by t^a,

$$(E^a)_{|p} = \begin{cases} C\sqrt{r_{|p}}(r^a)_{|p}, & \text{if } 0 \leq r_{|p} \leq R, \\ (CR^{5/2}/r_{|p}^2)(r^a)_{|p}, & \text{if } r_{|p} > R, \end{cases} \tag{11}$$

$$B^a = 0, \tag{12}$$

$$j^a = 0, \tag{13}$$

where r^a is the radial spatial vector field from γ. This yields that

$$F_{ab} = \begin{cases} C\sqrt{r}(r_a t_b - t_a r_b), & \text{if } 0 \leq r_{|p} \leq R, \\ (CR^{5/2}/r_{|p}^2)(r_a t_b - t_a r_b), & \text{if } r_{|p} > R, \end{cases} \tag{14}$$

$$J^a = \begin{cases} (5C/2\sqrt{r_{|p}})t^a, & \text{if } 0 < r_{|p} \leq R, \\ 0, & \text{if } r_{|p} = 0 \text{ or } r_{|p} > R. \end{cases} \tag{15}$$

To verify that these are indeed solutions of Maxwell's equations,

$$\nabla_{[a}F_{bc]} = 0, \tag{16}$$

$$\nabla_a F^{ab} = J^b, \tag{17}$$

note first that one can define

$$(A_a)_{|p} = \begin{cases} (2Cr_{|p}^{3/2}/3)t_a, & \text{if } 0 \leq r_{|p} \leq R, \\ -(CR^{5/2}/r_{|p})t_a, & \text{if } r_{|p} > R, \end{cases} \tag{18}$$

so that

$$F_{ab} = \nabla_a A_b - \nabla_b A_a. \tag{19}$$

Substitution of equation 14 into the left-hand side of equation 16 then yields the desired result.

For equation 17, substitution 14 and application of the product rule gives, for $0 \leq r \leq R$,

$$\begin{aligned} \nabla_a(C\sqrt{r}[r^a t^b - t^a r^b]) &= (C/2\sqrt{r})r_a(r^a t^b - t^a r^b) \\ &\quad + C\sqrt{r}([\nabla_a r^a]t^b + r^a\nabla_a t^b - [\nabla_a t^a]r^b - t^a\nabla_a r^b), \\ &= (C/2\sqrt{r})t^b + C\sqrt{r}([2/r]t^b + 0 + 0 + 0), \\ &= (5C/2\sqrt{r})t^b, \end{aligned}$$

and for $r > R$,

$$\begin{aligned} \nabla_a(CR^{5/2}/r^2[r^a t^b - t^a r^b]) &= (-2CR^{5/2}/r^3)r_a(r^a t^b - t^a r^b) \\ &\quad + (CR^{5/2}/r^2)([\nabla_a r^a]t^b + r^a\nabla_a t^b - [\nabla_a t^a]r^b - t^a\nabla_a r^b), \\ &= -(2CR^{5/2}/r^3)t^b + (CR^{5/2}/r^2)([2/r]t^b + 0 + 0 + 0), \\ &= 0, \end{aligned}$$

which matches equation 15 as required.

Lastly, consider a test particle of charge q and mass m initially co-moving with the charge distribution. If its initial position p is within the charge distribution (i.e., $r_{|p} < R$), then it experiences a Lorentz force $qF^a{}_b t^b = qC\sqrt{r_{|p}}r^a$, which is not Lipschitz continuous at $r = 0$. Thus, if the particle's initial location p satisfies $r_{|p} = 0$, then there are infinitely many worldlines which result as solutions to its equation of motion: $d^2r/d\tau^2 = (q/m)C\sqrt{r}$, where τ is the proper time along the particle's worldline. Note that these solutions are only locally in analogy with those of equation 3, for the latter in principle can result in unbounded velocities. In contrast, the relativistic equation of motion concerns proper time τ, not the coordinate time t, but for sufficiently small relative velocities with the inertial observers posited at the beginning of this section, they approximate each other arbitrarily well.

2.3 Indeterminism of Natural Motion in General Relativity

As with virtually all previous examples of locally nonunique solutions to a differential equation of motion, the one described above in Minkowski spacetime with a carefully chosen distribution of matter renders a force on a test particle that is non-Lipschitz at certain points. The same failure of uniqueness, however, can also arise in general relativity from the geodesic motion of a test particle—that is, from gravity alone.

Consider any relativistic spacetime (M, g_{ab}) with Levi-Civita connection ∇.[2] The (proper) acceleration for a test particle whose wordline has tangent vector ξ^a can be expressed using a locally flat derivative operator ∂ as

$$\xi^b\nabla_b\xi^a = \xi^b\partial_b\xi^a - \xi^b\xi^m C^a{}_{bm}, \tag{20}$$

where $C^a{}_{bm}$ is the connection tensor between ∇ and ∂ whose components in a particular coordinate basis are the Christoffel symbols (Malament, 2012, Prop. 1.7.3; Wald, 1984, p. 34). The connection tensor in turn can be expressed using the metric (Malament, 2012, eq. 1.9.6; Wald, 1984, eq. 3.1.28):

$$C^a{}_{bm} = \frac{1}{2}g^{an}(\partial_n g_{bm} - \partial_b g_{nm} - \partial_m g_{nb}). \tag{21}$$

Thus by substituting 21 we can rewrite equation 20 as

$$\xi^b\nabla_b\xi^a = \xi^b\partial_b\xi^a - \frac{1}{2}\xi^b\xi^m g^{an}(\partial_n g_{bm} - \partial_b g_{nm} - \partial_m g_{nb}). \tag{22}$$

In coordinates adapted to the flat connection ∂, the right-hand side expresses the "force" on the test particle (up to a factor of the mass of the particle). Now, the *true* force on a test particle is not a coordinate-dependent quantity. But what matters here is that equation 22 has the same form as Newton's second law, considered as a differential equation. Thus, through an appropriate choice of g_{ab}, one can design the same non-uniqueness to its solutions for appropriate choices of initial conditions.

Implementing this feature requires the first derivatives of the spacetime metric with respect to ∂ to be non-Lipschitz continuous at some point, so the metric cannot be everywhere smooth. While smoothness is often demanded of the spacetime metric, its full strength is not required to formulate general relativity adequately. As the example below will show, allowing it to be C^1 on

[2] From the present perspective, the stress-energy tensor T_{ab} is not an independent object for a relativistic spacetime once the metric has been specified, since the Riemann tensor $R^a{}_{bcd}$ associated with the Levi-Civita connection determines the Ricci tensor R_{ab} and curvature scalar R, which in turn determine T_{ab} through Einstein's equation, $T_{ab} = (1/8\pi)(R_{ab} - (1/2)Rg_{ab})$.

a one-dimensional line still enables one to define all the needed geometric objects of the theory, while still allowing for non-unique solutions to the initial value problem for some geodesics.

In analogy with the examples of this indeterminism already considered, I restrict attention to static, spherically symmetric spacetimes on $\mathbb{R}^4$, whose metrics and inverse metrics take on the general form (Wald, 1984, eq. 6.1.5)

$$g_{ab} = e^{2\nu} t_a t_b - e^{2\lambda}(r_a r_b + r^2(\theta_a \theta_b + \sin^2\theta \phi_a \phi_b)), \tag{23}$$

$$g^{ab} = e^{-2\nu} t^a t^b - e^{-2\lambda}(r^a r^b + r^{-2}(\theta^a \theta^b + \csc^2\theta \phi^a \phi^b)), \tag{24}$$

where I have used the abbreviations $x_a = \partial_a x$ and $x^a = (\partial/\partial x)$ for $x \in \{t, r, \theta, \phi\}$, spherical coordinate fields well-adapted to the symmetries of the spacetime, and ν and λ depend only on r. Further, I restrict attention to test particles whose initial four-velocity is t^a. The goal is to find a metric whose derivatives yield an acceleration field proportional to $\sqrt{r}$ in the right-hand side of equation 22.

To analyze this problem, first consider the acceleration of *any* test particle whose initial four-velocity is t^a, not necessarily one undergoing geodesic motion. Substitution of equation 23 into equation 22 gives that

$$t^b \nabla_b t^a = t^b \partial_b t^a - \frac{1}{2} t^b t^m g^{an}(\partial_n g_{bm} - \partial_b g_{nm} - \partial_m g_{nb}) = g^{an} t^b \partial_b (t^m g_{nm}) - \frac{1}{2} g^{an} \partial_n (t^b t^m g_{bm}). \tag{25}$$

Since ν and λ depend only on r, the product rule for differentiation then yields that

$$\partial_b(t^m g_{nm}) = 2\nu' e^{2\nu} t_n r_b, \tag{26}$$

$$\partial_n(t^b t^m g_{bm}) = 2\nu' e^{2\nu} r_n, \tag{27}$$

where $\nu' = d\nu/dr$. Combining these with equations 24 and 25,

$$t^b \nabla_b t^a = g^{an} t^b (2\nu' e^{2\nu} t_n r_b) - \frac{1}{2} g^{an}(2\nu' e^{2\nu} r_n) = \nu' e^{2\nu - 2\lambda} r^a. \tag{28}$$

There are many possible choices of ν and λ which will yield a function on the right-hand side that is not everywhere Lipschitz continuous. For simplicity, I will examine the case of

$$\nu = \lambda = A + (2B^2/3) r^{3/2}, \tag{29}$$

where A and B are constants. For this choice, the spacetime is conformally equivalent to Minkowski spacetime, i.e.,

$$g_{ab} = e^{2A + (4B^2/3) r^{3/2}} \eta_{ab}, \tag{30}$$

and

$$t^b \nabla_b t^a = B^2 \sqrt{r} r^a. \tag{31}$$

Thus a test particle initially at any point p with $r_{|p} = 0$ and four-velocity t^a undergoes geodesic motion if all the points of its wordline satisfy $r = 0$. But this is not the only geodesic passing through p with tangent vector t^a, for there are infinitely many solutions to the equation $dr^2/d\tau^2 = B^2\sqrt{r}$ with initial condition $r = 0$. Just as in the previous subsection, these solutions are only locally analogous to those of equation 3, due to the relativistically necessary proper time. But the way that the "gravitational field"—really, just spacetime curvature—emulates the Maxwellian field in the role as the source of spontaneous "acceleration" is similar.

The new metric given by equation 30 is clearly smooth everywhere except on the line of $r = 0$, where it is merely C^1. Thus the Riemann tensor $R^a{}_{bcd}$ associated with the Levi-Civita connection, which is necessary to define the Ricci tensor R_{ab} in Einstein's equation, is well-defined in the usual way everywhere except on the line of $r = 0$.[3] Here, it is most natural to set $(R^a{}_{bcd})_{|r=0} = 0$, for observers with worldlines on $r = 0$ measure the metric to be Minkowskian (up to an immaterial constant factor). The resulting curvature, while well-defined everywhere, does not vary continuously, much as the electric charge density did not in the previous sections.

3 Indeterminism and the Strong Energy Condition

The matter distributions and spacetimes considered above are unusual. Are there general criteria for deciding when a worldline of a test particle is not deterministic? A failure of smoothness in the metric or matter distribution is certainly necessary, but I suspect more can be said, in particular in connection with the SEC. Following Curiel (2017, §2.1), one can distinguish (at least) between a geometric version of the SEC, which states that

$$R_{ab}\xi^a\xi^b \geq 0 \tag{32}$$

for any timelike vector ξ^a, and a physical version of the SEC, which states that

$$(T_{ab} - \frac{1}{2}T g_{ab})\xi^a\xi^b \geq 0 \tag{33}$$

for any timelike vector ξ^a. The physical version is implied by the geometric version under the assumption of Einstein's equation, but the geometrical version has a natural interpretation, namely that timelike geodesics locally converge. Below I sketch an argument that a necessary condition for the nonuniqueness of a solution to a test particle's initial value problem at some point p is a violation of the SEC in a neighborhood of p.

Consider a spacetime with a C^1 metric in which the initial value problem for the geodesic motion of a point test particle at $p \in M$ has more than one solution, and let τ^a be the tangent vector field of these solutions, extended if necessary to a neighborhood around them, and pick a triple of vector fields x^a, y^a, z^a defined on the first solution that are orthogonal to and Lie derived by τ^a. Together, the four form a geodesic reference frame which we may use to define a coordinate system in a neighborhood of the initial point p, with the first local solution lying at the (x, y, z) coordinate origin. We may equip the other solution with a tetrad as well that is Lie derived by (a local extension to a neighborhood of) its tangent vector τ^a, and in particular has one component as the connecting field representing the relative acceleration between the two solutions. The contrapositive to the Peano uniqueness theorem (Agarwal and Lakshmikantham, 1993, §3.3) states that this must be on average increasing. But it is well known that the *average* radial acceleration amongst the three independent spatial directions is proportional to the Raychaudhuri scalar:

$$ARA = -\frac{1}{3}R_{ab}\tau^a\tau^b, \tag{34}$$

[3] One may apply the formulas of Wald (1984, p. 446) to calculate these explicitly, although doing so does not give any obvious insight into the nature of the indeterminism this spacetime exhibits.

which is the quantity bounded by the SEC. But ARA here is positive, so the geometric SEC is violated. Thus:

Proposition The geometric SEC is violated in the neighborhood of a point of a spacetime where the initial value problem for the geodesic equation does not have a unique solution.

In other words, gravitation must become "repulsive" as this point is approached. Moreover, since Einstein's equation sets $R_{ab} = T_{ab} - \frac{1}{2}T g_{ab}$, where T_{ab} is the stress-energy tensor and T is its trace, this implies the right-hand, contracted with the timelike vectors $\xi^a \xi^b$ becomes negative as this point is approached. And *this* in turn implies that the physical SEC is violated.

One of the interesting aspects of this argument is that substantive assumptions particular to relativity theory only entered at the end, in the invocation of Einstein's equation, and only there to connect the constraint, which is really on the Raychaudhuri scalar, with a well-known energy condition. Thus, one can apply a similar argument to other spacetime theories. In geometrized Newtonian gravitation, for example, the Raychaudhuri scalar is equated with the mass density through the geometrized version of Poisson's equation (Malament, 2012, p. 269). So the analog of the SEC in that theory is just the condition that mass be non-negative. Insofar as this is a central, not auxiliary, assumption of Newtonian gravitation, the above argument then yields that the sort of indeterminism considered in this paper is not possible with Newtonian gravitation alone. The reason for this comes again in the interpretation of the ARA: Newtonian gravitation is always attractive, never repulsive, while repulsivity is a necessary condition for indeterminism.

4 Discussion and Conclusions

Perhaps the first question raised by the foregoing examples is whether they should be excluded from legitimacy. It is easy to do so by reaffirming the demand for only smooth spacetime structures, for example. But this would rule out the discontinuous matter distributions and non-smooth spacetime metrics used in modeling stars (as object with compact boundaries), shock waves (Israel, 1960; Smoller and Temple, 1997), and colliding black holes (Merritt and Milosavljević, 2005). Demanding that the SEC seems in conflict with the use of successful models, especially in cosmology, that violate it persistently (Curiel, 2017). One could also restrict attention only to "physically reasonable" distributions of known or pedestrian matter, but the grounds for doing so may be questionable (Manchak, 2011).

Perhaps it is best, as Fletcher (2012) advocates with classical (non-relativistic) mechanics, to decline to make a decision and instead embrace the idea that there are many different versions of the theory of general relativity, some more delimited and other less so, some in which the above sorts of indeterminism are allowed and others in which they are eliminated. For, by allowing different versions of our theories to coexist, we can gain more insight into how their different parts fit together. In particular, further investigation of non-smooth spacetimes might perhaps allow for a new kind of response to the singularity theorems (Curiel and Bokulich, 2012) and other results about non-extendibility, in two ways: first, those results tend to assume that the collection of spacetimes that can be extended into must be smooth—indeed, on that mark the general relativistic spacetime I considered above would have to have its origin excised; and second, they also send to assume some version of the SEC (Curiel, 2017), whose violation, we saw in the previous section, is necessary for the failure of determinism. Is there some well-motivated way of performing non-smooth extensions to spacetimes to avoid singularities?

The other significant question raised by this investigation, to my mind, is about the exact role that the idealization of the point particle and its worldline are supposed to play in modern spacetime theory. In the use and interpretation of some of the most basic concepts of the theory, such as the classification of tangent vectors into timelike, null, and spacelike, and the definitions of singularities and causality conditions, invoke the notion of a worldline of a point particle. Some approaches to the foundations of relativity theory even take them as primitive. But in the above models, they are not determined by the usual sorts of data provided. One cannot maintain, in these models, that the events of the spacetime manifold encode all the goings on, here and elsewhere, past, present, and future, and at once hold that some particles' histories are not so determined. It reveals a tension between the "principal" nature of general relativity's foundation—interpretive principles about worldlines seemingly grounded in firm evidence—with the "constructive" nature of the particular matter fields we might wish to model upon it (Einstein, 1954).

One response to all this is to add structure, a further specification of the worldlines really occupied by particles. In some sense, the branching spacetimes framework (Placek, 2014) does something like this, although not quite in this context: again, it is not the spacetime structure that is indeterministic on the example considered above, but the test particles within them. The difficulty with this response is that it gives up on a sort of reductionism that, while not mandated by the spacetime picture, is friendly with it: namely, the view that all matter, particles included, are fields on spacetime. Once one fixes those fields, all is determined. Reifying these indeterministic examples through an addition to the theory would seemingly abandon that.

Another response, and the one I tentatively prefer, is a careful reevaluation of the notion of a test particle and the role it plays in the foundations of spacetime theory. On this view, it is an extremely convenient and expedient idealization, but one whose limits need to be more clearly addressed. As a sort of infinite idealization, test particles cannot share or well-approximate all the properties or features of their de-idealized, extended, internally interacting field-theoretic counterparts. Just as infinite systems in statistical mechanics have features that no finite systems share, so too do test point particles. This is not to say they should be extricated from the theory—far from it—but that further work is needed to understand their explanatory role, and the limits of their applicability. This paper has not attempted an answer to this question, but my hope is that it will stir the spirits of others to it.

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