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Dedekind's Map-theoretic Period

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ABSTRACT

In 1887–1894, Richard Dedekind explored a number of ideas within the project of placing mappings at the very center of pure mathematics. We review two such initiatives: the introduction in 1894 of groups into Galois theory intrinsically via field automorphisms, and a new attempt to define the continuum via maps from $\mathbb{N}$ to $\mathbb{N}$ (later called Baire space) in 1891. These represented the culmination of Dedekind's efforts to reconceive pure mathematics within a theory of sets and maps and throw new light onto the nature of his structuralism and its specificity in relation to the work of other mathematicians.

Richard Dedekind contributed a lot to establishing a mathematical style that has marked much of twentieth-century mathematics, being 'an inspiration to each succeeding generation'.[1] It is not merely that, as is well known, he provided set-theoretic foundations for the number system (particularly the reals and the naturals): he was also the great pioneer of the modern structural style of algebra, which would later be taken up by Noether, Artin, van der Waerden, and Bourbaki. In fact, one has the impression that Dedekind's style resonates not only with set-theoretic structures, but even with some aspects of category-theoretic mathematics. The main reason for this, I believe, is the central role that morphisms had in his mathematical reflections. This paper deals with morphisms, and with a period in Dedekind's active life that was particularly inspired by them.

[1] [Edwards, 1983]. It is noteworthy that this praise comes from somebody who has opposite preferences — Edwards favors constructivistic methods and regards Dedekind's ideals as an 'invention'. Van der Waerden wrote that Evariste Galois and Richard Dedekind 'gave modern algebra its structure; the weight-bearing skeleton of this structure comes from them' (quoted in [Reck, 2012] from the foreword to the 1964 edtion of [Dedekind, 1871; 1894])

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Experts on Dedekind's work know that he was extremely meticulous, to a surprising degree.[2] It is also a fact that he had a mathematician's attitude to avoiding long narratives or philosophical arguments, preferring brief indications of his insights accompanied by the real thing: careful chains of mathematical reasoning. This is linked to another fact: when he writes prose or motivating arguments, he aims at precision, so that sentences in one of his prefaces may be read with almost the same care that one would devote to the formulation of one of his theorems.

For all of these reasons, I have long been puzzled by a key passage in the Preface to *Was sind und was sollen die Zahlen?*, where Dedekind talks about the foundation of arithmetic, analysis, and algebra. He singles out a 'unique foundation' for these core areas, namely the general concept of an *Abbildung*, a mapping or function. (Incidentally, my own preference as a translation of the German *Abbildung* is the word 'representation', which captures some crucial connotations the idea had for Dedekind.[3] 'Abbildung' is a compound from 'Bild,' meaning image or figure — the result of copying something, but also a mental representation; the *Abbildung* of an object may be pictorial, but may also be a mental picture, or a conception, or even a word.[4] Thus the term has a broad meaning which is coherent with the presumed logical nature of the notion. Rendering it by 'representation' makes all of this more evident, I think, and helps to make Dedekind's logicist position easier to understand; to translate it into 'transformation', as Behman did in the first English edition, seems to me quite unfortunate.)

All of us, when discussing the basic system that underlies *Was sind und was sollen die Zahlen?*, like to say that Dedekind operates on the basis of two powerful basic concepts: set and mapping, in his terms *System* and *Abbildung*. He himself might occasionally refer to the '*Systemlehre der Logik*' (the theory of sets, from logic) in casual talk about the foundations of pure mathematics.[5] Yet when the time came for a precise formulation of the key idea, he chose to *forget* the concept of set and emphasize that of mapping. This happened in 1888, and the very good proof-reader that Dedekind boasted to be did not change that key passage in any of the editions of *Was sind...* authorized during his lifetime: 1893, 1911, nor in the English authorized translation of 1901.

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[2] See [Dugac, 1976; Scharlau, 1981]. Hilbert made jokes about the chronographic precision with which Dedekind could date his discoveries. This is because he kept a diary in which he would record all kinds of little details.

[3] See [Ferreirós, 1999, pp. 229, 250 f]; thus in my Spanish translation of Dedekind's work (1998) I employed 'representación', and the same was done by Benis-Sinaceur in the French translation (2008). However in this paper I shall use 'map' or 'mapping', following the choice employed in most secondary literature on the topic.

[4] Dedekind refers explicitly to the translation of an arithmetic theorem from one language to another, as given by a mapping or representation; see [Dedekind, 1888, no. 134]. One is reminded of Wittgenstein's *Tractatus*.

[5] For the reasons why he also regarded mappings as a quintessential logical notion, see [Ferreirós, 1999, pp. 249–253 (also pp. 228–229, 237–238); Ferreirós, 1996, pp. 44–46, 57–60]. A recent discussion of the point is in [Klev, 2015].

The riddle is even worsened when one considers the first draft of Dedekind's booklet, written in the early 1870s. Here, in contrast to the published edition, we find mention of *two* basic concepts:

> If we scrutinise closely what is done in counting the quantity or [cardinal] number of things, we are led by necessity to the concept of correspondence or mapping. The concepts of set and mapping, which will be introduced in the sequel in order to lay the foundations of the concepts of number and cardinal number [*Anzahl*], would still remain indispensable for arithmetic if one wished to presuppose the concept of cardinal number as immediately evident ('inner intuition').[6]

Compare with the corresponding sentence in the first Preface of *Was sind...*, which he prepared at least ten years later, in the summer of 1887:

> If we scrutinise closely what is done in counting an aggregate or [cardinal] number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent [*abzubilden*] a thing by a thing, an ability without which no thinking is possible at all. Upon this unique and in any event absolutely indispensable foundation [*einzigen, auch sonst ganz unentbehrlichen Grundlage*], as I have already affirmed in an announcement of this work,[7] must, in my judgment, the whole science of numbers be established.

The purpose of this paper is to offer an explanation of this remarkable feature of Dedekind's conception, and to explore its context.

As we shall see, there are indications that warrant the conclusion that Dedekind's reflections on mathematical method entered into a distinctive phase around 1887. It happened at the time of publication of his essay on natural numbers, *Was sind ...*, or maybe somewhat earlier, and it led him to lay extraordinary emphasis on mappings — and morphisms — as the basic notion in mathematics. This is in agreement with the above-mentioned surprising statement, that *Abbildung* is the 'unique' and 'indispensable' foundation for the whole science of numbers, *i.e.*, for pure mathematics. We shall thus call this phase Dedekind's *map-theoretic* period, in contrast to the set-theoretic period that went roughly from 1858 to 1887;[8] we may take it to encompass the years

[6] Verfolgt man genau, was wir beim Abzählen der Menge oder Anzahl von Dingen thun, so wird man nothwendig auf dem Begriff der Correspondenz oder Abbildung geführt. Die Begriffe des Systems, der Abbildung, welche im Folgenden eingeführt werden, um den Begriff der Zahl, der Anzahl zu begründen, bleiben auch dann für die Arithmetik unentbehrlich, selbst wenn man den Begriff der Anzahl als unmittelbar evident (Innere Anschauung) voraussetzen wollte. [Dugac, 1976, p. 224]

[7] Dedekind refers here to Dirichlet's *Vorlesungen*, 2nd ed., 1879, §163, p. 470.

[8] See [Dugac, 1976; Scharlau, 1981; Sieg and Schlimm, 2005], and my own presentation in [Ferreirós, 1999, pp. 82 ff.]. For a detailed reconstruction of the evolution of Dedekind's notion of a map, see [Sieg and Schlimm, 2014, §2].

1887 to 1894. The contrast should not be construed as an opposition, but it certainly was a radicalization of earlier ideas: maps were present already in the earlier work, but while his mathematical style around 1870 stresses sets and operations on sets, around 1890 he is doing the same with maps and morphisms.

Dedekind's map-theoretic period came rather late in his life, when he was around 60 years old, and it did not last long, partly because he would publish few papers in subsequent years (but see also the conclusion). Yet it was the period of some of his most influential contributions, not only *Was sind ...* but also the Supplement XI to the 1894 *Vorlesungen* that Emmy Noether insistently recommended to colleagues and students ('it's all in Dedekind already', she liked to say). It seems to me that this was his last ambitious project regarding the foundations of mathematics, even though it had to remain an unfinished project.

I should also make clear that the analysis and reconstruction offered here is intended both as an attempt to understand Dedekind's perspective on pure mathematics around 1890, and to reconstruct certain themes that emerge from his late writings. In fact, given the sparseness of available evidence, it is the combination of both elements that lends credibility and strength to my analysis.

1. SET-THEORETIC STRUCTURES, AND MAPS

The notion of a map (*Abbildung*, representation) emerged gradually in Dedekind's work, and it certainly did not appear for the first time in the 1880s. In fact, several pieces of evidence allow us to establish that it was in the 1870s when he articulated this general notion. It was presented publicly in the third, 1879, version of *Vorlesungen*: 'It happens very frequently, in mathematics *and other sciences*, that when we find a set [System] Ω of things or elements ω, each definite element ω is replaced by a definite element ω' which is made to correspond to it according to a certain law; ... [t]erminology becomes more convenient if, as we shall do, one conceives of that substitution as a mapping [*Abbildung*] of the system Ω, and accordingly one calls ω' the image [Bild] of ω, ...' [1879, p. 470; emphasis added]. In a footnote, Dedekind went on to announce the publication of *Was sind ...*, where he would show in detail how 'this mental faculty' of mapping or 'representation' (a thing being represented by an image, mapped into it), 'without which thinking is not at all possible', is the foundation for the entire science of numbers.[9]

In effect, Dedekind had been deepening his conception of maps in the course of writing a draft with his advanced ideas on the foundations of natural number. It was drafted from 1872 to 1878, and the very first part contains all of the basic concepts which later appeared in *Was sind ...*: set or 'system' and

[9] The text is quoted in full by Sieg and Schlimm [2014, §2.2]. It also indicates that Dedekind had previously used 'substitution' for what now became a mapping. In [Sieg and Schlimm, 2014], the authors explain the changes in Dedekind's notion of substitution from the 1850s onward.

the relevant operations, 'mapping' and 'chain', the definition of infinite sets, the new conception of the structure ℕ of natural numbers (see [Ferreirós, 1999, pp. 107–109]; original in [Dugac, 1976, pp. 293–297]). In particular, one finds this definition: 'A set S is called *distinctly mappable in a set* T [*deutlich abbildbar in einem System* T], when to every thing contained in S (original) one can determine a (corresponding) thing contained in T (image), so that different images correspond to different originals.' That notion is next employed to characterize finite and infinite sets via the famous notion of Dedekind-infinite: set S is infinite when there exists a proper part U of S, such that S is 'distinctly mappable in' U (see the original German in [Ferreirós, 1999, p. 109]).

The concept of injective mapping ('clear' or 'distinct representation') was an immediate outgrowth of the concept 'distinctly representable' or 'mappable', and Dedekind was soon considering maps in general. In the next part of the draft [Dugac, 1976, p. 296], which should be dated to the first half of the 1870s, perhaps 1873 or 1874, he starts an 'investigation of a (distinct or indistinct) mapping of a set S in itself'.

We should pause to ask why Dedekind was so quick in moving from injective or bijective maps, to general mappings. This was not at all common in his time; such notions would still take a long time before they became common property of the mathematical community. (As a relevant example, nothing similar happened in Cantor's work; indeed, Cantor employed different concepts in different contexts such as the study of cardinality (*eindeutig zuordnen*), of well-ordered sets (*Abbildung*), and defining the exponentiation of aleph (*Belegung*), which could all have been unified under Dedekind's notion of a bijective *Abbildung*.[10] Dedekind's elaboration of the map concept, like other aspects of his work related to set theory and its methods, was essentially independent of Cantor's contributions.[11]) The answer lies in the fact that Dedekind had already been considering homomorphisms in the context of algebra. This is even more interesting due to the role such ideas played in his presentation of structural concepts.

Consider Dedekind's first public presentation of the notion of (number) field, right at the beginning of his exposition of ideal theory in 1871 ('On the composition of binary quadratic forms', tenth supplement to Dirichlet's *Vorlesungen*). This text marks a turning point in the development of the structural conception in algebra. Although his immediate goal is to present work by Gauss and Dirichlet (on binary quadratic forms) and then to generalize it to all types of algebraic integers, Dedekind found it convenient 'to adopt a somewhat higher standpoint' and introduce a notion which seems 'adequate to serve as a foundation for higher algebra' and for those parts of number theory connected with it:

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[10] For further details and quotations, see [Ferreirós, 1999, pp. 188, 275, 288]. Interestingly, Cantor's use of the term 'Abbildung' came after he had come to know Dedekind's 1872/78 draft; see [Ferreirós, 1995] for details of this rather interesting story.

[11] I have written extensively on this topic elsewhere; see in particular see [Ferreirós, 1999, Ch. 7 and *passim*].

> By a *field* we shall understand every set of infinitely many real or complex numbers, which is so closed and complete in itself, that addition, subtraction, multiplication, and division of any two of those numbers yields always a number of the same set. The simplest field is formed by all rational numbers, the greatest field by all [complex] numbers. We call a field A *divisor* of field M ... [if $A \subseteq M$]; it is easily seen that the field of rational numbers is a divisor of all other fields. The collection of all numbers simultaneously contained in two fields A, B constitutes again a field D, ... Moreover, if to any number a in the field A, there corresponds a number $b = \varphi(a)$, in such a way that $\varphi(a + a') = \varphi(a) + \varphi(a')$, and $\varphi(aa') = \varphi(a)\varphi(a')$, the numbers b constitute also (if not all of them are zero)[12] a field $B = \varphi(A)$, which is *conjugate* to A and results from A through the *substitution* φ; inversely, in this case $A = \psi(B)$ is also a conjugate of B. Two fields conjugate to a third are also conjugates of each other, and every field is a conjugate of itself. [Dedekind, 1871, pp. 223 f.]

One can hardly overestimate the significance of this rich passage for the history of sets and structures. It incorporates crucial ideas related to the notions of set and map as used in algebra, and Dedekind goes on to give abundant proof of the mastery he had attained of the set-theoretic style by 1871. There is little doubt that contemporary readers found it difficult to follow, accustomed as they were to an algebra and 'higher' number theory formulated in terms of numbers and equations (or forms). As is clear from the definitions, and from Dedekind's subsequent use of them, a field 'substitution' is nothing but a homomorphism — in practice, for the reasons given in footnote 12, an isomorphism.

What we shall see, in his later work around 1890, is how Dedekind radicalized his early conception in attempting to place the concept of map (and morphism) at the very centre of pure mathematics. But before moving into that, it seems advisable to pause and reconsider his role in the early history of structuralism as a mathematical method.

It is well known that the emergence of set-theoretic structures in modern mathematics was a complex process, in which many actors played a role. No single mathematician can be regarded as 'the father' (or mother) of set-theoretic structuralism (see [Corry, 2003; McLarty, 2006; Ferreirós, 1999] among others). The process by which structures and axiom systems became central to mathematical thinking was long and followed different strands. As regards axiomatics, it is clear that work on the foundations of geometry had a leading role; figures such as Pasch and Hilbert must be emphasized here; yet one should not forget the axiomatic line going from Grassmann through Schröder to Peano. As

[12] The clause in parentheses indicates clearly that, in general, Dedekind allows for homomorphisms; compare [Sieg and Schlimm, 2014, fn. 35]. The requirements that there be an $x \neq 0$ in $\varphi(A)$, coupled with the other structural requirements, implies already that the correspondence is an isomorphism. Thus, Stillwell translates 'substitution' as 'isomorphism' in the corresponding passages of his edition of [Dedekind, 1996, §16, p. 108].

regards the notion of a structure, it seems to me that Grassmann (with his work on a 'general theory of extension' for which 3-dimensional space is a particular instance), Riemann (with his reflections on the reconception of pure mathematics around the notion of a 'manifold') and Dedekind contributed more than most other authors in the mid- to late-nineteenth century. The two strands of geometric axiomatics and structural reconception of algebraic ideas became linked and intertwined in the work of Hilbert (the algebraic work that was his specialty before 1899 left clear signs in the mathematical style with which he investigated the *Grundlagen der Geometrie*).

I should add two more crucial remarks, one related to the general definition of structures, the other to the role of maps as morphisms. The precise general concept of a structure, defined set-theoretically, can only be found in work of the 1930s by such figures as Birkhoff [1935], the Bourbaki group, and so on. Even van der Waerden's *Moderne Algebra* does not feature a general definition of structure, but only general definitions of field, ring, ideal, *etc.*[13] Yet one can say that it was a relatively easy exercise to forge this general definition on the basis of the previous work and examples in all kinds of cases from geometry through number systems to algebra and topology (Hausdorff), concepts of integral (Lebesgue), *etc.* Thus one can say that during the first third of the twentieth century the general idea of a structure was 'in the air', belonging in the *image* of modern mathematics shared by many mathematicians (not yet incorporated in the *body* of their work).[14] Furthermore, it is clear that Dedekind was a pivotal figure because of the depth of his impact through his writings on general number theory (algebraic numbers), Galois theory, algebraic curves ('function' fields in the 1882 joint paper with Weber), and foundations; these works promoted very forcefully the adoption of that structural image, offering clear examples of sets of elements characterized by the behavior of certain operations and relations on the elements.

Yet the crucial idea, in the context of this paper, is the second. As far as I know, *no other nineteenth-century mathematician emphasized maps (morphisms) like Dedekind* in their presentation of concrete set-theoretic structures. The text quoted above, on number fields (subsets of $\boldsymbol{C}$ with field structure), is paradigmatic. If we compare it with Hilbert's *Grundlagen der Geometrie*, or his work on algebraic number theory, or on the axiomatics of real numbers, around 1899, the simple fact is that we find nothing comparable — no focal role for maps. (Interestingly, the concepts of *Abbildung* and *Bild* can be found in Hilbert's 1902 work on the axiomatics of plane geometry, where he defines neighbourhoods as images of Jordan domains; yet this is still geometric and pictorial, far from Dedekind's abstract use of the concept.[15])

[13] See the detailed discussion in [Corry, 2003], which underscores the difference between Dedekind's more particular work (on number fields, 'function' fields, *etc.*) and van der Waerden's.

[14] This useful terminology is due to Leo Corry.

[15] Furthermore, although Hilbert's main approach in these papers is via groups of transformations, he uses two different notions, *Abbildung* (on the topological side) and

Even the presentations of the general notion of a structure in the 1930s or 1940s tend to be rather elementary, in the sense that they focus on a set S, its elements, and the operations $*$ and relations $<$ defined on them — without underscoring morphisms. By contrast, Dedekind's approach stands out precisely because of the central role he ascribes to the maps we call morphisms. Little wonder that Emil Artin would write that Dedekind's style 'is easy to read and elegant for us today, but at the time it was too modern' (see the full quotation, which refers to Hilbert, in the next section).

2. ON PERMUTATIONS: MORPHISMS AND ALGEBRA

In December 1893 the famous algebraist Georg Frobenius wrote to Göttingen professor Heinrich M. Weber congratulating him for the plans to publish the *Lehrbuch der Algebra*, which was to become a highly succesful textbook.[16] He immediately wrote:

> Hopefully you will often follow Dedekind's way, but avoid the all-too-abstract turn [*Winkel*] that he pursues so eagerly now. His newest edition [*Vorlesungen über Zahlentheorie*, 1894] contains so many beautiful things, §173 is a work of genius, but his permutations are too incorporeal, and it is certainly unnecessary to take the abstraction so far.[17]

Frobenius was famous for his strong and witty personality, which could seem 'choleric' at times, but was also the source of many jokes. Both of them were good friends of Dedekind, indeed Weber can be described as Dedekind's closest friend in mathematics. The sentence above is a very intentional joke on two central notions of Dedekind's reconception of algebra, 'permutation' and 'field' [*Körper*, from which *körperlos*, incorporeal].

As early as 1871 Dedekind began to insist on the idea that the concept of a (number) field must be regarded as the core notion of algebra, adequate to establish 'the foundation of higher algebra' [1871, p. 224].[18] His introduction

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Transformation (on the group-theoretic side) — this underscores again the fact that he remained one level of abstraction below Dedekind. See [Hilbert, 1902].

[16] [Weber, 1895/96]. On the significance of this book, see [Corry, 2003, pp. 33–43; 2005].

[17] In [Dugac, 1976, p. 269]:

> Hoffentlich gehen Sie vielfach die Wege von Dedekind, vermeiden aber die gar zu abstrakten Winkel die er jetzt so gern aufsucht. Seine neueste Auflage enthält so viel Schönheiten, der §173 ist hochgenial, aber seine Permutationen sind zu körperlos, und es ist doch auch unnöthig, die Abstraktion so weit zu treiben. Ich bin also fast froh, dass Sie die Algebra schreiben und nicht unser verehter Freund und Meister, der auch einmal mit solchen Gedanken sich trug.

[18] Dedekind [1871] emphasized that concepts such as *field* and *ideal* have the advantage that their definition requires absolutely no particular representation [*Darstellungsform*] of the numbers; the power of 'these extremely simple concepts' is demonstrated by the fact that the proof of the general laws of divisibility in algebraic number theory does not require 'any distinction of several cases', being perfectly uniform and general.

of the notion of a number field was prepared by studies of the algebraic theory of equations and very especially of Galois theory. (In fact, the notion is implicit in the original work of Galois.) Dedekind lectured on Galois theory at Göttingen in the winters of 1856/57 and 1857/58, and he always had plans to write an introduction to algebra presenting this theory.[19] From the early 1870s he entertained the view that 'the precise investigation of interrelations between different fields [*Verwandschaft zwischen den verschiedenen Körper*]' is the 'proper subject of modern Algebra' (as he repeated in [1894, p. 466]).

Finally, in the well-known Supplement XI of the fourth edition of *Vorlesungen*, he chose to devote several sections to a highly abstract presentation of Galois theory, based on the study of groups of 'permutations'. This word is redefined to denote a mapping π from field A to an image set A' that preserves the basic operations which define the structure of a number field [1894, pp. 457–458]. He requires that the map π be such that

$$\pi(u+v) = \pi(u) + \pi(v) \qquad \pi(u-v) = \pi(u) - \pi(v)$$
$$\pi(uv) = \pi(u)\pi(v) \qquad \pi(u/v) = \pi(u)/\pi(v);$$

and he shows how this entails that the image $A' = \pi(A)$ will be a field and indeed isomorphic to A. Dedekind then considers such 'permutations' (isomorphisms) in the particular case that $A = A'$, *i.e.*, he deals with automorphisms, and goes on to research *sets* of such morphisms which under map-composition form a group.

In order to do so, he starts from this definition:

> A set Π of n different field-permutations [*Körper-Permutationen*] π is called a group, when each can be composed with any other, so that the result is always contained in Π. [1894, p. 482]

Next he shows that this definition implies each π is a field automorphism, and also that the identity map belongs to Π — hence the set Π indeed forms a group. In modern terms, the Galois group is introduced as a group of field automorphisms — the automorphisms of a finite extension M of number field K that fix K itself — and to that extent Dedekind's presentation lays the groundwork for later work such as Emil Artin's famous book.[20]

The main goal of algebra, Dedekind says, is to research and determine completely all the fields K that lie between a certain number field A and its finite extension M, establishing their reciprocal relations. The solution was begun by Lagrange and brought to an end by Galois, thanks to the theory of groups

[19] See among others the Frobenius letter mentioned above. See also [Scharlau, 1981, pp. 101 ff.].

[20] [Artin, 1942]. See [Kiernan, 1970] and compare [Dean, 2009]. In particular, the realization that one can do Galois theory without relying on primitive elements is due to Dedekind. Artin followed him in avoiding that particular way of proceeding (which was preferred by van der Waerden in early editions of his *Moderne Algebra*).

[Dedekind, 1894, p. 482]. But Dedekind's presentation deemphasizes equations and their resolvents, to put all the emphasis on the interconnection between groups and fields; thus it puts emphasis on what would be regarded as the 'modern' understanding of Galois theory (in the twentieth century, *e.g.*, the 1920s).

In order to approach that problem, Dedekind makes use of the following result, his 'Fundamental theorem III' proved in §165:

> If field M is a finite extension of A, and φ is a permutation of A, then the degree (B, A) determines the number of permutations π of M, which extend φ. Then A is the field fixed by the set Π of all those permutations, and φ is their common restriction to A.[21] [Dedekind, 1894, p. 475]

The symbol (B, A) is introduced by Dedekind for the degree of B over A. In a later publication, Dedekind remarks that the theorem just given offers light on the 'deep' question concerning the possible extensions of an automorphism on a given field to its extension fields [1901, p. 276]. In fact, this late paper deals with extensions of K in general (finite or infinite) and tries to advance in the resolution of that deep question by studying automorphisms of the field A of all algebraic numbers.[22] As he remarks, up to then infinite extensions had been left untouched, 'Noli me tangere' (*Touch me not*) had been the motto. The way group theory of field permutations determines the network of subfields in a finite extension M of A is made clear on the basis of 'Fundamental theorem I' in §166:

> If a group Π consists of n different permutations π of the field M, and if A is the field [fixed by] Π, then one has $(M, A) = n$, and the rest [or common restriction] of Π is the identical permutation on A. [Dedekind, 1894, p. 483]

Now one also has the following consequences: for a subgroup of Π, call it Π', made by p permutations, the field A' corresponding to Π' is a subfield of M containing A (and one has $n = pq$ with $(M, A') = p$ and $(A', A) = q$). And if we have another subgroup Π'' and the field A'' corresponding to it, the intersection of Π' and Π'' is a group again, and its corresponding field is the product of A'

[21] His actual formulation is a bit more involved, due to technical details of his definitions that can be ignored here: 'If field B is finite over A, and φ is a permutation of A, then the degree (B, A) gives the number of permutations π of the product AB, which are multiples of [*i.e.*, extend] φ. Then A is the field of the set Π of all those permutations, and φ is the rest of the set Π.' As Dedekind defines it, B being 'finite over' A does not require it to be an extension of A, but AB is a finite extension.

[22] The methods employed by Dedekind here offered crucial ideas for the resolution of the question, when joined with Steinitz's ideas about transcendent extensions; but he wrongly believed that the field $\mathbb{R}$ of real numbers does not allow for non-trivial automorphisms. Noether in her editorial remarks refers to work by Ostrowski, Noether herself, and Kamke that solved the question (see [Dedekind, 1901, p. 292]).

and A'' [Dedekind, 1894, p. 484]. The different subgroups of group Π give us all the subfields of M — and thus the key problem of algebra can be reduced to a group-theoretic question:

> From this one recognizes that the complete determination of all these fields $A', A'', \ldots$ and the investigation of their interrelations is completely solved (*erledigt*) by determining all of the groups $\Pi', \Pi'', \ldots$ contained in group Π, and this problem belongs in the general theory of groups.[23] [Dedekind, 1894, p. 484]

It was precisely this very abstract approach to algebra, built on the notion of *Körper* permutations, that displeased Frobenius and led him to the accusation that all was unnecessarily abstract and 'too incorporeal'. It is a noteworthy fact about Dedekind's presentation of Galois-theoretic ideas that equations and resolvents are not mentioned — not even once. By contrast, the whole presentation of [Weber, 1895/96, part III] revolves around equations and the resolution of equations, and yet Weber's handbook was the most advanced text-book treatment of algebra in its time. It can be said that Weber's treatment is very Dedekindian, but it reflects what Dedekind might have done, say, in the 1870s.[24] The new abstract turn introduced by Dedekind in the fourth edition of *Vorlesungen* [1894] is clearly in the spirit of Emil Artin's later work [1942]. It is often said that the novelty in Artin was to leave in the background the whole problem of equations, considering that groups of permutations — in the old sense — can be replaced by groups of automorphisms defined autonomously. But all this 'is already in Dedekind'.[25]

Frobenius of course was a more traditional mathematician — despite his work on group characters — and one of his strengths was the ability to handle long and difficult calculations. Dedekind by contrast laid a lot of emphasis on structural considerations and tended to disregard explicit constructive control of the objects. That is to say, he looked for conceptual grasp of mathematical situations by means of structural considerations, and he also laid great emphasis on the invariant characterization of key notions and methods, while he tended to disregard the ability to control explicitly and handle the concrete objects and particular situations. His approach was thus oriented towards the most general features, investigated conceptually and structurally (the drive to

[23] Hieraus erkennt man, dass die vollständige Bestimmung aller dieser Körper $A', A'' \ldots$ und die Untersuchung ihrer gegenseitigen Beziehungen vollständig erledigt wird durch die Bestimmung aller in der Gruppe Π enthaltenen Gruppen $\Pi', \Pi'', \ldots,$ und diese Aufgabe gehört in die allgemeine Theorie der Gruppen.

[24] Around the late 1870s (when collaborating on a famous paper published in 1882 on the Riemann-Roch theorem), Weber had access to Dedekind's lecture notes from 1857/58 on higher algebra and Galois theory, and this aroused his interest in the matter (see [Weber, 1895/96, p. vii]). The notes are available in [Scharlau, 1981].

[25] I agree with Dean's evaluation of the situation, see his [2009]. The phrase 'es steht alles schon bei Dedekind' (it is all in Dedekind already) was famously coined by Emmy Noether; see above.

modern axiomatics is undeniable here), self-consciously avoiding calculation. This was, among other things, because explicit calculation often forces the mathematician to make choices of elements (representative objects) that are to a certain extent arbitrary.

As Hasse said, Dedekind's approach — like Hilbert's — was oriented 'towards the general and conceptual, towards existence and structure'; it inaugurated the axiomatic analysis of abstract structures.[26] In fact, when one compares Hilbert's *Zahlbericht* with Dedekind's Supplement XI to the *Vorlesungen* [1894], it is the latter which emerges as more modern and structuralist — even though Dedekind was the senior, by 31 years! This is not merely my impression, but coincides with what Emil Artin himself suggested:

> Dedekind's presentation is easy to read and elegant for us today, but at the time it was too modern. Thus the appearance of Hilbert's *Zahlbericht* in the *Jahresbericht der DMV* was greeted with great joy.[27] [Artin, 1942, p. 549]

The evaluation of this modern style of Dedekind may vary depending on the context; indeed it seems that key experts in modern algebra have received it with joy and excitement (Artin, Noether, van der Waerden), while number theorists have tended to lay more value on constructive methods (examples being Weyl, Hasse, Weil).

Notice that Dedekind's orientation and style was heavily shaped by certain crucial convictions and also some strong dislikes:

(1) a preference for invariant characterization, dislike of arbitrary choices of the kind made necessary by constructivist approaches,
(2) dislike of the introduction of extrinsic auxiliary means, to the point that even polynomials are perceived as extrinsic — insistence on intrinsic characterization,
(3) search for utmost generality of mathematical concepts (which ultimately are 'purely logical'); set and above all mapping are the core notions of pure mathematics,[28]
(4) arithmetization as a general orientation in pure mathematics, which implies that mathematics is not a general exploration of abstract structures, but only of number structures (mostly located in the lattice between $\mathbb{Q}$ and $\mathbb{C}$) and the corresponding mappings.

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[26] [Hasse, 1952, pp. vii–viii]. Cited (in German) in the introduction to [Hilbert, 1998], p. xxvii.

[27] Cited and translated by Stillwell in his [Dedekind, 1996, p. 46]. See also the remarkable, but perhaps contentious, indications given by Olga Taussky-Todd (cited in the introduction to [Hilbert, 1998], p. xxviii).

[28] See [Ferreirós, 1996; forthcoming].

There is an obvious tension between the great 'logical' generality of the basic concepts sought by Dedekind, and his decision to restrict mathematics to number structures (his peculiar version of arithmetization). Mathematics is not identical with a universal logic; it is rather the systematic study of those areas of general logic that crystalize around the concept of number. In my opinion, this move was a Solomonic solution, a way to remain close to his contemporaries' conception of mathematics; but probably it was also a way to avoid the puzzling, dizzying vision of mathematics as a general exploration of abstract structures.

The treatment of Galois theory on the basis of groups of automorphisms can be seen precisely as a beautiful embodiment of the above four maxims. Groups are not introduced extrinsically as a computing tool, nor even as a structure derivative from equations and the solutions they imply — but *intrinsically* as a *natural* structure of the set of morphisms. This makes it possible to deal with Galois theory without relying on primitive elements à la Galois, which Dedekind perceives as an arbitrary choice masking the invariant nature of the underlying structures.

Such options and the ensuing style were resented by many of Dedekind's contemporaries. Even a good friend who followed his work closely, such as Frobenius, could say that Dedekind was taking abstraction too far,[29] and could think that this style of algebra based on 'permutations' (morphisms) was too incorporeal — meaning unnatural, I believe, on the grounds that it is excessively far away from algebra conceived as the theory of the resolution of equations.

The reader may wonder how 'natural' it is to emphasize time and again maps and morphisms as the central concepts, when it is all supposed to be about number fields. Even granting that algebra studies the family relations between number fields, as Dedekind thought — is the introduction of maps not extrinsic to some extent? The next section will show that this is not the case for him.

3. MAPS ON $\mathbb{N}$ AND THE CONTINUUM

To begin with, Dedekind's treatment of the very foundations of arithmetic, the theory of $\mathbb{N}$, is completely map-theoretic. The natural-number structure is defined on the basis of 'simply infinite' sets, *i.e.*, sets S endowed with an ordering map σ which satisfies some closure conditions. Prominent among these is the 'chain'-condition that guarantees S to be the smallest set containing a base element e and its images, that is to say, that guarantees mathematical induction. To be more explicit, Dedekind's axiom is the chain condition: $\mathbb{N} = \sigma_0\{1\}$, that requires $\mathbb{N}$ to be the 'chain' of a singleton under map σ; thus $\mathbb{N}$ is the minimal closure of $\{1\}$ under σ, and this means that the principle of induction is valid for $\mathbb{N}$.[30] As a matter of fact, everything is done via mappings and

[29] This is reminiscent of the famous 'abstract nonsense' in category theory.

[30] Of course, σ is the successor function. For details on Dedekind's very abstract chain theory, on which he also based a proof of the Cantor-Bernstein theorem, see [Dedekind, 1888, §4 and *passim*; Ferreirós, 1999, pp. 230–237, 239 ff.; Sieg and Schlimm, 2005].

chains:[31] the elementary theory of finite and infinite sets, the theory of initial segments of ℕ and the order relation in ℕ, the theory of recursive functions, the study of models of Dedekind's axioms, *etc.* Little wonder, from this point of view, that Dedekind would emphasize the key idea of maps in the introduction to this work. But there is more.

On Jan. 2, 1891, Dedekind drafted a manuscript entitled 'Stetiges System aller Abbildungen der natürlichen Zahlenreihe N in sich selbst': *The continuous set of all mappings of the natural number sequence N in itself.* This unfinished manuscript offers a remarkable attempt to define a continuous structure directly upon the set ℕ of natural numbers, using the apparatus of mappings established as the basis for Dedekind's booklet [1888]. It represents a substantial move of liberation from traditional ideas about the number system and its so-called 'genetic construction', towards a purely set-theoretic approach to continuum structures. Indeed it is quite fascinating to find out that Dedekind was advancing toward the Baire space ${}^{\omega}\omega$ in this highly abstract way, many years before René Baire did so [1909].[32]

Baire space — in the sense of descriptive set theory — is the set of all maps $\mathbb{N} \to \mathbb{N}$ with the structure given by a topology that makes it homeomorphic to the set of irrational numbers.[33] This turned out to be — as Baire would make clear — the natural setting for study of descriptive set theory, and thus it is, 'next to the natural numbers, perhaps the most fundamental object of study of set theory' [Moschovakis, 2006, p. 135].

I should hasten to add that Dedekind himself did not define nor study the crucial topology on ${}^{\omega}\omega$, which makes it an abstract set-theoretic continuum.[34] His manuscript is very short, just the first page of a potential paper, and in it we only find a definition of an order relation on ${}^{\omega}\omega$ and the proof that this relation is transitive. He starts by considering a map α of $\boldsymbol{N}$ into itself, and he uses the notation $n\alpha$ for the image of number n generated by α; $n\alpha$ is a natural number. Then he defines:

> If α, β are mappings of N into itself and *different* from each other, there is at least one natural number n, for which the difference $n\alpha - n\beta$ is

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[31] Let me insist that the crucial notion of a chain is entirely map-theoretic (see [Dedekind, 1888, no. 44]). As the reader probably knows, this is a famous classic example of an impredicative definition.

[32] In his paper, Baire introduced ${}^{\omega}\omega$ as the set of all sequences of integers.

> Ce sont précisément ces deux notions: ensemble de points, point limite, qu'il nous sera utile, pour la suite de nos recherches, de remplacer par des notions un peu différentes. J'ai été conduit ainsi à la notion *d'ensemble de suites d'entiers,* que j'appelle aussi *espace à 0 dimension,* pour des raisons exposées dans le courant du mémoire, et j'étudie les fonctions définies sur ces nouveaux ensembles. [1909, p. 97]

[33] The irrationals with the topology inherited from ℝ. Famously, this morphism can be obtained using continued fractions.

[34] In fact, zero-dimensional; ${}^{\omega}\omega$ is homeomorphic to the product of any finite or countable number of copies of itself.

> different from zero, and among *these* numbers n there is a *least* one r. Then
>
> $$x\alpha = x\beta, \text{ in case } x < r,$$
>
> and $r\alpha - r\beta$ is either positive or negative. In the *first* case, $r\alpha > r\beta$, let us call α *greater than* β, β less than α, in symbols $\alpha > \beta$ and $\beta < \alpha$. In the *second* case $r\beta > r\alpha$, hence $\beta > \alpha$, $\alpha < \beta$. Therefore: of two different mappings α, β one and only one is always greater than the other.[35]

Then comes the *Theorem*: 'If α, β, γ are three different mappings of N in itself, and $\alpha > \beta$, $\beta > \gamma$, then it is also $\alpha > \gamma$.' After the proof, the carefully written draft breaks off.

Nevertheless it is clear (from the title) that Dedekind envisioned this structure as being continuous, and the natural way to proceed is to consider all maps $\mathbb{N} \to \mathbb{N}$ that coincide in an initial segment as forming an open set. (Dedekind had formulated the concept of an open ball, as a basis for topology, already before 1870; see [Ferreirós, 1999, 138 ff.].)

Just as the Galois group was presented (in 1894) as a group of automorphisms, the idea in this short 1891 note is to define a continuum whose elements are internal maps of $\mathbb{N}$ into itself — *all* the 'automaps' of $\mathbb{N}$. You may think of each one of these as a 'point' in an abstract space of mappings, a function space. This idea, which Dedekind had in 1891, is a 'very modern' idea, of a kind that would take still some decades to become usual for the mathematics community.

The very existence of this little piece, which proposes to obtain a continuum *directly* (without going through the usual constructions of integer and rational numbers) as made of *Abbildungen* of $\mathbb{N}$ into itself, appears to confirm that Dedekind had reasons to uphold that the notion of a mapping is 'the unique' foundation of pure mathematics. We see why it makes sense to speak of a *map-theoretic* period. The structure of $\mathbb{N}$ is established through the properties of an 'automap'; a continuum structure is obtained via the space of $\mathbb{N}$-automaps; algebra deals with fields, and its central question is approached through the Galois group, defined as a group of morphisms. In general, Dedekind prefers to characterize the (number) structures of pure mathematics by the behaviour of mappings, and so — it seems — would he approach the question of characterizing number fields like $\mathbb{Q}$ and $\mathbb{C}$.[36] Thus we can answer the question at the end of

[35] [Dedekind, 1891, pp. 371–372, my translation]. The manuscript is preserved in the Göttingen Library under the signature *Cod. Ms. Dedekind, III*, 2. It was published by E. Noether in Dedekind's *Werke*, Vol. II.

[36] It is clear, *e.g.*, from [Dedekind, 1888] that he conceives the operations sum and product as recursive *mappings*. Also one of the manuscripts devoted to the introduction of the set $\mathbb{Z}$ of integers, apparently drafted around 1890, introduces an ordering map before going into the definition of operations (this has been published in Spanish and French translation, see [Dedekind, 1890?].

the last section and say that an emphasis on maps and morphisms, in connection with number fields, is perfectly *intrinsic* from Dedekind's standpoint.

I surmise that the origins of the 1891 manuscript lie, most likely, in a reflection on the real numbers as given by digital expansions (like the one I offered in §3 of my [2011]), joined with the will to rethink the whole edifice of mathematics in the light of the basic analysis of $\mathbb{N}$ that Dedekind had just published. It is a noteworthy feature of Dedekind's attempt that he tries to work completely generally with all mappings $\varphi : \mathbb{N} \rightarrow \mathbb{N}$. It must have been obvious for him that the particular case of functions $\lambda : \mathbb{N} \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ was the immediate counterpart of real-number decimal expansions; even more simply, digital notation in the form $\mu : \mathbb{N} \rightarrow \{0, 1\}$ would have sufficed. But it was typical of Dedekind's way of thinking to consider that, since our starting point is $\mathbb{N}$, and mappings of $\mathbb{N}$ into itself, the 'natural' way to proceed is to define a continuous structure on the set of *all* mappings $\varphi : \mathbb{N} \rightarrow \mathbb{N}$.

The manuscript is nicely comparable with Cantor's paper 'Über eine elementare Frage der Mannigfaltigkeitslehre' [1892] in which he employed the diagonal method to show that the collection of all functions $f : \mathbb{N} \rightarrow \{0, 1\}$ is uncountable, and to prove Cantor's Theorem.[37] Interesting that the germ of Cantor space, and that of Baire space, can be found in works of the same year, one by Cantor and the other by Dedekind. It is again characteristic of Dedekind, and distinct from Cantor, that he aimed at replacing the traditional continuum with a new, abstract map-theoretic structure.

A cautionary remark is in order. As mentioned, Dedekind did not define the topology on Baire space ${}^{\omega}\omega$ although it is natural to think that he had the right notion of it — given that he had in mind a 'continuous set' structure, which the title clearly indicates. But if he had defined and investigated this topology, Dedekind would have found out that Baire space is totally disconnected, and in this sense quite unlike the real-number system. (Precisely, the fact that $\mathbb{R}$ is connected causes technical difficulties in descriptive set theory; because Baire space is disconnected, it has advantages in this setting.[38]) The warning is that we can have no idea of what Dedekind's reaction to this finding would have been. Could he have been ready to accept this peculiar structure as an abstract continuum? We shall never know.

4. THE CONCEPTUAL BASIS OF *SYSTEMLEHRE*—AN *ABBILDUNGSLEHRE*?

In a paper written some years ago (but [forthcoming]), I tried to reconstruct in a faithful but interpretive way the principles underlying Dedekind's conception of sets. His practical handling of sets and maps was very precise, but the basic principles on which the system rested should have been laid out

[37] In the form: given any set C, the cardinality of C is strictly less than the cardinal of the set of all functions $f : C \rightarrow \{0, 1\}$.

[38] Also, every (non-void) Polish space is the continuous image given by some map on Baire space; in this sense it is universal. Baire space is a perfect Polish space, and also a 'Baire space' in the topological sense.

clearly. Several indications (among them the famous theorem 66), make it clear that this was a version of the *dichotomy conception*, which holds that a set is determined top-down, by partitioning the class of everything into two parts [Gödel, 1947, p. 180]. I have also showed that Dedekind accepted the principle of comprehension early on, *e.g.*, in the 1870s (see [Ferreirós, 1999, p. 227; forthcoming]. However, by 1888 Dedekind had reasons to prefer an abstract formulation not tied to concepts expressed in a particular language (formal or not).

Thus in my forthcoming paper I emphasize two 'basic laws' as key ingredients of an *abstract logic* dealing with the theory of sets:

(A) *Subsets*: Any subcollection of a set S is itself a set, a 'thing' $S' \subseteq S$. One must consider here not only subcollections definable by expressions in a symbolic language, but any *arbitrary* collection of elements of S.[39]

(B) *Universal Set*: The totality of all things is itself a thing, a set V, the 'thought-world' (*Gedankenwelt*) or Universal Set.

The principle of comprehension becomes a theorem, an easy consequence of (A) and (B): Given any well-defined property $E(x)$ there exists $S = \{x : E(x)\}$, the set of all things that have (or fall under) the property. Conversely, comprehension suffices to prove principle (B), but one can argue[40] that a formal version of comprehension based on an explicit symbolic language is *not enough* to prove (A).

Now, on the basis of the previous discussion, and in light of the fact that Dedekind chose to underscore mappings *alone* as the foundation of arithmetic and pure mathematics, the question arises how this may affect the previous reconstruction. It is possible to consider how to revise the 'basic laws' in a map-theoretic spirit.

The principle of Universal Set cannot be simplified or modified, as far as I can see — it must be left untouched: the *Gedankenwelt* is a *thing*, a set (and this of course is the source of inconsistency). Notice also that Dedekind's [1888] has a 'definition' (*Erklärung*) introducing the notion of mapping, but interestingly does not mark the paragraph that introduces the notion of set with the same word [1888] (p. 344 *vs.* p. 348). This might be interpreted to mean that the basic theory of 'sets of elements (logic)' — as the 1887 draft presents it — belongs in the underlying logic that can be taken for granted. Meanwhile, the concept of a mapping or 'representation', while still purely logical, is a more

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[39] This principle of subsets thus includes but transcends the separation principle of first-order ZFC. Principle (A) entails Choice, as it licenses arbitrary subsets (see also [Ferreirós, 2011]). 'Thing' is a technical term which applies to any object of thought, and the criterion of identity for things is the Leibnizian one [Dedekind, 1888, p. 344].

[40] [Ferreirós, forthcoming]. Here one must consider the footnote [1888, p. 345] which forcefully expresses Dedekind's opposition to Kronecker's definabilist requirements — such requirements are 'totally irrelevant' (*gänzlich gleichgültig*) since the same 'general laws' apply in all cases.

advanced and specific logical notion which is introduced explicitly. Of course, there were also pragmatic reasons, as the general notion of map was uncommon for mathematicians or logicians at the time.

Perhaps it is advisable to restate briefly the reasons why Dedekind regarded mappings as a matter of logic. Logic was defined as the theory of the laws of thought. Thought and judgement depend not only on concepts (sets) but also on relations; relations can also be regarded as correspondences or representations [*Abbildungen*]. It seems likely that Dedekind intended to subsume under *Abbildungen* also the relations between object and word (concept), state of affairs and sentence (proposition). For all of these reasons, he states that without relations, correspondence, representations ('mappings' is the usual translation), no thought is possible.[41] Logicians of his time agreed — they admitted the theory of maps inside the theory of relations, a clear example being the work of Schröder (see the references in footnote 41). Even we today seem to agree with him insofar as we regard relations as an elementary logical notion; as a result, the notion of *functional relation* is expressible in elementary logic (first-order) with identity.[42]

There is a natural way to reformulate the idea of comprehension with maps, for each well-defined property $E(x)$ defines a map, a function from the Universe V with two values — say, following the usual convention, 1, 0. The image of any thing α under the map is 1 if $E(\alpha)$ is the case, and 0 whenever $\neg E(\alpha)$ is the case. Equivalently, Dedekind would have no problem in accepting (with Frege) that the True and the False are things in logical space, in the *Gedankenwelt*; call them '1' and '0'. (Any two designated things in V can be chosen to play these roles; he might have chosen 'Yes' and 'No'.) Now, it is for Dedekind a logical law that any such map defines two *things*, formed by: (a) those entities whose image is 1; and (b) those whose image is 0. These two things are sets, *Systeme*.

That is again the dichotomy conception of sets at work. But in order to capture the idea of 'basic law' (A) it is indispensable to go further, for the law here concerned does not depend on 'which way the determination comes into place (*zustande*), or even on whether we know a way to decide the question' [Dedekind, 1888, p. 345, fn]. Thus the general principle would have to make clear that each and every mapping from V to the two-element set $\{1, 0\}$ is given, regardless of the question whether it is definable through any particular theoretical or conceptual means.

And of course the restriction of the image set to $\{1, 0\}$, although natural in relation to the case of properties $E(x)$ considered above, is completely

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[41] Dedekind is explicit in establishing a parallelism between relations, correspondences, representations; see the text quoted in the introduction (from 1887). For further details, see [Ferreirós, 1996, pp. 44–46, 57–60; 1999, pp. 249–253].

[42] Yet of course first-order logic lacks the resources to yield all maps on a given domain. We, unlike Dedekind, establish a clear distinction between definable and arbitrary relations. This is why, in retrospect, Dedekind's 'logic' is but a theory of sets (but so is the case also with the logical systems of other logicians around 1900).

inessential. In such situations, Dedekind always tended to generalize. Thus we would have the following basic law:

(A′) *Maps*: Given any two sets S and T, the collection of all maps $\varphi : S \to T$ is given.[43]

Each and every 'representation' or map from a given set S to any other set T is given, regardless of issues of explicit determination. And finally, we have the principle suggested two paragraphs earlier, which in general could be expressed as follows:

(C) *Origin set*: Given a map $\varphi : S \to T$ and set $U \subseteq T$, the collection of all elements s of S such that $\varphi(s) = t \in U$ is a thing, a set.

In particular, for any element t in T, the collection of all elements s of S such that $\varphi(s) = t$ is a thing, a set.

To summarize, then, Dedekind's conception of map theory could be made explicit through the above principles (A′), (B), and (C). This dichotomy conception, one hardly needs to add, generates contradictions.

The history of principle (A′) goes back to Dirichlet, who emphasized that arbitrary functions are to be admitted in analysis, regardless of the possibility of explicit determination through any analytic expression. (For the case of piecewise continuous functions, he backed that idea through his proof that, in such cases, there is a Fourier series representing the function; also Riemann underscored this point.)[44] Nevertheless, the new set-theoretic context promoted that idea into a much stronger principle, the existence of the collection of *all* arbitrary maps $\varphi : S \to T$ as indicated above.

One might further speculate with the possibility of reducing the theory of sets to an axiom system based on maps alone — after all, set and map are *interdefinable* notions, as von Neumann showed in the 1920s. Yet this would be idle speculation here, as there is no sign in [Dedekind, 1888], nor in his related manuscripts, that he ever had that goal in mind. The two principles (A), (B) mentioned above, of Arbitrary Subsets, and of Universal Set, remained in all likelihood 'basic laws of logic' to him — at least until 1897 or 1899, when Bernstein and Cantor respectively made Dedekind aware of the antinomies.

5. CONCLUSION

Dedekind's idea of introducing the continuum directly on the basis of maps of ℕ into itself remained unpublished until 1931 (when volume II of his *Werke* appeared) and had no influence on the development of related ideas. René Baire made clear the importance of Baire space (as it was later called) for descriptive

[43] Notice that, of course, we may always take $T = V$ and/or $S = V$.

[44] Riemann was even more emphatic in 'mathematizing' arbitrary functions. For the history of the notion of function, see, *e.g.*, [Luzin, 1998].

set theory in 1909; he conceived it as a space of *sequences*, homeomorphic to the subspace of irrational numbers. But it was only much later, after World War II, that Baire space would be conceived and defined as $^{\omega}\omega$, the space of all maps of ω into itself. For our purposes here, the ideas contained in the manuscript 'The continuous set of all mappings of the natural-number sequence N into itself' are of importance insofar as they reinforce the idea of a *map-theoretic* period in Dedekind's thought around 1890. It clarifies the centrality of the notion of mapping in Dedekind's conception of pure mathematics, and the sense in which mappings or 'representations' are 'the unique foundation' of arithmetic, algebra and analysis.

In so doing, the manuscript sheds light on a much more influential product of Dedekind's map-theoretic period, his treatment of Galois theory on the basis of a definition of the Galois group via field automorphisms. These are the 'Körper-permutationen' (literally, 'body-permutations') that raised Frobenius's charge of excessive unmotivated abstraction. Fifty years later, a presentation of Galois theory delivered in this style [Artin, 1942] would be celebrated as a triumph of modern algebra. Now, it is well known that the generation of German algebraists working in the inter-war period paid quite a lot of attention to Dedekind's 1894 'Supplement XI'. Noether insisted on students and colleagues reading and getting acquainted first-hand with this work, and Artin too recommended it as a methodological model for modern algebraists.

The parallel between the ideas should be quite obvious, and is a clear sign of the systematic reworking of classical mathematics that Dedekind made his life work. There are differences, of course, as 'field-permutations' are morphisms that fix a base field K, while the number maps or number 'representations' (*Abbildungen*) of 1891 are arbitrary mappings. But in both cases, central structures of pure mathematics come to life by the action of mappings of certain structures (number fields here, simply infinite systems there) into themselves. The Galois-theoretic interrelations of groups and fields obtained in one case, via morphisms of a field into itself, are the key to 'the precise investigation of interrelations between different fields' which Dedekind conceived as the central question and 'proper subject of modern algebra'. In the other case, left unfinished and incomplete, the idea was to obtain the central structure of analysis, the continuum, directly as the space of mappings of $\mathbb{N}$ into itself.

I have mentioned that the map-theoretic period may not have lasted long. It is indeed noteworthy that Dedekind's pioneering contributions to lattice theory, inspired by Schröder's algebraic and axiomatic work, operate on a quite different style.[45] The key definitions here are not presented via maps and subsets, but directly as elementary properties of two operations $+, -$ defined on the elements of 'a dual group' (the commutative and associative laws, and a third characteristic law).[46]

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[45] See the definition of a lattice or 'Dualgruppe' offered in [Dedekind, 1900, p. 237]; the topic has been studied by [Mehrtens, 1979] and [Schlimm, 2011].

[46] $m + (m - n) = m$; and $m - (m + p) = m$. This concept of a *Dualgrupppe* emerged from Dedekind's study of the properties of modules m, n under the operations of union

Perhaps Dedekind presented things this way because it was the most natural approach for this particular research. But I would suggest that he was also trying to promote the reception of his novel ideas by making them attractive to like-minded scholars. Thus in the late 1890s he downplayed the radical idea of 'maps first' and adopted a style akin to Schröder's (which is also closer to Hilbert's). The radical map-theoretic project that he had started exploring around 1890 remained unfinished.

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