Forthcoming in The British Journal for the Philosophy of Science

On the Structure and Function of Scientific Perspectivism in Categorical Quantum Mechanics

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Abstract

Contemporary scientific perspectivism is primarily viewed as a methodological framework of how we obtain and form scientific knowledge of nature, through a broadly perspectivist process, especially, with reference to quantum mechanics. In the present study, this is implemented by representing categorically the global structure of a quantum algebra of events in terms of structured interconnected families of local Boolean probing frames, realized as suitable perspectives or contexts for measuring physical quantities. The essential philosophical meaning of the proposed approach implies that the quantum world can be consistently approached and comprehended through a multilevel structure of locally variable perspectives, which interlock, in a category-theoretical environment, to form a coherent picture of the whole in a nontrivial way.

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1 Contemporary Perspectivism as a Methodological Framework of Scientific Inquiry

Contemporary scientific perspectivism, invigorated initially in the work of Ronald Giere [2006], is primarily viewed in the present study as a methodological framework of how we obtain and form scientific knowledge of nature, through a broadly perspectivist process, constituting arguably a distinct part in scientific inquiry. In relation to philosophical matters, scientific perspectivism occupies a middle ground between the extremes of the context-free universals of metaphysical objectivism, the rigid reductive methodology favored by positivist considerations of science, and the inherent relativism entailed by certain sociological accounts of science (for instance, Giere [2006], Massimi [2012]). Especially, in the proposed form, scientific perspectivism amply recognizes the existence of a mind-independent world as being logically *prior* to experience and knowledge, constituting the overarching condition for the possibility of knowledge. It emphasizes, however, that scientific knowledge of the world can never be pure, direct or unmediated, since it requires pre-conceptualization or structural organization; it requires the adoption of a perspective. Any knowledge and comprehension of something, either through a process of perception and classification or of normative structuring or purposeful acting, any attempt of combining individual experimental data or incorporating phenomena under generic patterns, etc., necessitates the endorsement of a conceptual-interpretative scheme and the selection of specific perspectives. In this respect, a context-free and interpretation-free access to reality as such seems an illusion. The 'book of nature' proves too subtle and complex to be determined by just reading off reality. In our view, therefore, scientific perspectivism provides an alternative middle path at the philosophical spectrum that keeps a firm grasp on reality, but does not accept that the human mind mirrors nature or that access to the world, particularly for scientific purposes, is possible independently of any prior conceptualization.

Contemporary scientific perspectivism is often subsumed under the thesis that knowledge of nature is possible only within the boundaries of historically well-defined scientific perspectives comprised by data analysis, theoretical models and principles relative to the perspective adopted, so that one may refer, for instance, to the Newtonian perspective, the Maxwellian perspective, etc. (Giere [2006]). Alternatively, in a weaker sense, scientific perspectivism is viewed as a means for assessing or evaluating, at the same time, rival modeling practices or incompatible research programmes which may give rise to perspectival knowledge (Rueger [2005]). Consequently, the usual sense attributed to the notion of a "perspective" refers to the actual scientific practices of scientific communities either within the same historical period (synchronic version of perspectivism) or in different historical periods in which those scientific practices belong (diachronic version of perspectivism). Despite dissimilarities in the rationale for embracing the synchronic or diachronic version of perspectivism, the foregoing notion of a "perspective" is applicable to both variants. Being anchored to the actual scientific practices, the ordinary understanding of a perspective is broadly taken to include: firstly, the body of knowledge claims advanced by the scientific community at a given time, secondly, the community's resources available at the time for reliably generating such claims, and, thirdly, second-order assertions concerning the justification of the knowledge claims so advanced (Massimi [2018], p. 2).

In the present study, by closely associating contemporary perspectivism to a methodology of scientific inquiry, we introduce a complementary notion of "perspective" of an *endo-theoretic/interactive* nature, conceived as the primary vehicle of tracing and investigating the world, as the principal unit of probing the natural world. Thus, the proposed conception should be also differentiated from the common pictorial understanding of a perspective as a visual metaphor involving a viewing projection, which, by itself, depicts the process of knowing as a passive activity. Instead, according to the proposed methodological framework of scientific perspectivism, a perspective is characterized endo-theoretically, namely, within a specific discipline, by a set of variables that are used to describe systems or to partition objects into parts, which together give a systematic account of a domain of phenomena. The indicated conception of "perspective", broadly defined at this stage, is rigorously formulated in Section 3.3.

It is worth noting that in view of the considered perspectivist methodology, the separation between the knowing subject and the object to be known, the partition between the observer and the observed, required for an objective description of phenomena, is *neither absolute nor catholic* as Cartesian-like epistemological approaches advocate, thus promoting an allegedly context-free account of the world. The subject-object partition is accomplished upon the condition of the adopted perspective. The choice to adopt a particular perspective signifies also the approval of a conceptual scheme on the basis of which one may isolate which of the many available properties do, and which do not count for the purposes of description, since the world does not come with one preferred system of description. Consequently, scientific observation may be regarded as perspectival in the sense that claims about what is observed cannot be completely detached, in all circumstances, from the context of observation. The significance of this point is particularly pertinent to quantum theory due to the existence of *incompatible* physical quantities, represented by corresponding non-commuting self-adjoint operators, pertaining to any nontrivial quantum system; measuring apparatuses of such quantities cannot be held simultaneously in quantum mechanics. Thus, each mode of observation of incompatible quantum mechanical quantities gives rise in an ineliminable way to a particular kind of representation, encoding, or description of the system (Section 2).

Especially, in relation to empirical testing of theories in contemporary physics, theoretical and methodological considerations specify the perspective from which we articulate the elementary yes-no experimental propositions or questions associated with properties of physical systems, in the sense that, on the one hand, they supply with a well-defined meaning the question that is put to nature, and, on the other hand, they specify the *kind* of the operations to be performed in order to ascertain particular answers to them (Karakostas [2014]). In this sense, it is legitimate to say that the perspectival nature of experimental/empirical knowledge is an essential characteristic of acquiring scientific knowledge.

Accordingly, the proposed methodological framework of scientific perspectivism consists of the following broad desiderata, to be further specified when applied to particular scientific domains of physical systems, concrete theoretical models and actual scientific practices. Let us first note that a single perspective provides, by definition, a partial/local and, thus, an incomplete description of the system to which it applies. Yet, the systematization of knowledge requires that perspectives associated with all aspects of a system can be correlated forming a *synthesized unity*, but they cannot be simply combined as independent integral parts of a third perspective. Hence, a perspective of all perspectives or, equivalently, a panoptical perspective from nowhere does not exist. It is crucial, however, that a full-fledged analysis of a successful framework of perspectivism in science ought to provide a syntax of perspectives, illustrating how *locally* shared perspectives can (or cannot) be meaningfully combined at a higher theoretical level (Section 3). Consequently, nature can be grasped scientifically, through structured multitudes of local variable perspectives, forming a *coherent multilevel* theoretical structure, exemplified by experimental procedures that render possible specific access to specific aspects of physical reality. Precisely this demanding task is accomplished by our category-theoretic perspectivist approach to quantum mechanics (Section 4).

The paper is organized as follows. In Section 2 we provide succinct argumentation, on the basis of fundamental structural features of quantum theory, revealing the theory's affinity to perspectivist reasoning. Furthermore, we specify, through the endemic feature of quantum contextuality, the endo-theoretic notion of perspective in the case of Hilbert-space quantum mechanics. Taking a more general categorical standpoint of which the above notion is an instance, in Section 3 we advance the view that an object of inquiry in a category can be completely specified by the network of all possible relations, thought of as *partial* or *local perspectives*, targeting the object under investigation. Consequently, we develop in Section 3.3 a general methodology constitutive of the notion of a perspective as a primary unit of resolving a targeted object or, generally, of probing the physical world. On this account, the perspectivist synthesis of the investigated object, subject to natural normative requirements, is based on a *multiplicity of intertwined local perspectives* covering the object entirely under their joint action. In section 4 we implement the proposed perspectivist methodology to quantum mechanics by representing categorically the global structure of a quantum algebra of events in terms of structured interconnected families of Boolean probing frames, realized as locally variable perspectives on a quantum system, being capable of carrying jointly all the information encoded in the former. The crucial conceptual and technically distinguishing feature of the foregoing representation is based on the categorical notion of an *adjunction*, formed by a pair of adjoint functors, which has been proved to hold between the category of quantum event algebras and the category of presheaves of Boolean event algebras. This pair of adjoint functors formalizes categorically a *bi-directional process* of encoding information from the quantum to the Boolean level of event structure, and conversely, decoding back. As demonstrated in Section 4.3, it is precisely this category theoretic fact that makes the application of the proposed perspectivist methodology to the quantum case novel in the literature, by linking in perspectivist terms the variable and local Boolean with the quantum global level of structure of quantum systems. Finally, Section 5 epitomizes the philosophical bearing of the considered approach on an overall picture of quantum mechanics, offering further insights of conceptual and methodological nature in relation to scientific theorizing.

2 The Affinity of Perspectivist/Contextual Reasoning to Quantum Mechanics

Standard quantum mechanics is formulated on a separable, complex Hilbert space associated to a physical system. In this framework, *quantum events* or *elementary propositions*, that is, true/false questions concerning values of physical quantities, are represented by orthogonal projection operators $\{\hat{P}_i\}$ on the system's Hilbert space H or, equivalently, by the closed linear subspace $H_{\hat{P}_i}$ of H upon which the projection operator $\hat{P}_i$ projects. The one-to-one correspondence between the *set of all closed linear subspaces* of H and the *set* *of all projection operators*, denoted by L_H, allows a translation of the lattice structure of the subspaces of Hilbert space into the algebra of projections with the appropriate lattice theoretic characterizations (Varadarajan [2007]). Then, a *quantum algebra of events* is identified with the algebraic structure of all projection operators on Hilbert space, ordered by inclusion and carrying an orthocomplementation operation, thus forming a complete, atomic, orthomodular lattice. In effect, a non-classical, non-Boolean logical structure is induced which has its origin in quantum theory.

An immediate path for revealing the affinity of the perspectivist/contextual reasoning to quantum mechanics is provided through Kochen-Specker's celebrated theorem and its recent ramifications (for example, Karakostas and Zafiris [2017], Svozil [2017]). In view of the latter, for any quantum system associated to a Hilbert space of dimension greater than two, *there does not exist a two-valued homomorphism* or, equivalently, *a truth-functional assignment* $h : L_H \rightarrow \{0, 1\}$ on the set of projection operators, L_H, interpretable as quantum mechanical propositions, preserving the lattice operations and the orthocomplement, even if these lattice operations are carried out among commuting elements only. The essence of the theorem, when interpreted semantically, asserts the impossibility of assigning definite truth values to *all* propositions pertaining to a physical system at any one time, for any of its quantum states, without generating a contradiction.

The Kochen-Specker result shows, in physical terms, that in a system represented by a Hilbert space of three or more dimensions, there exist projection operators $\{\hat{P}_i\}$ such that it is not always possible to assign truth values 0 and 1 to all corresponding propositions pertaining to the system, so that the following conditions are fulfilled:

(i) For any orthogonal $i-$tuple of projection operators, $\{\hat{P}_i\}$, the assignment satisfies $\sum_i h(\hat{P}_i) = h(\hat{\mathbb{I}}) = 1$, that is, one projection operator is mapped onto 1 ('true') and the remaining $i - 1$ projection operators are mapped onto 0 ('false') (completeness of the basis condition).

(ii) If a projection operator, $\hat{P}_k$, belongs to multiple complete orthogonal bases, then, it is consistently assigned the *same* value in all bases (noncontextuality condition).

The initial proof of Kochen and Specker establishes that no such assignment of 1's and 0's is possible for a special case restricted to a finite sublattice of projection operators on a three-dimensional Hilbert space, associated to a spin-1 quantum system, in a way that preserves the noncontextuality condition. The ingenuity of the proof, essentially of a geometrical nature, and its far reaching consequences have gradually generated an overwhelming production of theoretical and experimental research on foundational issues in quantum mechanics related to the contextual character of the theory as a *structural* feature of the quantum mechanical formalism itself (see Cabello et al. [2010], Howard et al. [2014]).

A failure of the noncontextuality condition means that the value assigned to a quantum mechanical observable A, whose representing self-adjoint operator $\hat{A}$ is analyzed in terms of spectral projections $\hat{P}_i$, depends on the *context* in which it is considered. An equivalent way of expressing the above is to say that the value of A depends on what *other* compatible observables are assigned values at the same time; i.e., it depends on a choice that concerns operators that commute with $\hat{A}$. This dependence captures the endemic feature of *quantum contextuality* and may be highlighted by using an explicit example. Let us consider, for reasons of simplicity, a triad of observables $\{A, B, C\}$ representing physical quantities of a quantum system S. According to quantum theory, it is possible to simultaneously measure a set of observables reliably if and only if the corresponding operators are commutative. Let us, then, assume that the operator $\hat{A}$ pertaining to system S commutes with operators $\hat{B}$ and $\hat{C}$ ($[\hat{A}, \hat{B}] = 0 = [\hat{A}, \hat{C}]$), not however the operators $\hat{B}$ and $\hat{C}$ with each other ($[\hat{B}, \hat{C}] \neq 0$). Then, due to the non-commutativity of the last pair of operators, the result of a measurement of observable A depends on whether the system had previously been subjected to a measurement of B or a measurement of C or in none of them. Thus, the value of the observable A depends upon the set of mutually compatible observables one may consider it with, that is, the value of A depends upon the selected context of measurement. In other words, the value of the observable A cannot be thought of as pre-fixed, as being independent of the experimental context actually chosen, as specified, in our example, by the $\{\hat{A}, \hat{B}\}$ or $\{\hat{A}, \hat{C}\}$ frame of mutually commuting operators. It is worth noting that the formalism of quantum theory does not imply how such a contextual valuation might be obtained on the set L_H of all projection operators on a Hilbert space or what properties it should possess.

To this end, we resort to the powerful methods of category theory, which directly captures the idea of structures varying over contexts, thus providing a natural setting for investigating multilevel structures and studying contextual phenomena. In the proposed category-theoretic perspectival representation of a quantum algebra of events,[1] developed in Section 4, the notion of *perspective* that is applied on a quantum system is tantamount to a set of mutually compatible physical quantities, as in the preceding example, or, more precisely, to a complete Boolean algebra of commuting projection operators generated by this set. It should be underlined that such a complete Boolean algebra of projection operators bears the status of a *logical structural invariant* characterizing a whole commutative algebra of observables that can be simultaneously spectrally resolved and hence be co-measurable (Radjavi and Rosenthal [2003]). Since in the lattice of quantum events there exist incompatible physical magnitudes non-commuting with any considered commutative algebra of observables, there exists a *multiplicity* of possible Boolean algebras of projections furnishing an invariant of this kind only at the local level of discourse. Therefore, although a quantum event structure is globally non-Boolean, it can be qualified spectrally, and hence be accessed experimentally, only in terms of Boolean event structures operating locally as logical structural invariants of co-measurable families of physical magnitudes (cf. Svozil [2018], Section 12.9.11). Consequently, a complete Boolean algebra of projection operators furnishes the role of a *Boolean probing frame* with respect to which a quantum event can be qualified and lifted to the empirical level. Thus, the consideration of each local Boolean frame at any temporal moment serves as a natural pre-condition for establishing a local invariant logical structure for the event evaluation of all co-measurable observables forming this context. Due to the absence of a global, uniquely defined Boolean frame over a quantum event structure, it is necessary to consider all possible local ones together with their

[1] Categorical or topos-theoretic approaches to the foundations of quantum mechanics have also been considered, from a different viewpoint, for instance, by Isham and Butterfield [1998], Döring and Isham [2011], Abramsky and Brandenburger [2011], Heunen et al. [2011]. Apparently, this is not the appropriate place for reviewing the variety of categorical approaches appearing in the literature in relation to quantum mechanical considerations. Suffice it here to point out that the category-theoretic approach employed in the paper is based on the notion of a *categorical adjunction* for relating structured families of Boolean probing frames to quantum event algebras, whereas, none of the aforementioned categorical approaches make any use of this notion for expressing the Boolean-quantum relation (see, especially, Section 4.3). For a detailed technical comparison of this kind involving all major categorical approaches, the interested reader may consult the Appendix in (Zafiris and Karakostas [2013], pp. 1112-21).

interrelations.

This naturally leads, by extension, to a *horizon of perspectives* on the structure of quantum events with respect to a multiplicity of various Boolean frames realized as experimental contexts for measuring physical quantities. Be sure, no single context or perspective can deliver a complete picture of the quantum system, but, by applying category theoretic reasoning, it is possible to combine them suitably in an overall structure that will capture the entire system. It is also of great importance how the various contexts or contextual perspectives relate to each other. Categorically speaking, this consideration is naturally incorporated into our scheme, since the category theoretic representation of quantum event algebras in terms of Boolean probing contexts can be described by means of a multilevel structure, mathematically known as a *topos*, which stands for a *category of sheaves of variable, overlapping and interconnected families of local Boolean frames*, capable of carrying all the information encoded in the former.

3 The Categorical Imperative: A Novel Mode of Object Specification in Perspectivist Terms

3.1 Categorical principles and prerequisites

Category theory provides a general theoretical framework for the study of structured systems in terms of their *mutual relations* and *admissible transformations*. Contrary to the atomistic approach of set theory, which crucially depends on the concept of elements-points and the membership relationship of a variable x in a set X, $x \in X$, in category theory the notion of *morphism* or *arrow* undertakes primary role. A morphism, for instance, $f : A \to B$ in a category $\mathcal{C}$ expresses one of the many possible ways in which the object A relates to the object B within the context of category $\mathcal{C}$. Thus, an incoming morphism to B from any other object A in category $\mathcal{C}$ may be considered as a *perspective* targeting B whose source is A in the same category. The category theoretic mode of thinking incorporates internally the very nature of "pointing at" or "viewing from". In this vein, the notion of structure does not exclusively refer to a fixed universe of sets of pre-determined elements, but acquires a variable reference (Bell [2008]).

Concomitantly, category theory is suitably equipped to deal successfully with highly complex structural problems in the natural sciences, since, besides the consideration of morphisms between objects in a category, there also exist, at the next higher level, structure preserving mappings between categories, namely *functors*, and, further on, mappings between functors called *natural transformations*. It is apparent that the theory of categories proceeds inherently in an hierarchical manner. It is able to formally define and operate with both structures of systems and structures of structures encompassing layers of increasing abstraction and complexity. Thus, essential categorical notions and constructions present themselves in ascending levels of generality and depth: category, functor, natural transformation, adjointness, higher-order categories, etc. The innovative conceptual spirit of category theory provides the framework from within we can capture and analyze the *shared structure* existing between different kinds of complex systems in terms of the structure preserving transformations between them. The basic categorical principles that we adopt in the subsequent analysis are summarized as follows.

(i) To each kind of mathematical structure used to represent a system, there corresponds a *category* whose objects have that structure and whose morphisms or arrows preserve it.

A *category* $\mathcal{C}$ is an aggregate consisting of the following:

(a) A class $Ob(\mathcal{C})$, whose elements $A, B, \ldots$ are called *objects*. For each object A an element $id_A : A \to A$ is distinguished; it is called the *identity morphism* for A.

(b) A class $Hom(\mathcal{C})$, whose elements $f, g, \ldots$ are called *morphisms* or *arrows*. Each morphism $f : A \to B$ is associated with a pair of objects, known as its *domain* and *codomain* respectively. The expression $Hom_{\mathcal{C}}(A, B)$ denotes the *Hom-class* of all morphisms from A to B.

(c) A binary operation $\circ$, called *composition of morphisms*, such that, for given arrows $f : A \to B$ and $g : B \to E$, that is, with codomain of f equal to the domain of g, then f and g can be composed to give an arrow $g \circ f : A \to E$. Generally, for any three objects A, B, and E in $\mathcal{C}$, the set mapping is defined as:

$$Hom_{\mathcal{C}}(B, E) \times Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, E).$$

The operation of composition is associative, $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B$, $g : B \to E$, $h : E \to D$, satisfying also the property of identity, $f \circ id_A = f = id_B \circ f$, for all $f : A \to B$. The operation of composition of morphisms binds category $\mathcal{C}$ as an associatively closed universe of discourse.

For an arbitrary category $\mathcal{C}$ the *opposite* or *dual* category $\mathcal{C}^{op}$ is defined in the following way. The objects are the same, but $Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$, namely, all arrows are inverted. Hence, the opposite category $\mathcal{C}^{op}$ of a given category interchanges the source and target of each morphism. A category $\mathcal{C}$ is called *locally small* if for all objects A, B in $\mathcal{C}$, each class of morphisms $Hom_{\mathcal{C}}(A, B)$ is a set. Furthermore, if the class of objects forms a set, then $\mathcal{C}$ acquires the status of a *small* category.

(ii) To any canonical construction[2] on structures of one kind, yielding structures of another kind, there corresponds a *functor* from the category of the first specified kind to the category of the second by preserving the essential relationships among objects of the respective categories.

Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *covariant functor* $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ is a class mapping that:

(a) Associates to each object $A \in \mathcal{C}$ an object $\mathbb{F}(A) \in \mathcal{D}$.

(b) Associates to each morphism $f : A \to B \in \mathcal{C}$ a morphism $\mathbb{F}(f) : \mathbb{F}(A) \to \mathbb{F}(B) \in \mathcal{D}$.

(c) Preserves identity morphisms and compositions, i.e., $\mathbb{F}(id_A) = id_{\mathbb{F}(A)}$ and $\mathbb{F}(g \circ f) = \mathbb{F}(g) \circ \mathbb{F}(f)$.

[2]In category theoretic language the term "canonical" is used in this context to specify that a categorical construction, for instance, a categorical relation, explanation, proof, etc., is realized without making arbitrary choices. For an intuitive example, independently of category theory, consider the isomorphism between a finite dimensional vector space and its double dual; the latter isomorphic relation is "canonical" in the sense that no arbitrary choices are needed in order to define it. In other words, in the example on vector spaces, the specification of the isomorphism does not depend on a particular representational scheme or on ad hoc assumptions about the choice of a vector basis.

A *contravariant functor* $\hat{\mathbb{F}} : \mathcal{C} \to \mathcal{D}$ is, by definition, a covariant functor $\mathbb{F} : \mathcal{C}^{op} \to \mathcal{D}$. A functor, therefore, is a type of mapping between categories that associates to every object of one category an object of another category and to every morphism in the first category a morphism in the second by preserving domains and codomains, identity morphisms, and composition of morphisms. Thus, a functor $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ provides a sort of 'picture' of category $\mathcal{C}$ in $\mathcal{D}$ by preserving the structure of $\mathcal{C}$.

(iii) To each natural translation between two functors having identical domains and codomains, there corresponds a *natural transformation*.

Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let further $\mathbb{F}$, $\mathbb{G}$ be functors from the category $\mathcal{C}$ to the category $\mathcal{D}$. A *natural transformation* τ from $\mathbb{F}$ to $\mathbb{G}$, $\tau : \mathbb{F} \to \mathbb{G}$, is a mapping that assigns to each object A in $\mathcal{C}$ a morphism $\tau_A : \mathbb{F}(A) \to \mathbb{G}(A)$ in $\mathcal{D}$, called the *component* of τ at A, such that for every arrow $f : A \to B$ in $\mathcal{C}$ the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} \mathbb{F}(A) & \xrightarrow{\tau_A} & \mathbb{G}(A) \\ {\scriptstyle \mathbb{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathbb{G}(f)} \\ \mathbb{F}(B) & \xrightarrow{\tau_B} & \mathbb{G}(B) \end{array}$$

That is, for every arrow $f : A \to B$ in $\mathcal{C}$, we obtain: $\mathbb{G}(f) \circ \tau_A = \tau_B \circ \mathbb{F}(f)$. Thus, natural transformations define structure preserving mappings of functors, while respecting the internal structure of the categories involved. Pictorially, one may think of this situation as follows. If the parallel functors $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbb{G} : \mathcal{C} \to \mathcal{D}$, having both the same domain and codomain categories, are conceived of as projecting a 'picture' of category $\mathcal{C}$ in $\mathcal{D}$, then a natural transformation is a way to transform globally or systematically the 'picture' defined by $\mathbb{F}$ onto the 'picture' defined by $\mathbb{G}$. The specification "natural", particularly, in the notion of natural transformations refers to the comparison of two functorial processes, sharing the same source and target categories, in a way that captures the *shared structure* or *generic common properties* existing in different contexts. No doubt, the key concept of natural transformations acquires in category theory the status of a principle, analogous to general covariance in physics, that penetrates deeper than is initially discernible.

(iv) To any natural *bi-directional functorial correlation* between two kinds of mathematical structures, there corresponds an *adjunction*, consisting of a *pair of adjoint functors* between the corresponding categories.

Let $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbb{G} : \mathcal{D} \to \mathcal{C}$ be functors. We say that $\mathbb{F}$ is *left adjoint* to $\mathbb{G}$ (correspondingly, $\mathbb{G}$ is *right adjoint* to $\mathbb{F}$), if there exists a bijective correspondence between the arrows $\mathbb{F}(C) \to D$ in $\mathcal{D}$ and $C \to \mathbb{G}(D)$ in $\mathcal{C}$, which is natural in both C and D:

$$\mathbb{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathbb{G}.$$

This means that the objects of the categories $\mathcal{C}$ and $\mathcal{D}$ are related with each other through natural transformations. Then, the above pair of adjoint functors constitutes a *categorical adjunction*. The latter concept is of fundamental logical and mathematical importance contributed to mathematics by category theory (Goldblatt [2006], p. 438).

3.2 Perspectivity functors and representability of objects

The category theoretic framework signifies a remarkable conceptual change in the way of conceiving the form and function of mathematical objects, departing from the atomistic set-theoretic approach of analyzing objects in terms of pre-determined or pre-distinguishable elements endowed with some particular externally imposed structure. The emphasis now is placed on the specification of objects in terms of the relations they bear with other objects of the same category. However counterintuitive it may initially appear, in category theory the nature of the objects is a derivative aspect of the patterns described by the morphisms or mappings that connect the objects. In this respect, it is not the objects, but rather the relationships between them that determine the essence of any category.

A pertinent problem in this general setting refers to the category theoretic expression of the relations that any depicted object A of a category $\mathcal{A}$ bears with all other objects in $\mathcal{A}$. According to the preceding, for any object of a category it is essential to consider the class of all incoming morphisms to this object, or dually, the class of all outgoing morphisms from this object to any other object of the same category. If we consider that these classes are sets in both cases, so that our category $\mathcal{A}$ is a small category at least locally, it is necessary to define the notion of a covariant functor (or dually, a contravariant functor) from the category $\mathcal{A}$ of the depicted object to the category of sets, denoted hereafter as $\mathcal{S}et$. Let us recall that a covariant functor is a functor which preserves in the target category the directionality of an arrow in the domain category, whereas dually, a contravariant functor is a functor which reverses it.

Regarding the posed problem, we address the following functorial characterization of the relations that any object A of a category $\mathcal{A}$ bears with all other objects in $\mathcal{A}$. As argued in the sequel, it is of primary significance to the philosophy of scientific perspectivism that within the categorical framework the object of inquiry can be equivalently replaced by its associated functorial representation consisting of all morphisms coming into (or going out of) the object concerned.

Proposition 3.2.1. *Each object A in a category $\mathcal{A}$ gives rise to a covariant functor of morphisms emanating from A, denoted by $\curvearrowright_A : \mathcal{A} \to \mathcal{S}et$, called the covariant $Hom_{\mathcal{A}}$-functor,* represented *by A.*

The covariant functor $\curvearrowright_A$ describes how a fixed object A sees all other objects of the category in which it belongs.

Definition 3.2.1. The covariant $Hom_{\mathcal{A}}$-functor, $\curvearrowright_A : \mathcal{A} \to \mathcal{S}et$, from the source category $\mathcal{A}$ to the evaluation category of sets is defined as follows:

(1) For all objects X in $\mathcal{A}$, $\curvearrowright_A(X) := Hom_{\mathcal{A}}(A, X)$;

(2) For all morphisms $f : X \to Y$ in $\mathcal{A}$,

$$\curvearrowright_A(f) := Hom_{\mathcal{A}}(A, f) = Hom_{\mathcal{A}}(A, X) \to Hom_{\mathcal{A}}(A, Y) \tag{3.1}$$

is defined as post-composition with f, viz., $\curvearrowright_A(f)(g) := f \circ g$.

The covariant representable functor, $\curvearrowright_A : \mathcal{A} \to \mathcal{S}et$, can be thought of as constructing an image of $\mathcal{A}$ in the category of sets in a covariant way. Only set-valued functors, that is, functors with codomain $\mathcal{S}et$ can be representable (Johnstone [2002], Volume 1).

Corollary 3.2.1. *A functor* $\mathbb{G} : \mathcal{A} \to \mathcal{S}et$ *is representable if it is naturally isomorphic to* $\curvearrowright_M$ *for some object* M *in* $\mathcal{A}$*. A representation of* $\mathbb{G}$ *is a choice of an object* M *in* $\mathcal{A}$ *together with a natural isomorphism of functors* $\curvearrowright_M \cong \mathbb{G}$*.*

Thus, $\mathbb{G}$ is representable if and only if such a representing object exists. Notice that representations of set-valued functors are unique up to a unique isomorphism (Mac Lane [1998]).

Now, let us dually consider the *opposite category* $\mathcal{A}^{op}$, and let A be an object in $\mathcal{A}$.

Proposition 3.2.2. *Each object* A *in a category* $\mathcal{A}$ *gives rise to a contravariant functor of morphisms targeting* A*, denoted by* $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}et$*, called the contravariant* $Hom_{\mathcal{A}}$*-functor,* represented *by* A*.*

The contravariant functor $\curvearrowright^A$ expresses how a fixed object A is seen by all other objects of the category in which it is placed.

Definition 3.2.2. The contravariant $Hom_{\mathcal{A}}$-functor, $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}et$, is defined as follows:

(1) For all objects B in $\mathcal{A}$, $\curvearrowright^A(B) := Hom_{\mathcal{A}}(B, A)$;

(2) For all morphisms $f : C \to B$ in $\mathcal{A}$,

$$\curvearrowright^A(f) := Hom_{\mathcal{A}}(f, A) = Hom_{\mathcal{A}}(B, A) \to Hom_{\mathcal{A}}(C, A) \tag{3.2}$$

is defined as pre-composition with f, viz., $\curvearrowright^A(f)(g) := g \circ f$.

The contravariant representable functor, $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}et$, can be thought of as constructing an image of $\mathcal{A}$ in the category of sets in a contravariant way.

Corollary 3.2.2. *A functor* $\mathbb{F} : \mathcal{A}^{op} \to \mathcal{S}et$ *is representable if it is naturally isomorphic to* $\curvearrowright^K$ *for some object* K *in* $\mathcal{A}$*. A representation of* $\mathbb{F}$ *is a choice of an object* K *in* $\mathcal{A}$ *together with a natural isomorphism of functors* $\curvearrowright^K \cong \mathbb{F}$*.*

The contravariant $Hom_{\mathcal{A}}$-functor represented by A, $\curvearrowright^A : A^{op} \to \mathcal{S}et$, is alternatively called the *functor of generalized elements* (*incoming morphisms*) of A (Awodey [2010]). Dually, the covariant $Hom_{\mathcal{A}}$-functor represented by A, $\curvearrowright_A : \mathcal{A} \to \mathcal{S}et$, is called the *functor of generalized co-elements* (*outgoing morphisms*) of A. The functors of the form $\curvearrowright^A$ (equivalently, $\curvearrowright_A$), for all objects A of $\mathcal{A}$, provide us with a tool for determining the properties of the objects of a category $\mathcal{A}$.

At a provisional stage, if we consider an object A in $\mathcal{A}$, we may think of an incoming morphism to A from any other object X in $\mathcal{A}$ as a perspective targeting A whose source is X in the same category. This is a quite broad idea, since, as argued in Section 3.3, a full-fledged notion of a perspective applicable to contemporary science is admissible to further restrictive normative requirements. Moreover, if one confines attention to a singleton perspective, only the set-theoretic aspect can be seen, thus concealing the full structure of the category of local compositions. Recall, in this respect, that an object A in $\mathcal{A}$ gives rise to a contravariant functor of morphisms targeting A, i.e., the contravariant representable

$Hom_{\mathcal{A}}$-functor, which is *represented* by the object A, denoted by $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}et$, bearing in this manner the semantics of what may naturally be called the *perspectivity functor* on the object A. For, the functor $\curvearrowright^A$ can be thought as a function whose variable argument runs over all objects of the category $\mathcal{A}$, such that its evaluation at each one of these objects results in the set of all partial perspectives emanating from this object and always targeting A in the same category. Due to the representability of the functor $\curvearrowright^A$ by the object A, we obtain an assignment $A \mapsto \curvearrowright^A$. The crucial observation, attributed to Yoneda, is that this assignment can be extended to the embedding functor $\curvearrowright$, defined as follows.

Definition 3.2.3. Let $\mathcal{A}$ be a locally small category. The Yoneda embedding of $\mathcal{A}$ is the functor

$$\curvearrowright : \mathcal{A} \to \mathcal{S}et^{\mathcal{A}^{op}}, \tag{3.3}$$

defined on objects A by

$$A \mapsto \curvearrowright^A, \tag{3.4}$$

and on morphisms f by

$$(f : A \to B) \mapsto (\curvearrowright^f : \curvearrowright^A \mapsto \curvearrowright^B), \tag{3.5}$$

where $\curvearrowright^f$ is the natural transformation between $\curvearrowright^A$ and $\curvearrowright^B$.

The functor $\curvearrowright$ is called the Yoneda embedding, since it embeds the category $\mathcal{A}$ into the functor category $\mathcal{S}et^{\mathcal{A}^{op}}$. This fact allows one to investigate category $\mathcal{A}$ in a wider context with no loss of information.

At the next stage, we may consider an arbitrary natural transformation of contravariant functors $\kappa : \curvearrowright^A \to \mathbb{F}$. Then, the naturality requirement reads as follows:

$$\begin{array}{ccc} \curvearrowright^A(A) & \xrightarrow{\kappa_A} & \mathbb{F}(A) \\ {\scriptstyle \curvearrowright^A(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{F}(f)} \\ \curvearrowright^A(B) & \xrightarrow{\kappa_B} & \mathbb{F}(B) \end{array}$$

Let id_A be the identity element at the object A in $\curvearrowright^A(A) := Hom_{\mathcal{A}}(A, A)$. Taking into account that for all morphisms $f : B \to A$ in $\mathcal{A}$,

$$\curvearrowright^A(f) := Hom_{\mathcal{A}}(f, A) = Hom_{\mathcal{A}}(A, A) \to Hom_{\mathcal{A}}(B, A)$$

is defined as pre-composition with f, evaluation at id_A yields $\curvearrowright^A(f)(id_A) := id_A \circ f$. Thus, we finally obtain:

$$\kappa_B(id_A \circ f) = \mathbb{F}(f)(\kappa_A(id_A)).$$

Hence, the natural transformation κ is in fact completely determined by the element $\kappa_A(id_A)$ in $\mathbb{F}(A)$. In turn, this sets up a bijection between the elements of $\mathbb{F}(A)$ and the natural transformations $\kappa : \curvearrowright^A \to \mathbb{F}$. The obtained proposition is called the Yoneda lemma, formulated as follows.

Lemma 3.2.1 (Yoneda)**.** *Let $\mathcal{A}$ be a locally small category and $\mathbb{F} : \mathcal{A}^{op} \to \mathcal{S}et$ a contravariant set-valued functor. Then, for any object A in $\mathcal{A}$, there is an isomorphism,*

$$\mathbb{F}(A) \cong Nat(\curvearrowright^{A}, \mathbb{F}) := Hom_{\mathcal{S}et^{\mathcal{A}^{op}}}(\curvearrowright^{A}, \mathbb{F})\,, \tag{3.6}$$

which is natural in both A and $\mathbb{F}$, where $Nat(\curvearrowright^{A}, \mathbb{F})$ denotes the set of natural transformations of contravariant functors $\curvearrowright^{A} \to \mathbb{F}$ in the functor category $\mathcal{S}et^{\mathcal{A}^{op}}$.

As an immediate corollary of the Yoneda lemma, we observe that if the contravariant set-valued functor $\mathbb{F}$ is of the representable form $\mathbb{F} = \curvearrowright^{X} : \mathcal{A}^{op} \to \mathcal{S}et$ for some object X in $\mathcal{A}$, then

$$\curvearrowright^{X}(A) \cong Nat(\curvearrowright^{A}, \curvearrowright^{X}),$$

or equivalently,

$$Hom_{\mathcal{A}}(A, X) \cong Nat(\curvearrowright^{A}, \curvearrowright^{X}).$$

Corollary 3.2.3. *For any locally small category $\mathcal{A}$, the Yoneda embedding is full and faithful. Equivalently, the functor*

$$\curvearrowright : \mathcal{A} \to \mathcal{S}et^{\mathcal{A}^{op}} \tag{3.7}$$

constructs a full and faithful image of the category $\mathcal{A}$ into the functor category $\mathcal{S}et^{\mathcal{A}^{op}}$.

Bearing in mind that the Yoneda embedding of a category $\mathcal{A}$ into the category $\mathcal{S}et^{\mathcal{A}^{op}}$ sends each object A of $\mathcal{A}$ to its associated perspectivity functor, i.e., the contravariant $Hom_{\mathcal{A}}$-functor represented by A, Corollary 3.2.3 gives rise to the following proposition.

Proposition 3.2.3. *Let A, B be objects in a category $\mathcal{A}$. Suppose we are given an isomorphism of their associated perspectivity functors: $\curvearrowright^{A} \cong \curvearrowright^{B}$. Then there exists a* unique isomorphism *of the objects themselves, that is $A \cong B$, which gives rise to this isomorphism of functors.*

Henceforth we arrive at the following result, which is of central significance for scientific perspectivism.

Proposition 3.2.4. *The information contained in an arbitrary object A of a category $\mathcal{A}$ is entirely* classified *and retrieved by its associated perspectivity functor $\curvearrowright^{A}$, i.e., the representable functor of generalized elements-perspectives on A.*

We conclude that, according to the category-theoretic modeling framework, an object A of a category $\mathcal{A}$, describing a species of structure, is completely specified by the network of all incoming morphisms, thought of as partial perspectives, targeting this object by all other objects in $\mathcal{A}$. In other words, an object in a category can be completely classified and retrieved by all internalized structure-respecting relations targeting it within the same category. The complete classification asserted by this proposition refers to the fact that the whole network of targeting morphisms determines the object uniquely up to canonical isomorphism. Thus, it provides the universal means of specification of the object in relation to its categorial species, in the sense that this specification is unique up to equivalence, established by an explicitly demonstrable isomorphism, not being dependent on the particular objects and morphisms considered.

The ground breaking consequence of this proposition is that the investigated object can be legitimately subjugated or even conceptually substituted by the network of all internalized structure-respecting relations targeting it within the same category. In this manner, the object constitutes a *perspectival representation* of the whole network of relations directed to it within its categorial species, and inversely, this network becomes *uniquely representable* up to equivalence by the targeted object.

3.3 Structural adaptability of probes and methodological norms of perspectivism

In practice, the specification of an object in a category by the network of all possible relations directed to it within the same category is physically redundant. The underlying idea is that on the basis of theoretical, experimental or computational reasons, a *category of probes* is always delineated in relation to an investigated object of some categorial species. This category of probes may have the status of a subcategory of the category in which the investigated object is structurally placed, but generally speaking, the category of probes is not required to be a subcategory. It is of utmost importance, on the contrary, that the action of a probe can be *structurally adapted* as an *internalized* and, therefore, *structure-respecting directed morphism* within the category of the investigated object. Henceforth, it is exactly this qualification of a probing relation that preconditions the notion of a perspective on an investigated object of some category.

In this generalized setting, what is basic for the function of perspectivism is that the specification of an object of a particular structural species can be enunciated in terms of certain relations it bears with objects of another structural species, which are thought of as partial probing frames or resolving units of the former, under the proviso that these relations can be appropriately internalized within the former category and eventually interpreted as perspectives targeting this object (Section 4.1).

It is precisely this possibility that we consider fundamental for a consistent, fruitful re-evaluation of the norms of scientific perspectivism in relation to the novel mode of object-specification induced by the development of category theory in mathematical thinking. More precisely, on account of Section 1, the notion of a perspective on an object of inquiry is conceived as the principal means of probing or resolving this object. In view of the categorical framework, such a kind of a resolving probe may be formulated independently of any a priori requirement of analysis of the object in its set-theoretic elements. This is the case because the notion of a perspective is not subordinate to a set-theoretic function between a probe and an object but, on the contrary, subsumes a well-defined structural characterization derived from the internalization of a probing relation within the category deciphering the species of the object under investigation.

Proposition 3.3.1. *A probing relation is qualified as a potential perspective on an object of a category if and only if it can be internalized within this category, so that it can be expressed as a structure-respecting morphism targeting the object of inquiry.*

An immediate consequence of this characterization results in the following corollary.

Corollary 3.3.1. *The domain of a probing relation in its functionality to act as a probe of an object of a category should be structurally adaptable to the species of the investigated object.*

The notion of structural adaptability is fundamental not only for clarifying what should be considered as a perspective on an object of some categorial species, but it essentially characterizes what should be qualified as a probe that constitutes the structural domain of a perspective. The pertinent issue arising in this state of affairs is the detection of the conditions that cast a probe structurally adaptable to the object of inquiry, so that it can function as a source of a perspective on this object. Intuitively, since a probe should furnish a unit of resolution of the investigated object, this unit can be structurally adapted to the species of the object if and only if it encodes some structural invariant feature pertaining to the level of resolution of the investigated object with respect to the employed unit. In other terms:

Proposition 3.3.2. *A probe as the source of a potential perspective on an object of some categorial species is structurally adaptable to this species if and only if it encodes a structurally invariant context, grouping together all information that can be delineated from the investigated object in terms of the probe's resolving power.*

This is an essential requirement for the expression of the norms pertaining to the constitution of a perspective in coordination with the framework of category theory. The requirement of structural adaptability qualifies a perspective as a local or partial frame of analysis of the investigated object by virtue of the structurally invariant context of resolution this analysis is based on.

Definition 3.3.1. A perspective on an object of some categorial species is a structurally adaptable probing relation to this species bearing an invariant context of resolution.

Corollary 3.3.2. *A perspective on an object of some categorial species is co-extensive to a local or partial cover of this object if its covering capacity is, firstly, compatible under the operation of restriction to subcovers and, secondly, stable under pulling back or overlapping.*

Concomitantly, the action of a probe, engulfed in an applied perspective on an object of inquiry, is co-extensive to a local or partial cover of this object only under the natural constraints of cover compatibility under restriction to subcovers and stability. As an immediate consequence of the preceding considerations, we obtain the following proposition.

Proposition 3.3.3. *A perspective on an object of some categorial species, qualified as a local or partial cover of this object, instantiates a partial or local structural congruence of the corresponding probe source with the target object.*

It is worthy to underline that the implicated notion of localization with respect to a perspective on an object of a categorial species is derived internally and intrinsically merely from the specific invariant capacity of the probe source in its function as a local cover, and not from any spatial embedding environment of any external form. In view of Proposition 3.3.3, a single perspective, although incomplete in its capacity to resolve the investigated object globally or in its entirety, shapes the target locally or partially in a structurally adaptable and congruent manner, so that, beyond its compatible restriction, it can be internally extended as well, under the proviso that overlaps compatibly with some other perspective deciphering another local cover of the investigated object. A crucial feature of this local perspectival schematism of an object in a category is that it does not assume or require the existence of an all-encompassing perspective, meaning that local perspectives should not be thought of as independent parts of an overall perspective. In contradistinction, the perspectival schematism or formation of an object is based on the idea of a *multiplicity* of possible local perspectives, covering the object entirely only under their *joint action*, which is only constrained by the normative requirement of compatible interconnection on their pairwise overlaps whenever this is the case (Section 4.2).

It may be also instructive to point out that a multiplicity of local perspectives covering jointly a target object is not merely a set of local perspectives. In category theoretic terms, it may be thought of as a *colimit* (or inductive limit) object, which is *synthesized jointly*, in a compatible manner, out of all local perspectives directed towards it, without admitting them as independent parts. Rather, it may be visualized as a multi-layered granulation *sieve* of *intertwined local perspectives*, whose variable concatenated openings comprise the resolving power of the corresponding probes, and which becomes structurally congruent to the targeted object as a colimit object by virtue of the joint covering action of all involved local perspectives[3] (Section 4.3). In view of the preceding analysis the validity of Proposition 3.2.4, concerning object-specification in perspectivist terms, remains intact, but refined substantially. In this respect, the central importance of the perspectivist methodology advanced in Section 1 can be condensed, in the process of its formulation from the standpoint of category theory, in the following proposition.

Proposition 3.3.4. *An object of a categorial species can be specified, classified, and retrieved in a universal way by the colimit assemblage of all local perspectives directed towards it and jointly covering it in a compatible manner uniquely up to equivalence.*

Most significantly, Proposition 3.3.4 does not exclude the inter-level determination of objects belonging to *distinct* categorial species, under the condition that there exists a *bi-directional* functorial correlation between them, formulated in the language of *adjunctions*. It is precisely the latter development that gradually introduced into category theory a paradigm change in understanding structures of general types and paved the way for forming bridges between seemingly unrelated mathematical disciplines, revealing, at the same time, the philosophical significance of category theory.[4] As analyzed in Section 4, it is indeed the categorical notion of adjunction, consisting of a pair of adjoined functors, that allows us to produce a perspectival representation of a quantum event algebra by linking appropriately the Boolean and the quantum structural levels.

4 Perspectival Representation of a Quantum Event Structure via Adjunction of Boolean Frames

The conceptual basis of the proposed perspectival representation of a quantum structure of events L in terms of interconnected families of Boolean probing frames, realized as suitable perspectives on L, relies on the physically significant fact that it is possible to analyze or

[3] For readers not familiar with the categorical construction of colimits we note that their existence expresses in category-theoretic language the basic intuition that a complex object may be conceived as arising from the interconnection of partially or locally defined informational units within a category. In a nutshell, colimits may be viewed as binding factors ‘gluing’ parts together.

[4] Philosophical considerations of category theory and their application to a variety of contexts have been given recently, for instance, by Halvorson [2019], Eva [2017] and Landry [2017].

‘coordinatize’ the information contained in a quantum event algebra by means of structure preserving morphisms $B \to L$, having as their domains locally defined Boolean event algebras B. As alluded to in Section 2, any single map from a Boolean domain to a quantum event algebra does not suffice for a complete determination of the latter’s information content. In fact, it contains only the amount of information related to a particular Boolean frame, prepared for a specific kind of measurement, and inevitably is constrained to represent exclusively the abstractions associated with it. This problem is confronted by employing a sufficient amount of maps, organized in terms of *covering sieves*, from the coordinatizing Boolean domains to a quantum event algebra so as to cover it completely. These maps furnish the role of local *Boolean covers* for the filtration of the information associated with a quantum structure of events, in that, their domains B provide Boolean coefficients associated with typical measurement situations of quantum observables. The local Boolean covers capture, in essence, separate complementary features of the quantum system under investigation, thus providing a structural decomposition of a quantum event algebra in the descriptive terms of Boolean probing frames. In turn, the incomplete and complementary local Boolean descriptions, arising from a multiplicity of locally variable perspectives, can be smoothly *pasted* or *glued* together, by demanding the satisfaction of *partial compatibility* between *overlapping* local Boolean covers, so that one may arrive at the synthesis and actual determination of the global quantum event algebra itself. The implementation of the *perspectival representation of a quantum event algebra in terms of structured multitudes of interconnected Boolean probing frames* requires distinct notions of Boolean and quantum categorical event structures, respectively. The methodology involved in the realization of the suggested approach necessitates the application of categorical sheaf theory[5] to quantum structures.

4.1 Categories of Boolean and Quantum event structures

Definition 4.1.1. A *Boolean categorical event structure* is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. The objects of $\mathcal{B}$ are complete Boolean algebras of events and the morphisms are the corresponding Boolean algebraic homomorphisms.

Definition 4.1.2. A *quantum categorical event structure* is a locally small co-complete category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras. The objects of $\mathcal{L}$ are quantum event algebras and the morphisms are quantum algebraic homomorphisms.

A quantum event algebra L in $\mathcal{L}$ is defined as an *orthomodular σ-orthoposet* (Chiara et al. [2004]), that is, as a partially ordered set of quantum events, endowed with a maximal element 1 and with an operation of orthocomplementation $[-]^* : L \to L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \perp \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L$, $l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \perp \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually. The σ-completeness condition, meaning that the join of countable families of pairwise orthogonal

[5] For a systematic introduction to category-theoretic sheaf theory, the reader may consult, for instance, Mac Lane and Moerdijk [1992], Borceux [1994/2008], and Johnstone [2002, Volume 2].

events exists, is required in order to have a well defined theory of quantum observables over L.

Definition 4.1.3. A quantum algebraic homomorphism is a morphism $H : K \to L$, which satisfies, for all $k \in K$, the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$, [e] $H(\bigvee_n k_n) = \bigvee_n H(k_n)$, where $k_1, k_2, \ldots$ countable family of mutually orthogonal events.

It is important to note that any arbitrary pair of events l and $\acute{l}$, belonging to a quantum event algebra L in $\mathcal{L}$ are compatible, if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely, if it is a Boolean sublattice of L. In perspectivist terms, and in view of Section 3.3, this indicates that a Boolean event algebra is *structurally adaptable* to a quantum event algebra, since it encodes a structurally invariant context of co-measurable observables by means of their joint compatible spectral resolution. Furthermore, as pointed out in Section 2, the global orthomodular structure of a quantum event algebra is rendered empirically inert without the adjunction of such local spectral invariants to it. The role of these invariants is to induce partial or local structural congruences with Boolean event structures pertaining to all typical contexts of measurement. The multiplicity of applicable local Boolean frames allows the filtration and separation of several resolution sizes and types of quantum observable grain depending on the qualification of the corresponding spectral projection operators.

Consequently, the objective is to derive the non-directly accessible quantum kind of event structure in terms of all possible partial structural congruences with the directly accessible Boolean kind of event structure, via the literal adjunction of local spectral invariants as probing frames to the former. In this setting, the major role is subsumed by all possible structural relations allowed among the probing Boolean frames, the spectra of which may be disjoint or nested or overlapping and interlocking together nontrivially. The necessity of providing a solution to the posed problem in functorial terms, i.e., non-dependent on the artificial choice of particular Boolean frames adjoined to a quantum event algebra, requires a category-theoretic interpretative framework based on the aforementioned notion of partial structural congruence between the distinct kinds of Boolean and quantum categorical event structures.

It is natural, therefore, to consider a Boolean categorical event structure $\mathcal{B}$ as a *category of probes* for the quantum categorical event structure $\mathcal{L}$. The structural adaptability of the category of Boolean event algebras $\mathcal{B}$ to the category of quantum event algebras $\mathcal{L}$ gives rise to the Boolean probing or shaping functor of $\mathcal{L}$ by $\mathcal{B}$.

Definition 4.1.4. A *Boolean probing or shaping functor* of a quantum categorical event structure $\mathcal{L}$, $\mathbb{M} : \mathcal{B} \to \mathcal{L}$, assigns to each Boolean event algebra in $\mathcal{B}$ the underlying quantum event algebra from $\mathcal{L}$, and to each Boolean homomorphism the underlying quantum algebraic homomorphism.

The shaping functor $\mathbb{M} : \mathcal{B} \to \mathcal{L}$ is technically a forgetful functor. Let us note, in this respect, that any Boolean event algebra may be thought of as a quantum event algebra of commuting projection operators. Thus, the functor $\mathbb{M}$ qualifies functorially the requirement of structural adaptability of the category $\mathcal{B}$ of Boolean probes to the category of quantum event algebras $\mathcal{L}$. Moreover, a quantum algebra of events forms a weaker structure than a Boolean event algebra. In the former case, for instance, only an orthocomplementation operation is properly defined, whereas, in the latter case a complementation operation is in order, giving rise to the double negation property. Obviously, a complementation operation is also trivially an orthocomplementation operation as well. In this sense, for every Boolean event algebra there exists by default an underlying quantum event algebra, which views complementation as merely orthocomplementation. Hence, the shaping functor $\mathbb{M} : \mathcal{B} \to \mathcal{L}$ acts as a forgetful functor, not taking into account the extra Boolean structure of $\mathcal{B}$.

Corollary 4.1.1. *The quantum algebraic homomorphism, $\psi_B : \mathbb{M}(B) \to L$, constitutes a Boolean frame of a quantum event algebra L, or, equivalently, a Boolean perspective on L whose source is the Boolean probe B.*

The foregoing corollary is a consequence of the existence of the shaping functor $\mathbb{M}$ that renders the category of Boolean event algebras structurally adaptable to the category of quantum event algebras, and the consideration of essential features of the quantum mechanical formalism, for instance, the factor that a normalized unit vector in Hilbert space, representing the physical state of a quantum system, is completely analyzed and thus specified by the determination of a basis on the state Hilbert space of the system. We note that the consideration of a basis on a system's Hilbert space — be it an orthonormal basis of eigenvectors of a selected observable to be measured, or, more generally, an orthonormal basis of common eigenvectors of a set of mutually compatible observables — always gives rise to a Boolean event algebra B, viewed as a special quantum event algebra $\mathbb{M}(B)$ within the quantum categorical event structure $\mathcal{L}$. The special structures of the type $\mathbb{M}(B)$ function as variable local Boolean frames of a quantum event algebra L in $\mathcal{L}$, under their intended physical interpretation as probing frames or perspectives on L for the manifestation and subsequent classification (or contextualization) of quantum events.

Proposition 4.1.1. *The Boolean probing or shaping functor of a quantum categorical event structure $\mathcal{L}$, $\mathbb{M} : \mathcal{B} \to \mathcal{L}$, is not invertible, i.e., there is no opposite-directing functor from $\mathcal{L}$ to $\mathcal{B}$.*

The proof of the above proposition follows immediately from Kochen-Specker's theorem, analyzed in Section 2, and, according to which, there does not exist a global Boolean two-valued truth-functional assignment pertaining to a quantum event algebra.

Because of the fact that an opposite-directing functor from $\mathcal{L}$ to $\mathcal{B}$ is not feasible, since a quantum event algebra cannot be realized within any Boolean event algebra, we seek for an extension of $\mathcal{B}$ into a larger categorical environment where such a realization becomes possible. This extension is expected to conform with the intended perspectivist semantics of adjoining a multiplicity of Boolean probing frames to a quantum event algebra, understood equivalently as Boolean perspectives on the latter. For this reason, it is necessary to extend the categorical level of $\mathcal{B}$ to the categorical level of diagrams in $\mathcal{B}$, such that the global information encoded in a quantum event algebra may be recovered in a structure preserving way by an appropriate sheaf-theoretic construction gluing together categorical diagrams of locally variable Boolean frames.[6] In view of Section 3.2, this is accomplished by means of the categorical technique of Yoneda's embedding $\curvearrowright : \mathcal{B} \to \mathcal{S}et^{\mathcal{B}^{op}}$, which is a full and faithful functor.

[6] In general, a diagram $\mathbb{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$ is defined as an indexed family of objects $\{X_i\}_{i\in I}$ and a family of morphisms sets $\{F_{ij}\}_{i,j\in I} \subseteq Hom_{\mathcal{C}}(X_i, X_j)$.

4.2 Functor of Boolean frames and pasting maps

It is apparent, therefore, that the realization of this extension process requires the construction of the functor category, $\mathcal{S}et^{\mathcal{B}^{op}}$, called the *category of presheaves of sets on Boolean event algebras*, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$.

Definition 4.2.1. The functor category $\mathcal{S}et^{\mathcal{B}^{op}}$ has objects all contravariant functors $\mathbb{P} : \mathcal{B}^{op} \to \mathcal{S}et$ and morphisms all natural transformations between such functors.

Each object $\mathbb{P}$ in the functor category $\mathcal{S}et^{\mathcal{B}^{op}}$ is a contravariant set-valued functor on $\mathcal{B}$, called a *presheaf of sets* on $\mathcal{B}$ (Borceux [1994/2008], p. 195). In order to obtain a clear understanding of the structure of the functor category $\mathcal{S}et^{\mathcal{B}^{op}}$, it is useful to think of a presheaf of sets $\mathbb{P}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$ as a right action of the category $\mathcal{B}$ on a set of events, which is partitioned into a variety of Boolean spectral kinds parameterized by the Boolean event algebras B in $\mathcal{B}$.[7] Such an action $\mathbb{P}$ is equivalent to the specification of a *diagram* in $\mathcal{B}$, to be thought of as a $\mathcal{B}$-variable set forming a presheaf $\mathbb{P}(B)$ on $\mathcal{B}$.

For each Boolean algebra B of $\mathcal{B}$, $\mathbb{P}(\mathrm{B})$ is a set, and for each Boolean homomorphism $f : C \to B$, $\mathbb{P}(f) : \mathbb{P}(B) \to \mathbb{P}(\mathrm{C})$ is a set-theoretic function, such that, if $p \in \mathbb{P}(\mathrm{B})$, the value $\mathbb{P}(f)(p)$ for an arrow $f : C \to B$ in $\mathcal{B}$ is called the *restriction* of p along f and is denoted by $\mathbb{P}(f)(p) = p \cdot f$. From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbb{P}$ on $\mathcal{B}$, in the environment of the functor category $\mathcal{S}et^{\mathcal{B}^{op}}$, is to identify an element of $\mathbb{P}(\mathrm{B})$, that is, $p \in \mathbb{P}(\mathrm{B})$, with an event observed by means of a physical procedure over a Boolean domain cover for a quantum event algebra. As demonstrated in Proposition 4.2.2, this identification forces the interrelation of observed events, over all Boolean probing frames of the base category $\mathcal{B}$, to fulfil the requirements of a uniform and homologous fibred structure.

Definition 4.2.2. The *Boolean realization functor of a quantum categorical event structure* $\mathcal{L}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$, namely, the functor of generalized elements of $\mathcal{L}$ in the environment of the category of presheaves on Boolean event algebras, is defined as

$$\mathbb{R} : \mathcal{L} \to \mathcal{S}et^{\mathcal{B}^{op}}, \tag{4.1}$$

where the action on a Boolean algebra B in $\mathcal{B}$ is given by

$$\mathbb{R}(L)(B) = Hom_{\mathcal{L}}(\mathbb{M}(B), L). \tag{4.2}$$

The presheaf functor $\mathbb{R}(L)(-) = Hom_{\mathcal{L}}(\mathbb{M}(-), L)$ constitutes the image of $\mathbb{R}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$ and is called the *functor of Boolean frames* or *functor of Boolean perspectives* on a quantum

[7] It is instructive to notice that the reasoning deployed at this point is of a general nature and originates from the analogy between groups and categories, since a group may be considered as a category with a unique object, corresponding to the unity of the group. In an analogous manner that we consider the right action of a group $\mathcal{G}$ on a set, to get the notion of a $\mathcal{G}$-set, we may consider the right action of a category $\mathcal{C}$ on a set, to get the notion of a variable $\mathcal{C}$-set, i.e., a set which is variable with respect to the objects of $\mathcal{C}$. All variable sets over the objects of $\mathcal{C}$ are not disjoint, but they are interrelated under the operation of pulling-back, or equivalently, under the operation of presheaf restriction, to obtain a (contravariant) functor from $\mathcal{C}$ to the category of sets. This is equivalent to a presheaf of sets on $\mathcal{C}$ and thus clearly every presheaf gives rise to such a right action. Accordingly, all these variable sets together with their pull-back relations form a diagram in $\mathcal{C}$, appearing as an object of the category $\mathcal{S}et^{\mathcal{C}^{op}}$ of presheaves of sets on $\mathcal{C}$, which is comprehended as a generalization of the notion of a $\mathcal{G}$-set.

event algebra L in $\mathcal{L}$. Since the physical interpretation of the presheaf functor $\mathbb{R}(L)(-)$ refers to the functorial realization of a quantum event algebra L in $\mathcal{L}$ in terms of structured multitudes of local Boolean frames adjoined to it, intuitively, it is natural to think of $\mathbb{R}(L)(-)$ as comprising the *network of relationships* that L has with all admissible Boolean frames-perspectives on L.

Proposition 4.2.1. *The Boolean frames-perspectives on a quantum event algebra L,*

$$\psi_B : \mathbb{M}(B) \to L, \tag{4.3}$$

being instantiated by the evaluation of the functor $\mathbb{R}(L)(-)$ at each B in $\mathcal{B}$, are interrelated by the operation of presheaf restriction.

The functor of Boolean frames $\mathbb{R}(L)$ of a quantum event algebra L in $\mathcal{L}$ is actually an object in the category of presheaves $\mathcal{S}et^{\mathcal{B}^{op}}$, representing L in the environment of the topos of presheaves over the base category of probes $\mathcal{B}$. Thus, for a fixed quantum event algebra L, the evaluation of $\mathbb{R}(L)(-)$, at each Boolean event algebra B in $\mathcal{B}$, instantiates locally a Boolean probing frame of L, or equivalently, a (B-shaped) Boolean perspective on L, denoted by $\psi_B : \mathbb{M}(B) \to L$. Importantly, since $\mathbb{R}(L)(-)$ is a presheaf functor in $\mathcal{S}et^{\mathcal{B}^{op}}$, all resultant structured multitudes of local Boolean frames, adjoined to L, are not ad hoc, but they are interrelated by the functorial operation of presheaf restriction. In relation to the latter, it is sufficient to observe that for each Boolean homomorphism $f : C \to B$, $\mathbb{R}(L)(f) : \mathbb{R}(L)(B) \to \mathbb{R}(L)(C)$ is a function between sets of Boolean frames of L in the opposite direction, such that, if $\psi_B \in \mathbb{R}(L)(B)$ is a Boolean frame of L, the value of $\mathbb{R}(L)(f)(\psi_B)$, or equivalently, the corresponding Boolean frame $\psi_C : \mathbb{M}(C) \to L$ is given by the restriction or pullback of ψ_B along f, denoted by $\mathbb{R}(L)(f)(\psi_B) = \psi_B \cdot f = \psi_C$.

Corollary 4.2.1. *For a fixed quantum event algebra L in $\mathcal{L}$, the set of all pairs (B, ψ_B), where B is a Boolean event algebra and $\psi_B : \mathbb{M}(B) \to L$ a Boolean probing frame of L defined over B, has the structure of a category.*

On the basis of Definition 4.2.2, the functor of Boolean frames $\mathbb{R}(L)$ of a quantum event algebra L in $\mathcal{L}$ forms a presheaf of sets on Boolean event algebras B in $\mathcal{B}$. Thus, we can legitimately consider the *category of elements* corresponding to the functor $\mathbb{R}(L)$, denoted by $\int(\mathbb{R}(L), \mathcal{B})$, and specified as follows: it has objects all pairs (B, ψ_B) and morphisms $(\acute{B}, \psi_{\acute{B}}) \to (B, \psi_B)$ are those Boolean homomorphisms $u : \acute{B} \to B$ of category $\mathcal{B}$ for which $\psi_B \cdot u = \psi_{\acute{B}}$, that is, the restriction or pullback of the Boolean frame ψ_B along u is $\psi_{\acute{B}}$. This category is naturally called the *category of Boolean frames* of L, or, equivalently, the *category of Boolean perspectives* on L.

Proposition 4.2.2. *For a fixed quantum event algebra L in $\mathcal{L}$, the category of Boolean frames of L induces a split, discrete and uniform fibration of L over its Boolean probes, where $\mathcal{B}$ is the base category of the fibration.*

By projecting on the second coordinate of the category of Boolean frames of L, $\int(\mathbb{R}(L), \mathcal{B})$, we obtain a functor, $\int_{\mathbb{R}(L)} : \int(\mathbb{R}(L), \mathcal{B}) \to \mathcal{B}$, as in the diagram below:

$$\begin{array}{ccc} \int(\mathbb{R}(L),\mathcal{B}) & & \\ {\scriptstyle \int_{\mathbb{R}(L)}}\Big\downarrow & & \\ \mathcal{B} & \xrightarrow{\mathbb{R}(L)} & \mathcal{S}et \end{array}$$

We first note that, in the case under study, the fibers are categories in which the only arrows are identity arrows and, thus, the fibration induced by the presheaf functor $\mathbb{R}(L)(-)$ of Boolean frames of L is discrete. If B is a Boolean probe in $\mathcal{B}$, the inverse image under $\int(\mathbb{R}(L),\mathcal{B})$ of B is simply the set of Boolean frames of L, i.e. $\mathbb{R}(L)(B)$, although its elements are written as pairs (B,ψ_B) so as to form a disjoint union. The emergence of a measurement event $q \in \mathbb{M}(B)$ with respect to the Boolean frame $\psi_B : \mathbb{M}(B) \to L$ amounts to the choice of a projection $q \in B$. In this sense, the Boolean frame ψ_B becomes a pointed one. Therefore, choice of projections effected by measurement procedures with respect to Boolean frames of L make the fibration split. Finally, the fibration is uniform over the base category $\mathcal{B}$ because for any two measurement events over the same Boolean event algebra, the structure of all Boolean frames that relate to the first event cannot be distinguished in any possible way from the structure of Boolean frames relating to the second. Henceforth, all possible events with respect to any particular Boolean frame are uniformly equivalent to each other. Accordingly, the fibration $\int_{\mathbb{R}(L)} : \int(\mathbb{R}(L),\mathcal{B}) \to \mathcal{B}$ amounts to a *partitioning* of a quantum event algebra L into *partially congruent Boolean perspectives* parameterized by the Boolean probes of the base category $\mathcal{B}$ of the fibration.

Consequently, the explicit representation of a quantum event algebra L in $\mathcal{L}$ in terms of coherently interconnected families of Boolean probing frames, capable of carrying all the information encoded in the former, requires the formulation of a covering scheme of L induced by these local Boolean frames. For this purpose, we initially consider restricted families of Boolean frames in $\mathbb{R}(L)$, distinguished qualitatively by their function as local Boolean covers of L. The requirements qualifying such restricted families of Boolean frames as local Boolean covers of L are the following: First, they should constitute a *minimal generating class* of Boolean frames, instantiating a subfunctor $\mathbb{T}$ of the functor of Boolean frames $\mathbb{R}(L)$ of L. Second, they should *jointly* form an *epimorphic family* covering L entirely on their overlaps. Third, they should be *compatible* under refinement operations or, more generally, pullback conditions in L. Fourth, they should be *transitive*, such that, subcovers of covers of L can be qualified as covers themselves. We proceed to the formulation of the preceding conditions in functorial terms and thus independently of arbitrary choices of Boolean frames, as follows.

Definition 4.2.3. A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a subfunctor $\mathbb{T}$ of the functor of Boolean frames $\mathbb{R}(L)$ of L, i.e., $\mathbb{T} \hookrightarrow \mathbb{R}(L)$.

Corollary 4.2.2. *For each probe B in $\mathcal{B}$, the set of Boolean frames $\psi_B : \mathbb{M}(B) \to L$ in $\mathbb{T}(B) \subseteq [\mathbb{R}(L)](B)$ are called Boolean covers of L.*

Proposition 4.2.3. *A functor $\mathbb{T} \hookrightarrow \mathbb{R}(L)$ of Boolean coverings for a quantum event algebra L can be expressed in the form of a right ideal $\mathbb{T} \triangleright \mathbb{R}(L)$ consisting of Boolean covers $\psi_B : \mathbb{M}(B) \to L$ of L.*

This means that Boolean covers of L are characterized by the following property, fitting them into right ideals: If $[\psi_B : \mathbb{M}(B) \rightarrow L] \in \mathbb{T}(B)$ and $\mathbb{M}(v) : \mathbb{M}(\acute{B}) \rightarrow \mathbb{M}(B)$ is a quantum homomorphism in $\mathcal{L}$ via $\mathbb{M} : \mathcal{B} \rightarrow \mathcal{L}$, for $v : \acute{B} \rightarrow B$ in $\mathcal{B}$, then $[\psi_B \circ \mathbb{M}(v) : \mathbb{M}(\acute{B}) \rightarrow \mathcal{L}] \in \mathbb{T}(B)$. Consequently, if ψ_B is qualified as a Boolean cover of L, then the composite $\psi_B \circ \mathbb{M}(v)$ is also qualified as a Boolean cover of L.

Definition 4.2.4. A functor $\mathbb{T} \hookrightarrow \mathbb{R}(L)$ of Boolean coverings for a quantum event algebra L is equivalently called a *spectral sieve* of L.

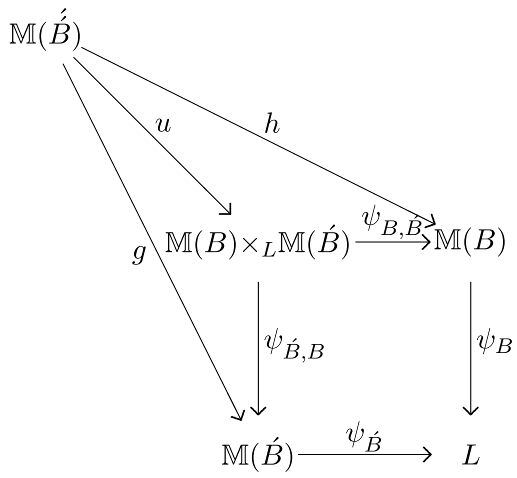
A spectral sieve adjoined to a quantum event algebra L can be intuitively conceived of as consisting of 'filtering holes' $\mathbb{M}(B)$, specified structurally by variable Boolean probing frames targeting L, permitting separation of several resolution sizes of observable grain, as well as their compatibility relations. For instance, if $\psi_B : \mathbb{M}(B) \rightarrow L$ denotes a Boolean frame of L belonging to a sieve $\mathbb{T}(L)$ adjoined to L, then any other Boolean frame of L whose spectral resolving power is coarser than ψ_B belongs to this sieve as well. Thus, spectral sieves adjoined to L encapsulate the process of multilayered sorting of the informational content of L via the 'filtering holes' $\mathbb{M}(B)$, for each probe B in $\mathcal{B}$. Consequently, spectral sieves of this form act primarily as the carriers of networks of internal relations among probing Boolean frames with respect to which the targeted quantum event structure is expected to be consistently and completely covered.

Proposition 4.2.4. *A family of Boolean covers $\psi_B : \mathbb{M}(B) \longrightarrow L$, B in $\mathcal{B}$, is the generator of a spectral sieve of Boolean coverings $\mathbb{T}$, if and only if, this sieve is the smallest among all containing that family.*

The spectral sieves of Boolean coverings for a quantum event algebra L in $\mathcal{L}$ constitute a partially ordered set under inclusion. The inclusion operation refers to subfunctors of the functor of Boolean frames $\mathbb{R}(L)$ of L. The minimal sieve is the empty one, $\mathbb{T}(B) = \emptyset$ for all B in $\mathcal{B}$, whereas the maximal sieve is the functor of Boolean frames $\mathbb{R}(L)$ of L itself.

The ordering relation between any two equivalence classes of Boolean frames in the set $[\mathbb{R}(L)](B)$, for variable B in $\mathcal{B}$, requires the existence of pullback compatibility conditions between the corresponding Boolean frames. Thus, if we consider a functor of Boolean coverings $\mathbb{T}(L)$ for a quantum event algebra L, we require that the generating family of Boolean covers they belong to is compatible under pullbacks.

Definition 4.2.5. The *pullback or categorical overlap* of any pair of Boolean covers $\psi_B : \mathbb{M}(B) \rightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbb{M}(\acute{B}) \rightarrow L$, $\acute{B}$ in $\mathcal{B}$, with common codomain a quantum event algebra L, consists of the common refinement $\mathbb{M}(B) \times_L \mathbb{M}(\acute{B})$ together with the two arrows $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, called projections, as shown in the diagram:



Proposition 4.2.5. *If the Boolean covers ψ_B and $\psi_{\acute{B}}$ of L are injective, then their pullback is given by their intersection.*

Note that the square in the preceding diagram *commutes* and for any Boolean domain object $\mathbb{M}(\acute{\acute{B}})$ or event algebra $\acute{\acute{B}}$ in $\mathcal{B}$ and arrows h and g that make the outer square commute, there is a unique $u : \mathbb{M}(\acute{\acute{B}}) \longrightarrow \mathbb{M}(B)\times_L\mathbb{M}(\acute{B})$ that makes the whole diagram commutative. Hence, we obtain the compatibility condition: $\psi_{\acute{B}} \circ g = \psi_B \circ h$. If, therefore, ψ_B and $\psi_{\acute{B}}$ are injective morphisms, then their pullback is isomorphic with the intersection $\mathbb{M}(B) \cap \mathbb{M}(\acute{B})$. Accordingly, we can define the *gluing or pasting map* between Boolean probing frames on their overlap, which is an isomorphism.

Definition 4.2.6. The *pairwise gluing isomorphism* of the Boolean covers ψ_B and $\psi_{\acute{B}}$ of L is defined as follows:

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(\mathbb{M}(B)\times_L\mathbb{M}(\acute{B})) \longrightarrow \psi_{B\acute{B}}(\mathbb{M}(B)\times_L\mathbb{M}(\acute{B})), \tag{4.4}$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ {\psi_{\acute{B}B}}^{-1}. \tag{4.5}$$

Proposition 4.2.6. *The Boolean coordinatizing maps $\psi_{\acute{B}B} : (\mathbb{M}(B)\times_L\mathbb{M}(\acute{B})) \to L$ and $\psi_{B\acute{B}} : (\mathbb{M}(B)\times_L\mathbb{M}(\acute{B})) \to L$ cover L in a compatible way on their intersection.*

An immediate consequence of Definition 4.2.6 is the satisfaction of the following *cocycle conditions*:

$$\Omega_{B,B} = 1_B \qquad 1_B : \text{identity of } B \tag{4.6}$$

$$\Omega_{B,\acute{B}} \circ \Omega_{\acute{B},\acute{\acute{B}}} = \Omega_{B,\acute{\acute{B}}} \qquad \text{if } \mathbb{M}(B) \cap \mathbb{M}(\acute{B}) \cap \mathbb{M}(\acute{\acute{B}}) \neq 0 \tag{4.7}$$

$$\Omega_{B,\acute{B}} = \Omega^{-1}{}_{\acute{B},B} \qquad \text{if } \mathbb{M}(B) \cap \mathbb{M}(\acute{B}) \neq 0. \tag{4.8}$$

Thus, the pairwise gluing isomorphism $\Omega_{B,\acute{B}}$ between any two injective Boolean covers in a spectral sieve $\mathbb{T}(L)$ of L assures that $\psi_{\acute{B}B} : (\mathbb{M}(B)\times_L\mathbb{M}(\acute{B}))$ and $\psi_{B\acute{B}} : (\mathbb{M}(B)\times_L\mathbb{M}(\acute{B}))$ probe L on their common refinement in a compatible way.

This concludes the covering scheme of a quantum event algebra L in $\mathcal{L}$ with respect to a spectral sieve $\mathbb{T}(L)$, provided that the family of all Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, for variable B in $\mathcal{B}$, generating this spectral sieve, jointly form an epimorphic family covering L completely,

$$T_L : \sum_{(B_j,\, \psi_j : \mathbf{M}(B_j) \to L)} \mathbf{M}(B_j) \twoheadrightarrow L, \tag{4.9}$$

where T_L is an epimorphism in $\mathcal{L}$ with codomain a quantum event algebra L.

4.3 The Boolean frames – Quantum adjunction

We constructed in Section 4.2 the Boolean realization functor of a quantum categorical event structure $\mathcal{L}$, $\mathbb{R} : \mathcal{L} \to \mathcal{S}et^{\mathcal{B}^{op}}$, realized for each L in $\mathcal{L}$ by its corresponding presheaf functor of Boolean frames, $\mathbb{R}(L)(B) = Hom_{\mathcal{L}}(\mathbb{M}(B), L)$, where the shaping functor $\mathbb{M} : \mathcal{B} \to \mathcal{L}$ fulfills the requirement of structural adaptability of the category of Boolean event algebras to the category of quantum event algebras, functioning as a category of probes to the latter. The semantics of the functor $\mathbb{R}(L)$ and, concomitantly, of any of its subfunctors $\mathbb{T}(L) \triangleright \mathbb{R}(L)$ amounts to the notion of a spectral sieve—comprised by an appropriately interconnected family of partially compatible, local Boolean frames—adjoined to L through processes of measurement of quantum observables.[8] The qualification of a functor $\mathbb{T}(L) \hookrightarrow \mathbb{R}(L)$ as a spectral sieve adjoined to L, according to Definition 4.2.4, implies that these processes effectuate a multilayered functorial classification of the informational content of L via the 'filtering holes' $\mathbb{M}(B)$, for each probe B in $\mathcal{B}$. In turn, this amounts to the capacity of separating and sorting compatibly all possible sizes of grain pertaining to quantum observable behaviour, solely in the perspectivist terms of such a spectral sieve. Thus, the physical significance of the adjunction of a spectral sieve on L is that it induces partial or local structural congruence relations between the Boolean and quantum levels of event structure in functorial terms, i.e., without the invocation of any *ad hoc* choices.

Importantly, this bi-directional dependence between the local Boolean and the global quantum structural level admits a rigorous categorical formulation in terms of a pair of adjoint functors, thus giving rise to a *categorical adjunction*. It is precisely this attribution that characterizes uniquely and differentiates conceptually and technically the proposed categorical approach in comparison to other ones, for instance, Isham and Butterfield [1998], Döring and Isham [2011], Abramsky and Brandenburger [2011], Heunen et al. [2011]. In a nutshell, as shown in Section 3.3, although a Boolean context may be intuitively conceived of as opening up a window on L, this is not sufficient for its unconditional qualification as a perspective on L. Equivalently, despite the heuristic value of this intuition, the perspectival qualification of such a context in its function as a local Boolean logical frame of L, with respect to which results of measurement of quantum observables are being consistently coordinatized, can be only established as part and parcel of a spectral sieve of L, i.e., only in functorial terms. The reason is that the notion of local or partial congruence between the Boolean and quantum levels of event structure pertains to a whole spectral sieve adjoined to L, and definitely not to ad hoc selected Boolean subalgebras of a quantum event algebra.

Henceforth, in the proposed categorical setting, the problem of establishing a perspectival representation of a quantum event algebra L via the appropriate adjunction of Boolean frames is solved precisely by functorially inverting the Boolean realization functor $\mathbb{R}$ of L, if such an inversion actually exists. This amounts to constructing explicitly the

[8] In his essay "The Theory of Groups", Arthur Eddington ([1956/2003], p. 1566) in a little known excerpt makes the following characteristic assertion: "In Einstein's theory of relativity the observer is a man who sets out in quest of truth armed with a measuring rod. In quantum theory he sets out armed with a sieve."

opposite-directing, left adjoint functor $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \to \mathcal{L}$, to the Boolean realization functor $\mathbb{R} : \mathcal{L} \to \mathcal{S}et^{\mathcal{B}^{op}}$. If, therefore, for a fixed quantum event algebra L in $\mathcal{L}$, the right adjoint functor $\mathbb{R}$ partitions or decomposes L in an orderly manner via the action of Boolean probing frames $\psi_B : \mathbb{M}(B) \to L$, comprising for variable B in $\mathcal{B}$ spectral sieves on L, and thus functioning as suitable perspectives or contexts for measurement of observables, then, inversely, the left adjoint functor $\mathbb{L}$ provides a *perspectival synthesis* of a quantum event algebra, in a structure preserving manner, by gluing compatibly together structured families or diagrams of variable local Boolean frames.

Thus, the existence of the functor $\mathbb{L}$, being the left adjoint to $\mathbb{R}$, gives rise to a *categorical adjunction* that has been recently proved to exist between the category of quantum event algebras $\mathcal{L}$ and the category of presheaves $\mathcal{S}et^{\mathcal{B}^{op}}$ on Boolean event algebras (Zafiris [2006]; see also Zafiris and Karakostas [2013] for a more detailed treatment including in addition the involved logical aspects). Since the proposed perspectivist interpretation of a quantum event structure is based on this pair of adjoint functors, it is useful to express their established bi-directional correspondence in the form of the following theorem.

Theorem 4.3.1. *There exists a categorical adjunction between the categories $\mathcal{S}et^{\mathcal{B}^{op}}$ and $\mathcal{L}$, called the Boolean frames–quantum adjunction, established by the pair of adjoint functors $\mathbb{L}$ and $\mathbb{R}$, as follows*

$$\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{L} : \mathbb{R} \tag{4.10}$$

where, the right adjoint,

$$\mathbb{R} : \mathcal{L} \to \mathcal{S}et^{\mathcal{B}^{op}}, \tag{4.11}$$

is the Boolean realization functor of a quantum categorical event structure $\mathcal{L}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$, whereas, the left adjoint,

$$\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \to \mathcal{L}, \tag{4.12}$$

is the colimit-preserving functor providing the perspectival synthesis of a quantum categorical event structure by means of diagrams of Boolean frames.

Equivalently, there exists a bijection, which is natural in both $\mathbb{P}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$ and L in $\mathcal{L}$,

$$Hom_{\mathcal{S}et^{\mathcal{B}^{op}}}(\mathbb{P}, \mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{L}\mathbb{P}, L), \tag{4.13}$$

abbreviated as

$$Nat(\mathbb{P}, \mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{L}\mathbb{P}, L). \tag{4.14}$$

In order to obtain some insight into the functioning of the above adjunction, it is required to highlight the basic ideas of the proof. To this effect, we consider a natural transformation τ between the presheaves $\mathbb{P}$ and $\mathbb{R}(L)$ on the category of probes $\mathcal{B}$ in relation to a quantum event algebra L in $\mathcal{L}$, i.e., $\tau : \mathbb{P} \to \mathbb{R}(L)$. This amounts to a family of maps τ_B, indexed by Boolean algebras B in $\mathcal{B}$, for which each τ_B is a map of sets,

$$\tau_B : \mathbb{P}(B) \to Hom_{\mathcal{L}}(\mathbb{M}(B), L), \tag{4.15}$$

such that the diagram below commutes for each Boolean homomorphism $u : \acute{B} \to B$ in $\mathcal{B}$:

$$\begin{array}{ccc} \mathbb{P}(B) & \xrightarrow{\tau_B} & Hom_{\mathcal{L}}(\mathbb{M}(B), L) \\ \Big\downarrow {\scriptstyle \mathbb{P}(u)} & & \Big\downarrow {\scriptstyle \mathbb{M}(u)^*} \\ \mathbb{P}(\acute{B}) & \xrightarrow{\tau_{\acute{B}}} & Hom_{\mathcal{L}}(\mathbb{M}(\acute{B}), L) \end{array}$$

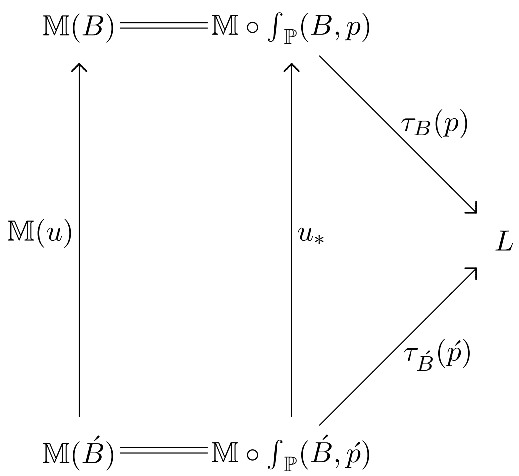
By utilizing the category of elements of the presheaf $\mathbb{P}$, the map τ_B, defined by relation (4.15), is identical with the map:

$$\tau_B : (B, p) \rightarrow Hom_{\mathcal{L}}(\mathbb{M} \circ \int_{\mathbb{P}} (B, p), L). \tag{4.16}$$

Therefore, the natural transformation $\tau : \mathbb{P} \to \mathbb{R}(L)$ can be equivalently represented as a family of arrows of $\mathcal{L}$ targeting L, which is being indexed by objects (B, p) of the category of elements of the presheaf $\mathbb{P}$, namely

$$\{\tau_B(p) : \mathbb{M}(B) \to L\}_{(B,p)}. \tag{4.17}$$

Thus, the condition of the commutativity of the preceding diagram is translated into the condition that for each arrow u the following diagram commutes:



Consequently, according to the above diagram, the arrows $\tau_B(p)$ form a cocone from the functor $\mathbb{M} \circ \int_{\mathbb{P}}$ to L. Furthermore, by taking into account the categorical definition of the colimit, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit $\mathbb{LP}$ to L. Equivalently, we deduce that there exists a bijection, which is natural in $\mathbb{P}$ and L,

$$Hom_{\mathcal{S}et^{\mathcal{B}^{op}}}(\mathbb{P}, Hom_{\mathcal{L}}(\mathbb{M}(-), L)) \cong Hom_{\mathcal{L}}(\mathbb{LP}, L), \tag{4.18}$$

$$Hom_{\mathcal{S}et^{\mathcal{B}^{op}}}(\mathbb{P}, \mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{LP}, L), \tag{4.19}$$

abbreviated as follows,

$$Nat(\mathbb{P}, \mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{LP}, L), \tag{4.20}$$

thus capturing the content of Theorem 4.3.1.

As a consequence, we obtain the following corollary:

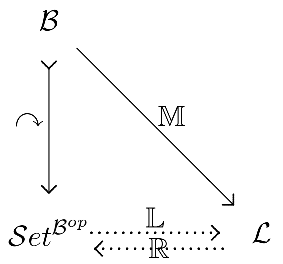
Corollary 4.3.1. *The left adjoint functor of the Boolean frames–quantum adjunction,* $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$*, is defined for each presheaf* $\mathbb{P}$ *in* $\mathcal{S}et^{\mathcal{B}^{op}}$ *as the colimit* $\mathbb{L}(\mathbb{P})$*, taken in the category of elements of* $\mathbb{P}$*:*

$$\mathbb{L}(\mathbb{P}) = Colim\{\int(\mathbb{P}, \mathcal{B}) \xrightarrow{\int_{\mathbb{P}}} \mathcal{B} \xrightarrow{\mathbb{M}} \mathcal{L}\} \ . \tag{4.21}$$

Hence, the Boolean realization functor of a quantum categorical event structure $\mathcal{L}$, realized for each L in $\mathcal{L}$ by the presheaf of Boolean probing frames or perspectives $\mathbb{R}(L) = Hom_{\mathcal{L}}(\mathbb{M}(-), L)$ in $\mathcal{S}et^{\mathcal{B}^{op}}$, has a left adjoint functor $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$, which is defined for each presheaf $\mathbb{P}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$ as the colimit $\mathbb{L}(\mathbb{P})$, providing the perspectival synthesis of a quantum categorical event structure by means of diagrams of Boolean frames.

The fundamental functioning of the Boolean frames–quantum adjunction, specified by the pair of adjoint functors $\mathbb{L} \dashv \mathbb{R}$, is made transparent if we consider that it provides a bi-directional mechanism of encoding and decoding information between diagrams of Boolean event algebras B and quantum event algebras L via the action of Boolean probing frames or perspectives $\psi_B : \mathbb{M}(B) \rightarrow L$. Thus, if we think of $\mathcal{S}et^{\mathcal{B}^{op}}$ as the categorical universe of variable local Boolean frames modeled in $\mathcal{S}et$, and of $\mathcal{L}$ as the categorical universe of quantum event structures, then the left adjoint functor $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$ signifies a translational code of information from the level of local Boolean algebras to the level of global quantum event algebras, whereas the Boolean realization functor $\mathbb{R} : \mathcal{L} \rightarrow \mathcal{S}et^{\mathcal{B}^{op}}$ signifies a translational code in the inverse direction. In general, the content of the information cannot remain completely invariant with respect to translating from one categorical universe to another, and conversely. However, as suggested by Theorem 4.3.1, there remain two alternatives for a variable set $\mathbb{P}$ over local Boolean frames, standing for a presheaf functor $\mathbb{P}$ in $\mathcal{S}et^{\mathcal{B}^{op}}$, to exchange information with a quantum algebra L. Either the content of information is transferred in quantum terms with the colimit in the category of elements of $\mathbb{P}$ translating, represented as the quantum morphism $\mathbb{L}\mathbb{P} \rightarrow L$, or the content of information is transferred in Boolean terms with the functor of Boolean frames of L translating, represented correspondingly as the natural transformation $\mathbb{P} \rightarrow \mathbb{R}(L)$. In the first case, from the setting of L, information is being received in quantum terms, while in the second, from the setting of $\mathbb{P}$, information is being sent in Boolean terms. Then, the natural bijection of Equation (4.14) corresponds to the assertion that these two distinct ways of information transfer are equivalent. Thus, the fact that these two functors are adjoint underlines an amphidromous dependent variation, safeguarding that the global information encoded in a quantum kind of event structure is retrievable in a structure-preserving manner by all possible partial structural congruences with the Boolean kind of event structure.

Importantly, by virtue of the existence of the Boolean frames–quantum adjunction, $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{L} : \mathbb{R}$, every probing relation from a Boolean event algebra B in $\mathcal{B}$ to a quantum event algebra L in $\mathcal{L}$ shaped by the functor $\mathbb{M} : \mathcal{B} \rightarrow \mathcal{L}$, or, equivalently, every Boolean frame-perspective on L, factors uniquely through the category of presheaves of sets $\mathcal{S}et^{\mathcal{B}^{op}}$, as revealed by the following commutative diagram:



This epitomizes the fact that there exists an exact solution to the problem of specifying a quantum event algebra L in perspectivist terms, by means of Boolean probing frames acting on it, which is provided by the left adjoint colimit functor $\mathbb{L} : \mathcal{S}et^{\mathcal{B}^{op}} \to \mathcal{L}$ of the Boolean frames-quantum adjunction. Henceforth, the specification of a quantum event algebra L in perspectivist terms, firstly, is not subordinate to ad hoc choices of Boolean frames adjoined to it, and, secondly, it is synthesized in the limit of the joint compatible action of all Boolean frames-perspectives acting on it.

5 Conceptual and Philosophical Implications

The key philosophical meaning of this approach implies, therefore, the view that the quantum world can be consistently approached and comprehended through a multilevel structure of overlapping Boolean frames, understood as locally variable perspectives applied on a quantum system, which interlock, in a category-theoretical environment, to form a coherent picture of the whole in a nontrivial way. In quantum mechanics the relation between the global theoretical structure and its various empirical sub-structures is indeed such that, depending on the type of experimental context a quantum system is brought to interact, different manifested aspects of the system are disclosed, impossible to be combined into a single picture as in classical physics, although only one type of system is concerned. Thus, by virtue of the proposed category-theoretic perspectivist approach to quantum mechanics, a quantum event structure can only be unfolded through structured interconnected families of Boolean probing frames capable of carrying all the information encoded in the former. And inversely, the non-directly accessible quantum event structure is uniquely constituted (up to equivalence) by a multiplicity of intertwined local perspectives directed towards it and covering the object of inquiry entirely under their joint action via the colimit-gluing process.

In view of the preceding considerations, and in relation to philosophical commitments of scientific perspectivism as stated in the introduction, it is worthy to note primarily that the suggested perspectivist approach to quantum mechanics elevates to an epistemological dictum the physically significant fact that values of quantum mechanical quantities cannot, in general, be attributed to a quantum object as inherently possessed, intrinsic properties. Whereas in classical physics, nothing prevented one from considering *as if* the phenomena reflected intrinsic properties, in quantum physics, even the *as if* is precluded. Indeed, quantum phenomena are not stable enough across series of measurements of non-commuting incompatible observables in order to be treated as direct reflections of predetermined inherent properties. As indicated in Section 2 via the Kochen and Specker theorem, in the quantum paradigm, it is no longer possible, not even *in principle*, to assign to a quantum system non-contextual properties corresponding to *all* possible measurements. This means that it is not possible to assign a definite unique answer to every single yes-no experimental question, represented by a projection operator, irrespective of which subset of mutually commuting projection operators one may consider it to be a member. In other words, within the formalism of Hilbert-space quantum mechanics, projection operators cannot be interpreted as defining *preexistent* to their measurement properties, possessing definite values, regardless of a referential context. Thus, according to the proposed view for quantum mechanics, well-defined values of quantum observables can, in general, be regarded as pertaining to an object under investigation only within a local Boolean probing frame or perspective from which the object-system is considered.

Adopting a Boolean probing frame and thus considering a preparatory Boolean environment for a quantum object to interact with a measuring arrangement does not determine which event will take place. It does determine, however, the *kind* of event that will take place. It forces the outcome, whatever it is, to belong to a certain Boolean sublattice of events, within the system's global non-Boolean logical structure, for which the standard measurement conditions are invariant. Such a set of standard conditions for a definite kind of measurement constitutes a set of necessary and sufficient conditions for the manifestation of an event of the selected kind. Upon the fulfillment of the latter conditions, we articulate meaningful objective statements that the properties attributed to quantum objects are part of physical reality. This equivalently means in the light of our approach that a complete Boolean algebra of projection operators in the lattice of quantum events picked by an observable to be measured instantiates locally a physical context, which serves as a Boolean reference frame relative to which results of measurement are being coordinatized. In this respect, Boolean probing frames or instances of concrete experimental arrangements in quantum mechanics play a role analogous to the reference frames of rods and clocks in relativity theory in establishing a perspectival aspect to microphysical reality. We further underline the fact that within the considered categorical scheme the variation of the local Boolean probing frames in the category $\mathcal{B}$ of probes, for each probe B in $\mathcal{B}$, is actually arising from any experimental praxis aiming to fix or prepare the state of a quantum system and corresponds, in this sense, to the variation of all possible Boolean preparatory contexts pertaining to the system for extracting information about it. Consequently, the demonstration of the Boolean frames–quantum adjunction makes it possible to constitute, in perspectivist terms, the global information content of a quantum event algebra from all possible properties associated with locally variable interconnected families of Boolean frames-perspectives on a quantum system, which are used in order to probe (or technically cover) the former.

It is instructive to emphasize in this regard that in the suggested category-theoretic perspectivist approach to quantum mechanics, a global quantum event structure is not conceptualized as an a priori existing set-theoretic structure, but it is constituted in a continuous process of extension from the local to the global level by actualization of new potential facts with respect to local Boolean frames. For, each quantum event actualized relative to a particular probing frame serves as a datum for subsequent potential actualizations, thus instantiating a bundle of potential relations referring to this frame. Importantly, all these potential relations, namely, all relations among observables at the local level, are captured by the internal relations among their underlined Boolean reference frames and are extended to the global quantum level through suitable sheaf-theoretic gluing conditions of structured families of partially ordered Boolean probing frames. In our view, therefore, quantum objects are definitely considered as carriers of inherent dispositional properties. This means that ascribing a property to a quantum object implies recognizing this object an *ontic potentiality* to produce effects whenever it is involved in various possible relations to other things in nature or whenever it is embedded within an appropriate experimental context. Thus, in contradistinction to a mechanistic or absolute objectivist perception, the following general conception of an object in quantum mechanics naturally arises out of our approach. According to this, a quantum object—as far as its state-dependent properties are concerned—constitutes a totality defined by all the possible relations in which this object may be involved.

At the epistemological level, the view that the properties possessed by observed quantum objects are, in general, context-dependent departs from the transcendent metaphysical vision of a world of self-autonomous objects, realized independently from their environment or their referential context. Be that as it may, such a view is perfectly compatible with the objectivity of scientific knowledge. Empirically confirmed manifestations of quantum systems cannot be meaningfully conceived of as absolute bare particulars of reality, enjoying intrinsic individuality. Instead, they represent local or partial carriers of patterns or properties which arise in interchange with their physical environment or, under the systematic and controllable procedures of scientific investigation, in interaction with their relevant experimental context. Thus, the resulting contextual object under study *is* the quantum object exhibiting a particular property with respect to a certain experimental situation. The contextual character of property ascription implies, however, that a state-dependent property of a quantum object is not a well-defined property that has been possessed *prior* to the object's entry into an appropriate context. This also means that not all contextual properties can be ascribed to an object at once. As already remarked, one and the same quantum object does exhibit several possible contextual manifestations in the sense that it can be assigned several definite incommensurable properties with respect to distinct incompatible quantum observables corresponding to different aspects of reality which, in principle, may not be considered simultaneously. Thus, according to our proposed view for quantum mechanics, the simplified assumption that knowledge of an object is achieved by forming a representation of that object as an immutable substance possessing intrinsic properties is rejected and, subsequently, replaced by the realistic possibility of formulating local or partial contextual theoretical structures enabling different or overlapping physical descriptions, grounded on the same actually existing object. Consequently, although possible in classical physics, in quantum mechanics we can no longer display the whole of nature in one view. It would be illusory to search for an overall frame by virtue of which one may utter 'this' or 'that', 'really is' independently of the adoption of a particular perspective or a context of reference.

In closing, it is only natural to assert that, in contrast to an Archimedean panoptical "view from nowhere" of the classical paradigm, the general epistemological implication of quantum theory acknowledges in an essential way a perspectival/contextual character of knowledge. Furthermore, the considered perspectivist approach to quantum mechanics provides the appropriate mathematical substratum for developing a post-classical, structured view of scientific theorizing in the sense of comprehending a theory not just as a class of empirical models simpliciter, as a structureless set of "models of the data", but also establishing mappings between these models allowing thereby their coherent embedding in a global theoretical structure. The aforementioned proposition refers to the actual scientific theorizing and practice of contemporary science, especially when dealing with complex trans-perspectival problems, the solution of which requires the use of information of a multi-scale variety, thus resulting from more than one perspective. The methodological framework of scientific perspectivism, developed in the present paper, aims at providing a suitable guide that is robust, encompassing and effective in the modeling (analysis-synthesis-representation) of multilevel phenomena and theoretical structures, standing in one-to-many relation with their multifarious empirical sub-structures, of which the proposed category theoretic approach to quantum mechanics constitutes a concrete manifestation.

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