Chapter 17
The Physics of Miniature Worlds

Susan G. Sterrett

After his father died, in early 1913, Ludwig Wittgenstein spent some time with various friends, family, and acquaintances. In late 1913, he began making plans to withdraw to Norway, away from Cambridge [UK],[1] where he had been working very closely with Bertrand Russell, and away from people, to work on solving the problems of logic. There was no looking back at aeronautics as an alternative career after that, it seems. The age-old problem of controlled, heavier-than-air flight on a scale that permitted humans to fly had [just recently] been solved, albeit by others (Fig. 17.1). There was still exciting and important work to do in aeronautics, but he had by then made the agonizing decision to become a philosopher, and, in working with Russell, he had found the age-old problem that he felt he was meant to solve instead: finding a correct theory of symbolism [105].[2]

Wittgenstein's own investigations into logic were bringing him around to notions of mirroring and corresponding. In notes expressing his views as of April 1914, he concludes "thus a language which *can* express everything *mirrors* certain properties of the world by these properties which it must have" (Wittgenstein 1979b, p. 107). And he struggles to accommodate his observation of the problematic fact that "in the case of different propositions, the way in which they correspond to the facts to which they correspond is quite different" (Wittgenstein, 1979b, p. 113). A year earlier, he had said there was no such thing as the form of a proposition ["the form

[1]Text in brackets indicates addition or change from original as it appeared in the First Edition of Sterrett (2005/2006).

[2]Numerals in brackets from 105 through 153 refer to the pagination in the original book, *Wittgenstein Flies A Kite* (Sterrett 2005/2006).

Adapted from Chapter 6 of *Wittgenstein Flies A Kite: A Story of Models of Wings and Models of the World* by Sterrett (2005/2006, pp. 105–153).

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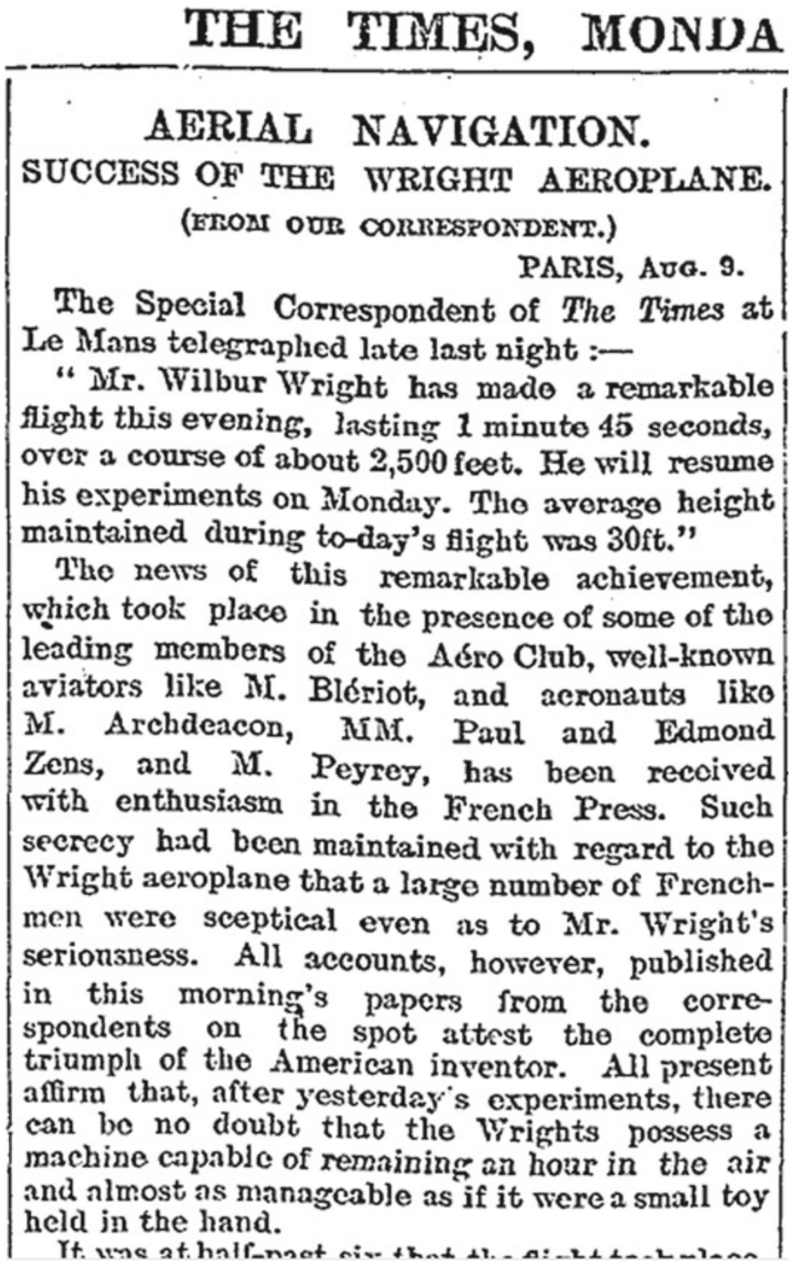

THE TIMES, MONDA

AERIAL NAVIGATION.

SUCCESS OF THE WRIGHT AEROPLANE.

(FROM OUR CORRESPONDENT.)

PARIS, AUG. 9.

The Special Correspondent of *The Times* at Le Mans telegraphed late last night :—

" Mr. Wilbur Wright has made a remarkable flight this evening, lasting 1 minute 45 seconds, over a course of about 2,500 feet. He will resume his experiments on Monday. The average height maintained during to-day's flight was 30ft."

The news of this remarkable achievement, which took place in the presence of some of the leading members of the Aéro Club, well-known aviators like M. Blériot, and aeronauts like M. Archdeacon, MM. Paul and Edmond Zens, and M. Peyrey, has been received with enthusiasm in the French Press. Such secrecy had been maintained with regard to the Wright aeroplane that a large number of Frenchmen were sceptical even as to Mr. Wright's seriousness. All accounts, however, published in this morning's papers from the correspondents on the spot attest the complete triumph of the American inventor. All present affirm that, after yesterday's experiments, there can be no doubt that the Wrights possess a machine capable of remaining an hour in the air and almost as manageable as if it were a small toy held in the hand.

Fig. 17.1 Several months after Ludwig Wittgenstein moved to England to pursue graduate study in aeronautical research, it was revealed in a public exhibition in Paris, France, that the problem of controlled, sustained, heavier-than air flight had been solved. From *The London Times*, August 9, 1908 of a proposition is not a thing" (Wittgenstein, 1979b, p. 105)]; his resolution of the issue of how propositions correspond to facts now is in terms of the general form of a proposition—something, he has decided, that all propositions do have in common. Then he says, "In giving the general form of a proposition, you are explaining what kind of ways of putting together the symbols of things and relations will correspond to (be analogous to) the things having those relations in reality" (Wittgenstein, 1979b, p. 113).

These exploratory thoughts about the notion of correspondence were the beginning steps toward an answer to one of the puzzles raised much earlier by musical scores and the gramophone records that had been such a striking arrival on the scene the year Wittgenstein was born: "What is the relationship between the symbols in the score and the patterns of grooves in the gramophone record?" That there was a mechanical process that could be used to make a gramophone record and one that could be used to play sound from it was well known. What about the process of creating a musical score, and the process by which a symphony could be imagined or [106] produced by a musician reading the score? Were these just as straightforward? Wittgenstein had already steered clear of simplistic accounts of a symbol as "sign of thing signified" a year earlier, at least in the case of words, in deciding that "Man possesses an innate capacity for constructing symbols with which some sense can be expressed, without having the slightest idea what each word signifies" (Wittgenstein 1979c, p. 100). But how did mirroring work, if not by a straightforward correspondence?

Wittgenstein was not alone in pondering how items of language could mirror a situation and how propositions could correspond to the world. The specific suggestion that equations function like pictures or models was made by Boltzmann in his *Lectures on the Principles of Mechanics*, a work in which he strove for an accurate exposition of mechanics that would be accessible to members of the general public. In explaining the role of pictures in physical theories, Boltzmann had there explained that even those who thought their approach had dispensed with pictures had not really done so: "[Partial differential equations] too are nothing more than rules for constructing alien mental pictures, namely of series of numbers. Partial differential equations require the construction of collections of numbers representing a manifold of dimensions" (Boltzmann, 1974, p. 226). Thus, he said, at the bottom Maxwell's equations "like all partial differential equations of mathematical physics…are likewise only inexact schematic pictures for definite areas of fact" (Boltzmann, 1974, p. 226). Boltzmann's suggestion, however, went just as far as claiming that symbolic equations could function like scientific models or pictures—it did not purport to explain exactly how either worked. It does seem, though, that picturing involved some imagined entities that may, but need not, correspond to something in reality. Boltzmann speaks of pictures almost interchangeably with mental pictures (Fig. 17.2).

Boltzmann became an extremely popular lecturer in Vienna around 1903, when, as mentioned earlier, Wittgenstein would have been about fourteen years old and would have known of Boltzmann's lectures. These lectures were so popular that the lecture hall in Vienna could not accommodate the audience, and Boltzmann was invited to give them at the palace instead. Boltzmann was present in Wittgen-

Fig. 17.2 Ludwig Boltzmann (1844–1906), with whom Wittgenstein had hoped to study. *Photo credit* University of Vienna, 1898. Public Domain

stein's youth through his prolific writing as well as through these lectures [which he] delivered a stone's throw from [Wittgenstein's family] home. The second volume of Boltzmann's *Lectures on the* [107] *Principles of Mechanics* was published in 1904, and a collection of his writings was published as *Popular Writings* in 1905, when Wittgenstein was sixteen. As we have seen, at that crucial time in his life, he was so interested in Boltzmann that, at least as he later recounted things to his friend and colleague von Wright, he had originally planned to study physics with him (McGuinness 1988, p. 54).

Boltzmann's *Popular Writings* anthology (Fig. 17.3) included an essay republished from a physics journal, "On the Indispensability of Atomism in Natural Sci-

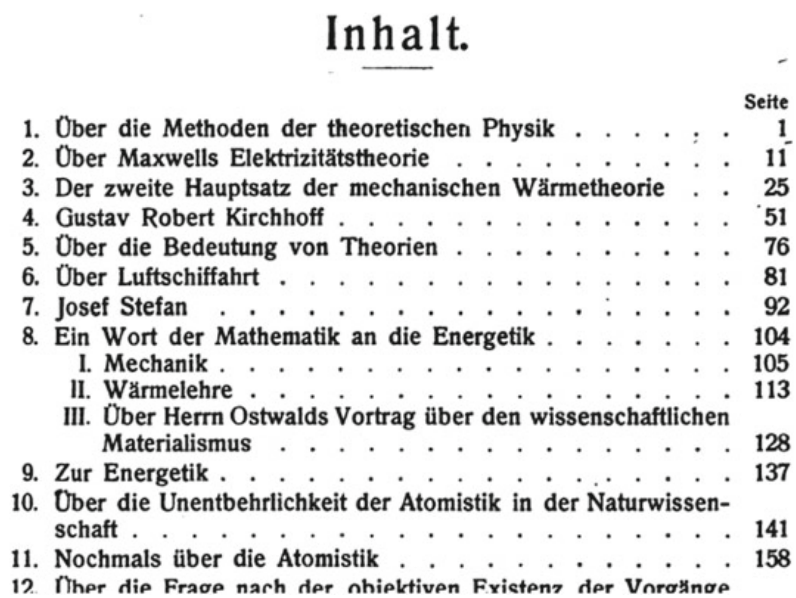

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Fig. 17.3 The Table of Contents to Boltzmann's Popular Writings. (1905) Essay 6 is "On Aeronautics" and Essay 10 is "On the Indispensibility of Atomism in Natural Science"

ence," in which he emphasizes the remarks about equations quoted earlier: "The differential equations of mathematico-physical phenomenology are evidently nothing but rules for forming and combining numbers and geometrical concepts, and these in turn are nothing but mental pictures from which appearances can be predicted". Here he refers the reader to Ernst Mach's *Principles of the Theory of Heat,* remarking that, as far as the use of models goes, there is no essential difference in the approach he takes and approaches such as energetics, in which equations rather than models of material points, are central: "Exactly the same holds for the conceptions of atomism, so that in this respect I cannot discern the least difference. In any case it seems to me that of a comprehensive area of fact we can never have a direct description but always only a mental picture". He had a rather precise criticism specific to differential equations and the corresponding assumption of a continuum, in that differential equations relied on the notion of a limit, and that observationally there was no distinguishing between systems of large numbers of finite particles and actual continuums. Thus, he said "those who imagine they have got rid of atomism by means of differential equations fail to see the wood for the trees" (Boltzmann, 1974, p. 45).[3]

Elsewhere, in his encyclopedia article on "Model," which was reprinted in the same anthology, Boltzmann again described the method he referred to as the theory of "mechanical analogies," remarking that, unlike in earlier days, "nowadays Philosophers postulate no more than a partial resemblance between the phenomena visible

[3] A number of commentators have pointed out the relevance of these and similar passages to Wittgenstein's early thoughts: McGuinness (1988), Janik and Toulmin (1973) and Barker (1980). "Hertz and Wittgenstein".

in such mechanisms and those which appear in nature" (Boltzmann 1974, p. 214). Looking closely at his remarks, though, it is clear he had run into a brick wall with this approach. He had to except from the models to which his remarks applied the kind of model that was used in [108] experimental engineering scale models. On the approach in which physical models constructed with our own hands are actually a continuation and integration of our process of thought, he says, "physical theory is merely a mental construction of mechanical models, the working of which we make plain to ourselves by the analogy of mechanisms we hold in our hands". In contrast, in his discussion of mental models, Boltzmann had explicitly described experimental models as of a different sort than the kind with which he was comparing mental models. Boltzmann even explained why they must be distinguished:

> A distinction must be observed between the models which have been described and those experimental models which present on a small scale a machine that is subsequently to be completed on a larger, so as to afford a trial of its capabilities. Here it must be noted that a mere alteration in dimensions is often sufficient to cause a material alteration in the action, since the various capabilities depend in various ways on the linear dimensions. Thus the weight varies as the cube of the linear dimensions, the surface of any single part and the phenomena that depend on such surfaces are proportionate to the square, while other effects — such as friction, expansion, and condition of heat, etc., vary according to other laws. Hence a flying-machine, which when made on a small scale is able to support its own weight, loses its power when its dimensions are increased. The theory, initiated by Sir Isaac Newton, of the dependence of various effects on the linear dimensions, is treated in the article UNITS, DIMENSIONS OF. (Boltzmann 1974, p. 220)

Thus, the experimental models represent a challenge: for experimental models, the relationship between model and what is modeled is in some ways unlike the relationship between a mental model and what is modeled by it.

Boltzmann committed suicide in the year 1906, the year after the anthology of his popular writings appeared, and Wittgenstein never did get to study with him. This remark of Boltzmann's might well have resonated with Wittgenstein's personal experience, even though he did not get to do experimental work under Boltzmann, for we know that Wittgenstein had built and played with a toy airplane (Spelt and McGuinness 2000), and these toys were quite serious affairs technically.[4] Boltzmann's remarks about experimental models, and his specific mention of a model [109] of a flying machine as a model that does not behave like the full-size machine it models, could scarcely fail to command Wittgenstein's attention. If an airplane design would only work the same way when enlarged, the problem of sustained, controlled heavier-than-air flight would have pretty much already been in the hands of countless children in Europe and America. As we saw earlier (Sterrett 2005/2006, pp. 11–15), Penaud had developed a rubber band-powered model airplane that was capable of sustained, stable flight (Fig. 17.4), and some of Penaud's designs were available as toys even before Wittgenstein was born. Boltzmann was especially aware of the fact that, in England, Maxim had shown that it was possible to design a full-

[4] Chanute's compendium (Chanute 1894) surveys an astounding number and variety of different proposed aeroplane designs, and identifies which of the designs were implemented as model aeroplanes that had undergone experimental trials.

In 1872 *Pénaud*, who had already succeeded (1870 and 1871) in compassing flight with the superposed screws and

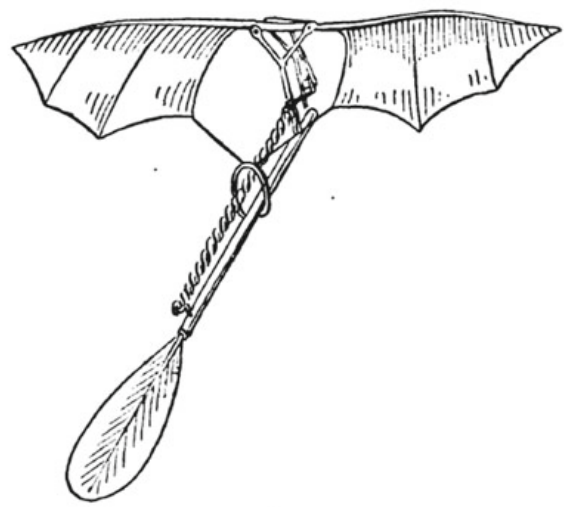


Fig. 17.—PÉNAUD—1872.

Fig. 17.4 Penaud had success with a rubber-band powered model airplane. Though some of his designs were intended as toys, larger versions were meant for researching specific full-size designs. The photograph of one of Penaud's model planophores being held by a man viewable online at the Science and Society Picture Library, Image No. 10438807 provides a sense of scale of these models

size steam-powered airplane capable of getting off the ground. In doing so, Maxim showed that an airfoil or kite could be powered—that is, that the power an engine produced could be large enough in proportion to its weight to get an airplane off the ground, which is all he was trying to establish at that point.

The unsurmounted obstacle was to get the sustained flight that Penaud had already achieved in a small-scale model, in a full-size airplane. Penaud's work had been the most promising, but as we saw earlier, he too had committed suicide (Fig. 17.5). In fact, he had done so upon receiving news that construction of the full-size model he had designed would not receive the funding he had been expecting. In 1905, the air was full of the promise of controlled heavier-than-air-flight, and there were some who believed the stories that two Americans had achieved it. Thus, Boltzmann's remarks describing significant and essential differences between mental models and models of flying machines would have had the effect of diminishing interest in mental models, because they made mental models seem like less robust representations of the world. And they may well have piqued interest in understanding whether and how models of flying machines could represent larger ones.

The puzzlement about the effect of size on the ability of machines to fly was common to just about anyone who played with toy flying machines. We saw earlier that the Wright Brothers recalled very clearly their puzzlement as children that the larger-sized models they built of exactly the same design didn't perform like the wonderful toy did. Even had Wittgenstein at age sixteen not recalled similar experiences with his childhood toys when reading this passage in Boltzmann's 1905 anthology,

Fig. 17.5 Alphonse Penaud (1850–1880) committed suicide in despair when funding for building a full-size version of his successful model was not granted. *Photo* Public Domain

Boltzmann's point about the effect of size on the strength and performance [110] of machines would almost certainly be remembered, given that Wittgenstein soon found himself enrolled in an engineering certificate program, and especially interested in aeronautics.

Certainly by the time he left his position as a research student in aeronautical engineering at Manchester in the fall of 1911 to show up at Cambridge asking to study logic with Bertrand Russell, Wittgenstein would have been familiar with the use of experimental scale models for specific types of engineering problems. Two wind tunnels were already in use for aeronautical research in England by that time. There was the tunnel that Wenham had convinced the Aeronautical Society of Great Britain to fund before anyone fully understood in general how to use the results on small-size models to predict the behavior of a full-size model (Baals and Corliss 1981, p. 3). Then there was the privately-funded wind tunnel that the inventor Maxim had built, which was constructed after Reynolds had shown how to use experiments to predict behavior in similar flow situations—at least for fluid flow in pipes. The significant thing about Reynolds' work was that it provided a way to determine similarity of flow regime for different-sized pipes, as well as for flow at different velocities and viscosities. Reynolds' work used liquids to investigate fluid flow, but air is a fluid, too, so Reynolds' work also bore on the questions of how wind tunnel results on models of aerodynamic surfaces could be used to predict the behavior of larger versions of the surfaces tested, and how that behavior would vary with different air velocities.

This is not to say that Wittgenstein would have then known exactly how to pick up where Boltzmann left off and fill in the story for experimental models, for it is not clear that there was an account of a general methodology of experimental models at that time. The way things stood with the practice of using engineering scale models might well have evoked almost exactly the puzzlement Wittgenstein had about propositions: "in the case of different propositions, the way in which they correspond to the facts to which they correspond is quite different". Given the approach then to experimental scale models, where the rules about how to scale from results on a model to results on a full-size object depended on whether you were talking about experiments in a towing canal or fluid flow in a pipe, one could just as well say that, in the case of different models, "the way in which they correspond to the things they model is quite different" [111].

What would the analogous point he had made for propositions be for experimental models? Wittgenstein's view in early 1914 about propositions was that "In giving the general form of a proposition you are explaining what kind of ways of putting together the symbols of things and relations will correspond to (be analogous to) the things having those relations in reality". What could it mean to give the general form of a model? Or, on Boltzmann's view that equations are really models of a sort, what does it mean to give the most general form of an equation?

In early 1914, Wittgenstein was asking these questions for propositions. Curiously, as we shall see, by the end of 1914, there would be a paper in the field of physics addressing analogous questions about empirical equations. The investigation in that physics paper involved finding the general form of an empirical equation, and it ended up addressing the question of what a universe built on a smaller scale would be like. There was more to be said in answering the question about the relationship between empirical equations and models than Wittgenstein was able to say about propositions in early 1914, the extra twist having to do with the fact that empirical equations involve measurement. The answer given for such equations would appear in late 1914 in a paper that also presented a formal basis for the methodology of experimental models. Though its author, Edgar Buckingham, was American, he had studied in Leipzig with Wilhelm Ostwald for his doctorate and had written a book on the foundations of thermodynamics. Thus, Buckingham's discussion was informed by the debates between Boltzmann and Ostwald about energetics, the kinetic theory of gases, and statistical thermodynamics (Figs. 17.6, 17.7 and 17.8).

Boltzmann had tried to tone down the strident claims of supporters of energetics such as Ostwald, who was antagonistic to the use of models. Ostwald's view, at least as Boltzmann understood Ostwald's emphasis, was not only that the use of models in thermodynamics and the kinetic theory of gases were so much extraneous and distracting baggage, but also that the use of models at all was suspect. In defending the use of models against such strident claims, we saw, Boltzmann pointed out that even proponents of energetics used models of a sort, inasmuch as they used equations as a sort of model—a model made of symbols. Boltzmann's suggestion that equations function like models may well have prodded Wittgenstein to think of a proposition as a model, and it may have even been implicit in some of Wittgenstein's statements in the manuscripts on logic he was [112] working on in 1913 and 1914. At any rate,

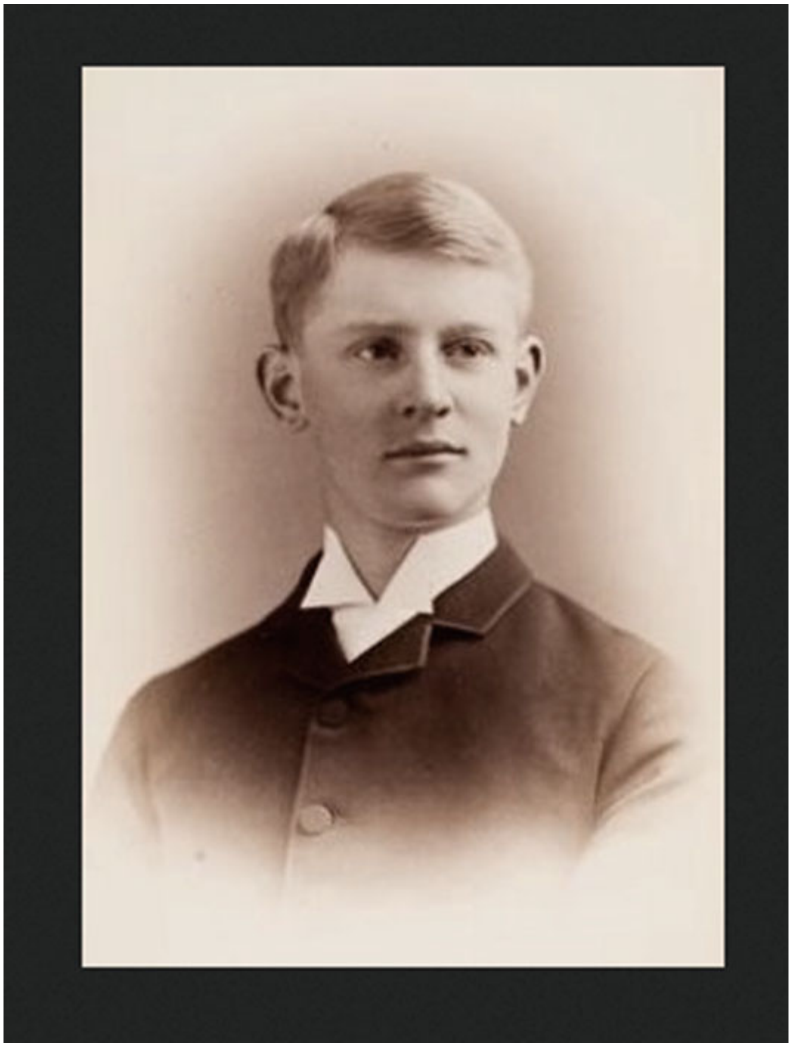
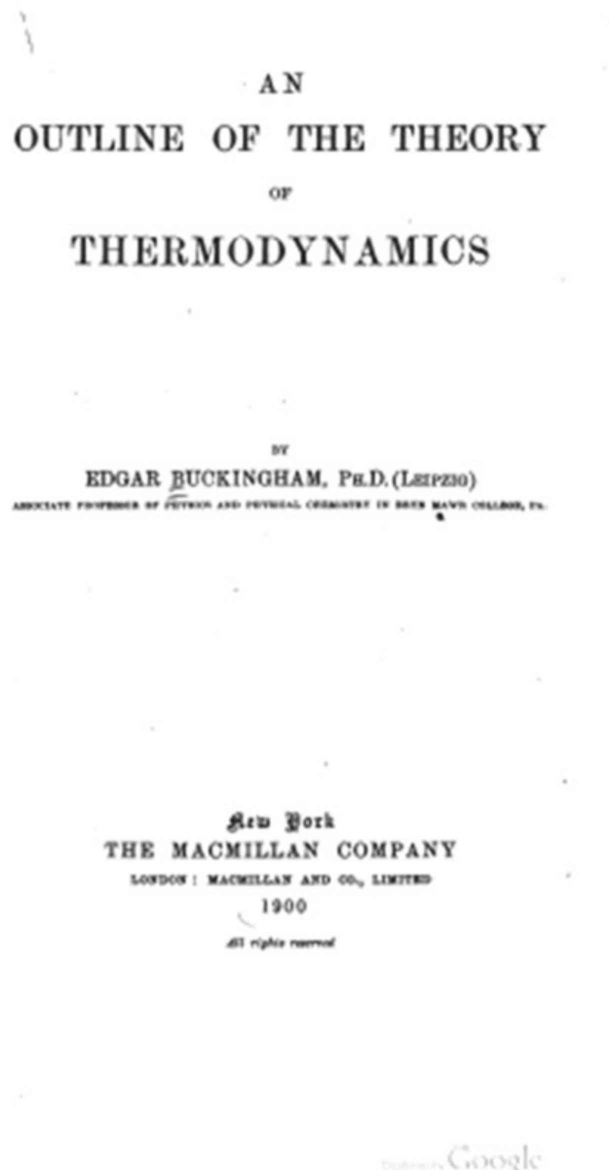


Fig. 17.6 The physicist Edgar Buckingham (1867–1940) as an undergraduate at Harvard. He later studied with W. Ostwald at Leipzig, earning a Ph.D. in 1893. *Photo* Harvard University Library

the notion of a proposition being like a model in some way was not explicit then. Wittgenstein just did not talk about propositions being models or pictures during his stay in Norway—that would come only after the crucial insight in late 1914 (Wittgenstein 1979a).

However, in early 1914, Wittgenstein *was* talking about propositions in terms of the facts to which they correspond, as was Russell. In the first manuscript on logic he produced in 1914, he writes "Proposition [which are symbols having reference to facts] are themselves facts: that this inkpot is on this table may express that I sit in this chair" (Wittgenstein 1979c, p. 97). Wittgenstein's move here about propositions and facts is at least vaguely reflective of Boltzmann's move in saying that manipulating symbols in an equation is using the equation like a model. Likewise, as indicated in the preceding quote from his manuscript, Wittgenstein had already, during his time


AN
OUTLINE OF THE THEORY
OF
THERMODYNAMICS

BY
EDGAR BUCKINGHAM, Ph.D. (Leipzig)
[illegible]

New York
THE MACMILLAN COMPANY
LONDON: MACMILLAN AND CO., LIMITED
1900
All rights reserved

Fig. 17.7 Title page of Buckingham's *An Outline of the Theory of Thermodynamics* (1900). *Image credit* Google Books

in Norway, made the move that propositions not only are symbols that correspond to facts, but are themselves facts—that is, they are the same sort of thing that they correspond to. Thinking of propositions as facts was not new. Frege had spoken of the marks on paper associated with a written sentence, and Russell [had spoken] of the varieties of fact that correspond to a proposition, including the example of the acoustic fact associated with a spoken sentence (Russell and Whitehead 1997, p. 402). Frege and Russell, though, tended to de-emphasize this kind of fact, to mention it

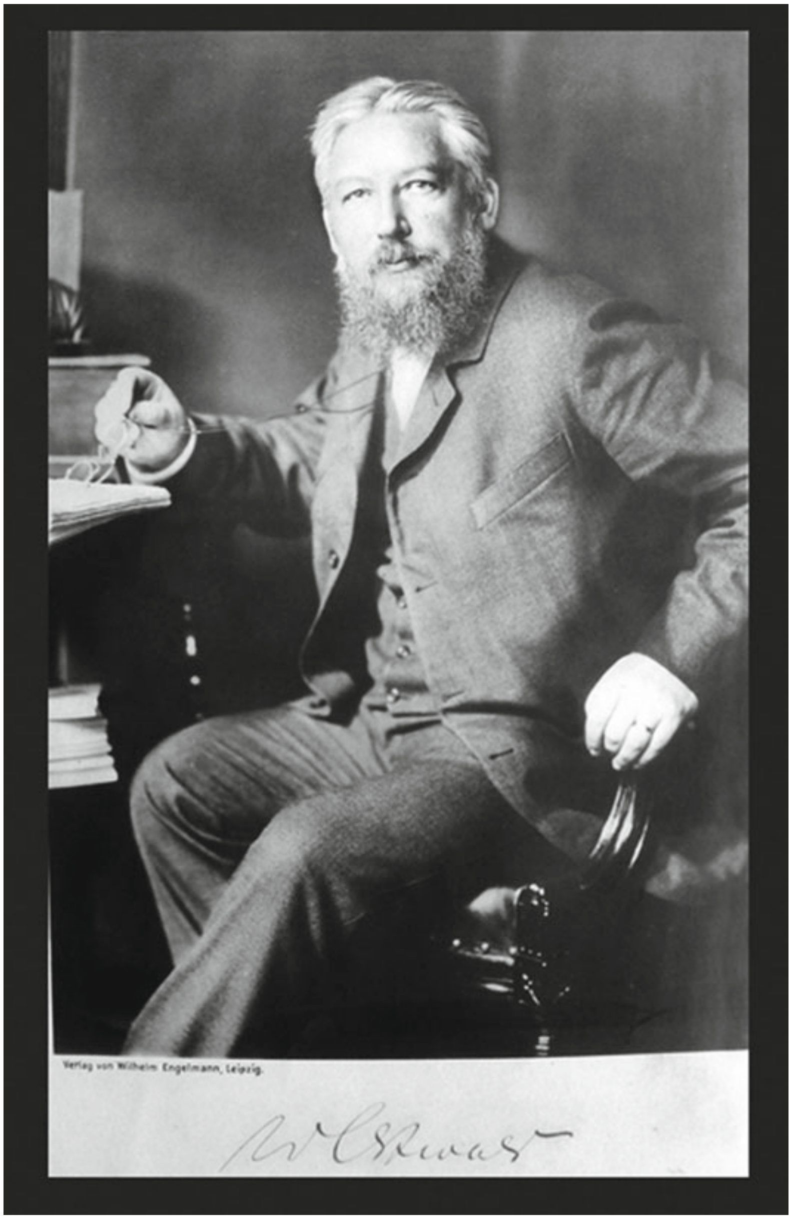


Fig. 17.8 W. Ostwald (1853–1937) Ostwald was Buckingham's Ph.D. advisor in Leipzig, Germany. He received the Nobel Prize in 1909. In 1919, Ostwald accepted Wittgenstein's manuscript for publication in a journal he edited, *Annalen der Naturphilosophie*. *Photo* Public Domain

only by way of contrasting the kind of fact that a proposition is with the kind of fact to which a proposition *corresponded*. But, in all fairness, even if Wittgenstein was tentatively exploring the possibilities of the observation that a proposition is a fact before the war (rather than merely mentioning it as a contrast to the kind of fact to which a proposition does correspond), he was not, pre-war, exploring the idea of a proposition being a fact *in terms of picturing or modeling* then. The key notion regarding propositions that shows up in the work Wittgenstein did during his stay in Norway just before the outbreak of World War I is the notion of correspondence rather than the notion of picture or model.

It was almost as though something in the atmosphere was stimulating people's appetites for a satisfying understanding of correspondence, similarity, and form. For, while Wittgenstein was living in Norway pondering problematic issues in logic such as the fact that "it seems as if, in the case of different propositions, the way in which they correspond to the facts to which they correspond is quite different," interest in similarity, [113] correspondence, and similarity transformations was appearing in a wide variety of contexts, especially in Britain.

Someone familiar with Boltzmann's *Popular Writings* might well find that these discussions of different kinds of correspondence, and, especially, the accompanying explorations of the consequences of similarity, brought to mind Boltzmann's remark about the methods of theoretical physics: "The new approach compensates the abandonment of complete congruence with nature by the correspondingly more striking appearance of the points of similarity. No doubt the future belongs to this new method" (Boltzmann 1974, p. 11). Boltzmann had written that in 1892, in the wake of Hertz's spectacular successes in electrodynamics, which in turn (according to Boltzmann) owed much to Maxwell's ingenious mechanical analogies for his equations describing electromagnetic phenomena.

Boltzmann's point here is that, although the analogies Maxwell came up with were crucial to Hertz's advances, Maxwell did not intend the analogies to be taken literally. Maxwell did not mean them to be regarded as hypotheses; Boltzmann felt that, just as with Maxwell's attitude toward his own equations, so things had become in general with the equations of theoretical physics: as he put it there, science speaks "merely in similes". Two decades after Boltzmann penned these remarks about the new methods in physics, we find that in the contexts of discussing thermodynamics, hydrodynamics, and biology (morphology), many other thinkers in many other fields were seeking definitive statements about similarity, too. Not all the ideas that sprang up were unprecedented, but ideas about the use of similarity, whether old or new, were now being explicitly reflected upon, talked about, and written about. In the years 1913–1914 in particular, simultaneous activity of this sort occurred in a number of very different disciplines. Looking back, the activity in the few years just prior to 1914 portends a convergence of ideas about similarity and correspondence.

Certainly some notion of correspondence was already familiar, from pure mathematics as well as from theoretical mechanics. Bertrand Russell's *Principia Mathematica*, published in England in three volumes from 1910 to 1913, reflected the approaches of pure mathematicians in Germany, such as Richard Dedekind and Georg Cantor. Dedekind and Cantor, friends and colleagues, became friends and published very different definitions of the real numbers in 1872, but both had employed the notions of [114] correspondence and similarity in formalizing and defining numbers and other mathematical concepts. Dedekind, for instance, had set up an analogy between the set of rational numbers and a straight line, which he then used to explicitly define a correspondence between rational numbers and points on the line (O'Connor and Robertson 1998). Rational numbers are numbers that can be expressed as the ratio of two whole numbers; irrational numbers cannot. Constructing a line segment corresponding to a rational number on a line uses only the most basic methods of geometrical construction, so Dedekind considered the rational numbers a logical starting point from which to define the rest of the real numbers. This correspondence was not merely illustrative, but was put to good use, for the fact that there were some points on the straight line to which no rational numbers corresponded motivated a definition of irrational numbers. Numbers, as Dedekind defined them, corresponded to "cuts" of the rational numbers that were analogous to "cuts" of the straight line. So analogy and correspondence were not of merely heuristic value in discovering the definition of number, but vestiges of them actually appear in the definition. Dedekind also showed how to use the definition of real numbers to construct a definition of complex numbers (numbers of the form $a + bi$, where a and b are real numbers and i is the square root of -1).

Cantor took a totally different approach. He did not presume the rational numbers, or any kind of numbers, as already familiar and known, and he aimed for a notion of number even more general than real numbers and complex numbers. Cantor's approach was to start with what he thought one of the most basic of mental activities—abstracting from individual properties of objects in a collection. This resulted in a definition of what he called "transfinite" numbers, which contrast with "finite" numbers. He gave a definition of infinite that distinguished infinite sets and finite sets, and in doing so discovered a whole world of infinite numbers. His definitions are based on two main notions: the notion of one-to-one correspondence and the notion of an order-preserving mapping. Roughly put, the notion of one-to-one correspondence captures the idea of how many, whereas the notion of an order-preserving mapping captures the notion of similarity of order structure. He found that these notions led to a way to delineate infinite sets. His definition of infinite says that infinite sets are sets that can be put in one-to-one correspondence with a proper [115] subset of themselves, something that is impossible for finite sets. On this definition of infinite, the set of whole numbers is easily seen to be infinite because it can be put into one-to-one correspondence with the even numbers (just map each number n to the number 2n), and the even numbers are contained within the set of whole numbers. He defined cardinal type, which expresses the informal idea of how many things are in a collection, and ordinal type, which expresses the informal idea of how things are ordered in a collection as well as how many things are in it. As he defined cardinal number and ordinal number, the two concepts coincide for finite numbers but are forced apart for transfinite (infinite) numbers. Russell's third volume of *Principia Mathematica*, published in 1913, dealt with transfinite cardinal and ordinal numbers and incorporated many of Cantor's ideas. Thus, these ideas became more widely known in England in 1913, especially among philosophers and people interested in philosophy and logic.

The ideas were the horizon-expanding kind, provoking the kind of exhilaration in thought that ballooning had for the senses. They made people feel that, using only simple, familiar ideas, they were transported to a realm where they were suddenly freed of things that had bound the before. Similarity was shown to be a very powerful, if unassuming, idea. Cantor argued that not only had he extended the notion of number, but that the notion of a transfinite ordinal number reflected the most general notion of number possible. The notion of "similar" aggregates (he used the term "aggregates" to refer to collections formed by the mind by abstracting from individual characteristics of the things in the collection) turned out to be exceedingly fruitful. It required a notion of something more structured conceptually than a mere aggregate. Thus Cantor was led to define ordered aggregates (where different notions of "less than" induce different orders) and then an even more important concept: a well-ordered aggregate, which has a "least element"—a starting point or end point. We can think of one aggregate being "transformed" into an aggregate to which it is similar by a similarity transformation, just as we can think of an aggregate being transformed into one to which it is equipollent (meaning that its elements and the transformed one can be put into a one-to-one correspondence) by a simple replacement of each element of the aggregate, one by one. The mathematical notion of a transformation can be used to discuss a [116] mapping. Instead of describing a mapping, which involves specifying a rule by which each element in one aggregate or object is paired with an element in the aggregate or object to which it is similar, we can talk about an aggregate or object being transformed into one to which it is similar.

Similarity is not just a mathematical notion, however. The notion of similarity is entwined in the thought and practice of just about any discipline you can think of, although it is not always talked about per se. Then, as now, notions of similarity were essential to much of scientific reasoning and engineering practice, even though discussion of the topic itself appears infrequently in scientific papers. However, in the years just prior to and including 1914, there was a cultural precipitation of papers that did explicitly reflect on the use of similarity.

In 1912, when James Thomson's *Collected Papers in Science and Engineering* appeared (twenty years after his death), it contained a paper about similar structures (Thomson and James 1875, 1912). These are structures in the most literal sense—structures such as bridges and columns. In that paper, "Comparison of similar structures as to elasticity, strength, and stability" he distinguishes two kinds of similarity between structures: similarity with respect to elasticity and bending, and similarity with respect to stability. The paper was written in 1875, just after Dedekind and Cantor published their accounts of number. James Thomson's style of reasoning illustrates that, in practical engineering, the method of similarity, though based on reasoning from principles of natural science, was still conceived of in terms of *specific kinds* of similarity, specific kinds of loads (wind on a surface versus attached weight), and specific disciplines (hydrodynamics versus mechanics of materials). The kinds of things that were then called "similarity principles" were statements covering a certain class of cases. The point of the "principle" was usually to state how one variable—the weight, size, elasticity—was to be varied as the linear dimension was varied—that is, as an object was increased or decreased in size but kept the same shape. James Thomson's examples are often about how to vary some quantity such that two structures of different sizes are similar in one of these respects. Here is one example of what is meant by a "similarity principle" taken from that work: "Similar structures, if strained similarly within limits of elasticity from their forms when free from applied forces, must have their systems of applied forces, similar in arrangement and of amounts, at homologous [117] places, proportional to the squares of their homologous linear dimensions" (Thomson 1875, p. 362) (Figs. 17.9 and 17.10).

Sometimes the reasoning is based on equations, but often it is not. Rather, some arguments from physical intuition are used: that weight increases as the cube of the linear dimension, and a cross-sectional area of a rope increases as the square of the linear dimension. They are very much like the statements, cited earlier, that Boltzmann used in describing the kind of model he called "those experimental models which present on a small scale a machine that is subsequently to be completed on a larger, so as to afford a trial of its capabilities".

At about the same time, the polymath D'Arcy Wentworth Thompson, then a professor of biology at the University of St. Andrews, was working out ideas that would soon appear in a compilation titled *On Growth and Form*. It is most well-known for its illustrations showing sketches of animals and animal parts that are "morphed" into others. Each sketch is overlaid with a grid; in the transformed sketch, the grid is stretched or slanted in some way so that the form of one species of animal looks as though it is obtained from another via a transformation mapping the points on the lines of one sketch to another by a mathematical function. Some commentators today regard this work as putting forth an alternative to Darwin's theory of natural selection, but this obscures the nature of D'Arcy Thompson's masterwork. Certainly it is true that D'Arcy Thompson wanted to put the brakes on the tendency of his contemporaries to use Darwin's theory of natural selection to explain everything, to the exclusion of other kinds of explanations. But careful readers of Darwin know that scientific explanations of animal forms according to Darwin's theory did not exclude the role of physics. Like Darwin, D'Arcy Thompson was a wonderful naturalist; unlike Darwin, he was a mathematician. D'Arcy Thompson wanted especially to ensure that the role of physics was not overlooked in explaining biological form. Likewise, the mathematical aspects of his work, which have to do with similarity and transformation, do not of themselves conflict with Darwin's theory of natural selection, either. D'Arcy Thompson's deeper mission was the mathematization of biology. The spirit of mathematics of the day was similarity, and it was reflected in his work on what might be called mathematical biology.

In a lecture he gave in 1911 to the British Association for the Advancement of Science titled "Magnalia Naturae: of The Greater [118] Problems of Biology," D'Arcy Thompson spoke of a tendency in recent biological work: "the desire to bring to bear upon our science, in grater measure than before, the methods and results of the other sciences, both those that in the hierarchy of knowledge are set above and below, and those that rank alongside our own" (Thompson 1911, p. 419).

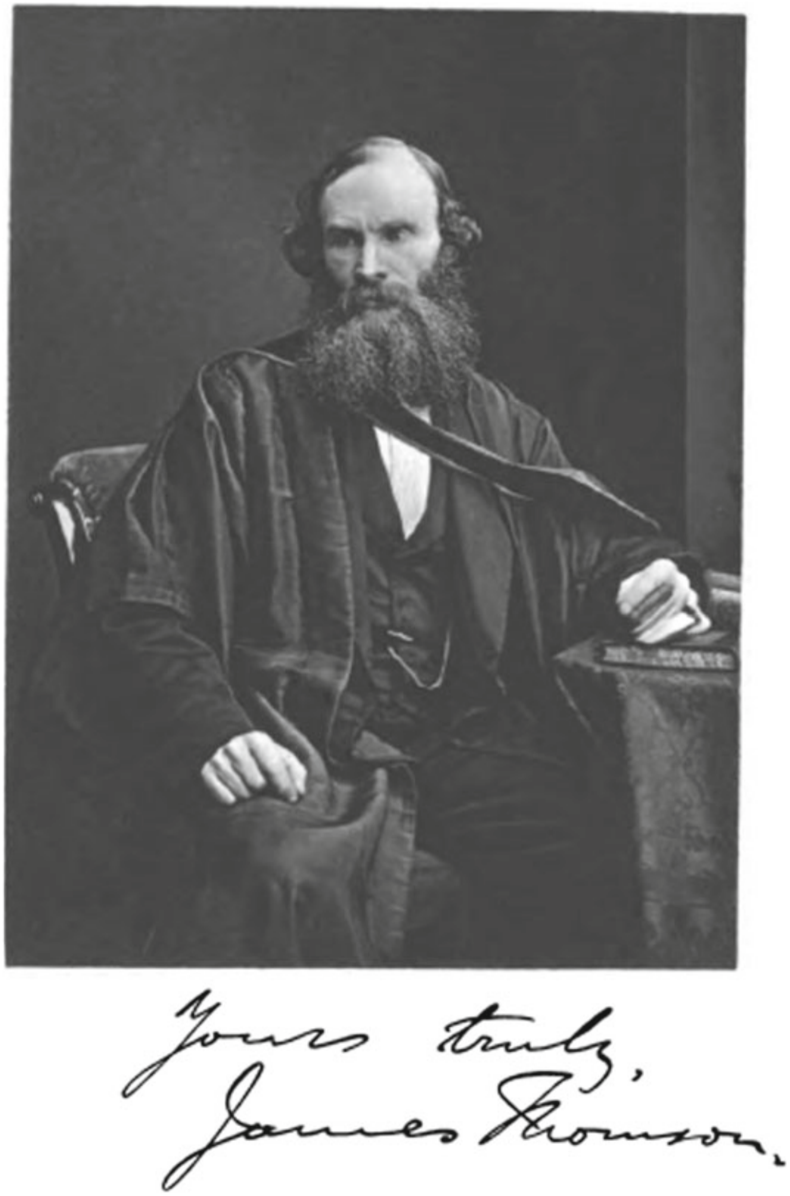
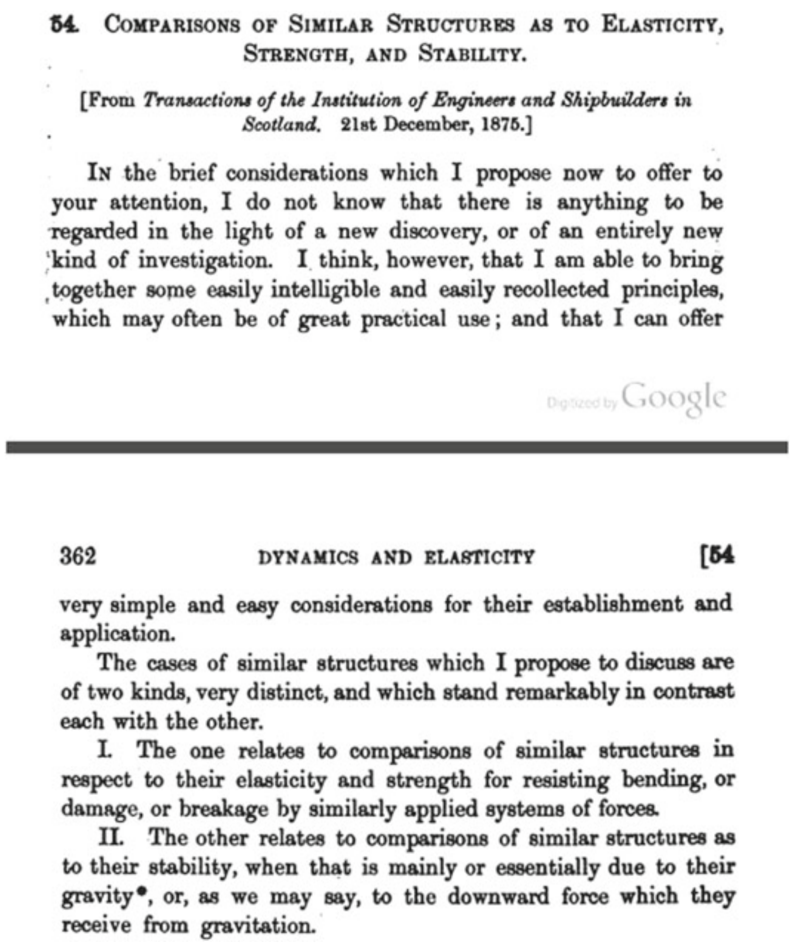


Fig. 17.9 James Thomson (1822–1892), Professor of Engineering, was the older brother of Lord Kelvin, and they collaborated on thermodynamics. He also wrote on different kinds of similarity in mechanics and on dimensional equations. *Image* Google Books (Thomson 1912)

He spoke of the unifying influence of physiology, with its focus on the living rather than the dead organism, and its amenability to being treated by the methods of the physical sciences, remarking "Even mathematics has been pressed into the service of the biologist, and the calculus of probabilities is not the only branch of mathematics to which he may usefully appeal". He spoke of the personal appeal that problems about morphology that were related to "mechanical considerations, to mathematical laws, or to physical and chemical processes" held for him. he also laid out reasons



54. COMPARISONS OF SIMILAR STRUCTURES AS TO ELASTICITY, STRENGTH, AND STABILITY.

[From *Transactions of the Institution of Engineers and Shipbuilders in Scotland.* 21st December, 1875.]

IN the brief considerations which I propose now to offer to your attention, I do not know that there is anything to be regarded in the light of a new discovery, or of an entirely new kind of investigation. I think, however, that I am able to bring together some easily intelligible and easily recollected principles, which may often be of great practical use; and that I can offer

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very simple and easy considerations for their establishment and application.

The cases of similar structures which I propose to discuss are of two kinds, very distinct, and which stand remarkably in contrast each with the other.

I. The one relates to comparisons of similar structures in respect to their elasticity and strength for resisting bending, or damage, or breakage by similarly applied systems of forces.

II. The other relates to comparisons of similar structures as to their stability, when that is mainly or essentially due to their gravity*, or, as we may say, to the downward force which they receive from gravitation.

Fig. 17.10 From James Thomson's 1875 "Comparison of Similar Structures as to Elasticity, Strength, and Stability" which is included in his *Collected Papers in Physics and Engineering* published in 1912. *Image* Google Books

supporting the possibility of "so far supporting the observed facts of organic form on mathematical principles, as to bring morphology within or very near to Kant's demand that a true natural science should be justified by its relation to mathematics" (Thompson 1911, p. 426).

On the first page of the compilation of his ideas into the large compendium *On Growth and Form*, which was published in 1917 and is even now regarded as a masterpiece, he opens the work with a reference to Kant's declaration, as he put it, "that the criterion of true science lay in its relation to mathematics" (Thompson 1992, p. 1). He goes on to say that:

> As soon as we adventure on the paths of the physicist, we learn to weigh and to measure, to deal with time and space and mass and their related concepts, and to find more and more our knowledge expressed and our needs satisfied through the concept of number, as in the dreams and visions of Plato and Pythagoras. (Thompson 1992, p. 2)

D'Arcy Thompson recognized Newton, too, of course, as a prime example of someone whose work had shown the tremendous fruitfulness of mathematizing a class of phenomena. The continuity D'Arcy Thompson saw between his own project of mathematizing biology and Newton, though, was in the use of similarity. "Newton did not shew the cause of the apple falling, but he shewed a similitude ('the more to increase our wonder, with an apple') between the apple and the stars" (Thompson 1992, p. 9). As in physics, so in the life sciences: "The search for differences or fundamental contrasts between the phenomena of organic and inorganic, of animate [119] and inanimate things, has occupied many men's minds, while the search for the community of principles or essential similitudes has been pursued by few" (Thompson 1992, p. 9). He compared the "slow, reluctant extension of physical laws to Tycho Brahe, Copernicus, Galileo and Newton (all in opposition to the Aristotelian cosmology), that the heavens are formed of like substance with the earth, and that the movements of both are subject to the selfsame laws" (Thompson 1992, p. 11). D'Arcy Thompson did not go so far as to claim [that] physics and mathematics were comprehensive, nor even to know how much they could explain. He recognized there were limits: "… nor do I ask of physics how goodness shines in one man's face, and evil betrays itself in another" (Thompson 1992, p. 13).

He did look to physics, though, for an explanation of lots of different kinds of behavior in creatures both living and nonliving—including an explanation of heavier-than-air flight. He reiterated the point Boltzmann made "that various capabilities depend in various ways on the linear dimensions," citing Helmholtz's 1873 lecture on similar motions and dirigibles for the reasoning supporting the by-then familiar conclusion that "the work which *can be done* varies with the available weight of muscle, that is to say, with the weight of the bird; but the work which *has to be done* varies with mass and distance; so the larger the bird grows, the greater the disadvantage under which all its work is done" (Thompson 1992, p. 42). But D'Arcy Thompson goes further and points out that, while this is true for a specific machine or animal form, it is not the whole story. Not all flight is powered by the sources assumed in these analyses. There is also, he says, "gliding flight, in which … neither muscular power nor engine power are employed; and we see that the larger birds, vulture, albatross or solan-goose, depend on gliding more and more" (Thompson 1992, p. 42). This is just one illustration of the fact that many factors other than size are involved in comparing flight capabilities of various birds. These other factors, he says, "vary so much in the complicated action of flight that it is hard indeed to compare one bird with another". In living things, we find that "Nature exhibits so many refinements and 'improvements' in the mechanism required, that a comparison based upon size alone becomes imaginary, and is little worth the making" (Thompson 1992, p. 44).

What can be said is that, in both how the fish swims and how the bird flies, streamlining is important. In properly streamlined wings, "a partial vacuum is formed above the wing and follows it wherever it goes, so long [120] as the stream-lining of the wing and its angle of incidence are suitable, and so long as the bird travels fast enough through the air" (Thompson 1992, p. 43). Here the kind of reasoning based on the observation Boltzmann had made in his "Models" essay ("that various capabilities depend in various ways on the linear dimensions") is informative; it tells us how the speed required to stay aloft increases with size. D'Arcy Thompson refers to this as a "principle of *necessary speed*," which he describes as "the inevitable relation between the dimensions of a flying object and the minimum velocity at which its flight is stable" (Thompson 1992, p. 45). That is, "in flight there is a certain necessary speed—a speed (relative to the air) which the bird *must attain* in order to maintain itself aloft, and which must increase as its size increases" (Thompson 1992, p. 41). This principle explains the qualitative differences between large and small birds. Large birds "must fly quickly, or not at all," whereas insects and very small birds such as hummingbirds, are capable of what appears to be "stationary flight," since, for them, "a very slight and scarcely perceptible velocity relatively to the air [is] sufficient for their support and stability" (Thompson 1992, p. 46).

Thus, a proper understanding of the significance of Boltzmann's observation about capabilities varying with dimension, in conjunction with finer distinctions about what is involved with flight, does permit conclusions based on size: "The ostrich has apparently reached a magnitude, and the moa certainly did so, at which flight by muscular action, according to the normal anatomy of the bird, becomes physiologically impossible. The same reasoning applies to the case of man" (Thompson 1992, p. 48). However, this is not to say that flight is impossible above a certain size—rather, that "gliding and soaring, by which energy is captured from the wind, are modes of flight little needed by the small birds, but more and more essential to the large". So the proper lesson to be drawn from considerations of the dependence of capabilities on size is not the one that had been drawn during the eighteenth century—that humans should not try to fly at all—but rather that humans should learn to *glide*. Thus, he observes, "It was in trying to glide that the pioneers of aviation, Cayley, Wenham and Mouillard, Langley, Lilienthal and the Wrights—all careful students of birds—renewed the attempt; and only after the Wrights had learned to glide did they seek to add power to their glider" (Thompson 1992, pp. 48–49) (Fig. 17.11).

What D'Arcy Thompson stressed in all this variety of phenomena was a principle that explained the variety: *the principle of similarity*. Recall that [121] he had mentioned that Newton's insight had involved discerning the similarity underlying the two very different cases of the apple's fall to the earth and the moon's hanging in the sky. In the case of flight, the hummingbird's hanging in the air and the difficulty large birds have in becoming airborne is explained by a principle as well: the principle of similarity. Galileo, too, had introduced the principle of similarity by remarking on the difference in performance of large and small creatures and machines Thompson draws out Galileo's point as it pertains to differences in animal form and to engineering design:

> But it was Galileo who, wellnigh three hundred years ago, had first laid down this general principle of similitude; and he did so with the utmost possible clearness, and with a great wealth of illustration drawn from structures living and dead. [citing 1914 translation, p. 130] He said that if we tried building ships, palaces or temples of enormous size, yards, beams and bolts would cease to hold together; nor can Nature grow a tree nor construct an animal beyond a certain size, while retaining the proportions and employing the materials which suffice in the case of a smaller structure. The thing will fall to pieces of its own weight unless

Fig. 17.11 Otto Lilienthal in a gliding experiment. Boltzmann commended the approach, and corresponded with him about the experiments. D'Arcy Wentworth Thompson notes the significance of the approach in his *On Growth and Form*. *Photo* Public Domain

> we either change its relative proportions, … or else we must find new material, harder and stronger than was used before. Both processes are familiar to us in Nature and in art, and practical applications, undreamed of by Galileo, meet us at every turn in this modern age of cement and steel". (Thompson 1992, p. 24)

To "change its relative proportions" is to change an animal's form. Thus, the form of animals are dependent on, or conditioned on, not only the material properties of the stuff of which they are made, but also on the force of gravity. Form is an effect of scale, but, in turn, "The effect of scale depends not on a thing in itself, but in relation to its whole environment or milieu" (Thompson 1992, p. 24). If this is so, the form of a land-based animal reflects the strength of the gravitational force. D'Arcy Thompson illustrates the point by asking what things would be like were the gravitational force different:

> Were the force of gravity to be doubled our bipedal form would be a failure, and the majority of terrestial animals would resemble short-legged saurians, or else serpents. Birds and insects would suffer likewise, though with some compensation in the increased density of the [122] air. On the other hand, if gravity were halved, we should get a lighter, slenderer, more active type, needing less energy, less heat, less heart, less lungs, less blood. Gravity not only controls the actions but also influences the forms of all save the least of organisms". (Thompson 1992, p. 51)

For very tiny organisms, the same general principles—that the effect of scale on form is a matter not only of the features of the organism itself, but of its whole environment – applies. However, for motions of such tiny animals, it is not gravity, but surface tension, that tends to be the dominant feature of the environment. “The small insects skating on a pool have their movements controlled and their freedom limited by the surface tension between water and air, and the measure of that tension determines the magnitude which they may attain”. There are other constraints on their size due to their form, too. In the respiratory system of insects, “blood does not carry oxygen to the tissues, but innumerable fine tubules or tracheae lead air into the interstices of the body”. There are natural limitations on the size of such a system; if they grew too much larger, “a vast complication of tracheal tubules would be necessary, within which friction would increase and fusion be retarded, and which would soon be an inefficient and inappropriate mechanism” (Thompson 1992, pp. 51–52).

Besides the limitations on size for insect forms, we can, conversely, see the insect’s form as constrained by its size: “we find that the form of all very small organisms is independent of gravity, and largely if not mainly due to the force of surface tension” (Thompson 1992, p. 57). One of D’Arcy Thompson’s well-known phrases comes from his point that the form of an object is a “diagram of forces”; the immediate context in which that phrase occurs is as follows:

> The form, then, of any portion of matter, whether it be living or dead, and the changes of form which are apparent in its movements an din its growth, may in all cases alike be described as due to the action of force. In short, the form of an object is a ‘diagram of forces,’ in this sense, at least, that from it we can judge of or deduce the forces that are acting or have acted upon it. (Thompson 1992, p. 16)

The point here is the effect on form of the environment, not the importance of forces. D’Arcy Thompson was clear that he was using forces only as a sort of shortcut expression: “… force, unlike matter, has no [123] independent objective existence. It is energy in its various forms, known or unknown, that acts upon matter” (Thompson 1992, p. 15). Here, we recognize his awareness of the view of energeticists (such as Ostwald and Hertz), of the problematic status of forces, and their tendency to replace explanations made in terms of force with explanations in terms of mass and energy. Throughout *On Growth and Form*, we find many explanations of animal behavior and form given in terms of energy available and expended. The reason D’Arcy Thompson used the notion of force in describing form was because form is abstract, rather than material, and he justifies his use of the term “force” as appropriate here *without* reifying force:

> But when we abstract our thoughts from the material to its form, or from the thing moved to its motions, when we deal with the subjective conceptions of form, or movement, or the movements that change of form implies, then Force is the appropriate term for our conception of the causes by which these forms and changes of form are brought about. When we use the term force, we use it, as the physicist always does, for the sake of brevity, using a symbol for the magnitude and direction of an action in reference to the symbol or diagram of a material thing. It is a term as subjective and symbolic as form itself, and so is used appropriately in connection therewith. (Thompson 1992, p. 16)

He elaborates on the interrelations of magnitude, ratio, and picture: “When we deal with magnitude in relation to the dimensions of space, our diagram plots magnitude in one direction against magnitude in another—length against height, for instance, or against breadth" (Thompson 1992, p. 78). What we get, he says there, is "what we call a picture or outline, or (more correctly) a 'plane projection' of the object". His emphasis on ratio is striking, and in fact he sums up the whole idea of form in terms of it: "what we call Form is a ratio of magnitudes referred to direction in space". This particular ratio is dimensionless, since it is a ratio of like magnitudes. A length, a height, a breadth, are all measured in dimensions of linear length, whatever units are used. Hence, whatever units are used to measure these magnitudes – inches, feet, millimeters, or centimeters—so long as the same units are used for both the magnitudes in the ration, the units cancel, and the resulting ratio has no units at all. So any ratio of like magnitudes is dimensionless [124].

However, Thompson does not restrict the ratios of interest to such ratios. When considering the variation of a length over time, as in studying growth, he points out that the ratio involved there has the dimensions of velocity: "We see that the phenomenon we are studying is a velocity (whose 'dimensions' are space/time, or L/T) and this phenomenon we shall speak of, simply, as *rate of growth*" (Thompson 1992, p. 78). The symbols L and T denote that the dimensions of the magnitudes being measured are length and time, respectively. They do not specify units of measurement, just what kind of measurement is being taken. Constructing a ratio of length to time gives rise to another kind of quantity, and thus we say that the quantity velocity has dimensions of L/T. So ratios of unlike magnitudes give rise to additional kinds of quantities, or kinds of magnitudes. D'Arcy Thompson's graphical representations also used contour-lines, or "isopleths," to represent a third dimension or magnitude on a two-dimensional surface. The contour-lines can show depth, or the third dimension, of a three-dimensional form.

So far, time is represented only insofar as each of these representations represents a form or configuration at a particular time. Then, Thompson explains, the outlines of an organism as it changes over time can be set out side by side (or, alternatively, overlaid on each other), and this series represents the organism's gradual change over time. Such a representation—somewhat like a series of comic strip frames—exhibits both the form of the organism and the growth of the organism. In addition, it shows how an organism's growth and form are interrelated: "it is obvious that the form of an organism is determined by its rate of growth in various directions" (Thompson 1992, p. 79).

As mentioned earlier, D'Arcy Thompson's goal was to mathematize biology—to treat it the way a physicist treats his subject. He had earlier remarked that "physics is passing through an empirical phase into a phase of pure mathematical reasoning," (Thompson 1992, p. 17n) and certainly the energeticists' emphasis on equations and energy balances was an example of the newer style of mathematical reasoning. Thompson had, however, identified the "old-fashioned empirical physics" as the one "which we endeavour, and are alone able, to apply [when we use physics to interpret and elucidate our biology]" (Thompson 1992, p. 15). That remark was made in the context of explaining the sense in which it was still appropriate to speak of forces as determining biological form, in spite of his recognition that forces do not exist [125].

The approaches he mentions in which the variables tracked are velocities seems to reflect at least some features of the Lagrangian approach. Lagrangian mechanics is a reformulation of Newton's formulation of mechanics. In Lagrangian mechanics, energy conservation principles, rather than force balances, are used to solve equations of motion. By that time, Lagrangian mechanics was generalized so that the quantities did not even need to be velocities and spatial coordinates; they were instead called "generalized velocities" and "generalized coordinates". It then became possible to express the equations of motion of a system using only variables for generalized velocities and time. The generalized velocities bore the same kind of relationship to generalized coordinates as velocities do to coordinates in classical formulations: they expressed the change in coordinates with respect to time. What was important was that the kinds of quantities that are arguments, or inputs, into the functions that express a system's equations of motion were independent of each other and together characterized the system.

The methods proposed for biology in On Growth and Form are easily generalized, so that biology appears as a case of more general principles that apply in physics as well. For, he said, many other things in the world can be seen as cases of the phenomenon of growth, if growth is seen as change in magnitude over time: "since the movement of matter must always involve an element of time, … in all cases the *rate of growth* is a phenomenon to be considered" (Thompson 1992, p. 81). If, as he also said, rate of growth is velocity, what he is saying here is akin to approaches in Lagrangian mechanics, where generalized velocities, rather than forces, are the variables considered important in addition to coordinates of position. We shall see these notions appear elsewhere, such as the use of side-by-side depictions of forms changing over time representing the dynamics of a situation, and the notion of form as consisting of ratios of magnitudes.

Thompson emphasizes the effects of magnitude or size: because different forces are predominant at different scales (gravity at one scale, and surface tension at another), animals on different-sized scales [have] very different kinds of forms. The mechanical principles that describe these forms and how the growth of these forms is constrained are different at different scales; the different mechanical principles are responsible for the [126] difference in form we observe in animals of very different sizes. In overview, he remarks:

> We found, to begin with, that "scale" had a marked effect on physical phenomena, and that increase or diminution of magnitude might mean a complete change of statical or dynamical equilibrium. In the end we begin to see that there are discontinuities in the scale, defining phases in which different forces predominate and different conditions prevail … [the range of magnitude of life] is wide enough to include three such discrepant conditions as those in which a man, an insect and a bacillus have their being and play their several roles. (Thompson 1992, p. 77)

He describes what life is like on three different scales, each such "world" smaller than the other:

> Man is ruled by gravitation, and rests on mother earth. A water-beetle finds the surface of a pool a matter of life and death, a perilous entanglement or an indispensable support. In a

> third world, where the bacillus lives, gravitation is forgotten, and the viscosity of the liquid, the resistance defined by Stokes's law, the molecular shocks of the Brownian movement, doubtless also the electric charges of the ionised medium, make up the physical environment and have their potent and immediate influence on the organism. The predominant factors are no longer those of our scale; we have come to the edge of a world of which we have no experience, and where all our preconceptions must be recast. (Thompson 1992, p. 77)

However, when Thompson looked for an underlying common principles of which these differing phenomena are illustrative, he found it in the principle of similitude, the same one he credited to Galileo in his work on mechanics of materials, and which he says is also recognizable in Newton's explanation of his discovery of the theory of gravitation.

The compilation of D'Arcy Thompson's works into a massive masterwork unified around the theme of growth and form was not published until 1917, but it is based on lots of scientific and engineering work done prior to 1914. He seems to have been especially interested in artificial and natural flight, even citing technical works from the years following the Wright Brothers' 1908 demonstrations in Europe, such as G. H. Bryan's 1911 *Stability in Aviation*, F.W. Lanchester's 1909 *Aerodynamics*, and [127] George Greenhill's 1912 *The Dynamics of Mechanical Flight*. He cited works more directly relevant to the biological emphasis of the work, too, such as E. H. Hankin's 1913 *Animal Flight*, and many, many scientific papers about insects and other animals.

In *On Growth and Form*, he cited papers from those who had been thinking about aviation well before 1900, too, including Helmholtz's 1873 paper on similar motions and dirigibles, mentioned earlier. Recall that, in that paper, Helmholtz had shown how, even when the differential equations governing the motions of dirigibles could not be solved, one could rewrite the equations in a form such that the coefficients were all dimensionless parameters. Then he showed that any two situations in which these dimensionless parameters were the same would have the same solutions. This gives a mathematically sound basis on which one can infer the motions of dirigibles (which were extremely large and unwieldy) from a model or from other observed cases (Fig. 17.12).

However, in citing this paper, D'Arcy Thompson does not draw from it anything more general than the kind of reasoning used for a specific case; recall that he was interested in showing that Helmholtz's conclusion held only assuming that the energy keeping a bird aloft came from muscular energy. He did not take issue with Helmholtz's method, only with his assumptions.

Still, it is telling that D'Arcy Thompson does not seem to be interested in the general theory of dimensions and similarity, or even in hydrodynamical similarity, which is rather presciently laid out in the remarkable paper by Helmholtz to which he refers, in which Helmholtz's methodology outpaces both his assumptions and conclusions. It was a paper ahead of its time. Helmholtz had directed his analysis of similar motions to the practical problem of steering air balloons, not gliders or airplanes, though it applied equally to both problems. When research into heavier-than-air flight was pursued in earnest, Helmholtz's paper was resurrected and recognized as containing the basis for all the important dimensionless numbers in hydrodynamics.

The attitude we see in Thompson's treatment of dimensional analysis in *On Growth and Form*—being interested in, even inspired by, the principle of similitude found in Galileo and Newton, (Thompson 1992, p. 79) yet being content to be led by that interest only so far as necessary to solve a problem at hand—seems to be representative of scientists of that era. Reasoning based on dimensional analysis was used to reach conclusions, and these conclusions were [128] considered basic principles of a general class of problems. Hence, we have various "laws" for specific kinds of situations, such as liquid flow in pipes, boats being towed in canals, streams flowing into lakes, and so on, with a corresponding rule about how a measurement taken on one scale has to be transformed to yield the corresponding value in the situation on another scale. So, for instance, in discussing the speed of aquatic animals, wherein the resistance is provided not by gravitational forces, but by "skin-friction," he reasons:

> Now we have seen that the dimensions of W are l^3 and of R are l^2; and by elementary mechanics W α RV^2, or V^2 α W/R. Therefore V^2 α l^3/l^2 and V α [square root of l]. This is what is known as Froude's Law, of the correspondence of speeds – a simple and most elegant instance of 'dimensional theory'. (Thompson 1992, p. 31)

He goes on to say that sometimes such questions about the effect of scale are "too complicated to answer in a word".

He points out that, depending on an engine's design, the design work can instead depend on the square, rather than the cube, of linear dimensions, and he mentions a different law in such a case: Froude's law of steamship comparison. In a footnote, he cites with approval Lanchester's wry remark that "the great designer was not hampered by a knowledge of the theory of dimensions," which reflects a respect for practical knowledge above this kind of theoretical principle (Thompson 1992, p. 31n) (Fig. 17.13).

Thompson goes on to show that there are subtleties involved that complicate such simplified generalizations. They usually have to do with details of the mechanisms by which different functions are accomplished, so they actually tend to be criticisms of the assumptions used rather than of the methodology of dimensional analysis. One

X.

Ueber ein Theorem, geometrisch ähnliche Bewegungen flüssiger Körper betreffend, nebst Anwendung auf das Problem, Luftballons zu lenken.

„Monatsberichte der Königl. Akademie der Wissenschaften zu Berlin" vom 26. Juni 1873. S. 501—514.

Fig. 17.12 Helmholtz's remarkable 1873 paper on similar motions and the steering of air balloons, cited by D'Arcy Wentworth Thompson's *On Growth and Form* for its method. The paper is now recognized for having identified the dimensionless parameters important in meteorological research, and hence to the work Wittgenstein and Eccles were doing at the Kite Flying Station in Glossop in 1908

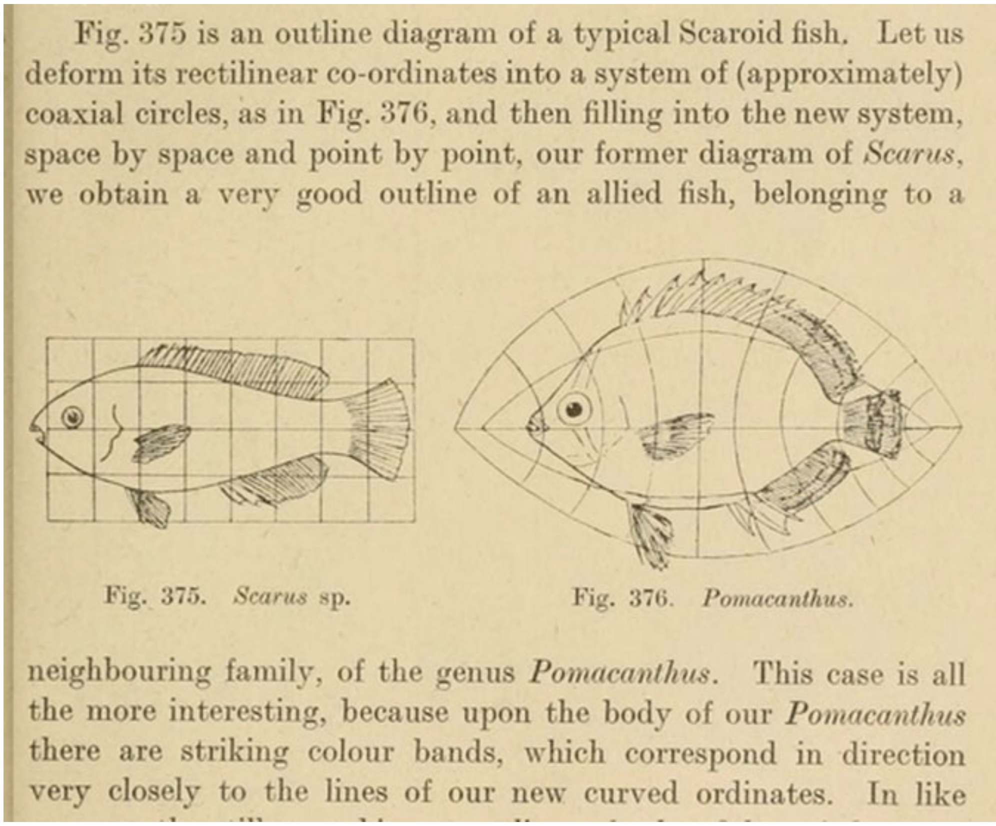
Fig. 375 is an outline diagram of a typical Scaroid fish. Let us deform its rectilinear co-ordinates into a system of (approximately) coaxial circles, as in Fig. 376, and then filling into the new system, space by space and point by point, our former diagram of *Scarus*, we obtain a very good outline of an allied fish, belonging to a

Fig. 375. *Scarus* sp.

Fig. 376. *Pomacanthus*.

neighbouring family, of the genus *Pomacanthus*. This case is all the more interesting, because upon the body of our *Pomacanthus* there are striking colour bands, which correspond in direction very closely to the lines of our new curved ordinates. In like

Fig. 17.13 An example of an illustration of the transformation of one form into another from D'Arcy Wentworth Thompson's *On Growth and Form*. Published in 1917, it reflects on and discusses many works on similarity and flight research around the same time Wittgenstein was writing the *Tractatus*

example of such a criticism is the point he had made about analyses based on dimensional reasoning that neglected the importance of gliding flight, which completely reversed the previous conclusion about the possibility of humans achieving heavier-than-air flight. The subtleties, anecdotes, and considerations Thompson brings up are meant to temper looking to any specific derivation using the principle of similitude as the arbiter of effects of scale—that is, to warn against regarding such laws of correspondence as themselves principles of nature. The validity of the particular laws of nature that are [129] consequences of the principle of similitude are very dependent on correct insight into the functions and forces relevant to the behavior of the machine or organism. The principle of similitude, though, is simply a principle about the behavior of forces and functions.

So despite the reservations expressed, the principle of similitude is called out as the underlying principle of D'Arcy Thompson's book, and with appropriate justification:

> In short, it often happens that of the forces in action in a system some vary as one power and some as another, of the masses, distances or other magnitudes involved; the "dimensions" remain the same in our equations of equilibrium, but the relative values alter with the scale. This is known as the "Principle of Similitude," or of dynamical similarity, and it and its consequences are of great importance. (Thompson 1992, p. 25)

Thompson's conclusion here about the significance of the principle of similarity, or dynamical similarity, follow his more specific observations that:

> A common effect of scale is due to the fact that, of the physical forces, some act either directly at the surface of the body, or otherwise in proportion to its surface or area; while others, and above all gravity, act on all particles, internal and external alike, and exert a force which is proportional to the mass, and so usually to the volume of the body. (Thompson 1992, p. 25)

Thus, the principle of similitude is a very general principle of science—of any science using measurement, in fact. The criticisms and qualifications D'Arcy Thompson raised were directed at inappropriate uses of the specific "laws" and "laws of correspondence" derived from considerations of similarity and theory of dimensions, not at the principle of similitude itself. Usually, the principle of similarity states conditions of similarity in terms of the constancy of a certain dimensionless parameter, and hence states when similar motions are expected, within a certain "scale" or "world". However, there are often discontinuities in behavior, such as when a flow transitions from smooth to turbulent, or as a substance transitions from a liquid to a gas. D'Arcy Thompson seems to cite the principle of similarity as an explanation of both the similarities that can be drawn within a particular scale and the discontinuities that exist between scales [130]. In fact, the way Thompson has stated the principle, it explains not only similarities and differences in behavior of animals and artificial machines within the same one of the three "worlds" he describes, but it also accounts for the discontinuities and very different kinds of forms and forces encountered between those three "worlds".

Physical similarity, dynamical similarity, and principles of similarity or similitude became more prominent topics in 1913 and early 1914. For one thing, an English translation of Galileo's *Dialogue Concerning Two New Sciences* was published in February 1914, after being practically unobtainable (Galileo 1914) (Fig. 17.14).

It was bound to be of great interest to anyone interested in the history or philosophy of science. According to the Translator's Preface by Henry Crew and Alfonso De Salvio, copies of the previous translation of Two New Sciences into English, done in 1730 by Thomas Weston, had become "scarce and expensive" by 1914 (Galileo 1914, p. vi). An even earlier English translation had "issued from the English press in 1665". But fate intervened and, they said, "It is supposed that most of the copies were destroyed in the great London fire which occurred in the year following, … even [the copy] belonging to the British Museum is an imperfect one" (Galileo 1914, pp. v–vi). The drama of the long struggle for this work to become available in English to twentieth-century readers made it a publishing event. At least, that is how the translators saw it:

> For more than a century English speaking students have been placed in the anomalous position of hearing Galileo constantly referred to as the founder of modern physical science, without having any chance to read, in their own language, what Galileo himself has to say. Archimedes has been made available by Heath; Huygens' Light has been turned into English by Thompson, while Motte has put the Principia of Newton back into the language in which it was conceived. To render the Physics of Galileo also accessible to English and American students is the purpose of the following translation. (Galileo 1914, p. v)

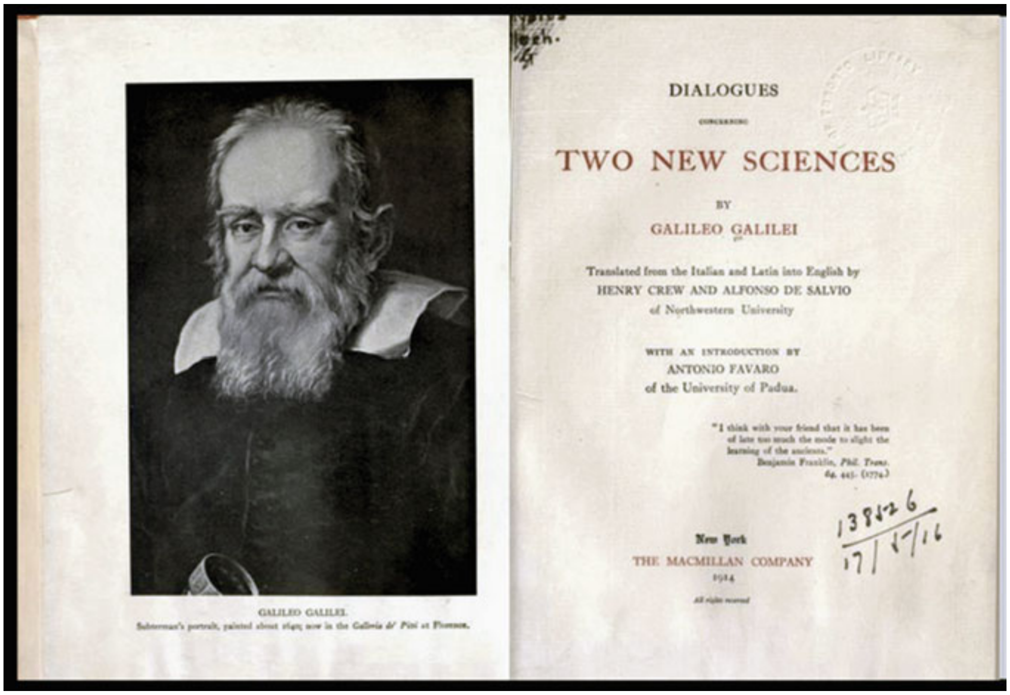
DIALOGUES
TWO NEW SCIENCES
BY
GALILEO GALILEI
Translated from the Italian and Latin into English by
HENRY CREW AND ALFONSO DE SALVIO
of Northwestern University
WITH AN INTRODUCTION BY
ANTONIO FAVARO
of the University of Padua.
"I think with your friend that it has been of late too much the mode to slight the learning of the ancients."
Benjamin Franklin, Phil. Trans. 64. 445. (1774.)
New York
THE MACMILLAN COMPANY
1914
All rights reserved

Fig. 17.14 The 1914 publication of a new English translation of Galileo's *Dialogues Concerning Two New Sciences* by the physicist Henry Crew was a major publishing event. English translations of the work had been "practically unobtainable"; however, Wittgenstein owned a rare and expensive 1730 edition translated by Thomas Weston. It is known for discussing the problem of scaling in mechanics and in biology

Crew lectured on the topic of the things to be learned from Galileo in 1913, and he mentioned the "theory of dimensions" in passing, even before the English translation of *Two New Sciences* was published (Crew 1913).

According to the recent study of Wittgenstein's books left with Russell [mentioned in a previous chapter of *Wittgenstein Flies A Kite*], which were books Wittgenstein owned during the period from 1905 to 1913, Wittgenstein owned one of the […] [131] editions of Galileo's *Discourses Concerning Two New Sciences* published in London in 1730. [Taking into account what we have just seen, i.e., that the edition would have been "scarce and expensive," this entry on the list takes on a special significance. A perusal of the list shows that] it is the *only* book by Galileo in the collection of Wittgenstein's books surveyed. Oystein Hide, the author of the study, remarks that "Wittgenstein later drew comparisons between his philosophical activity and the work of, for example, Galileo within his scientific field" (Hide 2004, p. 74).

The collection of books owned by Wittgenstein that Hide describes does not contain many other books of this sort, with two striking exceptions: a six-volume set of facsimilies of notebooks of "Leonardo De Vinci" published in Paris in 1891, and a copy of *Principia Philosophia* by "Isaco Newtono" published in 1728. Galileo and Newton stand out as the early scientists who wrote on similarity and similitude. We saw that D'Arcy Thompson cited Newton's use of similitude in discovering the law of gravitation, and that Boltzmann cited Newton in his encyclopedia article on models just after mentioning models of flying machines, in mentioning "The theory, initiated by Sir Isaac Newton, of the dependence of various effects on the linear dimensions". And we shall see that the physicist Heike Kamerlingh Onnes mentions Newton's use of mechanical similitude in his Nobel lecture in 1913. Leonardo da Vinci [was, and] is known not only for his work on flying machines, including parachutes and helicopters, but for his work in hydrodynamics and his explicit discussions and illustrations of the use of proportion in art as well as in science. His notebooks used both his artistic capabilities and his mechanical genius. His sketch of the human body with geometric proportions overlaid on it is well-known, but he investigated many topics, and his work in hydrodynamics was striking. His sketches of fluid flows, especially turbulent flows, show a remarkable talent for observation of the phenomena that were being investigated with such intensity in the years prior to 1914. He is also known for his use of proportion in art as well as in science. It's possible that the theft of the Mona Lisa from the Louvre in 1911 and its recovery in 1913 put Leonardo's works in the public eye during this time. At any rate, the fact that among the books Wittgenstein owned prior to 1914, we find expensive copies of works by Newton and Galileo in which they discuss similitude may reflect interests stimulated by the resurgence of interest in their works on similitude in England (and perhaps elsewhere) in the years preceding 1914. The interest in Leonardo's notebooks can be accounted for by the aeronautical work in [132] his notebooks alone, especially his invention of a helicopter, but it also fits with an interest in ratio and proportion.

The publishing event of a new English translation of Galileo's *Two New Sciences* in 1914, which was being written about and discussed in 1913 prior to its publication, is significant to our story because of the prominence of the discussion of similarity it contains and the accessible and memorable way the principle is explained and its consequences depicted. One of the translators, Henry Crew, was a physicist, a professor of physics at Northwestern University, and had studied with Hermann von Helmholtz in Berlin in 1883–1884; (Cahan 2004) this would have been after Helmholtz had already developed the criteria for similar motions, and perhaps it accounts for Crew's interest in translating this particular work of Galileo's. In a nutshell, the dialogue begins with a conversant using a limited notion of similarity (geometric similarity), puzzling over the invalidity of the consequences one can draw from it, and then proceeds to a discussion in which the same conversant comes to use a more general notion of similarity, from which the conclusions and experimental predictions drawn are in fact valid. There are not, as in D'Arcy Thompson's treatment of the principle, examples about flight, but there is more emphasis on correct, rather than incorrect, uses of the principle.

Galileo's dialogue begins with Salviati, usually taken to be the voice of Galileo, recounting numerous examples of a large structure that has the same proportions and ratios as a smaller structure but that is not proportionately strong. In these opening pages of the dialogue, the wise and seasoned Salviati explains to the earnest but puzzled Sagredo that "if a piece of scantling [corrente] will carry the weight of ten similar to itself, a beam [trave] having the same proportions will not be able to support ten similar beams". The phenomenon of the effect of size on function of machines of similar design holds among natural as well as artificial forms, Salviati explains: "just as smaller animals are proportionately stronger and more robust than the larger, so also smaller plants are able to stand up better than larger". Perhaps the most well-known of Salviati's illustrations is about giants:

> … an oak two hundred cubits high would not be able to sustain its own branches if they were distributed as in a tree of ordinary size; […] and nature cannot produce a horse as large as twenty ordinary horses [133] or a giant ten times taller than an ordinary man unless by miracle or by greatly altering the proportions of his limbs and especially his bones, which would have to be considerably enlarged over the ordinary. Likewise the current belief that, in the case of artificial machines the very large and the small are equally feasible and lasting is a manifest error. (Galileo 1914, pp. 52–53)

It is this point about scale we saw reflected throughout D'Arcy Thompson's work, applied especially to the case of flight. Yet it was well known in 1914 that engineers used scale models. The question was whether a certain use was valid, and the problem was that determining this often seemed a matter of engineering knowledge and skill, not a matter of pure science Recall that there was some ambivalence in D'Arcy Thompson's discussion; he criticized a number of analyses based on dimensional considerations, although he is clear that his reservations were not about the validity of the principle of similitude itself. He credited Galileo as the first to articulate the principle of similitude, or dynamical similarity.

Although Galileo's work opens with the wise participant in the dialogue reminding the others of the reasons for the lack of giant versions of naturally occurring life-forms, it soon proceeds to a case of a valid use of a small (artificial) machine to infer the behavior of a large (artificial) machine. As in Helmholtz's reasoning in his paper on similar motions and dirigibles, the basis for similarity is found not in mere geometric similarity, but, more deeply, in dimensional considerations drawn from an equation of motion. At a later point in the dialogue, Sagredo makes use of Salviati's statement that the "times of vibration" (period of oscillation) of bodies suspended by threads of different lengths "bear to each other the same proportion as the square roots of the lengths of the thread; or one might say the lengths are to each other as the squares of the times" (Galileo 1914, p. 139). From this, Sagredo uses one physical pendulum to infer the length of another physical pendulum:

> Then if I understand you correctly, I can easily measure the length of a string whose upper end is attached at any height whatever even if this end were invisible and I could see only the lower extremity. For if I attach to the lower end of this string a rather heavy weight and give [134] it a to-and-fro motion, and if I ask a friend to count a number of its vibrations, while I, during the same time-interval, count the number of vibrations of a pendulum which is exactly one cubit in length, then knowing the number of vibrations which each pendulum makes in the given interval of time one can determine the length of the string. Suppose, for example, that my friend counts 20 vibrations of the long cord during the same time in which I count 240 of my string which is one cubit in length, taking the squares of the two numbers, 20 and 240, namely 400 and 57600, then, I say, the long string contains 57600 units of such length that my pendulum will contain 400 of them; and since the length of my string is one cubit, I shall divide 57600 by 400 and thus obtain 144. Accordingly I shall call the length of the string 144 cubits. (Galileo 1914, p. 140)

The basis on which Sagredo infers the length of the larger from the smaller is a fundamental relationship—the relationship between length and period that describes the behavior of any pendulum, which can be expressed in terms of the constancy of the value of a certain ratio containing them. What he derives from it is a *law of correspondence* telling him how to find the corresponding length in the large pendulum from the length of the small. Salviati (the voice of Galileo) responds approvingly to his claim that this method will yield the length of the string: "Nor will you miss it by as much as a hand's breadth, especially if you observe a large number of vibrations" (Galileo 1914, p. 140). Henry Crew, the physicist who co-translated the work, thought the emphasis on experiment in *Two New Sciences* was one of the important contributions of Galileo's work.

This work of Galileo's, which is generally credited with giving not only the first, but probably the best ever, exposition of physical similarity, appeared just as scientists from Britain's National Physical Laboratory (NPL) presented a compendium of their own about similarity (Stanton and Pannell 1914). The paper, "Similarity of Motion in Relation to the Surface Friction of Fluids," by T. E. Stanton and J. R. Pannell, was submitted to the Royal Society of London in December 1913 and was read to the Society in January of 1914. Stanton was superintendent of NPL's Engineering Department and was interested in the possibilities of using small-scale models in wind tunnels for engineering research. Except for the work of Osborne Reynolds, experimental study of similar motions in fluids was, according to the authors, only done since 1909: [135]

> Apart from the researches on similarity of motion of fluids, which have been in progress in the Aeronautical Department of the National Physical Laboratory during the last four years, the only previous experimental investigation on the subject, as far as the authors are aware, has been that of Osborne Reynolds …. (Stanton and Pannell 1914, p. 200)

Osborne Reynolds was the celebrated professor of engineering at Manchester who had retired a few years prior to Wittgenstein's arrival and enrollment there as an engineering student. What Stanton and Pannel cite as Reynolds' major discoveries were that there was a critical point at which fluid flow suddenly changed from "lamellar motion" to "eddying motion" (today we would say "from laminar flow to turbulent flow"), that the critical velocity was directly proportional to the kinematical viscosity of the water and inversely proportional to the diameter of the tube, and that for geometrically similar tubes, the dimensionless product:

$$\text{(critial velocity)} \times \text{(diameter)}/\text{(kinematic viscosity of water)}$$

was constant (Stanton and Pannell 1914, p. 200). They also noted that, no matter what the conditions of flow, whether above or below the critical velocity, whenever the values of the dimensionless product:

$$\text{(velocity)} \times \text{(diameter)}/\text{(kinematic viscosity of water)},\ \text{or}\, \mathrm{vd}/\nu$$

were the same [in two different setups], so were the corresponding values of another dimensionless parameter:

$$(\text{density}) \times (\text{diameter})^3/(\text{coefficient of viscosity of water})^2$$
$$\times\ (\text{rate of fall of pressure along the length of the pipe})$$

As is often the case, there was a complication: experiment showed that the surface roughness of the pipe wall (or of whatever surface forms the flow boundary) needed to be taken into account as well. This is a matter of geometry on a much smaller scale making a difference. However, the overall approach of the use of dimensionless parameters to establish similar situations was still seen to be valid, as their experiments illustrated:

> From the foregoing it appears that similarity of motion in fluids at constant values of the variable vd/ν will exist, provided the surfaces relative to which the fluids move are geometrically similar, which similarity, as Lord RAYLEIGH has pointed out, must extend to those [136] irregularities in the surfaces which constitute roughness. In view of the practical value of the ability to apply this principle to the prediction of the resistance of aircraft from experiments on models, experimental investigation of the conditions under which similar motions can be produced under practical conditions becomes of considerable importance … By the use of colouring matter to reveal the eddy systems at the back of similar inclined plates in streams of air and water, photographs of the systems existing in the two fluids when the value of vd/ν was the same for each, have been obtained, and their comparison has revealed a remarkable similarity in the motions. [ref: Report of the Advisory Committee for Aeronautics, 1911–1912, p. 97] (Stanton and Pannell 1914, p. 201)

The authors here refer to the dimensionless parameter "vd/ν" as a variable. This variable is a product of several measurable quantities and is a dimensionless parameter. That is, the units all cancel out, and its value is independent of the choice of units of measurement. To see this, we can talk about the dimensions of each contributor to the dimensionless parameter, also known as a dimensionless product: the dimensions of v are length divided by time; the dimensions of d are length, and the dimensions of ν are (length time length) divided by time. What Stanton and Pannell meant in referring to it as a variable was that their equation for the resistance R includes a function of this dimensionless parameter:

$$\text{resistance R} = (\text{density}) \times (\text{velocity})^2 \times (\text{some function of vd}/\nu)$$

Or, as they put it, $R = \rho v^2$ F(vd/ν, where "F (vd/ν" indicates some unspecified function of vd/ν Hence, vd/ν is a variable in the sense that the relation for resistance includes an unspecified function of vd/ν It is also a variable in the more practical sense: it can be physically manipulated.

Stanton and Pannell present this relation as a consequence of the Principle of Dynamical Similarity, in conjunction with assumptions about what "the resistance of bodies immersed in fluids moving relatively to them" depends on (Stanton and Pannell 1914, p. 201). Evidently, it was Rayleigh who suggested the generalization; they cite Rayleigh's contribution [on p. 38] of the *Report of the Advisory Committee* *for Aeronautics, 1909–1910*. Rayleigh had there spoken of the possibility of taking a more general approach than current researchers were taking in applying “the principal of dynamical similarity”. He explained his ‘more general’ approach as follows [137].

> … We will commence by supposing the plane of the plate perpendicular to the stream and inquire as to the dependence of the forces upon the linear dimension (l) of the plate and upon the density (ρ), velocity (v) and kinematic viscosity (ν) of the fluid. Geometrical similarity is presupposed, and until the necessity is disproved it must be assumed to extend to the thickness of the plate as well as to the irregularities of surface which constitute roughness.
>
> If the above-mentioned quantities suffice to determine the effects, the expression for the mean force per unit area normal to the plate (P), analogous to a pressure, is
>
> $$P = rv^2 \times f(n/vl) \qquad \text{(A)}$$
>
> where f is an arbitrary function of the one variable n/vl.
>
> It is for experiment to determine the form of this function, or in the alternative to show that the facts cannot be represented at all by an equation of form (A). (Rayleigh 1910, pp. 532–533)

Rayleigh does not here say exactly how the relation (A) is obtained from the assumptions he lists, which was not uncommon at that time when invoking the principle of similarity. Referring to his other papers, and papers by others who invoked it, such as Reynolds and Helmholtz, it is fairly clear that invoking the principle of similarity meant using dimensional considerations, sometimes using the specific fact that a scientific equation required consistent units.

Stanton and Pannell present the results obtained at the National Physical Laboratory in the paper. It is interesting to note that the results are presented in graphs where one of the variables plotted is the term $R/\rho v^2$. [Since, as explained above, “$R = \rho v^2$ F(vd/ν), where “F(vd/ν)” indicates some unspecified function of vd/ν”, the term $R/\rho v^2$ is equal to F(vd/ν)] and so it just another expression for the unspecified function, and is dimensionless. What this implies is that the laboratory experiments are not conceived of in terms of the values of individual measurable quantities such as velocity but are classified in terms of the value of a dimensionless parameter.

Rayleigh, too, published a kind of survey paper in early 1914 (Rayleigh 1914). Here he actively campaigned for wider appreciation and use of the principle, which he credited Stokes with having “laid down in all its completeness”.

Rayleigh wrote: “There is a general law, called the law of dynamical similarity, which is often of great service. In the past this law has been unaccountably [138] neglected, and not only in the present field. It allows us to infer what will happen upon one scale of operations from what has been observed at another” (Rayleigh 1914). He then discussed the example of a sphere moving uniformly through air, remarking that, if the kinematic viscosity can be assumed to be the same across the cases considered:

> When a solid sphere moves uniformly through air, the character of the motion of the fluid round it may depend upon the size of the sphere and upon the velocity with which it travels. But we may infer that the motions remain similar, if only the product of diameter and velocity be given. Thus if we know the motion for a particular diameter and velocity of the sphere, we can infer what it will be when the velocity is halved and the diameter doubled. The fluid velocities also will everywhere be halved at the *corresponding* places. (Rayleigh 1914)

Fig. 17.15 Britain's National Physical Laboratory in 1914, as featured in an article in *Nature* magazine. In 1908, Thomas Stanton, Superintendent of NPL's Engineering Department since 1901, was directed to apply techniques of wind research to the study of flight, especially the efficiency and safety of the aeroplane. Stanton had gone to study with Osborne Reynolds at Owens College in Manchester in 1888, the same institution where Wittgenstein would later enroll as a student in aeronautical research in 1908 (though it later became Victoria University, then the University of Manchester). In 1914, Stanton and Pannell published the results of a major study in hydrodynamics, "Similarity of Motion in Relation to the Surface Friction of Fluids"

So we see one use of the principle is to be able to use one observation or experiment as representative of a whole class of actual cases: all the other cases to which it is similar, even though the cases may have very different values of measurable quantities such as velocity. The important fact of the situation is the dimensionless parameter just mentioned: "It appears that similar motions may take place provided a certain condition be satisfied, viz., that the product of the linear dimension and the velocity, divided by the kinematic viscosity of the fluid, remain unchanged" (Figs. 17.15 and 17.16).

The important feature of the situation is the value of this dimensionless parameter; that would mean that, even in cases of a *different fluid*, so long as this dimensionless product is the same, the motions will be similar! Can that be right? Yes, and Rayleigh points out that it is a particularly useful application of the principle: "If we know what happens on a certain scale and at a certain velocity in *water*, we can infer what will happen in *air* on any other scale, provided the velocity is chosen suitably".

There is a qualification he adds here, one that the reader familiar with the qualifications needed since the advent of special relativity (that relativistic phenomena do not appear only so long as the velocities involved are much smaller than the velocity of light) might see as an analogue for sound: "It is assumed here that the compressibility of the air does not come into account, an assumption which is admissible so long as the velocities are small in comparison with that of sound" (Rayleigh 1914, p. 246). Neglecting the compressibility of air in certain velocity ranges and not others is another illustration of [139] the discontinuity of scale D'Arcy Thompson emphasized. It does not mean that the method of similarity requires neglecting the compressibility of air, just that in the low-velocity range, establishing similarity of motions between two different situations does not require accounting for it.

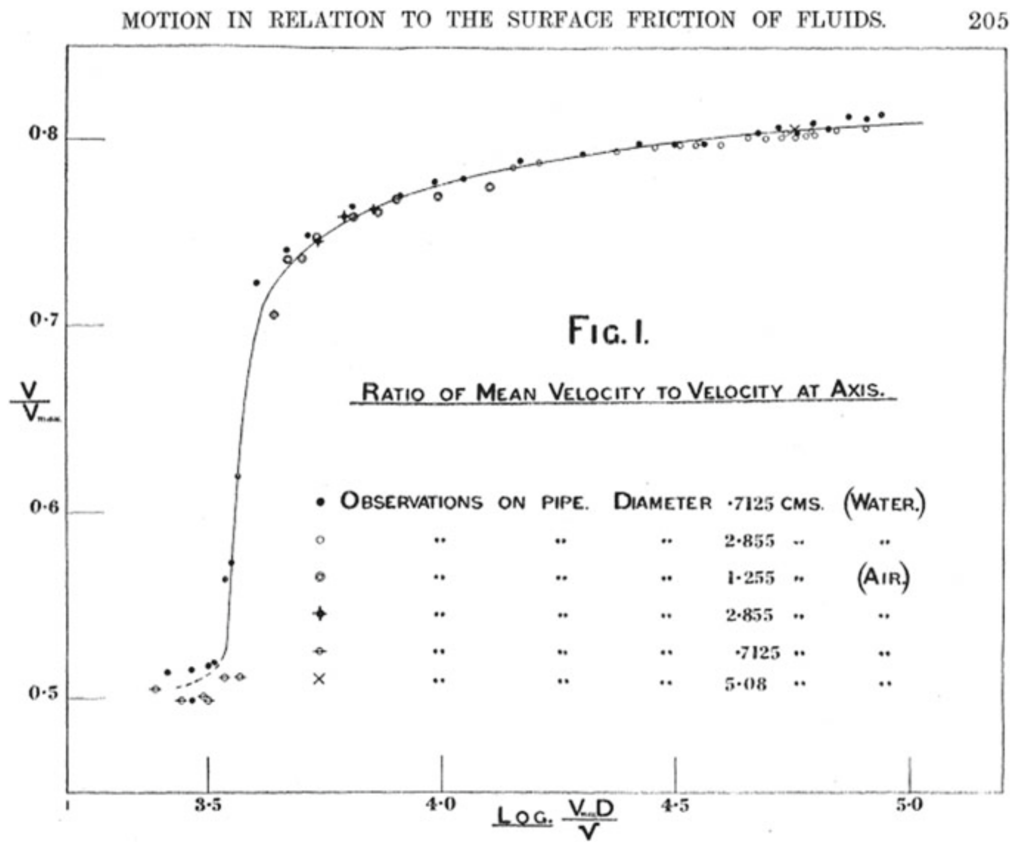

Fig. 17.16 Figure I from Stanton and Pannell's 1914 "Similarity of Motion…" paper

However, Rayleigh also pointed out that, in contrast to the compressibility of air, it appeared that viscosity *was* important in many cases where it was so small that it seemed improbable that it should matter (Rayleigh 1914, p. 237). When viscosities were low, as in water, one would not expect that the actual value of viscosity would be a significant factor in water's qualitative behavior Osborne Reynolds' results on fluid flow in pipes had shown that it is; Reynolds began to suspect that viscosity was important even in water when he observed unexpected changes in fluid flow as the temperature was varied. Since viscosity varies with temperature, he investigated the effect of viscosity and found that it was indeed important for fluid flow through pipes, even for nonviscous fluids such as water. He also investigated cases where viscosity was the "leading consideration," as Rayleigh put it, in remarking that "It appears that in the extreme cases, when viscosity can be neglected and again when it is paramount, we are able to give a pretty good account of what passes. It is in the intermediate region, where both inertia and viscosity are of influence, that the difficulty is greatest (Rayleigh 1914, pp. 245–246).

These unexpected experimental results showing that viscosity needed to be taken into account in these intermediate regions were as unwelcome as they were unexpected, since there was already on hand a well-developed theory of hydrodynamics that neglected viscosity—on that lent itself to closed-form mathematical solutions of the equations. This discipline, hydrodynamics, had produced many beautiful mathematical theorems and solutions. If fluid friction were included, though, the mathematical equations became intractable, as Helmholtz had remarked in his papers on hydrodynamics. Helmholtz had lain out and responded to this problem in his 1873 paper on similar motions of air balloons; his response had been to show how to use the intractable equations to find similarity conditions – the relevant dimensionless parameters for dynamical similarity—to permit inferring the behavior of a large balloon from experiments on smaller ones. This was an experimental alternative to the impossible task of finding a solution of the intractable hydrodynamical equations that accounted for fluid friction, or viscosity. As the authors of an historical survey put it: "Helmholtz, in fact, presented to the Berlin Academy of Sciences in 1873 (Fig. 17.12) a [140] dimensional analysis of the equations of fluid motion that already encompassed what we know today s the Froude, Reynolds, and Mach criteria for model-prototype similarity" (Rouse and Ince 1957, p. 200). [The guidance about dynamical similarity in Stokes' (Fig. 17.17) 1850 paper and Helmholtz' 1873 paper] may be what Rayleigh was referring to when he added that, even in the intermediate region, where the difficulty is the greatest, "we are not wholly without guidance," lamenting the neglect of "the law of dynamical similarity" (Rayleigh 1914, p. 246).

One might think that, by 1914, when the use of wind tunnels had become recognized as essential to practical aeronautical research, this principle would have become accepted and would no longer be in question, at least among aeronautical researchers. But if Rayleigh's estimation of the state of the profession is correct, even as late as 1914 this wasn't so! He writes that "although the principle of similarity is well established on the theoretical side and has met with some confirmation in experiment, there has been much hesitation in applying it, … (Rayleigh 1914, p. 246). He especially mentions problems in its acceptance in aeronautics due to skepticism that viscosity, which is extremely small in air, should be considered an important parameter: "In order to remove these doubts it is very desirable to experiment with different viscosities, but this is not easy to do on a moderately large scale, as in the wind channels used for aeronautical purposes".

Rayleigh tries to persuade the reader of the significance of the effects of viscosity on the velocity of fluid flow by relating some experiments he performed with a cleverly designed apparatus in his laboratory. The apparatus consisted of two bottles containing fluid at different heights, connected by a tube with a constriction, through which fluid flowed due to the difference in "head," or height of fluid, in the two bottles. The tube with the constriction contained fittings that allow measurement of pressure head at the constriction, and on either side of it. To investigate the effects of viscosity, Rayleigh varied the temperature of the fluid, which changes the fluid viscosity, and he observed how the velocity of the fluid flowing between the two bottles was affected. The kind of relationship he establishes and uses is of the form Galileo employed in reasoning from one pendulum to another. In other words, he worked in terms of ratios (ratios of velocities, ratios of viscosities, ratios of heads), and he employed the fact that some ratios are the square root of others. He took the

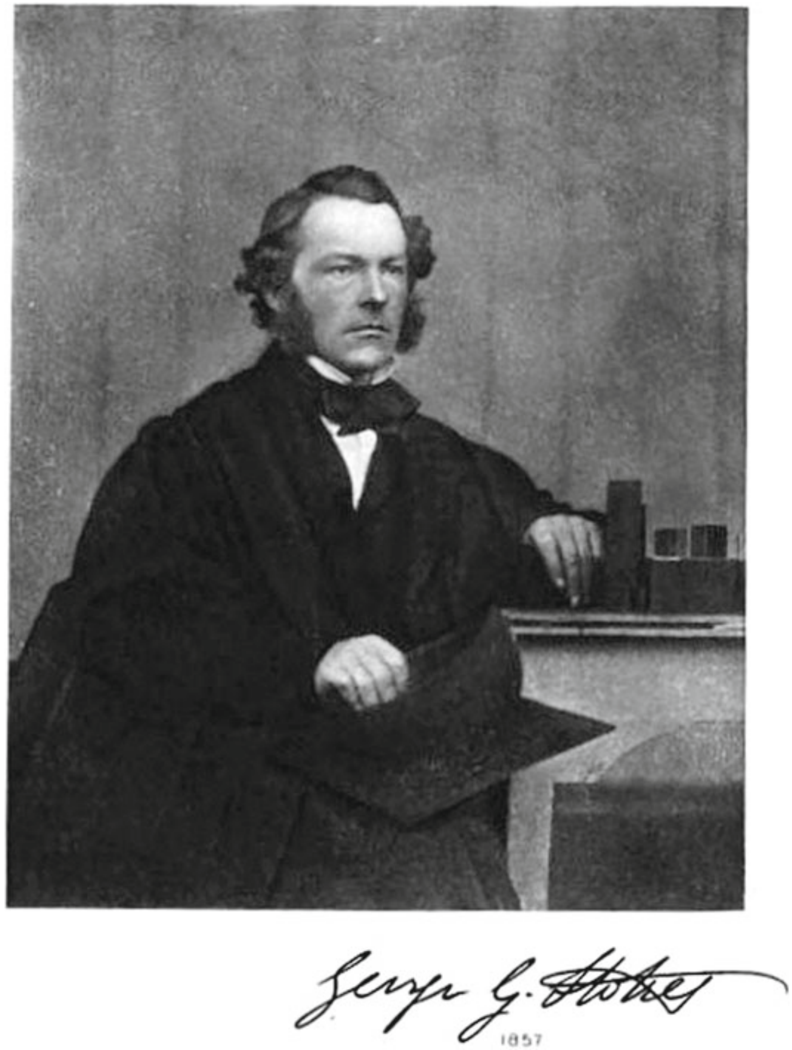

Fig. 17.17 Sir George Gabriel Stokes (1819–1903) Rayleigh credited Stokes for the Principle of Dynamical Similarity. In 1850 Stokes had read a paper "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums" to the Cambridge Philosophical Society, which identified the dimensionless parameters important for hydrodynamical similarity. *Image* Google Books

experimental results he reported in this 1914 paper to conclusively settle the question of the relevance of viscosity to fluid motions [141].

Thus, in early 1914, there was first a report form Britain's National Physical Laboratory on the experimental work performed there recently on similarity of fluid motions, presented in January; then, in March, Rayleigh's paper on Fluid motions appeared. Rayleigh's paper explained the need fulfilled by such experimental work, which employed the principle of dynamical similarity and reported his own experiments meant to eradicate any remaining skepticism about the principle. In these works, there is a move toward generalization of the specific relationships then in use for specific applied problems in hydraulics. The work at the National Physical

Laboratory was aimed at carrying out a "systematic series of experiments" for "the purpose of establishing a general relations which would be applicable to all fluids and conditions of flow". The dual purpose of both establishing the relation and exploring its limits is seen in how Stanton and Pannell stated their purpose in the paper submitted in December 1913:

> The object of the present paper is to furnish evidence confirming the existence, under certain conditions, of the similarity in motions of fluids of widely differing viscosities and densities which has been predicted, and further by extending the observations through a range in velocity of flow which has not hitherto been attempted to investigate the limits of accuracy of the generally accepted formulae used in calculations of surface friction. (Stanton and Pannell 1914, pp. 199–200)

The immediate motivation for performing experiments to furnish this evidence was practical, though there was no hope of producing an equation from which predictions could be directly generated. Recall that the general relation the paper proposed contained an undetermined function. So the goal was to find out how to find situations similar to the one you needed to know about that could be carried out in the laboratory and to use the correspondence between them to determine what you wanted to know. It was the question of similar motions that was of special interest in the investigation: "experimental investigation of the conditions under which similar motions can be produced under practical conditions becomes of considerable importance" because of "the practical value of the ability to apply this principle to the prediction of the resistance of aircraft from experiments on models" (Stanton and Pannell 1914, p. 201). To be brief, what they were really [142] interested in was providing a methodology for experimental engineering scale models in hydrodynamics and aerodynamics. Somehow the key lay not in a fully detailed equation, but in an equation that enabled them to determine when certain situations were similar and told them how to determine corresponding motions in one situation from observations made on a similar situation.

Adding to this swirl of ideas gathering in late 1913 and early 1914 was the attention being paid to a wonderful experimental result based on similarity and correspondence in the area of thermodynamics: the successful liquefication of helium. The same month that Stanton and Pannell submitted their paper from the National Physical Laboratory, December 1913, Heike Kamerlingh Onnes delivered his Nobel lecture. Again, the ideas were not brand-new in 1913, but the worldwide public attention being paid to them was. Onnes had published a paper in 1881 called "General Theory of Liquids," in which he argued that van der Waals' "Law of Corresponding States," which had just been published the prior year, could be derived from scaling arguments, in conjunction with assumptions about how molecules behaved (Onnes 1881). Van der Waals was impressed with the paper, and a long friendship between the two ensued. Van der Waals won the Nobel Prize in 1910 for "The equation of state for gases and liquids," and he mentioned in his lecture that the "law of corresponding states" had become "universally known," though he said nothing more about it there (van der Waals 1910). The Nobel Prize awarded to Onnes in 1913 brought wider attention to Onnes's account of the foundations of the law of corresponding states—though you might not guess it from the title of Onnes's award, given for

"Investigations into the properties of substances at low temperatures, which have led, amongst other things, to the preparation of liquid helium".

Onnes's lecture highlighted the connection between his investigations into properties of substances at low temperatures and similarity principles:

> [F]rom the very beginning … I allowed myself to be led by Van der Waals' theories, particularly by the law of corresponding states which at that time had just been deduced by Van der Waals.
>
> This law had a particular attraction for me because I thought to find the basis for it in the stationary mechanical similarity of substances and from this point of view the study of deviations in substances of simple chemical structure with low critical temperatures seemed particularly important". [143] (Onnes 1913, p. 306)

What's special about low temperatures, of course, is that, according to the kinetic theory of gases on which van der Waals' equation of state was based, there would be much less molecular motion. Onnes's approach in looking for the foundation of the law of corresponding states has a slightly different emphasis than the kinetic theory of gases. Boyle's Law (often called the ideal gas law) and van der Waals' equation were based on investigating the relationship between the microscale (the molecular level) and the macroscale (the properties of the substance, such as temperature and density.) but Onnes was instead looking at the foundation for the similarity of states. Like van der Waals, he looked to mechanics and physics for governing principles, but Onnes pointed out that it was also useful to look at principles of similarity. At low enough temperatures, where motion of the molecules was not the predominant factor, the relevant principles of similarity would be principles of static mechanical similarity, as opposed to dynamical similarity.

As had happened in Osborne Reynolds' work in hydrodynamics, a criterion for similarity had arisen out of investigations into the transition from one regime to another. For Reynolds, it as the critical point at which fluid flow underwent a transition from laminar to turbulent flow (or, in his terminology, from "lamellar" to "eddying" flow) that led to the identification of the dimensionless parameter that later became known as Reynolds Number. The Reynolds Number is in a way a criterion of similarity: fluid systems with the same Reynolds Number will be in the same flow regime, regardless of the fluid. So it was with thermodynamics: the critical point at which a substance undergoes a transition from the gaseous to liquid state led to the identification of a criterion of similarity of states that held for all substances.

Van der Waals was very interested in the continuity of states, particularly the continuity between the gas and liquid states. Although his prize was awarded for his work on "The equation of state for gases and liquids," the doctoral dissertation in which he presented the equation had been titled "On the continuity of the gas and liquid state," and he emphasized the important role that the idea of continuity had played in developing the equation:

> … I conceived the idea that there is no essential difference between the gaseous and the liquid state of matter — that the factors which, apart from the motion of the molecules, act to determine the [144] pressure must be regarded as quantitatively different when the density changes and perhaps also when the temperature changes, but that they must be the very

factors which exercise their influence throughout. And so the idea of continuity occurred to me. (Onnes 1913, p. 255)

Van der Waals thought that the force of attraction between molecules would be a big part of the story about the continuity of the gas and liquid states, so he wanted to revise the equation to account for it. He added two more variables to Boyle's Law, which he called "a" and "b" and which are characteristic of a particular substance. Instead of Boyle's ideal gas law—(pressure) × (volume) = (gas constant) × (temperature)—van der Waals' equation can, as Johann Levelt Sengers points out, be written as a cubic equation—that is, as a third-degree polynomial in volume, with coefficients depending on temperature and pressure. There are many variations on the equation he proposed, but the specific form of the equation and the details about the various ways in which it was revised are not important to us here. What is significant about it for us here is that van der Waals could use the equation to explicitly solve for the values of volume, temperature, and pressure at the critical point (Levelt-Sengers 2002). The solution of the equation at the critical point is thus for the trio and is in terms of the parameters a and b introduced by van der Waals:

$$P_c = a/\left(27b^2\right)$$

$$V_c = 3b$$

$$RT_c = 8a/(27b)$$

The specific expressions [above] are not important to our story here; the important thing to notice about them is that the critical values of pressure, volume, and temperature [indicated by the subscript 'c' in the expressions above] can be expressed in terms of the parameters a and b that van der Waals introduced, and that [the specific values of] a and b are characteristic of a [particular] substance. Once these are in hand, the next conceptual step is to use these expressions to eliminate the constants a, b, and R from his equation of state he does this by using the critical values of these measurable quantities as units of measurement. Or, as we might say today, he normalizes pressure, volume, and temperature using their critical values, just as Mach number is a ratio of the velocity of something (such as a bullet) in a medium (such as air) to the celerity [(velocity of sound in)] in the medium. It is only because their critical values are given as functions of the constants a [145] and b that his reduction is possible. Thus, reduced pressure, denoted by P*, is the pressure of the gas or liquid divided by P_c, and V* and T* are similarly defined. This yields an equation of state in which neither a, b, nor R appears. Levelt Sengers writes, "This is a truly remarkable result". The equation:

… is universal: all characteristics of individual fluids have disappeared from it or, rather, have been hidden in the reduction factors. The reduced pressures of two fluids are the same if the fluids are in corresponding states, that is, at the same reduced volume and pressure.

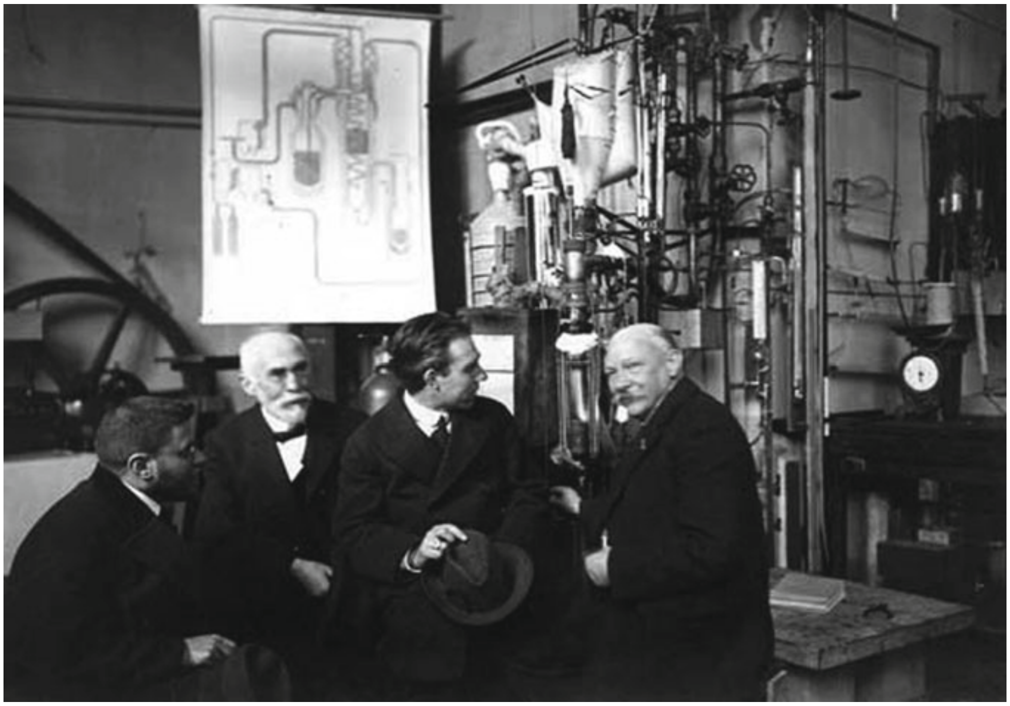

Fig. 17.18 Paul Ehrenfest, Hendrik Lorentz, Neils Bohr, and Kamerlingh Onnes. Onnes' "Principle of Corresponding States" was a remarkable result about fluids, allowing scientists to deduce behavior of one liquid from experiments on another. Onnes was inspired by Newton's use of mechanical similarity in developing it. *Photo credit* nobelprize.org

In his presentations to the Academy, van der Waals deduces straightforwardly that in reduced coordinates, the vapor pressure curve and the coexistence curve must be the same ("fall on top of each other") for all fluids (Levelt-Sengers 2002, p. 25) (Fig. 17.18).

The principle of corresponding states allowed scientists to produce curves representative of all substances from experiments on a particular substance:

> The principle of corresponding states … frees the scientist from the particular constraints of the van der Waals equation. The properties of a fluid can now be predicted if only its critical parameters are known, simply from correspondence with the properties of a well characterized reference fluid. Alternatively, unknown critical properties of a fluid can be predicted if its properties are known in a region not necessarily close to criticality, based on the behavior of the reference fluid. (Levelt-Sengers 2002, p. 26)

If we reflect on how this method of prediction works, we see that the same could be said of Reynolds' discoveries about the critical point of transition from laminar to turbulent flow. Although people today tend to think in terms of a critical Reynolds Number—that is, the value the dimensionless parameter has at the point of transition—in their 1914 paper, Stanton and Pannell put Reynolds' discovery in terms of a critical velocity. By using v_c as an abbreviation for critical velocity, this can be conveniently abbreviated by saying, as he put it, that "Reynold's discovery was that for geometrically similar tubes $v_c d/\nu$ was constant," where d denotes a diameter or other chosen distance in the situation, and ν denotes viscosity. This is stating Reynold's discovery as the kind of constraint [146] provided in thermodynamics by the statement that the dimensionless parameter $(P_c \times V_c)/(R \times T_c)$ is constant. Don't be misled by the fact that these look like the kind of equations you are familiar with using to plug in values and predict the [value of the] one chosen as the unknown. As in thermodynamics, so in hydrodynamics; their value lies not in using the equation directly, but in telling how experimental curves for one substance can be used to predict the behavior of another. The curves for fluid flow were meant to apply to any fluid—hence Rayleigh's comments that experiments on oil as a reference fluid could be used to predict the critical velocity of another fluid from its properties in a noncritical region—just as the law of corresponding states allows one to make predictions about the critical points of other substances from experiments performed on a reference substance.

Onnes used this insight about corresponding states to set up an experimental apparatus to liquefy helium, which has an extremely low critical temperature. What is so exciting about his story is that he had to rely on the law of corresponding states to estimate the critical temperature so that he would know where to look—that is, so that he would know what condition to create in order for the helium to liquefy. What is especially relevant to our story is that he did more than just use van der Waals' law of corresponding states. He also gave a foundation for it that was independent of the exact form of van der Waals' equation and did not depend on results in statistical mechanics. Instead, he used *mechanical* similarity:

> Kamerlingh Onnes's (1881) purpose is to demonstrate that the principle of corresponding states can be derived on the basis of what he calls the principle of similarity of motion, which he ascribes to Newton. he assumes, with Van der Waals, that the molecules are elastic bodies of constant size, which are subjected to attractive forces only when in the boundary layer near a wall, since the attractive forces in the interior of the volume are assumed to balance each other … He realizes this can be valid only if there is a large number of molecules within the range of attraction … [Onnes] considered a state in which N molecules occupy a volume v, and al have the same speed u (no Maxwellian distribution). The problem is to express the external pressure p required to keep the system of moving particles in balance, as a function of the five parameters. He solves this problem by deriving a set of scaling relations for M, A, v, u and p, which pertain if the units of length, mass, and time are changed". [147] (Levelt-Sengers 2002, p. 30)

The 'scaling relations' Onnes developed are another way of bringing in dimensional considerations, or the "theory of dimensions" that we saw earlier as key to D'Arcy Wentworth Thompson's work in biology on the importance of size, to Galileo's arguments about similarity in mechanics of materials, to Helmholtz's paper on similar motions and dirigibles, to the work by Reynolds, Stanton, and Pannell on similar motions in fluids, and to Rayleigh's crusade for a proper appreciation and more widespread use of the principle of dynamical similarity. As Sengers notes, scaling relations are supposed to hold no matter what units are used for the measureable quantities involved. Onnes provides a criterion for corresponding states based on these scaling relations, along with assumptions abut what the molecular-sized objects are like. Sengers remarks:

> Two fluids are in corresponding states if, by proper scaling of length, time and mass for each fluid, they can be brought into the same "state of motion". It is not clearly stated what he means by this, but he must have had in mind an exact mapping of the molecular motion in one system into that of another system if the systems are in corresponding states.

Then she gives her own suggestion of what it means to be in the same "state of motion":

> In modern terms: suppose a movie is made of the molecular motions in one fluid Then, after setting the initial positions and speed of the molecules, choosing the temperature and volume of a second fluid appropriately, and adjusting the film speed, a movie of the molecular motion in a second fluid can be made to be an exact replica of that in the first fluid. (Levelt-Sengers 2002, p. 30)

Shortly after Onnes's Nobel lecture, Richard Chase Tolman, a physicist at the California Institute of Technology, published a paper titled "The Principle of Similitude" in *Physical Review*, a major physics journal in the U.S. What it suggested sounded a lot like the idea of being able to make movies of one situation that were replicas of movies of other situations except for film speed. Tolman's paper proposed the following:

> The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe. [148] (Tolman 1914, p. 244)

Tolman then proceeded to show that he could derive a variety of laws, including the ideal gas law, from the principle of similitude he proposed. He proceeded in much the same way as Onnes had done in showing that the principle of corresponding states was a consequence of mechanical similarity. Tolman first developed scaling laws, laying out a transformation rule for how each quantity from a short list he had constructed—length, time, velocity, acceleration, electrical charge, and mass—should be scaled from the present universe to the miniature universe. He laid out and justified a transformation rule for each quantity individually, but some quantities were dependent on others (the transformation equation for velocity is a consequence of the equations for length and time). His justifications seem to be based on the criterion that the two universes would be observationally equivalent from the standpoint of an observer located in one of them:

> … let us consider two observers, O and O', provided with instruments for making physical measurements. O is provided with ordinary meter sticks, clocks and other measuring apparatus of the kind and size which we now possess, and makes measurements in our present physical universe. O', however, is provided with a shorter meter stick, and correspondingly altered clocks and other apparatus so that he could make measurements in the miniature universe of which we have spoken, and in accordance with our postulate obtain exactly the same numerical results in all his experiments as does O in the analogous measurements made in the real universe. (Tolman 1914, p. 244)

Examples are that if O measures a length to be I, O' measures it to be xl, where O' has a meter stick that is shorter than O's and x is a number less than l. In obtaining the transformation equation for time, however, Tolman appeals to the physical fact that "the velocity of light in free space must measure the same for O and O'," (Tolman 1914, p. 245) and he concludes that if O measures t, O' will measure xt. So the equations for length and time are $l' = xl$ and $t' = xt$. In obtaining the transformation equation for mass, he appeals to Coulomb's law as setting a constraint that must be satisfied, in conjunction with the requirement that O and O' measure the same charge, to obtain $m' = m/x$. Once the equations for the "fundamental magnitudes, length, time and mass" have been obtained, he says, we "can hence obtain a whole series of further equations for force, temperature, etc., by merely [149] considering the dimensions of the quantity in question" (Tolman 1914, p. 247). He then tries to show how various physical relations, such as the ideal gas law, can be deduced from simple physical assumptions and his proposed principle of similitude. for relations about gravitation, however, a contradiction arises, which he embraces and uses to propose new criteria for an acceptable theory of gravitation. He ends feeling triumphant about his proposed principle. It's a new relativity principle, he concludes: "the principle of the relativity of size"!

> … in the transformation equations which we have developed we have shown just what changes have to be made in lengths, masses, time intervals, energy quantities, etc., in order to construct such a miniature world. If, now, throughout the universe a simultaneous change in all physical magnitudes of just the nature required by these transformation equations should suddenly occur, it is evident that to any observer the universe would appear entirely unchanged The length of any physical object would still appear to him as before, since his meter sticks would all be changed in the same ratio as the dimensions of the object, and similar considerations would apply to intervals of time, etc. From this point of view we see that it is meaningless to speak of the absolute length of an object, all we can talk about are the relative lengths of objects, the relative duration of intervals of time, etc., etc. The principle of similitude is thus identical with the principle of the relativity of size. (Tolman 1914, p. 255)

Einstein had shown that the principle of relativity of uniform motion led to the conclusion that it was meaningless to speak of two events occurring simultaneously—that one could talk only about relative simultaneity, never absolute simultaneity. Tolman structures his claim along the same lines, in saying that the principle of relativity of size leads to the conclusion that it is meaningless to speak of absolute length (Fig. 17.19).

[Tolman's claim that it is meaningless to speak of absolute length] seems at least on the face of it at odds with D'Arcy Thompson's view about the importance of size—that different laws govern at different size scales We shall see that Tolman did not have the last word on the issue of the principle of similarity he proposed. [A later chapter of *Wittgenstein Flies a Kite*, "Models of Wings and Models of the World", relates how, later that year, Buckingham published a paper in *The Physical Review* in which he develops the background needed for a comprehensive response to the problem of observationally indistinguishable universes that Tolman raised. He shows where Tolman's reasoning errs, and presents a correct statement of the relevant similarity criterion.] Still, we may ask why [Tolman] addressed the topic of the significance of size in physics at all. It is clear why a physicist would be interested in principles of relativity of motion, for relative velocities [150] were essential to analyzing any dynamic situation in physics. But why an interest in size and scale?

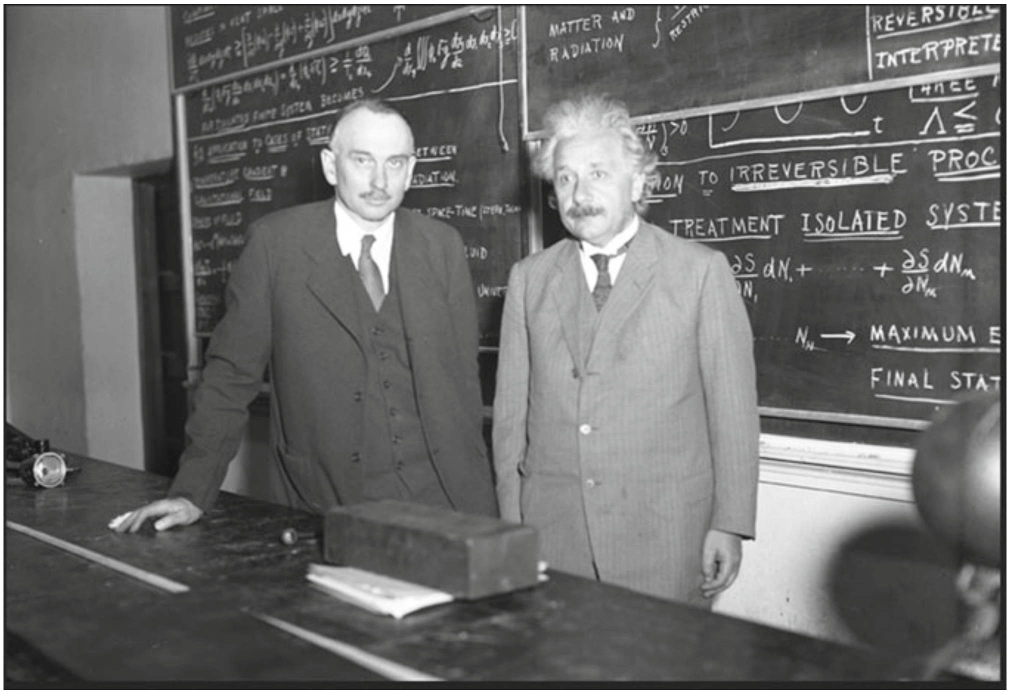

Fig. 17.19 Richard C. Tolman (1881–1948) and Albert Einstein (1879–1955) at the California Institute of Technology, where Tolman was Professor of Physical Chemistry and Mathematical Physics from 1922 to 1948. *Photo credit* Los Angeles Times Photographic Archive, UCLA Library

It does seem that the equation of the effect of size in scientific works was on people's minds even earlier, and when answered in one context, seems to rise in another. The question of size had been raised by Newton, who endorsed the notion of replica miniature worlds, at least in terms of their dynamic similarity, and by Galileo, who pointed out that, although there are certainly rules that inform us how to build replicas of different sizes, sometimes there are so many things you have to take into account in making a replica of a different size that you might not be able to address all the considerations: the rules are more complicated than just keeping relative magnitudes of the same kind the same. This is the point D'Arcy Thompson picks up on, and so there is at least an apparent tension between, on the one hand, Thompson's insight about how different life is for organisms that live in worlds at different size scales, and on the other hand, his conviction that physical laws are universal, and that universality has to extend across worlds of different size scales.

Actually, in Osborne Reynolds' most famous work, the 1883 "An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," he expresses his conviction that physical laws are universal, when faced with an apparent challenge to it. In a subsection of the paper he called "Space and Velocity," he expresses this conviction and hints at an unexpected resolution to the seeming paradox:

> As there is no such thing as absolute space or absolute time recognised in mechanical philosophy, to suppose that the character of motion of fluids in any way depended on absolute size or absolute velocity, would be to suppose such motion without the pale of the laws of motion. If then fluids in their motions are subject to these laws, what appears to be the dependance of the character of the motion on the absolute size of the tube and on the absolute velocity of the immersed body, must in reality be a dependance on the size of the tube *as compared with* the size of some other object, and on the velocity of the body *as compared with* some other velocity. What is [151] the standard object and what the standard velocity which come into comparison with the size of the tube and the velocity of an immersed body, are questions to which the answers were not obvious. (emphasis added) (Reynolds 1883, p. 937)

Reynolds considered this the philosophical, as opposed to the practical, aspect of his "experimental investigation" and the primary result of it (Reynolds 1883, p. 935). An idea about how the apparent paradox might be resolved had come to him only after he had figured out that the law of transpiration depends on an unusual relation of the magnitude: "the size of the channel [the opening the gas has to flow through] and the *mean range* of the gaseous molecules". This discovery arose when he realized that a change in temperature affected the rate of transpiration, or gas flow, through the pores. Then he looked at Stokes' equation for further clues, upon which he realized that the form of Stokes' equation did contain the information that there was a relation of a sort that had been hitherto overlooked: a relation between what he called "dimensional properties"— properties that did depend on size (velocity, size of the tube) and "the external circumstances of motion". Deriving and then equating different expressions for fluid acceleration yielded what later became known as the Reynolds Number. He reported his result at the outset of the paper as follows:

> In their philosophical aspect these results related to the fundamental principles of fluid motion; inasmuch as they afford for the case of pipes a definite verification of two principles, which are – that the general character of the motion of fluids in contact with solid surfaces depends on the relation between a physical constant of the fluid and the product of the linear dimensions of the space occupied by the fluid and the velocity. (Reynolds 1883, p. 935)

This seems to be saying that the way to resolve the paradox between the conviction that size can only be relative, and our experience with phenomena that do seem dependent on size, is to expand the notion of "relative size" to include relations of linear magnitudes to other magnitudes. The crucial ratio is still dimensionless, as a ratio of two linear magnitudes would be, but it involves relations [to quantities] such as the viscosity of a fluid, not just the geometry of the situation. This still doesn't answer the question of [152] whether there can be a miniature universe or not, for the question of what is meant by a miniature universe becomes more complex. Preserving the [magnitudes of] things that a length is relative to is no longer a matter of size, no longer [merely] a matter of geometry. Thus it is natural that the question of whether a miniature universe can be a replica universe was bound to be asked again—as Tolman did, in 1914.

There may be some historical reasons for the intensified interest in scale, similarity, and transformation equations (correspondence rules) among physicists in the years leading up to 1914. Certainly there are different scales of magnitudes described by the kinetic theory of gases. Onnes's 1913 Nobel lecture in which he credits use of the law of corresponding states for his success in liquefying helium may have brought attention to the power of using scaling laws and principles of mechanical similarity, since they were involved in Onnes's derivation of that law. That alone would explain Tolman's interest: although he worked in a number of areas, he had a special interest in foundations of statistical mechanics, and he later authored [textbooks] on it (Tolman 1927, 1938).

An impetus for the interest in scale and similarity may have come from other areas of physics as well. For instance, did the discovery in 1911 that there was something much, much smaller than an atom—its nucleus—raise questions about what things were like inside the nucleus, or about what sorts of laws governed at such tiny scales? That is, did it raise the sort of image for physics that D'Arcy Thompson had evoked for biology: the image of how different things are in the world within a world within "our" world experienced by a bacillus? Or, alternatively, might the discovery of atomic nuclei have spurred reflection on, and helped renew commitment to, the universality of physical laws at all scales, as in Tolman's paper? Einstein's special theory of relativity emphasized the difference between cases in which velocities were very small in comparison to the velocity of light and those that were not. It was interesting that Einstein presented that theory as continuous with, rather than an overthrowing of, classical mechanics, and, emphatically, as a consequence of universal laws. The time around 1914 was also a time when there were new techniques allowing measurement of distances between molecules, meaning that assumptions that previously could only be debated as theoretical or inferred could be more directly determined experimentally (AIP 1990). [153]

Whatever the historical reasons for the interest, principles of similarity were in fact relevant to discussions and active research programs in fields as diverse as zoology and atomic physics, and to endeavors as practical as applied aerodynamics. Another historical fact in 1914 was that the U. S. lagged behind almost every European country in aerodynamic research. Some in the U. S. were trying to change that. The ensuing bureaucratic struggle set the stage for the writing of Edgar Buckingham's remarkable paper on physically similar systems that had something to say in response to all these claims about similarity that were swirling around in 1914 (Fig. 17.20).[5]

[5] Later chapters of the book from which this paper is excerpted delineate the structure of the explanation given in Buckingham's 1914 paper describing how a physical system can be constructed so as to model another physical system, which follows in part from examining the logical consequences of the "most general form" of a "physical equation". I then show how it can be seen as analogous to the structure of the explanation presented in the *Tractatus*, on which a proposition can be regarded as "a model of reality as we imagine it".

ON PHYSICALLY SIMILAR SYSTEMS; ILLUSTRATIONS OF THE USE OF DIMENSIONAL EQUATIONS.

By E. Buckingham.

1. *The Most General Form of Physical Equations.*—Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of n different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain n symbols $Q_1 \cdots Q_n$, one for each kind of quantity, and also, in general, a number of ratios r', r'', etc., so that it may be written

$$f(Q_1, Q_2, \cdots Q_n, r', r'', \cdots) = 0. \quad (1)$$

Let us suppose, for the present only, that the ratios r do not vary during the phenomenon described by the equation: for example, if the equation describes a property of a material system and involves lengths, the system shall remain geometrically similar to itself during any changes of size which may occur. Under this condition equation (1) reduces to

$$F(Q_1, Q_2, \cdots Q_n) = 0. \quad (2)$$

If none of the quantities involved in the relation has been overlooked, the equation will give a complete description of the relation subsisting among the quantities represented in it, and will be a complete equation. The coefficients of a complete equation are dimensionless numbers, *i. e.*, if the quantities Q are measured by an absolute system of units, the coefficients of the equation do not depend on the sizes of the fundamental units but only on the fixed interrelations of the units which characterize the system and differentiate it from any other absolute system.

Fig. 17.20 First page of Edgar Buckingham's 1914 *Physical Review* paper "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations". A brief summary of the work appeared in May 1914 in the Journal of the Washington Academy of Sciences, as "Physically Similar Systems" (pp. 347–353)

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