Suppes predicate for classes of structures and the notion of transportability*

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Abstract

Patrick Suppes' maxim "to axiomatize a theory is to define a set-theoretical predicate" is usually taking as entailing that the formula that defines the predicate needs to be transportable in the sense of Bourbaki. We argue that this holds for *theories*, where we need to cope with *all* structures (the models) satisfying the predicate. For instance, in axiomatizing the theory of groups, we need to grasp all groups. But we may be interested in catching not all structures of a species, but just some of them. In this case, the formula that defines the predicate doesn't need to be transportable. The study of this question has lead us to a careful consideration of Bourbaki's definition of transportability, usually not found in the literature. In this paper we discuss this topic with examples, recall the notion of transportable formulas and show that we can have significant set-theoretical predicates for classes of structures defined by non transportable formulas as well.

Keywords: Suppes predicates. Set-theoretical predicates. Axiomatization of theories. Classes of structures. Bourbaki. Species of structures. Transportable formulas.

*Dedicated to Edelcio Gonçalves de Souza, who always had demonstrated interest in these matters.

1 Introduction

A real revolution in the discussion of scientific theories arose in the 1950s having Patrick Suppes as one and perhaps the most important responsible. The 'revolution' was directed to the logical empiricist view (started in the 1920s) that a scientific theory would be seen as a formal calculus to which an interpretation is ascribed via what Carnap termed *correspondence rules* (other philosophers used other names for the very same thing). The axiomatics would be, in principle, within classical first order logic, but later they have acknowledged that modal operators could also be used; Federick Suppe's (not Suppes) article presents us three steps in the development of the empiricists' ideas, the last one involving modal logics (see his article in [18]).

The precise nature of the correspondence rules (CR) is not clear at all. They look as informal associations (Carnap recalls that N. R. Campbell called this set of rules a 'Dictionary' [7, Chap.24]) connecting theoretical terms (of the language) with observable terms. These two concepts, roughly speaking, mean the following. Observable terms are those terms either directly perceived by the senses (such as 'hard' and 'hot') or those that can be measured by "a simple apparatus" [7, Chap.23], such as the temperature of a certain body. Carnap's own example of a CR is the following: "The temperature (measured by a thermometer and, therefore, an observable in the wider sense explained earlier) of a gas is proportional to the mean kinetic energy of the molecules" (loc. cit.). That is, the connection is given by an informal (almost *ad hoc*) association. Theoretical terms, as the name indicates, are 'theoretical' and cannot be availated as above; a typical example is the kinetic energy in the above example. Despite Carnap had written his book after the raising of modern model theory [8], he doesn't consider *formal semantics*, where the association of language-terms with 'the reality' is given by formal (mathematical) rules, and this 'reality' is taken as a set-theoretical structure. This brings important distinctions we shall made later.

Several criticisms were posed to this view, some of them by Suppes himself. It is enough to remember here that in providing this kind of approach to a huge theory such as general relativity, which requires several 'step theories' such as tensorial calculus, Riemannian geometry, partial differential equations, real analysis and so on, would turn the axiomatics something rather difficult to follow, for all these step theories would be in need of being axiomatized too.

Suppes started by considering all these step theories as done in advance, presupposing them as already given by (informal) set theory and directed the efforts to the interested theory itself. This seems simple today, but, as we have said, constituted a real advance in the axiomatic approach to scientific theories, a program that was suggested by David Hilbert in the sixth of his celebrated 23 Problems of Mathematics [14]. The new resulting program was named the *semantic approach* to theories, in contrast to the 'syntactical' approach of the logical empiricists. Suppes refers to these two views as constituting the *extrinsic* approach (his semantic view) and the *intrinsic* one [19].

Why informal set theory? Precisely because he didn't wish to discuss the foundational issues; if we need tensors, they can be defined set-theoretically. Do we need partial differential equations? Set theory does the job. Do we need to be more specific about proofs? We proceed as the mathematicians usually do. But of course if it is necessary to make this base explicit, we can choose an axiomatic set theory that gives the desired results, say (for most physical theories) the ZFC first-order system [8, pp.592-3]. The important thing is that we don't need any more to be occupied with the details of these step theories, but just assume them, as for instance physicists do in most cases.[1]

Let us give a more detailed simple example, to be used also later. This is the case of semi-groups. As it is known from any basic book on algebra, a semi-group is a (non-empty) set endowed with a binary operation which is associative. We can write this in terms of structures (see more on this below) of the form

$$\mathcal{S} = \langle M, \star \rangle, \tag{1}$$

where $M \neq \emptyset$ and $\star$ a function from $M \times M$ to M, that is, an element of $\mathcal{P}(M \times M \times M)$. As said before, the only axiom being this one (the quantifiers range over M):

$$\forall x \forall y \forall z (x \star (y \star z) = (x \star y) \star z). \tag{2}$$

Examples of semi-groups, or models of this axiomatics, abound. The set of real numbers endowed with addition of these numbers, the set of natural numbers with multiplication, the set of all $n \times n$ matrices with multiplication of matrices, etc. Suppes' account was to show that this move can be summed up by a certain formula of the language of set theory,[2] namely, the predicate (if we need to say, a formula with just one free

[1]Notice that in informal set theory we can do practically everything we wish in mathematical terms.

[2]Notice that the informal set theory has not a well defined language, but if necessary we can reason as if the only specific predicate is membership, $\in$. All other symbols are either logical symbols or defined ones.

variable) $\mathsf{P}(X)$ defined as follows:

$$\mathsf{P}(X) := \exists M \exists \star \, (X = \langle M, \star \rangle \wedge \star \in \mathcal{P}(M \times M \times M) \wedge (2)). \tag{3}$$

Important to realize that those who analise the definition claim for instance that the formula is in the free variable X, as we see, and doesn't depend on any specific property of the set M, referring to only to the way it enters in the formula by means of the whole expression [9]; so, it is *transportable* in the sense of Bourbaki (see below). This means that there cannot be imposed restrictions whatever on the principal sets occurring in the formula (in the case, in the set M), for instance, the requirement that M must be different from another non-empty set N (see below). But, could we use the predicate below as the axiom, as in almost all standard books? That is,

$$\mathsf{P}(X) := \exists M \exists \star \, (X = \langle M, \star \rangle \wedge M \neq \emptyset \wedge \star \in \mathcal{P}(M \times M \times M) \wedge (2)). \tag{4}$$

Since this is important for that what follows, and since neither Bourbaki nor those who mention his definitions are clear, we shall try to make this claim precise.[3] In fact, at §4 of his *Algebra I* book [4, p.30], Bourbaki introduces the definition of group this way:

> "Definition 1 – A set with an associative law of composition, possessing an identity element and under which every element is invertible, is called a *group*."

The adaptation to semi-groups is immediate, just by requiring that the operation (law of composition) be only associative. We see that the empty set is not excluded of being a group (or a semi-group), and there is no justification for that. We guess this has to be with his notion of transportable formula, although as we shall see soon, the adding of something like $M \neq \emptyset$ does not violates the transportability condition.

But sometimes we would be interested in collecting not all semi-groups, but just some of them. Thus, we must add some restriction to the above predicate, say by avoiding that some specific structure (or structures) enter in the range of the predicate. In this case, some care is to be taken into account.

Thus, we see that if we wish to axiomatize all models of a certain kind, the formula must of course be transportable, which is the case when we

[3] Really, people in general simply adopt Bourbaki's definitions taking into account that they are clear. In our opinion, they are not.

are (in Suppes' sense) axiomatizing a certain *theory*, as the theory of semi-groups, for in this case we are interested in collecting (as models of the predicate) all semi-groups. But we can also use set-theoretical predicates for collecting a certain classes of models, which doesn't require the formula to be transportable. Let us go to the details.

2 Mathematics: a world of structures

Nicholas (or 'Nicolas') Bourbaki is the pseudonym of a group of (mainly) French mathematicians who, in the 1930s, intended to rescue French mathematics to actuality, that is, to keep it in the level and dealing with the methods (the axiomatic method) that were being developed mainly in Germany (the book by van der Waerden, *Modern Algebra* [22], was taken as a paradigm).[4] It is known that France lost many of its more important scientists during the first world war, so that the mathematics taught in the universities during the post-war times were still the 'old' mathematics of the XIX century, and did not cover the most recent subjects with new methods (the axiomatic method), such as abstract algebra. The group, initially formed by Jean Dieudonné, Henri Cartan, André Weil, Claude Chevalley and Jean Delsarte, started in 1934 a project termed *Éléments de Mathématique* and, according to Dieudonné, planed to finish it in three years. Dieudonné recalls that this was a plan of young and ill informed students and that they never had planed such a thing if they were more informed [13].[5] Important to notice that the members of the group change from time to time, so that the group is still alive today. The initial objective was yet not achieved at all (which gives an idea of the wide task they ascribed to themselves). A very nice historical account about the group and its realizations can be seen in [16]; further considerations in [9], [17].

The idea was to see all mathematical theories as formed by structures of a kind.[6] These structures were to be build from some fundamental structures, termed *mother structures*, which are the *algebraic*, *ordering* and *topological* structures, Bourbaki also acknowledges that this expressed a stage

[4]In explaining (p.ix) the purpose of his book, van der Waerden said that he wished "to introduce the reader into the whole world of [algebraic] concepts", by considering the "recent expansion" in this field, "due to the 'abstract', 'formal', or 'axiomatic' school".

[5]Meaning that they realized later the huge work they intended to cover in so few years.

[6]Important to notice that Bourbaki didn't deal with all fields of the mathematics of the day, for instance leaving number theory and geometry aside. There is no apparent reason for that.

in the development of mathematics, so that further developments could suggest other 'mother' structures. According to Corry [9], since Bourbaki reputed ordering structures as fundamental, his axiomatics for sets initially took the notion of ordered pair as primitive [6], what was modified in later versions, when he turned to the usual way of taking the unordered pair as given by a specific axion, from which the ordered pair results by definition. All other mathematical structures would arise from suitable 'combinations' of these mother structures. So, the field of the real numbers was characterized by being a complete ordered field, these three words indicating the (topological) completeness, ordering, and an algebraic structure [5, p.264]. A semi-group is an algebraic structure, so are the monoids, groups, rings, etc.

We clearly see the formalistic purely syntactical approach of Bourbaki. Mathematics is got by writing symbols in the paper according to the rules stated in the Theory of Sets. If something was not written yet, it does not belong to the field of mathematics. So, his notion of truth is quite peculiar, in a certain sense constructive: something is *true* if we have a proof of it, even an indirect proof, which shows that his mathematics is 'classic', that is, the excluded middle law, so as *reductio at absurdum*, among other 'classical' procedures, hold. In the same vein, something is false if there is a proof for its negation. Thus, something for which there is no proof neither by the affirmation nor to its negation, is neither true nor false. In other words, the *mathematics* is classic, by the *metamathematics* is not, being constructive. Bourbaki doesn't think of semantics as we are accustomed nowadays.

2.1 Species of structures

In Chapter 4 of the book on set theory, Bourbaki develops his 'theory of structures', which interests us here. In [11], the authors proposed a modification and adaptation of Bourbaki's notions, grounding them in a set theory with atoms. Here we shall follow Bourbaki, but without the technical details and subtleties. As said before, most of the content of present day mathematics, according to Bourbaki, fall under the notion of structure.[7] But, what is a structure? Bourbaki speaks of *species of structures* and

[7]Interesting to mention that the theory of categories [16] was never mentioned by Bourbaki [9] (another omission are the Gödel's incompleteness theorems [17]), although the concept of category arose with one of its members, Samuel Eilenberg (together with Saunders MacLane), and further developed for instance by another of members, Alexander Grothendieck. Categories are 'big enough' to be treated as usual sets, so it would be necessary to expand the logical (ZF) basis to cope with them, something that perhaps

of *structures of a certain species*; intuitively speaking, using a terminology which departs from him, a (mathematical) structure is precisely that we are spectating it to be since our logical courses: a set, or a collection of sets, endowed with relations and operations not only among their members (which would characterize *order-1* structures [15]) but also among collections of elements of these sets (the elements of the basic sets are called *individuals*), relations among them, etc. So, we may have relations whose relata are also relations, that is, the structures may by of *order-n*, $n > 1$.[8]

To give an idea of how things proceed, let us start with a finite collection of sets $E_1, E_2, \ldots, E_n$ which will be the *principal* sets and a finite collection $A_1, A_2, \ldots, A_m$ of *auxiliary* sets (in our examples, we shall use just three principal sets, F, G, and H). The auxiliary sets must not contain any references to the principal sets. For instance, the vector space structure comprises a principal set V of vectors and an auxiliary set K of scalars, while semi-groups have just one principal set M and no auxiliary sets. Using the set-theoretical operations of taking the power set and the Cartesian products, we can obtain a sequence of new sets $\mathcal{P}(F)$, $F \times H$, $\mathcal{P}(G) \times H$, $\ldots$. These sets are constructed from a very subtle schema S he terms an *echelon construction schema* we will not recall here (but see [5, chap.4], [11], [9]). The last set in the sequence is the echelon of scheme S. For instance, to get a binary operation over a set F, we build a sequence of sets with a certain schema S (omitted here), for instance (there is not just one schema, so different sequences can be obtained) F, $F \times F$, $F \times F \times F$, and finally we get the echelon of scheme S, $\mathcal{P}(F \times F \times F)$. The binary operation will be an element $\mathfrak{s} \in \mathcal{P}(F \times F \times F)$ (before, in our given example of semi-groups, we have termed $\star$ such an $\mathfrak{s}$), to which we impose the restrictions we wish, written in the form of postulates, say that $\mathfrak{s}$ must be associative, that is [5, p.60],

$$\mathfrak{s}(\mathfrak{s}(a, b), c) = \mathfrak{s}(a, \mathfrak{s}(b, c)), \tag{5}$$

which in our above terminology means $\star(\star(a, b), c) = \star(a, \star(b, c))$, or simply $(a \star b) \star c = a \star (b \star c)$.

Another important concept is that of *canonical extensions of mappings*. Given an echelon construction schema S and two collections of sets $E_1, \ldots,$

Bourbaki was not being up to do.

[8] We use the terminology 'order-n' instead of 'first-order', 'second-order' etc. to avoid confusion with the order of the languages. Really, we can define order-n structures ($n > 1$) in first-order languages, say in first-order ZF; for instance, well-ordering structures are not order-1, as it is easy to see (the postulate requires the reference to 'all non-empty subsets', which is not a sentence of first-order.

E_n and $E'_1, \ldots, E'_n$, let us consider mappings (functions) $f_i : E_i \to E'_i$. Bourbaki defines extensions of these mappings from the sets in an echelon based in S constructed over the E_i to the corresponding sets in the echelon also based in S but now on the E'_i, until getting a mapping from the echelon of scheme S based on the E_i to the echelon of scheme S based on E'_i (the reader must be attentive with the terminology). This last mapping is the *canonical extension* of the f_i, written $\langle f_1, \ldots, f_n \rangle^S$. If the f_i are injective (surjective, bijective), then $\langle f_1, \ldots, f_n \rangle^S$ will be injective (surjective, bijective) [5, p.261].

A *species of structure* Σ is defined this way. We take a collection of principal base sets $x_1, \ldots, x_n$, a collection of auxiliary bases sets $A_1, \ldots, A_m$ and a specific echelon construction schema $S(x_1, \ldots, x_n, A_1, \ldots, A_m)$. An element $\mathfrak{s} \in S(x_1, \ldots, x_n, A_1, \ldots, A_m)$ is the typification of Σ. The typification is written by Bourbaki as a formula $T(x_1, \ldots, x_n, \mathfrak{s})$. Let now $R(x_1, \ldots, x_n, \mathfrak{s})$ be a transportable formula (see below) with respect to the given typification, with the x_i as the principal sets and the A_j as the auxiliary sets. This formula will be the *axiom* of the species of structures with typification T. If we select some particular sets $E_1, \ldots, E_n, U$ so that both $T(E_1, \ldots, E_n, U)$ and $R(E_1, \ldots, E_n, U)$ hold, then U is said to be *a structure of species* Σ. His first example is that of the species of structures of ordered sets, where from a set A, we get (by a suitable echelon construction schema S) the set $\mathcal{P}(A \times A)$ and the typification $\mathfrak{s} \in \mathcal{P}(A \times A)$ (a binary relation on A), with the axiom $\mathfrak{s} \circ \mathfrak{s} = \mathfrak{s}$ (reflexivity) and $\mathfrak{s} \cap \mathfrak{s}^{-1} = \Delta_A$ (transitivity), being Δ_A the diagonal of A (informally, the set $\Delta_A = \{(x, x) : x \in A\}$). Other examples can be found in [5, pp.263ff].

Thus, the restrictions imposed to $\mathfrak{s}$ constitute the axioms of the species of structure (in the case of semigroups, the restriction is that $\mathfrak{s}$ must be associative). As we see, all of this relies on the notion of transportability, for the axiom (the conjunction of the formulas we standardly use) must be transportable. So, the predicate (4) defines the species of structure of semigroups, the semi-groups (the structures that satisfy the predicate) being the structures of that species.

2.2 Transportable formulas

The notion of transportable formulas is important for our account, so that it deserves a particular subsection. The definition is marked by Bourbaki with the symbol '¶', which means 'difficult exercise'. So, let us go slow, even without providing all the details. Important to remark that it does not constitute a simple and easy definition. Interesting enough that people who mention Bourbaki's account do not discuss it in full and, in our opinion, neglect important aspects of it. This is why we shall give some attention to it.

Think again of a semi-group. Remember that we (by hypothesis) intend to develop the *theory* of semi-groups, which requires an axiomatization of *all* of them (by the way, this is one of the main advantages of the axiomatic method).[9] So, we need to provide a definition that does not exclude any semi-group from the list, which requires that our definition should not refer to any particularity of the domain M that could leave some semi-group out, being not covered by the definition. So, we cannot characterize semi-groups by saying things like 'a semi-group is a set M distinct from the set of the natural numbers so that blah-blah-blah', for in doing this we would be eliminating important semi-groups, such as $\langle \mathbb{N}, + \rangle$. In other words, the formula which characterizes the species of structure must be *transportable*, or invariant by substitutions of the principal set(s), as we shall see soon.

Bourbaki calls (in our terminology) formulas 'relations'. Suppose we have an echelon construction schema S for $n + m$ terms (sets), where there are n principal sets $x_1, \ldots, x_n$ and m auxiliary sets $A_1, \ldots, A_m$, which, as before, is written $S(x_1, \ldots, x_n, A_1, \ldots, A_m)$; let us abbreviate by $S(x_i, A_j)$. As we have seen, an element $\mathfrak{s} \in S(x_i, A_j)$ characterizes a *typification* of $\mathfrak{s}$. Notice that to typify something is just to select if from a certain set constructed by set-theoretical operations from base sets (principal and auxiliary). So, as seen before, $\star \in \mathcal{P}(M \times M \times M)$ is a typification of a binary operation on the base set M. Another example is useful. Let us consider vector spaces again, with V as principal set and K as the auxiliary set. We form the Cartesian product $K \times V$ and chose an element $\cdot \in K \times V$. This element can be written as

$$\cdot = \{\langle k, \alpha \rangle : k \in K \wedge \alpha \in V\}. \tag{6}$$

If we write $\langle k, \alpha \rangle$ as $k \cdot \alpha$ of simply $k\alpha$ for short, we see that the typification characterizes the operation of multiplication of vectors by scalars.

The typification could involve several choices say $s_1 \in S_1(x_i, A_j)$, ..., $\mathfrak{s}_p \in S_p(x_i, A_j)$ if we have also several echelon construction schemes S_1, ..., S_p. This of course defines a formula, which we write, as above, adapting Bourbaki's notation, $T(x_1, \ldots, x_n, \mathfrak{s}_1, \ldots, \mathfrak{s}_p)$. Now comes the ¶ part.

[9] Bourbaki emphasizes this. He says that the main task of axiomatization is that enable us to study non-categorical theories (*multivalent* in his terminology). As he says, "The study of multivalent theories is the most striking feature which distinguishes modern mathematics from classical mathematics" [5, p.385].

Let $R(x_1, \ldots, x_n, \mathfrak{s}_1, \ldots, \mathfrak{s}_p)$ be a formula, and let $y_1, \ldots, y_n, f_1, \ldots, f_n$ be variables other than the x_i and the $\mathfrak{s}_j$. The f_i are bijections from x_i onto y_i, and Id_j are the identity functions of the auxiliary sets A_j. Once we have the canonical extension

$$\langle f_1, \ldots, f_n, Id_1, \ldots, Id_m \rangle^S, \tag{7}$$

we can get the $\mathfrak{s}'_j$ by applying these extensions to the $\mathfrak{s}_j$, namely,

$$\mathfrak{s}'_j = \langle f_1, \ldots, f_n, Id_1, \ldots, Id_m \rangle^{S_j}(\mathfrak{s}_j), \tag{8}$$

so that the formula $R(x_1, \ldots, x_n, \mathfrak{s}_1, \ldots, \mathfrak{s}_n)$, by bijective mappings, gives $R(y_1, \ldots, y_n, \mathfrak{s}'_1, \ldots, \mathfrak{s}'_m)$. Then, the formula R is *transportable* if these two formulas are equivalent, that is, iff we can prove in the system that

$$R(x_1, \ldots, x_n, \mathfrak{s}_1, \ldots, \mathfrak{s}_n) \leftrightarrow R(y_1, \ldots, y_n, \mathfrak{s}'_1, \ldots, \mathfrak{s}'_m). \tag{9}$$

The notation is in fact far-fetched. So, let us try to translate the definition to a language closer to that we use today. Let $R(x_1, \ldots, x_n, \mathfrak{s})$ a formula (we take just one $\mathfrak{s}$) and S be an echelon construction schema. If $f_i : x_i \to y_i$ $(i = 1, \ldots, n)$ are bijections, any canonical extension $\langle f_1, \ldots, f_n \rangle^S$ is also a bijection. So, we get that

$$\langle f_1 \ldots, f_n \rangle^S \Big(S(x_1, \ldots, x_n, \mathfrak{s}) \Big) = S(y_1, \ldots, y_n, \mathfrak{s}'), \tag{10}$$

being $\mathfrak{s}' = \langle f_1, \ldots, f_n \rangle^S(\mathfrak{s})$. Then, if $R(y_1, \ldots, y_n, \mathfrak{s}')$ also holds, the formula $R(x_1, \ldots, x_n, \mathfrak{s})$ is transportable. Let us remark that (9) is speaking in syntactical terms, that is, in *proof*. The equivalence must be shown on syntactical grounds.[10]

But nowadays we are accustomed with semantics, so some authors prefer to express the idea on semantical groundings [11]. In this case, we can grasp the concept by considering two isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$. Let α be a sentence of the language appropriated for both structures.[11] In the present day 'semantical' language, α is transportable if and only if

[10] We emphasize once more the purely syntactical aspect of Bourbaki's approach.

[11] We leave the formal definition of 'appropriate' out, keeping only with its intuitive aspect. But see [11], [15].

$$\mathfrak{A} \models \alpha \text{ iff } \mathfrak{B} \models \alpha, \tag{11}$$

that is, if and only if α is preserved under isomorphisms [11]. Let us take an example. Think of the Peano's axioms for arithmetics (within set theory). The axioms can be written, in a standard language, as follows, where the quantifiers range over the set of natural numbers $\mathbb{N}$, except for the third one, where quantification over subsets of $\mathbb{N}$ is also allowed, and where 0 means 'zero' and n' stands for the sucessor of n:

1. $\forall n(0 \neq n')$
2. $\forall n \forall m(n' = m' \to n = m)$
3. $\forall A(A \subseteq \mathbb{N} \to (0 \in A \land \forall n(n \in A \to n' \in A) \to A = \mathbb{N}))$

Thus, $\mathcal{N} = \langle \mathbb{N}, 0, ' \rangle$ is a model of these axioms, the *standard* model. According to Bourbaki, the axioms must be transportable. Let us prove that using the semantic approach, by considering the first axiom. It is easy to show that it is transportable. Really, let us consider another set $\mathbb{N}_1$ such that $\mathcal{N}_1 = \langle \mathbb{N}_1, 0_1, '' \rangle$ is also a model of the above axioms (that is, it is also a structure of *that* species). Thus, let $f : \mathbb{N} \to \mathbb{N}_1$ be a bijection such that $f(0) = 0_1$ and $f(n') = (f(n))''$, the sucessor of $f(n)$ in the second structure. If $m \in \mathbb{N}_1$, let $n = f^{-1}(m) \in \mathbb{N}$, so that since $n' \neq 0$, then $f(n') \neq f(0) = 0_1$. Thus $m'' \neq 0_1$. In other words, $\mathcal{N} \models \forall n(0 \neq n')$ entails $\mathcal{N}_1 \models \forall m(0_1 \neq m'')$. The converse is also easy to prove. With more patience, we can prove that the other two axioms are also transportable.

Notice that this 'semantic' account is an *interpretation* of Bourbaki's notions and, although we agree that most mathematicians will take it for granted, it can't t be shown to be equivalent to the original approach, for there is no way of comparison between then: one is syntactical, the other is semantical, and we know that set theory is not a complete theory (when syntax agrees with semantics). So, we must take care.

We also remark that neither axiom poses a restriction on the principal set (namely, $\mathbb{N}$). The restriction of being different of 0 is ascribed to the sucessor of n, and this does not violate the definition of transportability.[12] But let us take the following formula $\mathfrak{s}(\mathfrak{s}(0)) = \{\{\emptyset\}\}$ (Zermelo's 'two'). Notice that now we have something different, namely, the presence of the

[12]Really, the formula is a particular case of Bourbaki's own example shown in the quotation below, namely, that (the negation of) $\mathfrak{s}_1 = \mathfrak{s}_2$ is transportable, just taking $\mathfrak{s}_1$ as n' and $\mathfrak{s}_2$ being 0, both in $\mathbb{N}$.

set $\{\{\emptyset\}\}$ which is not part of the formal language. And, of course, taking another definition of 'two', we could have $\mathfrak{s}(\mathfrak{s}(0))$ being associated to another set, say $\{\emptyset, \{\emptyset\}\}$ (von Neuman's 'two'). Thus, $\mathfrak{s}(\mathfrak{s}(0)) = \{\{\emptyset\}\}$ is not transportable.

Bourbaki gives us the following not so clear example, as fas as we know, never discussed elsewhere:

> "For example, if $n = p = 2$ and if the typification (...) is '$\mathfrak{s}_1 \in x_1$ and $\mathfrak{s}_2 \in x_1$', [then] the relation $\mathfrak{s}_1 = \mathfrak{s}_2$ is transportable. On the other hand, the relation $x_1 = x_2$ is not transportable." [5, p.262].

Our explanation is as follows. The typification takes elements of a same set x_1, hence we need no more than this set in our echelon. The (only) bijection will be some $f : x_1 \to y_1$, being y_1 a set whatever. Hence the canonic extension $\langle f \rangle^S$ is f itself. Then, the formula (1) $\mathfrak{s}_1 = \mathfrak{s}_2$ conduces to (2) $f(\mathfrak{s}_1) = f(\mathfrak{s}_2)$ by the bijection. Obviously, if (1) holds, so does (2). For the second case, we have two sets x_1 and x_2, and two bijections $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$. But $x_1 = x_2$ doesn't entail that the set x_1 (or x_2, since they are equal) is lead by the two bijections f_1 and f_2 in the same set, so that not necessarily y_1 and y_2 are equal.

This last remark and the example may suggest something already mentioned earlier, namely, why Bourbaki didn't made the exigence that the domain of a group (and this applies also to semi-groups) needs to be not empty. Apparently, this could be due to the fact that the formula $x \neq \emptyset$ seems to be not transportable, for the negation of a transportable formula is also transportable and we could just take $x_2 = \emptyset$ above. But this is false. The emptyset has specific properties; let us see. Suppose we have a set M (to go along with our example) for defining the species of structures of semi-groups; should we use (1) or (4) as the axiom? It is indifferent, and this is due to the restriction. Really, suppose we have again another N (which plays the role of y_1 in the definition), and let $f : M \to N$ be a bijection. Since $M \neq \emptyset$, we conclude that $N \neq \emptyset$, so the restriction doesn't impede the transportability of the formula (as we shall see with more details below, this will be not the case with other non-empty sets). The difference in using (1) or (4) is that, as we have remarked, with the first we enable the emptyset to be a semi-group, something that is avoided in the second case.

Of course the above reasoning is grounded on the following immediate theorem:

METATHEOREM 1 *A formula α is transportable if and only if all subformulas of α are transportable.*
Proof: If α has some subformula β that is not transportable, then β will be not invariant under isomorphisms, so not will be α. The converse is trivial. ▮

So, from the perspective of Bourbaki, we cannot select *some* structures to be the models of some set-theoretical predicate; we must consider *all* structures that satisfy the predicate. As we shall see, this is precisely the case of Suppes' set-theoretical predicates when used to axiomatize theories. But, as anticipated, sometimes we are interested in selecting some particular model or some class of models. Next, let us consider this.

3 Selecting classes of models

According to one of most widespread characterization of the semantic approach to scientific theories, a theory is specified by a family of structures, the *models* of the theory [21, p.77]. Models of most scientific theories are, as said already, set-theoretical structures. In first-order logic, models (of first-order languages or theories) are *order-1* structures, or structures comprising sets as their domain(s) and the relations (and operations) are defined among the elements of the domain(s). No relation (operations are particular cases of relations, so we shall speak of relations only) can have as arguments other relations or sets of such elements, as the case of topological spaces illustrates.[13] With respect of such languages, we have an important theorem which goes as follows. Given a certain collection of order-1 structures, there exists a necessary and sufficient condition for axiomatizing such a collection, that is, a condition that says that there can exist a theory (set of postulates) whose models are exactly the chosen structures, namely, the collection must be closed by elementary equivalence and by ultraproducts [8, Thm.4.1.12, p.220]. The right definitions are not important here, but just the fact that there is such a criterium for first-order languages. Concerning higher-order languages or classes of structures, there is no a similar theorem; given classes of such structures, we need to study them case by case. The importance of this fact is that most structures that model postulates of a scientific theory are higher-order structures, or order-n structures with $n > 1$. A typical example is that of classical particle mechanics, which can be summarized as follows (this is one of the

[13] A topological space (in terms of structures) is an ordered pair $\mathcal{T} = \langle X, \tau \rangle$ where τ is a collection of subsets of X satisfying certain axioms [5, p.263], that is, τ comprises *sets* of elements of the domain.

most simple examples we have, so it is explored to exhaustion by several authors too).

According to Suppes, going to the characterization of a theory in terms of structures, "A system of classical (particle) mechanics is a mathematical structure of the following sort ...", and then specifies a basic finite domain P of entities, the *particles*, a set T of instants of time (usually an interval of the real number line), and some other elements which are not of our interest here (but see [20, p.320]). All of this is collected in a structure $\mathcal{P} = \langle P, T, \ldots \rangle$, subjected to suitable postulates.

The set-theoretical predicate would be something saying that some set X is a classical particle mechanics iff it obeys the predicate

$$\mathsf{CPM}(X) := \exists P \exists T \ldots (X = \langle P, T, \ldots \rangle \wedge P \neq \emptyset \wedge T = [a, b] \subseteq \mathbb{R} \wedge \ldots \quad (12)$$

But there are two important restrictions in this definition: the principal set P must be finite and non empty, and the auxiliary set T must be an interval of real numbers (*ibidem*). So, as we have seen already, the set-theoretical predicate is a transportable formula (the other elements of the structure are typifications), and so it defines a specie of structures in the sense of Bourbaki, selecting a huge class of structures that satisfy it, the models of the predicate or, as Suppes suggests, the 'classical particle mechanics'.

But sometimes we may wish to consider just a small part of the whole class of models. Let us take a simple example. Suppose again that we have a set-theoretical predicate for semi-groups, but we wish, by some hidden reason, to avoid considering all semi-groups having the set of real numbers as domain, say $\mathcal{A} = \langle \mathbb{R}, + \rangle$, the semi-group of the real numbers with usual addition. How should we proceed? This is simple, one may say. Just take the predicate (4) and impose that the domain must be different from $\mathbb{R}$, that is, something like

$$\mathsf{P}(X) := \exists M \exists \star (X = \langle M, \star \rangle \wedge M \neq \mathbb{R} \wedge \star \in \mathcal{P}(M \times M \times M) \wedge (2)). \quad (13)$$

It is easy to see that the models of such a predicate are all semi-groups (the empty set included) *except* those that have $\mathbb{R}$ as the domain. But wait! The formula, as we have seen, is not transportable due to the imposed restriction. So, it doesn't axiomatize the theory of semi-groups, for in this case no semi-group should be leaved out.

A more relevant example should be the following. Suppose we wish to consider just those models characterized by a *representation theorem* our theory may admit. Let us say something on this point. Suppes emphasizes that sometimes a theory is so that there is a subclass of models with the following property: for every model of the theory, there is in this class a model which is isomorphic to the given model. In mathematics, it is simple to give examples, namely, every group is isomorphic to a group of permutations (Cayley's representation theorem) or Stone's representation theorem, which says that every Boolean algebra is isomorphic to a a certain field of sets. Thus, in a certain sense, it is enough to have this subclass in order to know all possible models of a theory (up to isomorphisms). (Just to comment, as Suppes recalls, in the case of empirical theories it may be quite difficult to find a representation theorem, or to prove it. But let us move on without discussing the details.)

The important fact is that we may be interested in some specific class of models, or then we wish at to disregard some specific model of the theory which is not interesting to us. In the case of the example, we need to impose that the models of the predicate will be precisely those we wish and not others. Although we shall not provide the details here, it is to be acknowledged that this can be done. The problem is that this step, which seems to be justified by the interest of the scientist, finds problems with the above definition of a set-theoretical predicate or Bourbaki's species of structures. Really, as we have seen, the formula which stand for the predicate that defines the theory must be transportable, that is, we cannot impose arbitrary restrictions on the principal sets, which need to be instantiated by any sets whatever (with possible exceptions such as the empty set, as seen earlier).

So, we see that set-theoretical predicates can be used both to define theories and also to select classes of structures, yet that sometimes things may go to not useful results, as in the case when we take the predicate $\mathsf{P}(x) := x = \emptyset$ [11], which apparently does not define any theory whatsoever, yet selects a structure having the emptyset as its domain (and vacuous typifications, of course).

Two things need to be enlighten: the first is that, as we have said, in considering such (set-theoretical) predicates for defining theories or classes of structures, all the step theories are being presupposed. Secondly, remember that (in general) we are working within a set theory such as the ZF system, and we could be interested in finding a set-theoretical predicate for ZF itself. In this case, as it results from Gödel's incompleteness theorems, being consistent, ZF does not admit ZFC-sets as models, that is, models that are sets of ZF. For doing that, we need to strengthen ZF with additional postulates, say by assuming the existence of *universes*,[14] or go-

[14]Universes, initially introduced by Alexander Grothendieck (who was a member of

ing to another stronger theory. But this (apparently) is not necessary for most theories in the empirical domain.

The fact is that there is no *one* solution to all problems. The use of set-theoretical predicates will depend on the set-theory being used and the needs of the scientist. Important is to be aware of the technique and of its importance. The details must be fulfilled in each particular case.

4 Conclusion

In this paper we have shown the dependence of Bourbaki's notion of species of structures to the concept of transportable formulas. Furthermore, we have enlighten that his definition of transportable formulas does not enable us to introduce arbitrary restrictions on the principal sets, for once some restriction is made, the formulas may not be invariant by isomorphisms. So, there is a strong difference between Bourbaki's species of structures and Suppes' set-theoretical predicates, which also characterize certain structures, the models of the predicate, but enabling us to introduce restrictions on the principal sets, thus allowing the selection of just the models we may be interested in. But, when the set-theoretical predicate is a transportable formula, Suppes' approach coincides with Bourbaki's.

Summing up, Bourbaki's approach and Suppes' account using transportable formulas are directed to the axiomatization of *theories*, where no model can be left out, while the use of set-theoretical predicates without such a restriction (of being a transportable formula) is more general, for it enables also to grasp just *some* relevant models of a certain class of models.

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the Bourbaki group) to cope with categories in set theoretical terms; see [3]. The existence of universes is equivalent to the existence of inaccessible cardinals.

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