# The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory $ZFC$ as it refers to our current knowledge on $\mathcal{X}$

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**Abstract.** Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2+1$ is infinite. Let $\beta = (((24!)!)!)!$, and let $\Phi$ denote the implication: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement $\Phi$ without invoking Landau's conjecture. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions `(1)`-`(4)`. `(1)` There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. `(2)` No known algorithm decides the finiteness/infiniteness of $\mathcal{X}$. `(3)` There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. `(4)` There is an explicitly known integer $n$ such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(5)` There is an explicitly known integer $n$ such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ and some known definition of $\mathcal{X}$ is much simpler than every known definition of $\mathcal{X} \setminus (-\infty, n]$. The following problem is open: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1)`-`(3)` *and* `(5)`*?* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions `(1)`-`(3)`. The set $\mathcal{X} = \{k \in \mathbb{N}$ : the number of digits of $k$ belongs to $\mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions `(1)`-`(3)`. The statement $\Phi$ implies that both sets $\mathcal{X}$ satisfy condition `(5)`.

**Key words and phrases:** complexity of a mathematical definition, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on $\mathcal{X}$, explicitly known integer $n$ bounds $\mathcal{X}$ from above when $\mathcal{X}$ is finite, infiniteness of $\mathcal{X}$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of $\mathcal{X}$, mathematical statement that cannot be formalized in the set theory $ZFC$, no known algorithm decides the finiteness/infiniteness of $\mathcal{X}$, physical impossibility of machine computations on sufficiently large integers.

## 1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *There are a large number of elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of* $\mathcal{X}$.

(3) *There is a known algorithm that for every* $n \in \mathbb{N}$ *decides whether or not* $n \in \mathcal{X}$.

(4) *There is an explicitly known integer* $n$ *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) *There is an explicitly known integer* $n$ *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ *and some known definition of* $\mathcal{X}$ *is much simpler than every known definition of* $\mathcal{X} \setminus (-\infty, n]$.

**Definition 2.** *We say that an integer* $n$ *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$, *cf.* [8] *and* [9].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [5] and [6].

**Definition 3.** *Let* $\Phi$ *denote the implication:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement $\Phi$. In Section 4, we heuristically justify the statement $\Phi$ without invoking Landau's conjecture.

**Statement 1.** *There is no explicitly known threshold number of* $\mathcal{P}_{n^2+1}$. *It means that there is no explicitly known integer* $k$ *such that* $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.

Proving the statement $\Phi$ will falsify Statement 1. Statement 1 cannot be formalized in the set theory *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2 and 3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$.

## 2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

**Definition 4.** *Let* $\beta = (((24!)!)!)!$.

**Lemma 1.** $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

*Proof.* We ask Wolfram Alpha at `http://wolframalpha.com`. □

**Statement 2.** *The set* $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ *satisfies conditions* `(1)-(4)`.

*Proof.* Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $\beta$, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer $\beta$ is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

In Statement 2,

$$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, \beta]$$

and the sets

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

and

$$\mathcal{X} \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

have definitions of similar complexity. The following problem arises:

**Open Problem 1.** *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1)-(3)` *and* `(5)`*?*

## 3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

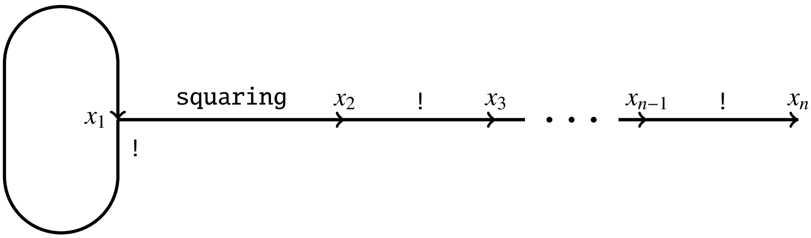


**Fig. 1** Construction of the system $\mathcal{U}_n$

**Lemma 2.** *For every positive integer n, the system* $\mathcal{U}_n$ *has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(f(1), \ldots, f(n)\big)$.

Let

$$B_n = \left\{x_i! = x_k :\ i, k \in \{1, \dots, n\}\right\} \cup \left\{x_i \cdot x_j = x_k :\ i, j, k \in \{1, \dots, n\}\right\}$$

For a positive integer $n$, let $\Psi_n$ denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant f(n)$.* The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\Psi_9$, cf. Conjecture 1 in Section 4.

**Theorem 1.** *For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.*

*Proof.* It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

**Theorem 2.** *For every integer $n \geqslant 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.*

*Proof.* If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \dots, x_n$, then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \dots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

**Theorem 3.** *Every statement $\Psi_n$ is true with an unknown integer bound that depends on n.*

*Proof.* For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □

## 4. A conjectural solution to Open Problem 1

**Lemma 3.** *For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 4.** *(Wilson's theorem,* [1, p. 89]*). For every integer $x \geqslant 2$, $x$ is prime if and only if $x$ divides $(x-1)! + 1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

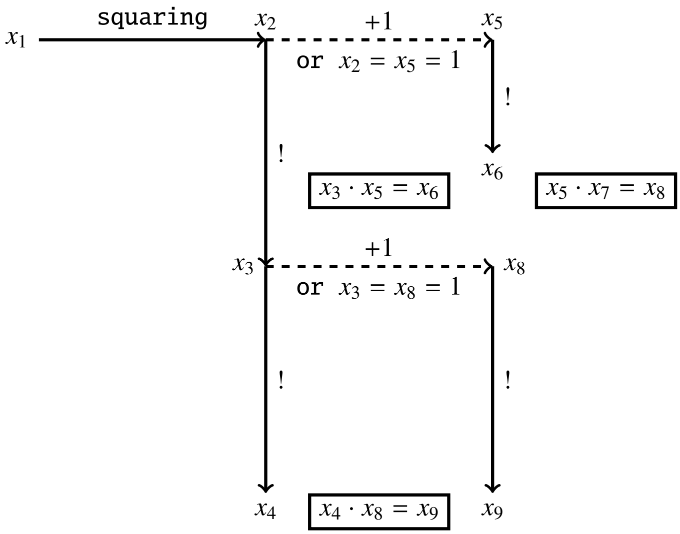


**Fig. 2** Construction of the system $\mathcal{A}$

**Lemma 5.** *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}
$$

*Proof.* By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

**Lemma 6.** *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

*Proof.* The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

**Conjecture 1.** *The statement $\Psi_9$ is true when is restricted to the system $\mathcal{A}$.*

**Theorem 4.** *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

*Proof.* Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

**Theorem 5.** *Conjecture 1 implies the statement* $\Phi$*.*

*Proof.* It follows from Theorem 4 and the equality $f(7) = (((24!)!)!)!$. □

**Theorem 6.** *The statement* $\Phi$ *implies Conjecture 1.*

*Proof.* By Lemmas 5 and 6, if positive integers $x_1, \dots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement $\Phi$ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$. □

**Definition 5.** *Let* $\mathcal{K} = \{k \in \mathbb{N} :$ *the number of digits of k belongs to* $\mathcal{P}_{n^2+1}\}$.

**Lemma 7.** $\text{card}(\mathcal{K}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}} \approx 10^{10^{450.6930560314272}}$.

*Proof.* The following `PARI/GP` ([4]) command

```
isprime(1+9*4^747,{flag=2})
```

returns $\%1 = 1$. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([7, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10\hat{}(9 * 4\hat{}747))$. □

**Statement 3.** *The sets* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *and* $\mathcal{X} = \mathcal{K}$ *satisfy conditions* `(1)`-`(3)`. *The statement* $\Phi$ *implies that both sets* $\mathcal{X}$ *satisfy condition* `(5)`.

*Proof.* Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 7 implies condition `(1)` for both sets $\mathcal{X}$. Condition `(3)` holds trivially for both sets $\mathcal{X}$. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition `(2)` holds for both sets $\mathcal{X}$. Suppose that the statement $\Phi$ holds. This implies two facts:

$$\beta \text{ is a threshold number of } \mathcal{X} = \mathcal{P}_{n^2+1} \tag{6}$$

and

$$\underbrace{9\dots9}_{\beta \text{ digits}} \text{ is a threshold number of } \mathcal{X} = \mathcal{K} \tag{7}$$

Thus condition (4) holds for both sets $\mathcal{X}$. The definition of $\mathcal{P}_{n^2+1}$ is much simpler than the definition of $\mathcal{P}_{n^2+1} \setminus (-\infty, \beta]$. The definition of $\mathcal{K}$ is much simpler than the definition of $\mathcal{K} \setminus (-\infty, \underbrace{9 \ldots 9}_{\beta \text{ digits}}]$. The last three sentences imply that condition (5) holds for both sets $\mathcal{X}$. □

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