Derivation of Classical Mechanics in an Energetic Framework via Conservation and Relativity

Philip Goyal*
University at Albany (SUNY), NY, USA
(Dated: July 27, 2020)

The notions of conservation and relativity lie at the heart of classical mechanics, and were critical to its early development. However, in Newton's theory of mechanics, these symmetry principles were eclipsed with domain-specific laws. In view of the importance of symmetry principles in elucidating the structure of physical theories, it is natural to ask to what extent conservation and relativity determine the structure of mechanics. In this paper, we address this question by deriving classical mechanics—both nonrelativistic and relativistic—using relativity and conservation as the primary guiding principles. The derivation proceeds in three distinct steps. First, conservation and relativity are used to derive the asymptotically conserved quantities of motion. Second, in order that energy and momentum be continuously conserved, the mechanical system is embedded in a larger *energetic framework* containing a massless component that is capable of bearing energy (as well as momentum in the relativistic case). Imposition of conservation and relativity then results, in the nonrelativistic case, in the conservation of mass and in the frame-invariance of massless energy; and, in the relativistic case, in the rules for transforming massless energy and momentum between frames. Third, a force framework for handling continuously interacting particles is established, wherein Newton's second law is derived on the basis of relativity and a staccato model of motion-change. Finally, in light of the derivation, we elucidate the structure of mechanics by classifying the principles and assumptions that have been employed according to their explanatory role, distinguishing between symmetry principles and other types of principles (such as compositional principles) that are needed to build up the theoretical edifice.

I. INTRODUCTION

Two key notions, namely conservation and relativity, lie at the heart of classical mechanics. Each is an instance of a fundamental physical symmetry—*conservation* of the symmetry that temporal evolution of a system is underlain by changelessness in some of its properties; *relativity* of the symmetry that, although observations are necessarily perspectival, there are classes of observers which are, in some fundamental sense, *physically equivalent*.

These notions played a vital role in the early development of mechanics. The notion of conservation was first formalized by Descartes through the principle that a system of colliding bodies conserves its total scalar 'quantity of motion', a principle which he then used to guide the formulation of laws of collision. Galileo, in his principle of relativity, posited the equivalence of inertial frames in uniform relative motion, which enabled his deduction of parabolic motion from vertical free fall. And, using the principle of relativity and a principle of conservation in tandem, Huygens deduced a new conservation law (the conservation of relative speed) for bodies in head-on collision[1].

However, in Newton's theory of mechanics, the notion of force took centre stage, and these principles were eclipsed[2] by domain-specific laws (rather than symmetry principles), particularly Newton's second law.

Subsequent developments in physics—beginning with the emergence of the conservation of energy as a meta-theoretic principle for coordinating physical theories of specific classes of phenomena, and the development of Einstein's special theory of relativity—have again brought symmetry principles firmly into the foreground. Today, symmetry principles are regarded as *meta-laws* that shape the theoretical landscape within which *specific* theories—with their domain-specific laws—take root. As Wigner put it: just as the laws of physics express regularities in events, symmetry principles express regularities in laws of physics [1].

From this perspective, even though the specific laws of a theory may well have been arrived at in a complex manner by combining theoretical and empirical inputs, as well as inspired guesswork, it is desirable to reformulate the theory in a way that reflects Wigner's hierarchy and accordingly clearly separates the symmetry princi-

*Electronic address: pgoyal@albany.edu

[1] See Sec. V C 1 for further details.
[2] See Sec. V C 1 for background on this shift.

ples (and other meta-theoretic inputs) from any specific laws, and shows how these symmetry principles shape its specific laws.

Previous efforts to better understand mechanics in terms of symmetry principles have tended to focus on deriving *parts* of mechanics, rather than on deriving the whole—for example, on deriving the conserved quantities of motion (reviewed in Sec. IV), or on deriving Newton's second law [2]. Consequently, these efforts have had relatively little impact and do not appear to be widely known.

In this paper, we show how both nonrelativistic and relativistic mechanics can be derived in their entirety on the basis of symmetry principles, particularly on the basis of the principle of relativity and the notion of conservation. This is carried out in three distinct steps[3]:

I. *Asymptotic conservation.* First, by considering a specific collision whose existence is underpinned by basic physical symmetries, we derive the mathematical forms of the asymptotically conserved scalar quantities of motion (namely corpuscular energy) by appeal to the principle of relativity. A further argument (due to Schütz [3]) is then used to derive asymptotic momentum conservation from asymptotic energy conservation.

II. *Energetic framework.* Next, guided by the desideratum that conservation be *continuous* rather than merely asymptotic, we embed a system of interacting bodies in an energetic framework containing a massless component that can bear energy (and, in the relativistic case, also momentum). This framework allows for the passage of energy (and possibly momentum) between its massive component (consisting of bodies in motion) and its massless component. Relativity and conservation are then used to determine (i) which interconversions are possible, and (ii) how the energy (and, in the relativistic case, the momentum) associated with the massless component transforms between inertial frames.

III. *Staccato model of motion change.* Finally, we posit a specific model of how a body undergoes change of motion due to the influence of another. Using this model, Newton's second law is derived via relativity. In the nonrelativistic case, the first part of Newton's third law (*viz.* that interparticle forces in a two-body system are antiparallel) then follows from conservation of momentum, while the second part (the centrality of those two-body interparticle forces that depend only upon position) follows from a symmetry-based argument.

In this three-fold process, the energetic framework (in Step II) provides a crucial link between the asymptotically conserved quantities and the force framework.

By building up mechanics in this layered manner, the distinct types of principles out of which mechanics is built up—ranging from the most general to the most specific—become clearly visible. For example, in addition to conservation and relativity, we find that compositional principles—often underlain by symmetries—also play a fundamental role. It also becomes apparent that other meta-theoretic desiderata, such as continuity, play a pivotal role. For example, in our development, the move from the first step to the second, where (in the nonrelativistic case) a massless form of energy is posited, is driven by the desideratum that total energy in an elastic collision be *continuously*—not just asymptotically—conserved. Similarly, in the relativistic case, a massless form of momentum must be posited in order that momentum be continuously conserved. The latter contrasts with the historical development, in which massless momentum was first introduced via Maxwell's equations.

Second, since our approach depends primarily on relativity and conservation, the parallelism between nonrelativistic and relativistic mechanics can be clearly exhibited. The shift from nonrelativistic to relativistic mechanics is straightforwardly achieved by changing the kinematical group by which relativity is implemented, and by allowing the massless component of the energetic system to bear momentum as well as energy. The possibility of interconversion between rest energy and other energetic forms directly follows from these changes, without the customary appeal to other special considerations (such as the laws governing the behaviour of photons).

Third, by approaching mechanics using the notion of the energetic system—governed by its own overarching conservation law—certain results are obtained with surprising ease. For example, an important consequence of relativistic dynamics is that massless energy-momentum transforms in the same way as corpuscular energy-momentum. In the standard approach, this fact is proved for the special case of a physical system describable via a stress-energy-momentum tensor [4] (see also [5, 6]). However, in our approach, in the context of the energetic framework, conservation and relativity jointly imply that massless energy and momentum transform as a four-vector, without recourse to any *specific* model of the massless component. In addition, in the nonrelativistic case, the corresponding argument shows that massless energy is frame-invariant, a fact that is

[3] The principles that are employed, and results obtained, in each step are summarized in Tables I and II (pages 22 and 23).

generally taken as axiomatic in nonrelativistic thermodynamics.

Finally, many features that are usually taken as axiomatic in standard presentations of nonrelativistic mechanics—such as the mathematical form of the quantities of motion (momentum, kinetic energy), the invariance of total mass, the frame-invariance of noncorpuscular energy, Newton's second and third laws, and the frame-invariance of force—are systematically derived, with the benefit of the insight that results from the symmetry-based approach. Similar benefits accrue through the treatment of special relativistic dynamics. For example, interconvertibility of mass and massless energy is seen to directly arise through the interplay of conservation and relativity, without appeal to any explicit model (such as electromagnetic) of the massless energy.

The paper is organized as follows. In Sec. II, we derive nonrelativistic mechanics in an energetic framework, beginning with the derivation of the nonrelativistic quantities of motion (Sec. II A), and then building up the energetic framework (Sec. II B). The development of the Newtonian framework within the resulting structure, and the insights to which this leads, are described in Sec. II C. A comparison of the derivation with the standard presentation of nonrelativistic mechanics in given in Sec. II D. A parallel treatment of relativistic mechanics is carried out in Sec. III. In Sec. IV, we briefly describe and analyze a selection of other derivations of quantities of motion from the literature.

In Sec. V, we clarify the structure of classical mechanics in light of our derivation by classifying and analyzing the physical principles employed, and examining the insights that our approach provides about the subtle issues that arose connected with conservation and relativity in the historical development of mechanics. We conclude in Sec. VI with a discussion of the relation between symmetry transformations and conservation laws, and of pedagogical approaches to mechanics in light of the present derivation.

II. NONRELATIVISTIC MECHANICS

A. Conserved quantities of motion

1. *Derivation of particle energy as asymptotically conserved scalar quantity of motion*

In its simplest form, a *scalar* conservation principle (as that posited by Descartes) applied to a set of bodies in motion asserts that their total (scalar) quantity of motion is conserved in any elastic collision. Here, the mathematical form of a body's quantity of motion is as yet unspecified, but it is presumed to be a function of its speed. However, without knowing anything further about the form of the quantity of motion, we can classify a collision as elastic if the bodies' initial and final speeds are the same.

Now, if one supposes that, during a collision, a body undergoes a *continuous* change of motion, it follows that two equal bodies in head-on elastic collision can be momentarily stilled (in some inertial frame). Thus, one can only hope to conserve their total quantity of motion if one compares the collision's pre- and post-collisional states. Accordingly, we posit that the total quantity of motion is *asymptotically* conserved. We then derive the mathematical *form* of the quantity of motion by considering a specific elastic collision observed from two different inertial frames, namely the lab frame, S, and a moving frame, S'.

We assume that a particle of mass[4] m with speed u has a scalar quantity of motion $f_m(u)$, to which we henceforth refer as its energy. By hypothesis, this function f is independent of the specific situation in which the particle finds itself. Hence, if we can determine the form of f by considering specific situations that we presume to be possible, then that form of f must apply to all situations. We further assume that the total energy of a system of widely-separated particles is the sum of their separate energies[5].

Suppose that, as observed from inertial frame S, two particles of equal mass approach from opposite directions, moving at the same speed, u, along the x-axis, and collide elastically at the origin, O (see Fig. 1)[6]. We assume that it is possible for the particles, after collision, to recede in opposite directions along the y-axis with their speeds undiminished[7]. Suppose that frame S'

---

[4] The *mass* of a body is here taken as a measure of substance from which a body is composed. In particular, no connection between mass and inertia (degree of resistance to force) is implied. The mass is assumed to be independent of the body's state of motion, and hence frame-independent.

[5] For simplicity, this energy-additivity is taken as given here, but is in fact a manifestation of the compositional symmetry of associativity (see Sec. V B).

[6] We assume here that isolated bodies move at constant velocity.

[7] Although the indicated collision is a premise of the following argument, it can be traced to more primitive symmetry requirements: (a) The possibility of post-collisional motion of at least one body along the y-axis (vertical) can be traced to the requirement of continuity together with the fact that both grazing and head-on collisions are possible. (b) The fact that the post-collisional velocities must then be antiparallel can be traced to the requirement that relatively-rotated reference frames are physically equivalent and the requirement that the same initial conditions lead to the same final conditions. Specifically, suppose that, as viewed in frame S, one particle post-collisionally travels along the positive y-axis, but the other *not* along the negative y-axis.

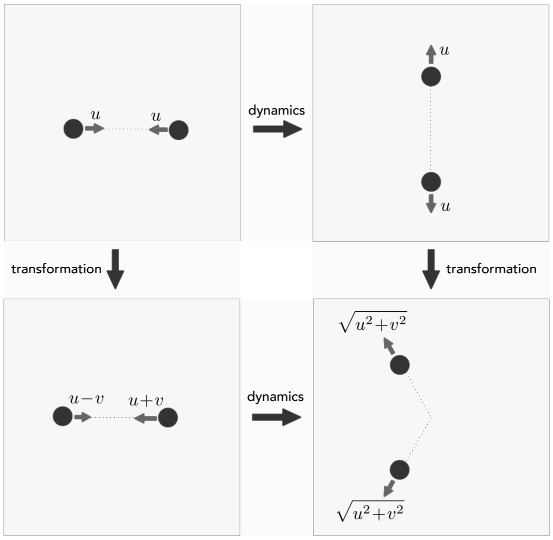

FIG. 1: *Derivation of nonrelativistic particle energy via asymptotic scalar conservation and relativity.* Consider an elastic collision of two equal-mass particles, as viewed in frames S and S'. In frame S, the particles, each of mass m, approach at equal speed u, and recede along the y-axis at the same speeds. In this frame, the total pre- and post-collisional energy of the system (assumed, for widely-separated particles, to be sum of the energies of the separate particles) is trivially conserved since the pre- and post-collisional speeds are the same. By relativity, frames S and S' are physically equivalent, so that, if the situation in S is physically possible (as is here assumed), the situation in S' is also. Therefore, since energy is asymptotically conserved in S, it must also be conserved in S'. That condition leads to a functional equation, $f_m(u+v) + f_m(u-v) = 2f_m(\sqrt{u^2+v^2})$, for the energy, $f_m(u)$, of a particle of mass m. Its solution is $f_m(u) = au^2 + b$, where a, b are functions of m.

moves at speed v along the x-axis of S. Since asymptotic energy conservation trivially holds in frame S, and since frames S and S' are (by the principle of relativity) *physically equivalent*, asymptotic energy conservation must also hold in frame S', which leads to the condition, for all $u \geq 0$ and all $v \in [-u, u]$,

$$f_m(u+v) + f_m(u-v) = 2f_m(\sqrt{u^2+v^2}), \qquad (1)$$

whose general solution is

$$f_m(u) = a(m)u^2 + b(m), \qquad (2)$$

where a, b are undetermined functions of m (see Appendix A 1).

To determine the forms of functions $a(m)$ and $b(m)$, assume that an object of mass m moving at speed u can equally be regarded as a composite[8] of two noninteracting masses, m_1 and m_2, such that $m = m_1 + m_2$, moving together at speed u. This composite has total energy

$$f_{m_1}(u) + f_{m_2}(u) = [a(m_1) + a(m_2)]\, u^2 + [b(m_1) + b(m_2)]\,.$$

Thus, for any m_1, m_2 and any u,

$$\begin{aligned} a(m_1+m_2)u^2 + b(m_1+m_2) = {} & [a(m_1) + a(m_2)]\, u^2 \\ & + [b(m_1) + b(m_2)]\,. \end{aligned}$$

Therefore, functions $a(m)$ and $b(m)$ both obey Cauchy's additive functional equation,

$$\begin{aligned} a(m_1+m_2) &= a(m_1) + a(m_2) \\ b(m_1+m_2) &= b(m_1) + b(m_2), \end{aligned}$$

which have general solutions $a(m) = \alpha m$ and $b(m) = \beta m$, where α, β are numerical constants. Hence,

$$f_m(u) = \alpha m u^2 + \beta m. \qquad (3)$$

The kinetic energy is $\alpha m u^2$, and the rest energy βm. As the energy-scale is arbitrary up to a multiplicative factor, we can set $\alpha = 1/2$ to conform with convention.

2. Derivation of particle momentum as asymptotically conserved vectorial quantity of motion

Using an argument due to Schütz [3], one now finds that the scalar asymptotic conservation principle implies—via another application of relativity—a vector asymptotic conservation principle.

Consider a general elastic collision in which, in frame S', two objects of mass m_1 and m_2 undergo elastic

---

Then an observer in a reference frame ($\widetilde{S}$) rotated about the z-axis (perpendicular to plane of collision, and passing through the midpoint of the particles' initial positions) of S by π would see a collision with the same initial positions and velocities, but with one of the post-collisional velocities along the *negative* y-axis and the other *not* along the positive y-axis. That is, the *same* initial positions and initial velocities would lead to *different* post-collisional velocities for observers in S and $\widetilde{S}$. This inconsistency can be avoided if the post-collisional velocities are along the positive and negative y-axes. Moreover, the same requirements imply that the post-collisional *speeds* of the two particles are equal. (c) The fact that, additionally, the post-collisional speeds coincide with the pre-collisional speeds then follows from the assumptions that (i) relatively-rotated reference frames are physically equivalent, and (ii) the time-reversed version of an elastic collision (*viz.* a collision that asymptotically conserves the total scalar quantity of motion) is also possible.

[8] That the mass of the composite is equal to the sum $m_1 + m_2$ can either be assumed, or derived from the compositional symmetry of associativity (see Sec. V B).

collision with initial velocities $\mathbf{u}_1, \mathbf{u}_2$, and separate at velocities $\mathbf{u}_1', \mathbf{u}_2'$. Energy conservation in frames S and S' implies that

$$m_1 u_1^2 + m_2 u_2^2 = m_1 u_1'^2 + m_2 u_2'^2,$$

and that, for all $\mathbf{v}$,

$$m_1|\mathbf{u}_1 + \mathbf{v}|^2 + m_2|\mathbf{u}_2 + \mathbf{v}|^2 = m_1|\mathbf{u}_1' + \mathbf{v}|^2 + m_2|\mathbf{u}_2' + \mathbf{v}|^2.$$

Subtracting the foregoing equations,

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{u}_1' + m_2\mathbf{u}_2', \qquad (4)$$

which is asymptotic momentum conservation. It follows that the asymptotically conserved vectorial quantity of motion, $g_m(u)\hat{\mathbf{u}}$, associated with a particle of mass m moving at velocity $\mathbf{u}$ is $\alpha m\mathbf{u}$ up to an additive vectorial constant. Consideration of the elastic collision above shows that the vectorial constant is zero. Finally, by regarding a mass m as a composite of masses m_1 and m_2 (as in Sec. II A 1, above) implies that $a'(m) = \alpha' m$, where α' is a numerical constant. Hence, a mass m moving at speed u has vectorial quantity of motion

$$g_m(u)\hat{\mathbf{u}} = \alpha' m\mathbf{u}. \qquad (5)$$

Following convention, we set $\alpha' = 1$, yielding the momentum $m\mathbf{u}$.

Remark. One can also derive the form of the momentum in a manner parallel to that used to derive the form of corpuscular energy by *positing* a vector asymptotic conservation principle, and then considering the elastic collision above. The disadvantage of this approach is two-fold: (i) the intuition behind a scalar conservation principle—that quantity of motion is not lost, merely redistributed—seems more compelling than that underlying a vector conservation principle; and (ii) the relationship between the two conservation principles is not made evident. Nevertheless, this approach does work, and yields the functional equation

$$g(v+u) - g(v-u) = 2g(w)\cdot\frac{v}{w}, \qquad (6)$$

where $w = \sqrt{u^2+v^2}$. As shown in Appendix A 2, this equation has general solution

$$g(u) = a'(m)u, \qquad (7)$$

where $a'(m)$ is an undetermined function.

One might imagine considering instead the initial state and stillpoint of an elastic head-on collision between two equal masses initially travelling at equal speed u. Conservation of the vectorial quantity of motion in frame S' would presumably then yield the equation $g(v+u) - g(v-u) = 2g(v)$, with solution $g(u) = a'(m)u$. However, since conservation of energy in S' implies that there is some non-motive energy present at the stillpoint, one is here making an implicit assumption, namely that there is no momentum associated with this non-motive energy. While this happens to be true in the nonrelativistic case, it is *not* true in the relativistic one. More importantly, this (implicit) assumption constitutes an assumption about the larger energetic framework (see Sec. II B), which deserves considered justification. Hence, in deriving the form of the vectorial quantity of motion, it is advisable to consider only the *asymptotic* states of an elastic collision, where (since the initial and final motive energies are the same) no such assumption is required.

B. Continuous conservation of energy and momentum in the energetic framework

We have noted above that conservation of a total scalar quantity of motion can only hold asymptotically, and only then for elastic collisions. In order to generalize this conservation law so that it applies continuously, and to all collisions, we must posit that every system of bodies exists within a larger energetic framework that can contain a massless component capable of bearing energy.

Since the imposition of continuous momentum conservation encounters no obvious obstacles when applied to bodies undergoing inelastic collisions, there is no specific need to assume that the massless component is also capable of bearing momentum.

The question then arises as to what kinds of interconversions of rest energy, kinetic energy, and massless energy[9] are possible, and how the energy of the massless component transforms between frames. By requiring continuous conservation of total energy and momentum, and by imposing relativity, we shall see that (see Fig. 2):

1. System mass is conserved. Therefore, rest energy cannot be dynamically converted into kinetic energy or massless energy.

2. Interconversion of kinetic energy and massless energy is permitted.

[9] By definition, massless energy refers to any form of energy other than the two forms—namely, rest energy and kinetic energy—that are explicitly associated with a massive body.

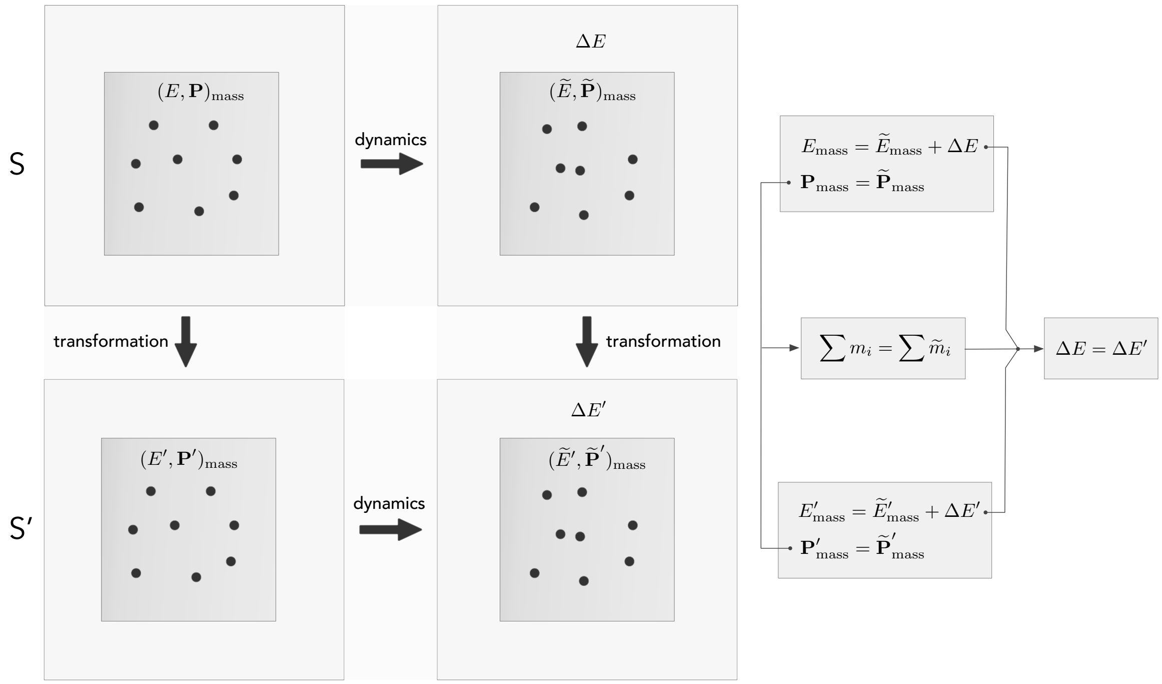

FIG. 2: *Mass conservation and frame-invariance of massless energy in the energetic framework.* The energetic framework posits that, in addition to a system of bodies with their energies and momenta, there exists a *massless* component capable of bearing energy. Within such a framework, energy can be *continuously* (not just asymptotically) conserved. In this example, in frame S, we consider a process in which the system initially has no massless energy, but dynamically evolves to a state in which the massless energy is ΔE. If conservation of *total* energy and momentum is applied to this dynamical process as seen in frames S and S' (Eqs. (8), (9), (11), and (12)), one finds that momentum conservation implies total mass conservation (Eq. (10)), and energy conservation then implies (Eq. (13)) the frame-invariance of massless energy, $\Delta E = \Delta E'$.

3. Massless energy takes the same value in all frames.

4. If two states—possibly of different systems—have equal values of total mass, energy, and momentum as observed in frame S, then this equality holds true if the states are observed in any other inertial frame.

1. *Interconversion between different forms of energy*

Consider two states of a system. Observed in frame S, the first contains masses m_i moving with velocities $\mathbf{u}_i$, but with the massless component bearing no energy. The second state consists of (i) masses $\widetilde{m}_i$ moving with velocities $\widetilde{\mathbf{u}}_i$, as well as (ii) a massless component which has energy ΔE. In what follows, quantities observed in frame S' are primed.

Imposition of momentum conservation in frames S, S' yields, respectively,

$$\sum_i m_i \mathbf{u}_i = \sum_i \widetilde{m}_i \widetilde{\mathbf{u}}_i \tag{8}$$

and, for any $\mathbf{v}$,

$$\sum_i m_i (\mathbf{u}_i - \mathbf{v}) = \sum_i \widetilde{m}_i (\widetilde{\mathbf{u}}_i - \mathbf{v}), \tag{9}$$

which together imply that

$$\sum_i m_i = \sum_i \widetilde{m}_i. \tag{10}$$

Thus, the conservation of momentum implies that no interconversion of rest energy into either kinetic energy or massless energy is possible.

We now impose energy conservation in frames S, S'. In frame S, we obtain

$$\frac{1}{2} \sum_i m_i u_i^2 + \beta \sum_i m_i = \frac{1}{2} \sum_i \widetilde{m}_i \widetilde{u}_i^2 + \beta \sum_i \widetilde{m}_i + \Delta E, \tag{11}$$

and, in frame S', for all $\mathbf{v}$,

$$\frac{1}{2}\sum_i m_i \left|\mathbf{u}_i - \mathbf{v}\right|^2 + \beta\sum_i m_i = \frac{1}{2}\sum_i \widetilde{m}_i \left|\widetilde{\mathbf{u}}_i - \mathbf{v}\right|^2 + \beta\sum_i \widetilde{m}_i + \Delta E', \quad (12)$$

which, together with Eq. (10), imply that

$$\Delta E' = \Delta E. \quad (13)$$

That is, the massless energy is frame-invariant[10], and so does not transform in the same way as the total energy of the mass component, whose energy and momentum transform as:

$$E' = E - \mathbf{P}\cdot\mathbf{v} + \frac{1}{2}Mv^2 \quad (14)$$

$$P' = \mathbf{P} - M\mathbf{v} \quad (15)$$

As we shall see later, this difference vanishes when one employs the Lorentz—rather than Galilean—transformations to relate inertial frames.

Due to these results, the application of continuous conservation of energy and momentum conservation to an energetic system (consisting of mass- and massless-components) proceeds as follows:

1. When imposing energy conservation to states involving a massless component, one must take into account the energy, ΔE, of this component. That is, in frame S, Eq. (11) becomes

$$\frac{1}{2}\sum_i m_i u_i^2 = \frac{1}{2}\sum_i \widetilde{m}_i \widetilde{u}_i^2 + \Delta E, \quad (16)$$

   subject to total mass conservation, Eq. (10). And, in writing down the corresponding conservation statement in frame S', the energy of the massless component $\Delta E'$ takes the *same* value as in S, namely ΔE.

2. As previously discussed, the momentum of the massless component is presumed to be zero in all frames, so that one need only consider momenta associated with masses. That is, momentum is 'internally' continuously conserved.

[10] In Sec. IV B, we discuss a derivation of nonrelativistic kinetic energy due to Maimon in which the frame-invariance of massless energy is implicitly assumed.

2. Frame-invariance equality of $(M, E, \mathbf{P})$

Consider two systems, possibly containing differing numbers of particles in different states of motion. Let the systems be in states that, in frame S, are described by the tuples $(M, E, \mathbf{P})$ and $(\overline{M}, \overline{E}, \overline{\mathbf{P}})$. We show that, if these tuples are equal in frame S, then the corresponding tuples $(M', E', \mathbf{P}')$ and $(\overline{M}', \overline{E}', \overline{\mathbf{P}}')$ of the two systems as described in S' are also equal.

The preservation of equality of M and $\overline{M}$ is immediate from the frame-invariance of total mass, Eq. (10). Consider the quantities E and $\mathbf{P}$ describing the first system,

$$\begin{aligned} E &= \frac{1}{2}\sum_i m_i u_i^2 + \beta\sum m_i + \Delta E, \\ \mathbf{P} &= \sum m_i \mathbf{u}_i, \end{aligned} \quad (17)$$

where ΔE is the energy of the massless component.

Using Eq. (13), E and $\mathbf{P}$ transform as

$$\begin{aligned} E' &= \frac{1}{2}\sum_i m_i \left|\mathbf{u}_i - \mathbf{v}\right|^2 + \beta\sum m_i + \Delta E' \\ &= E - \mathbf{P}\cdot\mathbf{v} + \frac{1}{2}Mv^2 \end{aligned} \quad (18)$$

and

$$\begin{aligned} \mathbf{P}' &= \sum m_i(\mathbf{u}_i - \mathbf{v}) \\ &= \mathbf{P} - M\mathbf{v} \end{aligned} \quad (19)$$

Thus, the tuple $(M', E', \mathbf{P}')$ is determined by $(M, E, \mathbf{P})$. It follows that, if $(M, E, \mathbf{P}) = (\overline{M}, \overline{E}, \overline{\mathbf{P}})$, then the tuples $(M', E', \mathbf{P}')$ and $(\overline{M}', \overline{E}', \overline{\mathbf{P}}')$ of the systems observed in frame S' are also equal. Consequently, the tuple $(M, E, \mathbf{P})$ can be thought of as the *macrostate* of the energetic system (composed of a massive and massless component).

In particular, note that, as $(M', E', \mathbf{P}')$ explicitly depends upon M, one can find systems such that $(E, \mathbf{P}) = (\overline{E}, \overline{\mathbf{P}})$ but $(E', \mathbf{P}') \neq (\overline{E}', \overline{\mathbf{P}}')$. For example, consider two systems, each containing two particles of equal mass, moving at equal speeds in opposite directions along the x-axis. Let the first system contain particles of mass m, moving at speed u; and the second with particles of mass $m/4$ moving at speed $2u$. These systems have equal energy $(2mu^2)$ and momentum (zero) in S, but unequal energy and momentum in S'. In a relativistic framework, however, the equality of $(E, \mathbf{P})$ tuples of two systems *is* frame-invariant, irrespective of whether or not these systems have equal masses (see Sec. III B 2).

C. Development of Newton's dynamical theory

The development above is based on the consideration of collisions. However, a *dynamical theory* must allow for interactions between bodies even when separated, and further allow for *ongoing* changes in motion. To build such a theory, one requires an explicit model for motion-change which is sufficiently broad as to be applicable to widely-separated bodies in ongoing interaction.

1. Staccato model of motion change

The simplest generalization of the previous collision-based considerations is to assume that continuous interaction between, say, two bodies (that are, in general, separated from one another) can be arbitrarily well approximated by a staccato model in which each body suffers a rapid succession of small *abrupt* changes of its motion. Between these changes—by the principle of inertia—these two bodies move at constant velocity[11]. Due to relativity, one can—without loss of generality—consider the effect of each body's change of motion in its initial rest frame. In a body's initial rest frame, $\overline{S}$, an abrupt change of motion causes the body, initially at rest, to move off at velocity $\Delta\bar{\mathbf{u}}$. Thus, the effect of the influence on the body's change of motion is completely characterized by $\Delta\bar{\mathbf{u}}$.

Now, over a small time interval, Δt, suppose that a body undergoes n abrupt velocity changes $\Delta\bar{\mathbf{u}}^{(1)}, \Delta\bar{\mathbf{u}}^{(2)}, \ldots, \Delta\bar{\mathbf{u}}^{(n)}$, with the ith change referred to frame $\overline{S}^{(i)}$ in which the body is at rest immediately prior to this change. Due to Galilean kinematics, the net velocity change, $\Delta\bar{\mathbf{u}}$, in the frame, $\overline{S}$, in which the body is at rest immediately prior all of these changes, is the sum of these velocity changes,

$$\Delta\bar{\mathbf{u}} = \Delta\bar{\mathbf{u}}^{(1)} + \Delta\bar{\mathbf{u}}^{(2)} + \cdots + \Delta\bar{\mathbf{u}}^{(n)}. \qquad (20)$$

By Galilean kinematics, velocity changes are frame-independent. Thus, as viewed in the lab frame, S, the cumulative effect of these change is to cause the body to undergo a change in velocity from $\mathbf{u}$ to $\mathbf{u} + \Delta\bar{\mathbf{u}}$.

2. Motion in a two-body system

Now, in a two-body system, it follows from the conservation of momentum that, if one body undergoes abrupt velocity changes due to the influence of the other, then the other must undergo corresponding abrupt velocity changes. Therefore, the only time-dependent quantities that can be attributed to body i *between* velocity changes are its position, $\mathbf{r}_i$, and velocity, $\mathbf{u}_i$. Over the interval $[t, t + \Delta t]$, one also can compute an average acceleration $\mathbf{a}_i = \Delta\mathbf{u}_i/\Delta t$.

Guided by the requirement of determinism, we now postulate that each body's average acceleration in Δt is determined by the bodies' masses and their time-dependent properties at the instant prior to the velocity changes that occur during this interval. Since the bodies move inertially between velocity jumps, these time-dependent properties consist in the bodies' positions and velocities only. Hence,

$$\mathbf{a}_1 = \mathbf{f}_{12}(m_1, m_2; \mathbf{r}_1, \mathbf{r}_2; \mathbf{u}_1, \mathbf{u}_2) \qquad (21)$$

and

$$\mathbf{a}_2 = \mathbf{f}_{21}(m_1, m_2; \mathbf{r}_1, \mathbf{r}_2; \mathbf{u}_1, \mathbf{u}_2). \qquad (22)$$

Here, the *influence function* $\mathbf{f}_{ij}$ encodes the influence on body i due to body j. As previously shown (Eq. (10)), total mass is conserved. Here we additionally assume that the m_i of separated particles remain constant during the interval.

Since the velocity change of each body takes the same value for two inertial frames in uniform relative motion, the $\mathbf{f}_{ij}$ must be do so also. Hence the latter can depend only on the bodies' frame-independent intrinsic properties, m_1, m_2, together with their relative position, $\mathbf{r}_{12} \equiv \mathbf{r}_2 - \mathbf{r}_1$, and relative velocity, $\mathbf{u}_{12} \equiv \mathbf{u}_2 - \mathbf{u}_1$. Furthermore, due to the momentum conservation, namely

$$m_1\Delta\mathbf{u}_1 + m_2\Delta\mathbf{u}_2 = \mathbf{0}, \qquad (23)$$

the influence functions must satisfy the constraint

$$m_1\mathbf{f}_{12} + m_2\mathbf{f}_{21} = \mathbf{0}. \qquad (24)$$

If one defines the *force functions* $\mathbf{F}_{12} \equiv \mathbf{f}_{12}/m_1$ and $\mathbf{F}_{21} \equiv \mathbf{f}_{21}/m_2$, then the above constraint can be re-expressed in terms of the force functions,

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0}, \qquad (25)$$

---

[11] Considering the motion of a body under the influence of a sequence of discrete impulses, between which the particle moves inertially, as a way of deducing results concerning the motion of the body when under the influence of a corresponding continuous force is a tactic employed extensively by Huygens, Newton, and others (see, for example, Ref. [2], where it is noted that "From Newton to Laplace, impulses were usually regarded as more fundamental, and continuous forces were assumed to be equivalent, in their observable effects, to a very rapid succession of impulses"). The specific argument given here is inspired by the derivation given in §2.1 of Ref. [2]

while the motion-change of body 1 can be expressed as

$$m_1\mathbf{a}_1 = \mathbf{F}_{12}, \tag{26}$$

with $\mathbf{F}_{12} = \mathbf{F}(m_1, m_2; \mathbf{r}_{12}, \mathbf{u}_{12})$, where $\mathbf{F}$ is a vector-valued function.

Body-centered forces in a two-body system. As indicated above, in the two-body system, the force, $\mathbf{F}_{12}$, exerted on body 1 by body 2 can depend upon their relative position vector, $\mathbf{r}_{12}$ and upon their relative velocity $\mathbf{u}_{12}$. We now show that, if the force does not depend upon $\mathbf{u}_{12}$, the equivalence of relatively-rotated inertial frames implies that $\mathbf{F}_{12}$ lies along $\mathbf{r}_{12}$.

Consider a frame S' that, relatively to the lab frame, S, is rotated by π about $\mathbf{r}_{12}$. In frame S', the particles' relative position vector, $\mathbf{r}'_{12}$, is the same as that in S—that is, $\mathbf{r}'_{12} = \mathbf{r}_{12}$—but the acceleration, $\mathbf{a}'_1$, of body 1 is $R\mathbf{a}_1$, where R is a rotation by π about $\mathbf{r}_{12}$. But, since $\mathbf{r}'_{12} = \mathbf{r}_{12}$, the application of Eq. (26) in frame S'—permitted because S' is physically equivalent to S—implies that the acceleration $\mathbf{a}'_1$ must be equal to $\mathbf{a}_1$. Hence, $R\mathbf{a}_1 = \mathbf{a}_1$, which implies that $\mathbf{a}_1$—and thus also $\mathbf{F}_{12}$—lies along $\mathbf{r}_{12}$.

Thus, if a force acts between two bodies which does not depend upon their relative velocity, it is necessarily a *central* force, which implies (via Eq. (26)) that the total angular momentum of the system is also conserved.

3. *Composition of forces*

We can extend the above model to a system of three or more bodies by assuming that, during each interval Δt, a body suffers many small changes in velocity due to each of the other bodies considered *separately*. Due to Galilean kinematics, these velocity changes add vectorially. Thus, denoting the velocity change of body i in interval Δt due to the presence of body $j \neq i$ as $\Delta\mathbf{u}_i^{(j)}$, the actual velocity change of body i due to the presence of all other bodies is

$$\Delta\mathbf{u}_i = \sum_{i\neq j} \Delta\mathbf{u}_i^{(j)} \tag{27}$$

$$= \sum_{i\neq j} \mathbf{F}(m_i, m_j; \mathbf{r}_{ij}, \mathbf{u}_{ij})\Delta t/m_i. \tag{28}$$

If one writes

$$\mathbf{F}_i = m_i \Delta\mathbf{u}_i/\Delta t, \tag{29}$$

where $\Delta\mathbf{u}_i/\Delta t$ is the average acceleration, this relation can alternatively be expressed as

$$\mathbf{F}_i = \sum_{i\neq j} \mathbf{F}(m_i, m_j; \mathbf{r}_{ij}, \mathbf{u}_{ij}). \tag{30}$$

That is, the net force on body i is the vector sum—for all $i \neq j$—of the force exerted by j on i.

4. *Smooth motion change*

One can further assert that the ideal of *smooth* motion change can be arbitrarily well by a time-average of the motion of a system which is subject to infinitesimally-small velocity jumps that are packed infinitely-densely in time[12]. In that limit, the instantaneous acceleration $\mathbf{a}_i = \lim_{\Delta t\to 0} \Delta\mathbf{u}_i/\Delta t$, so that

$$\mathbf{F}_i = m_i\mathbf{a}_i \tag{31}$$

where $\mathbf{F}_i$ is determined through Eq. (30).

In summary, Newton's framework can be seen to arise the following assumptions (beyond those made earlier):

- *Abruptness of change.* Change in the motion of a body, due to interaction with another, is well-approximated as arising through a rapid succession of small, abrupt velocity changes.

- *Influence is a function of the $m_i, \mathbf{r}_i, \mathbf{u}_i$.* In a two-body system, the change in velocity of a body in a given time-interval due to the influence of the other is a function of the bodies' positions, and velocities, together with their masses.

- *Composition of influences.* In a many-body system, the change in motion of a body is the resultant of that due to each of the other bodies considered separately.

D. Comparison with standard presentations of nonrelativistic classical mechanics.

Typical presentations of nonrelativistic classical mechanics are based around Newton's laws of motion, together with a separate statement of the conservation of energy and the expression for the work done by a force. These may be summarized as follows:

I. *Newton's laws of motion*

---

[12] In Ref. [2], this is referred to as the *secular principle.*

1. *First law.* An isolated body moves at constant velocity as observed in an inertial frame.
2. *Second law.* A body subject to net force $\mathbf{F}$ experiences a rate of change in its momentum given by $\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p} \equiv m\mathbf{u}$.
3. *Third law.* In a two-body system, the forces exerted by one body on another are equal and opposite ($\mathbf{F}_{12} = -\mathbf{F}_{21}$), and are directed from one body to the other ($\mathbf{F}_{12} = \alpha_{12}\mathbf{r}_{12}$).
4. *Composition of forces.* A body subject to forces $\mathbf{F}_\mathrm{a}, \mathbf{F}_\mathrm{b}, \mathbf{F}_\mathrm{c}, \ldots$ experiences net force $\mathbf{F} = \mathbf{F}_\mathrm{a} + \mathbf{F}_\mathrm{b} + \mathbf{F}_\mathrm{c} + \ldots$.

II. *Energy*

1. *Energy conservation.* The total energy—composed of the sum of kinetic energy and non-corpuscular energy—of any isolated system is conserved.
2. *Change in kinetic energy due to a force.* A body that moves $d\mathbf{x}$ whilst subject to force $\mathbf{F}$ experiences a change in its kinetic energy $dE_\mathrm{k} = \mathbf{F}.d\mathbf{x}$.

A complete statement requires a number of additional statements, such as the additivity of mass, the additivity of energy, the conservation of total mass, the frame-independence of force, and the frame-independence of massless (non-corpuscular) energy.

Such a presentation of mechanics raises a number of questions. For example:

1. *Quantities of motion.*
   (i) Why does the scalar quantity of motion take the form $mu^2/2$? Equivalently, why is the change in kinetic energy given by the expression $dE_\mathrm{k} = \mathbf{F}.d\mathbf{x}$?
   (ii) Why does the vectorial quantity of motion take the form $m\mathbf{u}$?
   (iii) Why are there *two* distinct conservation laws, one involving the scalar quantity of motion, the other a vector quantity of motion? And what is the relationship between these conservation laws?
2. *Interrelation and interpretation of the laws of motion.*
   (i) Is Newton's first law to be interpreted as a special case of second? If not, what is its role?
   (ii) Does Newton's second law give a *definition* of force, or does it embody a specific physical model of particle motion?
   (iii) Is the law of momentum conservation more fundamental than Newton's laws, or rather to be regarded as their consequence?

It is difficult to provide compelling answers to many of these questions within the confines of the standard presentation of mechanics. It is important to recall that many of these questions were the source of historical debate in the two centuries following the formulation of Newton's laws, prior to the shift of attention (in the first quarter of the twentieth century) to the foundations of modern physics. For example, as indicated in the introduction to Section IV and in Sec. V C 1, the historical pathway to the mathematical form of the quantities of motion was complex and indirect. This gave rise to lingering doubt as to the proper means to quantify the degree of motion of a body, as to whether kinetic energy was a quantity of motion on a par with momentum, and as to the relation between the laws of conservation of momentum and energy [7–9]. Another example concerns the interpretation of Newton's second law, which has often been regarded as simply providing a *definition* of force rather than being a genuine physical law that embodies a specific model of particle dynamics[13].

One of the advantages of the present derivation is that, by building up mechanics from a different standpoint—in particular by exploiting symmetry principles to a greater degree than was the case historically—it is possible to formulate compelling responses to most of the above questions. In brief:

1. *Quantities of motion.*
   (i) As shown in Sec. II A, by considering a particular collision (whose existence is justifiable by means of general symmetry requirements) and making use of the principle of relativity, the requirement that the sum total of a scalar quantity of motion be asymptotically conserved determines the quantity $mu^2/2$ as the scalar quantity of motion. That is, the mathematical form of the quantity is determined by general principles and symmetry requirements.
   (ii) Another application of the principle of relativity to a general elastic collision then implies that the quantity $m\mathbf{u}$ is also asymptotically conserved. Thus, in the elastic case, the conservation of one quantity (kinetic energy) implies conservation of the other.

---

[13] See, for instance, [10, p. 901], [11, p. 60]. See also the discussion of Newtonian principles given in [12, Ch. 10] due to Poincaré (§8) and Painlevé (§9); and also [2, §6–7].

(iii) As detailed in Sec. V C 2, the present approach enables a nuanced understanding of the relationship between momentum and energy conservation. In brief, for elastic collisions, asymptotic energy conservation plus relativity implies asymptotic momentum conservation. However, in the energetic framework, *continuous* conservation of energy and momentum must be separately postulated.

2. *Interrelation and interpretation of the laws of motion.*

(i) The derivation of the quantities of motion in Sec. II A presumes Newton's first law, which indicates that its status is more fundamental than the second law.

(ii) As shown in Sec. II C, Newton's second law, viz. $\mathbf{F} = m\mathbf{a}$, together with the fact that $\mathbf{F}$ is a frame-independent function, arises from a specific model of motion-change—specifically, from the assumption that motion-change in a system of interacting bodies occurs via a rapid succession of abrupt changes—as well as the above-mentioned assumptions concerning the functional form of two-body influence, and the composition of influences in a many-body system. It is this model, together with relativity, which implies that a function of the body's average rate of change of velocity (excluding any higher temporal derivatives of position) provides the *measure* of the influence exerted upon it. In the absence of such a simplifying model, the measure of influence could conceivably depend upon a finite number of temporal derivatives of $\mathbf{r}$. Thus, from this standpoint, the second law embodies a specific model of motion-change and is therefore not simply a definition.

(iii) In the present derivation, mechanics is built up in three distinct steps, the first two of which involves imposing conservation and relativity to derive the quantities of motion, the conservation of mass, and the frame-independence of massless energy. Thus, from this perspective, the general notion of conservation shapes the landscape in which Newton's second and third laws subsequently take root.

In addition, the present derivation provides a clear understanding of a number of subsidiary statements that are a necessary part of the standard presentation but are generally taken as axiomatic. For example, continuous energy and momentum conservation in the energetic framework (see Sec. II B) provides a principled understanding of why total mass is conserved, and why massless energy (such as heat) is frame-independent.

A final advantage of the present approach becomes apparent when one seeks to formulate a special relativistic dynamics that is consistent with the Lorentz transformations. The standard presentation of nonrelativistic mechanics is based around Newton's laws of motion, takes the specific quantities of motion as axiomatic, and gives a peripheral role to the principle of relativity. Consequently, when one transitions from the Galilean transformations to the Lorentz transformations, it is far from obvious what features of the above framework must be changed (and how they must be changed) and what features can be retained.

In contrast, as shown in Sec. III, the present derivation enables a transparent and systematic transition to special relativistic dynamics. In particular, the change of transformation group is implemented at the outset, the consideration of an elastic collision immediately giving rise to the relativistic expressions for corpuscular energy and momentum. Already at this stage, one can see that total corpuscular momentum cannot be continuously conserved. The requirement of continuous energy and momentum conservation accordingly forces the introduction of massless momentum in addition to massless energy into the energetic framework. Within the energetic framework thus formulated, conservation of energy and momentum then show that mass is no longer necessarily conserved, and that massless energy-momentum transforms in the same way as massive energy-momentum. Finally, the notion of force can be introduced in a manner that initially parallels the nonrelativistic case, while the complexities that subsequently emerge are clearly traceable to the change in transformation group.

III. RELATIVISTIC MECHANICS

A. Conserved Quantities

We first derive the forms of the energy and momentum of a particle by assuming asymptotic conservation of energy for an elastic collision.

1. *Kinetic & Rest Energy*

In parallel to the nonrelativistic case (Sec. II A 1), we assume that a particle of mass m with speed u has a scalar quantity of motion $F(u)$, to which we henceforth refer as its energy. We assume that m is a frame-invariant parameter, and that the total energy of a system of widely-separated particles is the sum of their separate energies.

Energy conservation for the collision of Fig. 1 as seen in frame S' implies that

$$F(u \oplus v) + F(u \oplus -v) = 2F(w), \tag{32}$$

where $w = \left[(u/\gamma(v))^2 + v^2\right]^{1/2}$ and $\oplus$ denotes collinear relativistic velocity addition.

Defining function $\widetilde{F}$ via the relation $\widetilde{F}(\gamma(u)) = F(u)$, and using the identities

$$\begin{aligned} \gamma(u \oplus v) &= \gamma(u)\gamma(v)\left[1 + \frac{uv}{c^2}\right] \\ \gamma(w) &= \gamma(u)\gamma(v), \end{aligned} \tag{33}$$

this conservation equation can be rewritten

$$\widetilde{F}(x) + \widetilde{F}(y) = 2\widetilde{F}\left(\frac{x+y}{2}\right), \tag{34}$$

where $x = \gamma(u \oplus v)$ and $y = \gamma(u \oplus -v)$. This is Jensen's functional equation, with general solution (see Appendix A 3)

$$\widetilde{F}(x) = ax + b. \tag{35}$$

Hence, for a particle of mass m, the conserved scalar quantity of motion is $F_m(u) = a(m)\gamma(u) + b(m)$, where $a(m)$ and $b(m)$ are undetermined functions of m.

To determine the forms of $a(m)$ and $b(m)$, write $F_m(u) = a(m)\left(\gamma(u) - 1\right) + \left(a(m) + b(m)\right)$, and consider the energy of a mass $m = m_1 + m_2$. The energy can be computed in two different ways, which must agree: $F_m(u) = F_{m_1}(u) + F_{m_2}(u)$. Defining $c(m) = a(m) + b(m)$, one thus obtains

$$\begin{aligned} \big(a(m) - [a(m_1) + a(m_2)]\big)\left(\gamma(u) - 1\right) + \\ \big(c(m) - [c(m_1) + c(m_2)]\big) = 0. \end{aligned} \tag{36}$$

Setting $u = 0$ shows that $c(m)$ satisfies Cauchy's additivity equation. The case $u \neq 0$ then shows that $a(m)$ also satisfies the additivity equation. Hence, $a(m) = a_0 m$ and $c(m) = c_0 m$, which imply $b(m) = b_0 m$, where a_0, b_0, c_0 are all constants.

Correspondence with the non-relativistic expression for energy then requires that $a_0 = c^2$, so that

$$F(u) = \gamma(u)mc^2 + b_0 m. \tag{37}$$

A non-zero value of b_0 would imply that there were *two* distinct contributions to rest energy, namely $b_0 m$ and mc^2. It does not seem possible to show that $b_0 = 0$ using considerations involving conservation and symmetry[14]. However, as $b_0 = 0$ is empirically well-supported, we assume at this point that $b_0 = 0$ in order to avoid undue complexity in what follows.

2. Momentum

The most direct way to derive the form of relativistic momentum is via Schütz's argument. Consider masses m_i moving in frame S at velocities $\mathbf{u}_i$, which then collide elastically and separate to yield masses m_i moving at velocities $\widetilde{\mathbf{u}}_i$, with no massless energy. Energy conservation in frames S, S' yield

$$\begin{aligned} \sum_i \gamma(u_i) m_i c^2 &= \sum_i \gamma(\tilde{u}_i) m_i c^2 \\ \sum_i \gamma(u_i') m_i c^2 &= \sum_i \gamma(\tilde{u}_i') m_i c^2. \end{aligned} \tag{38}$$

Using the relation $\gamma(u') = \gamma(u)\gamma(v)(1 - u_x v/c^2)$, the latter can be rewritten

$$\begin{aligned} \gamma(v) \sum_i \gamma(u_i)\left[1 - \frac{u_{ix} v}{c^2}\right] m_i c^2 \\ = \gamma(v) \sum_i \gamma(\tilde{u}_i)\left[1 - \frac{\tilde{u}_{ix} v}{c^2}\right] m_i c^2. \end{aligned} \tag{39}$$

Using the former, one thus obtains

$$\sum_i \gamma(u_i) m_i u_{ix} = \sum_i \gamma(\tilde{u}_i) m_i \tilde{u}_{ix}, \tag{40}$$

which is momentum conservation in the x-direction. Momentum conservation in the y- and z-directions follows similarly by considering frame S' moving in those directions. Thus, the vectorial conserved quantity of motion of a particle of mass m and velocity $\mathbf{u}$ is $\gamma(u) m \mathbf{u}$ up to a multiplicative constant. Requiring correspondence with the nonrelativistic momentum fixes this constant to unity.

3. Photons

The relationship, $E = pc$, between the energy, E, and momentum, $\mathbf{p}$, of massless particles that travel at light speed can be derived as the limiting case ($m \to 0$ with E held fixed) of the expressions for energy and momentum of massive particles. The relationship between energy of such a particle (a 'photon') and the frequency of a

[14] In Sec. IV E, we discuss an argument due to Einstein [13] which purports to show that $b_0 = 0$.

light wave can be obtained by applying conservation and relativity to a process (a 'collision') in which two waves of equal frequency f are incident along the y-axis, and then scatter without change of frequency, receding along a line inclined at angle θ to the line of incidence.

If one assumes that a luminous plane wave has associated particles, each of whose energy, E, is a function of the wave frequency, f, such that $E = H(f)$, and one then applies conservation of energy in frame S', taking the Doppler effect into account, one obtains the functional equation

$$2H(\gamma f) = H(\gamma(1-\beta\cos\theta)f) + H(\gamma(1+\beta\cos\theta)f)\,, \tag{41}$$

which holds for all β, θ. This yields the solution $E = hf$, up to an additive constant, where h is some constant.

B. Continuous energy and momentum conservation in an energetic framework

Let us now consider how to fit relativistic mechanics into the energetic framework. If we continue to assume that the massless component bears energy but not momentum, we run into an immediate problem. To see this, consider a system of masses m_i moving at velocity u_i (for simplicity, in one dimension) that interact and give rise to masses $\widetilde{m}_i$ moving at velocity $\widetilde{u}_i$. Conservation of momentum in frames S and S' yields:

$$\sum_i \gamma(u_i) m_i u_i = \sum_i \gamma(\widetilde{u}_i)\widetilde{m}_i \widetilde{u}_i, \tag{42}$$

and

$$\sum_i \gamma(u_i \oplus -v) m_i (u_i \oplus -v) = \sum_i \gamma(\widetilde{u}_i \oplus -v)\widetilde{m}_i(\widetilde{u}_i, \oplus - v) \tag{43}$$

which holds for any $\mathbf{v}$. The latter becomes

$$\gamma(v)\sum_i \gamma(u_i) m_i (u_i - v) = \gamma(v)\sum_i \gamma(\widetilde{u}_i)\widetilde{m}_i(\widetilde{u}_i - v)\,, \tag{44}$$

which, via Eq. (42), implies that

$$\sum_i \gamma(u_i) m_i = \sum_i \gamma(\widetilde{u}_i)\widetilde{m}_i. \tag{45}$$

That is, the total mass energy (rest energy plus kinetic energy) is conserved. This has two striking consequences:

1. In an elastic collision in which two equal bodies collide head on, momentum cannot be conserved at the stillpoint if the bodies' masses remain unchanged. That is, momentum is no longer continuously conserved.
2. If there is no additional contribution to a particle's rest energy apart from mc^2 (that is $b_0 = 0$ in Eq. (37)), the conversion of kinetic energy to massless energy is not possible.

The second of these consequences is at odds with the nonrelativistic case (where conversion from kinetic energy to massless energy is possible), and thus violates the minimal requirement of correspondence. In order to remove both of these difficulties, we modify the energetic framework so that the massless component can bear *momentum* as well as energy. This change restores the continuous conservation of momentum, and removes the second difficulty above.

1. *Interconversion of energy and momentum between massive and massless forms*

Consider again a system of masses—now in three dimension, with velocities $\mathbf{u}_i$ in S—but allowing for a massless component that can bear momentum as well energy. Momentum conservation in frames S, S' yields

$$\sum_i \gamma(u_i) m_i \mathbf{u}_i = \sum_i \gamma(\widetilde{u}_i)\widetilde{m}_i \widetilde{\mathbf{u}}_i + \Delta\mathbf{P}, \tag{46}$$

where $\Delta\mathbf{P}$ is the massless momentum, and

$$\sum_i \gamma(u'_i) m_i \mathbf{u}'_i = \sum_i \gamma(\widetilde{u}'_i)\widetilde{m}_i \widetilde{\mathbf{u}}'_i + \Delta\mathbf{P}'. \tag{47}$$

Energy conservation in frames S and S' additionally yields

$$\sum_i \gamma(u_i) m_i c^2 = \sum_i \gamma(\widetilde{u}_i)\widetilde{m}_i c^2 + \Delta E, \tag{48}$$

and

$$\sum_i \gamma(u'_i) m_i c^2 = \sum_i \gamma(\widetilde{u}'_i)\widetilde{m}_i c^2 + \Delta E'. \tag{49}$$

Using the relation $\gamma(u') = \gamma(u)\gamma(v)(1 - u_x v/c^2)$, the latter can be rewritten

$$\gamma(v)\sum_i \gamma(u_i)\left[1 - \frac{u_{ix}v}{c^2}\right] m_i c^2 = \gamma(v)\sum_i \gamma(\widetilde{u}_i)\left[1 - \frac{\widetilde{u}_{ix}v}{c^2}\right]\widetilde{m}_i c^2 + \Delta E'. \tag{50}$$

Using Eqs. (48) and (46), this reduces to

$$\Delta E' = \gamma(v)\left(\Delta E - v\Delta P_x\right). \tag{51}$$

Similarly, using the relations

$$\begin{aligned}\gamma(u')u'_x &= \gamma(u)\gamma(v)(u_x - v)\\ \gamma(u')u'_y &= \gamma(u)u_y\\ \gamma(u')u'_z &= \gamma(u)u_z,\end{aligned} \tag{52}$$

together with the Eqs. (46) and (48), Eq. (47) becomes

$$\begin{aligned}\Delta P'_x &= \gamma(v)\left(\Delta P_x - \frac{v\Delta E}{c^2}\right)\\ \Delta P'_y &= \Delta P_y\\ \Delta P'_z &= \Delta P_z.\end{aligned} \tag{53}$$

Thus, the transition to the relativistic case—which involves change of the transformation group *and* allowing for massless momentum—brings about two major changes as compared with the nonrelativistic case. First, the energy, ΔE, and momentum, $\Delta \mathbf{P}$, of the massless component of the system transform in precisely the same way as the energy and momentum of the mass-component. Unlike the argument given by Laue [4], this conclusion is reached without positing any specific model of the massless energy (see Sec. III D 1 d for more discussion on this point).

Second, variability of the system's total mass is not disallowed, so that the conversion of rest energy to kinetic and massless energy is, in principle, possible. In particular, generalizing a line of argument due to Einstein [14], suppose that a body, initially of mass m, at rest in S emits massless energy ΔE with zero total massless momentum. Then, conservation of momentum implies that the body is at rest in S after emission, while energy conservation in frames S and S' yield, respectively,

$$E = \Delta E + \widetilde{E} \tag{54}$$

$$E' = \gamma(v)\Delta E + \widetilde{E}', \tag{55}$$

where $E, \widetilde{E}$ are the initial and final energies of the body in S, and the primed energies are likewise as observed in S'. Subtraction gives

$$(E' - E) = (\gamma(v) - 1)\,\Delta E + (\widetilde{E}' - \widetilde{E}). \tag{56}$$

The energy differences $(E' - E) = (\gamma(v) - 1)\,mc^2$ and $(\widetilde{E}' - \widetilde{E}) = (\gamma(v) - 1)\,\widetilde{m}c^2$ are the initial and final kinetic energies of the body, where $\widetilde{m}$ is the body's post-emission mass. Since the body is at rest before and after emission in S' and is moving at speed v before and after emission in S', the above equation implies that the mass of the body changes due to the emission:

$$m - \widetilde{m} = \Delta E/c^2. \tag{57}$$

Note that, whereas Einstein's original argument presumes that massless energy is electromagnetic in origin, the conclusion has been reached without any specific model of the massless energy, and thus constitutes a generalization of the original argument (see Sec. III D 2 a for more discussion).

2. *Frame-invariance equality of* $(E, \mathbf{P})$

Since the mass and massless components' energy and momentum transform in the same way, the *total* energy and momentum, $(E, \mathbf{P})$, of the energetic system transform according to Eqs. (51) and (53). Hence, two systems with equal values of $(E, \mathbf{P})$ in frame S will also have equal values in frame S', even if they have unequal mass[15]. Thus, $(E, \mathbf{P})$ can be regarded as the macrostate of a relativistic energetic system.

C. Force and work in relativistic mechanics

Unlike the case in nonrelativistic physics, the continuous conservation of momentum applied to the energetic framework in the relativistic context requires that one allow a form of momentum other than that associated with the masses. Consequently, it is not possible to formulate a dynamical theory of the masses without taking into account the larger energetic framework (in which the masses are embedded) and explicitly tracking the energy and momentum of the massless component of the energetic system.

Nevertheless, the notion of a force acting on a particle can be developed in a manner parallel to that presented in Sec. II C. As in Sec. II C, our analysis is based on the following model of motion change: (i) a body's response to an influence takes the form of a rapid succession of small abrupt changes of its motion; and (ii) the body moves at constant velocity in between these motion changes. Again, due to relativity, there is no loss of generality in considering the effect of a body's change of motion within its instantaneous rest frame, $\overline{S}$, so that the abrupt change is characterized by the body's change of velocity, $\Delta\overline{\mathbf{u}}$ in $\overline{S}$.

Now, suppose that, over a small time interval, $\Delta\tau$, referred to $\overline{S}$, the body undergoes n velocity changes $\Delta\overline{\mathbf{u}}^{(2)}, \Delta\overline{\mathbf{u}}^{(2)}, \ldots, \Delta\overline{\mathbf{u}}^{(n)}$, with the ith change referred to frame $\overline{S}^{(i)}$ in which the body is at rest im-

[15] As described in Sec. IV C, this fact is used as an axiom in the derivations of relativistic energy and momentum due to both Ehlers et al. [15] and to Lalan [16].

mediately prior to this change. Due to the governing Lorentzian kinematics, the net velocity change, $\Delta\bar{\mathbf{u}}$, in the frame, $\bar{S}$, in which the body is at rest prior to *all* of these changes, is *approximately* the sum of these velocity changes,

$$\Delta\bar{\mathbf{u}} = \Delta\bar{\mathbf{u}}^{(1)} + \Delta\bar{\mathbf{u}}^{(2)} + \cdots + \Delta\bar{\mathbf{u}}^{(n)} + \mathcal{O}\left(\delta^3/c^2\right), \quad (58)$$

where δ is of the order of the $\Delta\bar{u}^{(i)}$, and $|\delta/c| << 1$. However, the error term vanishes in the limit where the $\Delta\bar{u}^{(i)} \to 0$. We will henceforth work in this limiting case, neglecting this error term. Specifically, we will suppose that, in frame $\bar{S}$, infinitesimal velocity changes $d\bar{\mathbf{u}}_1, d\bar{\mathbf{u}}_2, \ldots$ referred to instantaneous rest frames $\bar{S}^{(1)}, \bar{S}^{(2)}, \ldots$ combine additively to yield infinitesimal velocity change $d\bar{\mathbf{u}}$ (referred to frame $\bar{S}$) in the interval $d\tau$.

Accordingly, the influence on the body can be quantified instantaneously via the *proper acceleration*, $\bar{\mathbf{a}} = d\bar{\mathbf{u}}/d\tau$, which can be transformed to give the acceleration in any other frame. The cause of the proper acceleration $\bar{\mathbf{a}}$ can then be posited as being due to an influence, $\bar{\mathbf{f}}$ on the body:

$$\bar{\mathbf{f}} = \frac{d\bar{\mathbf{u}}}{d\tau}, \quad (59)$$

where $\bar{\mathbf{f}}$ is some heretofore unspecified function.

Above, the notion of influence has been quantified in the body's instantaneous rest frame, $\bar{S}$. This quantification is special in the sense that influences are *additive* in this frame due to the additivity of infinitesimal velocity changes, and furthermore suffices for dynamical predictions, provided $\bar{\mathbf{f}}$ is known. The quantification of influence in other frames, however, involves some degree of arbitrariness[16]. One choice is to simply posit that

$$\mathbf{f} = \frac{d\mathbf{u}}{dt}, \quad (60)$$

where $\mathbf{f}$ is the influence on the body as observed in frame S. The transformational relation between $\bar{\mathbf{f}}$ and $\mathbf{f}$ is then determined by the kinematical transformation of acceleration between frames $\bar{S}$ and S. However, suppose that, in frame S, two bodies interact elastically, and that the resulting change of velocity occurs when the bodies are so close that the propagation of influence between them occurs virtually instantaneously. In that case, in an interval of time Δt that includes the interaction, the change of momentum can be entirely attributed to the bodies. Thus, the bodies' average accelerations, $\mathbf{a}_i = \Delta\mathbf{u}_i/\Delta t$, are constrained by the conservation of momentum,

$$m_1 d\left(\gamma(u_1)\mathbf{u}_1\right) + m_2 d\left(\gamma(u_2)\mathbf{u}_2\right) = \mathbf{0}. \quad (61)$$

In order to harmonize the definition of influence (Eq. (60)) and the above constraint on accelerations due to conservation of momentum, one can instead choose to measure the influence on the body via $d\mathbf{p}/dt$, whose measure is given by the expression

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \quad (62)$$

We can accordingly speak of a *force*—with dynamical measure $d\mathbf{p}/dt$—acting on the body. In terms of force, the conservation of momentum—for the special case of elastic interaction between minimally-separated bodies—reduces to $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$. Another reason, unrelated to the conservation of momentum, for working with force (rather than influence) is that it yields the correct $\mathbf{E}$ and $\mathbf{B}$ field transformations when Lorentz's force law is assumed to hold in all frames [18].

Nevertheless, the above choice between measuring influence via $\mathbf{f}$ or via $\mathbf{F}$ is nontrivial since, unlike the situation in nonrelativistic mechanics, one is not determined by the other[17] given the body's rest mass, m.

In general, although one can measure the force on a particle through $\mathbf{F} = d\mathbf{p}/dt$, the lack of continuous conservation of total particle momentum means that there is no general analogue to Newton's third law. Note that this conclusion does not rest upon suppositions concerning the finite speed of motion of the massless component's momentum.

If a body is subject to influences due to many sources, then one can assert the composition of influence (in analogy to Newtonian mechanics). Then, in the instantaneous rest frame $\bar{S}$, due to the additivity of infinitesimal velocity changes due to each of these influences, the corresponding influences (due to each of the sources) add vectorially. Since force and influence coincide in $\bar{S}$, it follows that the corresponding (proper) *forces* in this frame also add vectorially.

Finally, it follows from the expressions for relativistic energy and momentum derived above that $\mathbf{F}\cdot d\mathbf{x}$ quantifies the increase in kinetic energy of a particle moving through $d\mathbf{x}$ as it is acted upon by force $\mathbf{F}$. A clearer

[16] Einstein alludes to the arbitrariness that is involved in extending 'force = mass × acceleration' to the relativistic setting in [17, §10].

[17] Specifically, $\mathbf{F} = \gamma^3 m\mathbf{a}_\parallel + \gamma m\mathbf{a}_\perp$, while $\mathbf{f} = \mathbf{a}_\parallel + \mathbf{a}_\perp$. Thus, for a particle of given mass, the velocity $\mathbf{u}$ must be given in order to convert $\mathbf{F}$ to $\mathbf{f}$ or $\mathbf{f}$ to $\mathbf{F}$.

understanding—which traces back more directly to the notions of conservation and relativity—would be desirable of why the same relation, namely $dT = \mathbf{F} \cdot d\mathbf{x} = \mathbf{u} \cdot d\mathbf{p}$, holds irrespective of the transformation group by which relativity is implemented.

D. Comparison with other presentations of relativistic classical mechanics.

The derivation of relativistic mechanics given above shows that certain fundamental results can be derived more generally and/or more simply that previously shown. In this section, we compare the above derivation with Einstein's original derivation, and other key arguments or theorems as appropriate.

Whereas presentations of nonrelativistic classical mechanics are quite standardized (typically based around Newton's laws and the conservation of energy), presentations of relativistic mechanics differ considerably from each other as well as from Einstein's original development. Nevertheless, their key content can be summarized as follows:

1. *Energy and momentum*
    (a) Energy and momentum of massive particles.
    (b) Energy-momentum relationship for massless energy.
    (c) Energy-frequency relationship of a photon.
    (d) Transformation properties of massless energy and momentum.
2. *Equivalence of mass and energy*
    (a) Inconvertibility of mass and massless energy.
    (b) Inertial behaviour of confined massless energy.
3. *Generalization of Newton's laws and work done by a force*
    (a) Newton's second law.
    (b) Newton's third law.
    (c) Composition of forces.
    (d) Work done by a force.

1. *Energy and momentum*

a. Energy and momentum of a massive particle. Einstein's original derivation [17, §10] of relativistic kinetic energy of a particle is based on Newton's equations of motion for a slowly accelerated electron in an electric field, on the transformation properties of the electric field (the latter derived by requiring the form-invariance of Maxwell's equations under Lorentz transformations), and on the presumption that $\mathbf{F} = m\mathbf{a}$ has the same form in any inertial frame, and on the work-energy relationship.

Subsequent derivations by Tolman [19], Einstein [13], and many others (see Sec. IV), do not make use of equations of motion and thus avoid any presumption regarding the form that Newton's second law takes in an inertial frame, and also avoid invoking any specific (electromagnetic) model of interaction. Instead, they adopt a more general kinematic approach based around the principle of relativity.

The derivation given in Step I (Sec. III A) adopts such a kinematical approach, invoking both the asymptotic conservation of energy as well as the principle of relativity, with relativistic momentum then being derived via Schütz's argument.

b. Energy-momentum relationship for massless energy. In the energetic framework, massless energy and momentum have been introduced in an abstract manner, namely without positing any *specific* model—for example, an electromagnetic model—of the massless component of the energetic system. As such, at this abstract level, there is no *necessary* relationship between ΔE and $\Delta\mathbf{P}$.

One can, however, establish that the relationship $\Delta E = c\Delta P$ holds for massless energy in specific cases. For example, this relationship can be derived if one considers the case of massless energy that is *particulate* on the assumption that this type of massless energy can be described by the limiting case ($m \to 0$, with E held constant) of the expressions for the energy and momentum of a massive particle. Alternatively, one could consider massless energy in the form of a monochromatic electromagnetic plane wave, and derive the above relationship using Maxwell's equations and the standard expressions for energy and momentum density.

An alternative approach, which we do not detail here, shows that this relationship follows from the requirement that, if massless energy ΔE is confined to an accelerating massless box, then the box behaves as if it possesses mass $\Delta E/c^2$. This approach has the benefit of not assuming that massless energy can be regarded as particulate or be describable via Maxwell's equations, but is instead based on the idea that massless energy must—when confined—have inertia; or, in short, on the idea that what we call *mass* is confined massless energy.

c. Energy-frequency relationship for a photon. The energy-frequency relationship for a photon may be based on the observation that the energy of a packet of electromagnetic energy-momentum and the frequency of a plane wave transform in the same manner, the former being derived from the transformation properties of the electric and magnetic fields [17, §8]. For, if one then posits that there exist quanta of electromagnetic energy whose energy is a function of frequency, *viz.* $E = H(f)$ where H is a function to be determined, it then follows that $E = hf$, where h is a constant.

In Sec. III A 3, we have shown that this relation can alternatively be derived using the same approach as used to derive the energy of a massive particle, namely by positing the conservation of energy and the principle of relativity as applied to a 'collision' between plane waves. This argument rests on the assumption that the relationship $\Delta E = c\Delta P$ holds between the energy and momentum of a packet of massless energy-momentum, and the above-mentioned assumption that there exist quanta of electromagnetic energy whose energy is a function of frequency. A benefit of this approach is avoidance of any use of Maxwell's equations.

d. Transformation properties of non-corpuscular energy and momentum. The transformation properties of energy and momentum have been established in the special case of a physical system describable via a stress-energy-momentum tensor, $T^{\mu\nu}$, by Laue [4] (see also [5, 6]).

However, as we have shown in Step II (Sec. III B), it is possible to derive the transformation properties of massless energy quite generally, without needing to specify any particular model of the massless component of the energetic system. In addition, the simplicity of the derivation enables one to clearly see that these transformation properties follow as a *direct consequence* of the transformation properties of corpuscular energy and momentum, as well as enabling a direct comparison with the corresponding results in the nonrelativistic case (see Sec. II B).

2. Equivalence of mass and energy

a. Interconvertibility of mass and massless energy. Einstein's 1905 argument to show the interconvertibility of mass and massless energy [14] is based on a thought experiment in which a body, initially at rest in S emits two equal packets electromagnetic energy in opposite directions. The conclusion, that the body suffers a decrease in mass of $\Delta E/c^2$ due to the emission of energy ΔE, is thus based on the use of a specific model (an electromagnetic model) of the massless energy.

However, as shown in Sec. III B, it is possible to derive this result on the basis of the transformation properties of massless energy-momentum alone, which in turn can be derived without recourse to any specific model of massless energy-momentum.

b. Inertial behaviour of confined massless energy. The inertial behaviour of confined massless energy is the second leg of mass-energy equivalence, and underpins the idea that mass is a form of trapped (or latent) directed massless energy. As we have mentioned above (Sec. III D 1 b), Einstein's argument to show such inertial behaviour can be turned around to argue that, if confined massless energy has the expected inertial behaviour, then $\Delta E = c\Delta P$ holds for the confined massless energy.

3. Generalization of Newton's laws and work done by a force

a. Newton's second law. The generalizations of Newton's second law to the relativistic regime carried out by Einstein [17, §10] and Planck [18] consider a charged particle interacting with an electromagnetic field, employ the Lorentz force law, and assume the transformation properties of the electromagnetic field. Planck's proposal, namely that one generalize Newton's second law to $\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p}$ is the relativistic momentum, is nowadays chosen in preference to Einstein's (who proposed to generalize the expression $\mathbf{F} = m\mathbf{a}$, which had the undesirable feature of requiring that m be generalized to take into account the direction of the acceleration relative to that of the velocity).

In contrast, the approach taken in Step III (Sec. III C) parallels that employed in deriving Newton's second law in the nonrelativistic case, and considers a particle subject to discrete impulses. The relativistic generalisation of the dynamical measure of force ($\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p}$ is relativistic momentum) is arrived at by appealing to the validity of the conservation of momentum in the special case of two bodies that are so close that the propagation of influence between them occurs virtually instantaneously. In particular, no appeal to an explicit model of particle interaction is required.

b. Newton's third law. The non-generalizability of Newton's third law to the relativistic regime is typically argued on the basis of the finite propagation of influences between bodies, which implies that corpuscular momentum cannot be continuously conserved.

In the present approach, the fact that corpuscular momentum is not continuously conserved (which prompts the introduction of the massless momentum in the energetic framework) *directly* implies that Newton's third law does not carry over to the relativistic regime.

c. Composition of forces. Newton's principle of the composition of forces is typically inferred to hold in a body's instantaneous rest from on the basis of the generalized form of Newton's second law.

In the approach taken in Step III (Sec. III C) to generalize Newton's second law, the instantaneous rest frame of a body plays a privileged role from the outset since infinitesimal velocity changes are additive only in this frame. Consequently, one finds that the natural generalization of the corresponding nonrelativistic argument is to take the proper acceleration as a measure of the *influence* on the body. Since infinitesimal velocity changes are additive in this frame, the influences on the body also compose additively. And, since force and influence are proportional to one another in this frame, forces in this frame compose additively. In this manner, the underlying reason for the specific way in which the principle of composition of forces must be generalized is made clear.

d. Work done by a force. In Einstein's original development [17, §10], the nonrelativistic formula for the work done by a force, $dW = \mathbf{F} \cdot d\mathbf{x}$, is assumed to hold in the relativistic domain, forming the basis for the derivation of the relativistic kinetic energy of a particle.

In contrast, the above approach avoids such an assumption, showing that the expression $dW = \mathbf{F} \cdot d\mathbf{x}$ follows from the previously-derived expressions for the relativistic energy and momentum of a particle and the generalized form of Newton's second law.

IV. PRINCIPLED DERIVATIONS OF QUANTITIES OF MOTION

The notion that a body in motion has an associated quantity of motion dependent upon both its speed and its quantity of matter (henceforth referred to as 'mass', on the understanding that the Newtonian distinction between mass and weight is not implied) occurs as early as the fourteenth century in Buridan's penetrating analysis of the motion of projectiles and other bodies (such as ships and grindstones)[18]. Buridan first argued that, as a body moves through the air, the air acts to *resist* (rather than, as Aristotle asserted, to *maintain*) the motion of the body. He then remarks that, given two projectiles of identical external shape and material form but differing mass—say, a hollow brass sphere and a solid brass ball of identical size and outer appearance—moving at the same speed, the heavier projectile suffers less diminution in speed than the lighter[19].

Buridan observes that this phenomenon can be explained if one assumes that each body has a quantity of motion, its *impetus*, an increasing function of both its mass and speed, and that it is this quantity that is degraded by air resistance. For, on the assumption that the resistance of a body depends on its external size, shape, and texture, but not its mass, the two projectiles would experience the same rate of diminution of their quantities of motion, but the heavier would suffer a lower rate of reduction in speed. He further asserts, presumably on the grounds of mathematical simplicity, that the impetus of a body is a *linear* function of its mass and speed, mu.

Descartes subsequently echoed Buridan's assertion that the quantity of motion is mu, which Newton and others subsequently vectorialized in order to handle inelastic collisions. The first principled derivation of a quantity of motion—mu^2—appears to have been due to Huygens, which was based on Galileo's law of free fall and on Torricelli's principle (see Sec. V C 1). The importance of removing the dependency on specific laws (such as Galileo's law of free fall) in favour of general principles was recognized, for example by Jean Bernoulli (leading to a submission to the Académie des Sciences in 1724), but not resolved[20].

As far as we have been able to ascertain, the first systematic derivation of the expressions of both momentum and energy from broad symmetry principles (for example, those based on the concepts of conservation and relativity), rather than specific laws, did not appear until the start of the twentieth century—Mach [23] and Dugas [12], for example, in their historically-minded analyses of the development of mechanics, make no mention of such derivations. However, such derivations began to appear soon after the beginning of the twentieth century (see, for example, Ref. [19]), apparently spurred by Einstein's special theory of relativity. Similar derivations have continued to appear, with many variations, until the present day.

The common feature of these derivations is the use of the principle of relativity to view a mechanical situation (most commonly a collision) from two different, but physically equivalent, standpoints. However, these derivations differ in the additional main idea that they employ. For example, some assume the conservation of a

[18] See [20, 21] for illuminating discussions. Buridan's 'The impetus theory of projectile motion' (from 'Questions on the Eight Books of the Physics of Aristotle') is available in [22] (see particularly p. 275).

[19] Buridan speculates that, if such resistance were entirely absent, a body would continue its motion indefinitely, and that such a condition might obtain with heavenly bodies.

[20] For details of the broader context of Bernoulli's submission, see [7], Chapter 7. Bernoulli's alternative derivations are discussed in Chapter 8.

scalar quantity of motion, while others introduce a principle that relates the total energy and/or momentum in different frames. Those derivations that consider a collision (rather than some other mechanical situation) differ in the particular collision that they consider—whether one-dimensional or two-dimensional; whether specially chosen (for example, possessing special symmetries) or not; whether elastic, inelastic, or completely inelastic. If inelastic, some additional considerations concerning non-motive energy are involved.

Below, we briefly describe and analyze a few selected derivations of particular interest.

A. Desloge (1976)

Desloge [24, 25] considers elastic collisions similar to that which we have done, except the masses after collision recede from one another along any line. Specifically, identical particles approach one another from opposite directions at speed u along a line of incidence represented by unit vector $\hat{\mathbf{n}}$, and emerge from their collision moving at their original speeds in opposite directions along a line of recession $\hat{\mathbf{n}}'$; and all $u, \hat{\mathbf{n}}, \hat{\mathbf{n}}'$ are possible. Rather than separately seeking a scalar conserved quantity which is a function of *speed* as we have done, Desloge seeks an additive function, h, of *velocity*. In a frame S' moving at velocity $\mathbf{v}$, he thus obtains

$$h(\mathbf{v}+u\hat{\mathbf{n}})+h(\mathbf{v}-u\hat{\mathbf{n}})=h(\mathbf{v}+u\hat{\mathbf{n}}')+h(\mathbf{v}-u\hat{\mathbf{n}}'), \quad (63)$$

which is to hold for all $u, \hat{\mathbf{n}}, \hat{\mathbf{n}}', \mathbf{v}$.

This equation has a rather elegant geometric interpretation. Consider a sphere of radius u with centre at $\mathbf{v}$. Then the sum of the h-values at a pair of antipodal points is the same as that at any other pair, and is also independent of the sphere's radius and centre. One could regard this equation as a variation of Jensen's functional equation. As such a view would lead one to expect, the general solution contains terms linear in the components of the vector argument. However, the general solution also contains a quadratic term, so that

$$h(\mathbf{u})=a+\mathbf{b}\cdot\mathbf{u}+cu^2, \quad (64)$$

with arbitrary $a, \mathbf{b}, c$, whose values could depend upon particle properties.

Additional arguments show that $\mu\mathbf{u}$ and $a+\frac{1}{2}\mu u^2$ are separately conserved, where a, μ are particle parameters. The connection of μ to mass is made ([11], Chapter 8) by *defining* μ as the relative mass of a particle (so that mass is operationally measured via Weyl's procedure), but the relation of parameter a to mass is not investigated. The treatment of relativistic quantities of motion is analogous.

Remarks. Compared with our approach, Desloge requires that one consider a more general collision (one with an arbitrary line of recession), as well as arbitrary relative direction of movement of frames S, S'. Mathematically, the approach employs a functional equation whose solution is rather intricate (owing to the vector argument of the unknown function), and requires additional, lengthy arguments to pare down the number of particle parameters. The payoff of this greater complexity is (i) a derivation of both energy (up to an additive particle parameter, in the nonrelativistic case) and momentum via a single functional equation, and (ii) a demonstration that these are the *only* quantities of motion that are independently conserved in an elastic collision.

B. Maimon's derivation of nonrelativistic kinetic energy (2011)

Maimon's derivation[21] of nonrelativistic kinetic energy is noteworthy as it considers an *inelastic* collision, specifically a completely inelastic head-on collision of equal masses moving at the equal speeds. An additive scalar conserved quantity is assumed to exist to which two types of contribution can occur—one due to mass (in which case it is assumed to be a function of speed), and the other a non-mass type referred to as 'heat'. The latter is implicitly taken to be frame-independent. When viewed in frames S, S', one obtains respectively

$$f(u)+f(u)=\Delta \quad (65)$$
$$f(u+v)+f(u-v)=2f(v)+\Delta'. \quad (66)$$

Assuming $\Delta=\Delta'$ (that is, quantity of 'heat' is frame-independent), one can eliminate Δ to obtain

$$f(u+v)+f(u-v)=2f(u)+2f(v). \quad (67)$$

Although the author (correctly) guesses its solution (in the special case where $u=v$), this equation is known as the quadratic functional equation[22], and has general solution $f(u)=au^2$. We remark that this solution lacks a rest energy term due to the inelastic nature of the collision that is considered.

Remarks. The derivation is brief and elegant, reducing to a well-known functional equation. However, the derivation cannot be immediately generalized to the rel-

---

[21] See http://www.physics.stackexchange.com/questions/535/
[22] See, for example, Ref. [26], Chapter 9

ativistic case since the assumption that 'heat' is frame-invariant no longer holds true. This makes clear that the assumption is not as trivial as it may initially appear. As we show in Sec. II B 1, the fact that the quantity of 'heat' is frame-invariant in the nonrelativistic case can be derived by applying conservation and relativity in an energetic framework.

C. Ehlers, Rindler, and Penrose (1965)

The derivation of Ehlers *et. al.* [15] of relativistic and nonrelativistic energy is based not on a consideration of collisions, but on the following assumptions (that together constitute their *Assumption II*):

1. *Direction-independence of energy of a two-particle system.* The sum of the energies of a pair of equal-mass particles approaching each other at equal and opposite speeds along a line is independent of the direction of this line.

2. *Frame-invariance of equality of total energy.* If two such systems, differing only in their lines of approach, have equal energy, then that equality holds even when the systems are observed in another inertial frame.

The first of these assumptions follows from the general notion of the isotropy of space, so the real weight is borne by the second. The nontriviality of the second assumption can be seen by noting that the assumption fails if the two particles instead move in the *same* direction.

In any case, once these assumptions are granted, the authors consider two systems, each composed of two equal-mass particles approaching each other along a line at speed u, where the lines of approach are along the x- and y-axes. Equating the sum of the energies of these two systems as seen in a moving frame (speed v), they obtain (in the nonrelativistic case):

$$f(u+v) + f(u-v) = 2f(\sqrt{u^2+v^2}), \tag{68}$$

which is the same as our Eq. (1). The equation is solved by reduction to Jensen's equation by writing $E(w^2) = f(w)$ and noting that $u^2 + v^2 = \frac{1}{2}(u_1'^2 + u_2'^2)$, where $u_1 = u + v$ and $u_2 = u - v$:

$$E(u_1'^2) + E(u_2'^2) = 2E\left(\frac{1}{2}(u_1'^2 + u_2'^2)\right). \tag{69}$$

The relativistic case follows the same pattern.

Remarks. The derivation is based not on conservation, but on an assumption (frame-invariance of equality of total energy for a given mechanical situation) which does not appear to follow naturally from elementary considerations. As we point out in Secs. II B 2 and III B 2, this assumption can itself be obtained as a by-product of deriving mechanics within an energetic framework.

A similar derivation by Lalan [16] (discussed in [27, §24]) obtains expressions for relativistic energy and momentum of a particle from the assumption that, if two systems have the same energy *and* momentum in one frame, then they also have the same energy and momentum in any other inertial frame. Like Ehlers *et. al.*, Lalan considers two systems, each consisting of pair of identical particles approaching each other at equal speeds, with the lines of approach along the x- and y-axes. He thereby obtains separate functional equations for relativistic energy and momentum, which, rather than being solved explicitly, are shown to be consistent with the known expressions for these quantities.

D. Sonego and Pin (2005)

Sonego and Pin [28] consider two bodies colliding elastically in one dimension. No special symmetries are assumed. The kinetic energy of a body is taken to be a function $T(u)$ of its speed u, and the asymptotic conservation of total kinetic energy is assumed in frame S:

$$T(u_1) + T(u_2) = T(\tilde{u}_1) + T(\tilde{u}_2), \tag{70}$$

where u_i and $\tilde{u}_i$ are the pre- and post-collisional speeds of body i. Schütz's argument is then used to obtain an expression for the momentum, p, in terms of the unknown function T. The authors then assume that $dT = udp$, which yields an equation that can be solved for T, for both the nonrelativistic and relativistic cases.

Remarks. The argument is innovative in its combination of Schütz's argument (to obtain momentum in terms of kinetic energy) with the positing of a relationship between momentum and energy as a way of fixing these quantities.

The main weakness of the argument is the lack of justification of the specific relation between dT and dp which is posited. The authors point out that this relation follows from $dT = Fdx$, which they regard as axiomatic (as the *definition* of kinetic energy). But it is unclear why one should regard $dT = Fdx$ as more fundamental than the relationship between, say, kinetic energy and speed, which one seeks to derive. Furthermore, as we have pointed out in Sec. III C, in view of the changes sustained by the expressions for energy and momentum in moving from nonrelativistic to relativistic mechanics, it is remarkable that $dT = Fdx$ should hold in relativistic and nonrelativistic mechanics alike—we know of no simple argument for why this should be so.

E. Einstein (1935)

Einstein's derivation [13] considers an elastic collision of equal bodies which, in frame S, approach along a line at equal speed u, and recede along another line at speed u. He shows that, viewed in frame S', it follows from velocity addition formulae that:

$$\begin{aligned} \gamma(u_1') + \gamma(u_2') &= \gamma(\tilde{u}_1') + \gamma(\tilde{u}_2') \\ u_1'\gamma(u_1') + u_2'\gamma(u_2') &= \tilde{u}_1'\gamma(\tilde{u}_1') + \tilde{u}_2'\gamma(\tilde{u}_2'). \end{aligned} \tag{71}$$

On this basis, the quantities $(\gamma(u)-1)mc^2$ and $\gamma(u)mu$ are taken as the kinetic energy and momentum, respectively.

A second argument is then given which aims to show that the rest energy is (or can be taken to be) mc^2. A variation of the above collision is considered in which the bodies collide inelastically, with the kinetic energy lost in the collision presumed to result in an equal mass-increase of the two bodies.. Thus, in frame S, the bodies, each initially of mass m, approach, both moving at speed u; and then recede (now each of mass $\widetilde{m}$), both moving at speed $\tilde{u}$. Taking the rest energy of each mass to be E_0, the conservation of energy in frames S and S' then yields:

$$2E_0 + 2mc^2\left[\gamma(u) - 1\right] = 2\widetilde{E}_0 + 2\widetilde{m}c^2\left[\gamma(\tilde{u}) - 1\right] \tag{72}$$

and

$$\begin{aligned} 2E_0 &+ mc^2\left[\gamma(u_1') + \gamma(u_2') - 2\right] \\ &= 2\widetilde{E}_0 + \widetilde{m}c^2\left[\gamma(\tilde{u}_1') + \gamma(\tilde{u}_2') - 2\right]. \end{aligned} \tag{73}$$

Using the fact that $\gamma(u_1')+\gamma(u_2') = 2\gamma(u)\gamma(v)$ and $\gamma(\tilde{u}_1')+\gamma(\tilde{u}_2') = 2\gamma(\tilde{u})\gamma(v)$, these yield

$$E_0 - mc^2 = \widetilde{E}_0 - \widetilde{m}c^2, \tag{74}$$

from which it follows that the change in rest energy of each body, $(E_0 - \widetilde{E}_0)$, is proportional to its change of mass, $(m - \widetilde{m})$. Einstein then argues that, since rest-energy changes are only determined to within an additive constant, "one can stipulate that E_0 should vanish together with m", hence that $E_0 = mc^2$.

Remarks. The first part of the argument is similar to that we have used, although the collision under consideration is more general, and explicit functional equations are not formulated. The second part of the argument presumes that conversion of kinetic energy to mass energy is possible. However, as we have seen in Sec. II B 1 and III B 1, whether or not this is the case depends on the form of the energy of a mass, and upon other assumptions concerning the wider energetic system; and is indeed *not* true in the nonrelativistic energetic framework. If the presumption is nevertheless granted (which risks inadvertently assuming what is to be proved), then the conclusion of the argument can be strengthened by using the result of Eq. (37), according to which $E_0 = mc^2 + b_0 m$ for mass m. Insertion into Eq. (74) implies that either $b_0 = 0$ or $m = \widetilde{m}$. But, by hypothesis, the collision is inelastic, so that $m \neq \widetilde{m}$, which implies that only the former possibility ($b_0 = 0$) survives. Thus, $E_0 = mc^2$.

V. STRUCTURE OF CLASSICAL MECHANICS

A. Overview

In the previous sections, classical mechanics has been reconstructed in three distinct steps:

I. Derivation of the asymptotically conserved quantities of motion via conservation and relativity.

II. Construction of the energetic framework (motivated by continuous conservation of energy and momentum).

III. Construction of the force framework (motivated by treatment of continuous interaction between separated particles).

These steps—and the principles employed, and results obtained, therein—are summarized in Tables I and II.

1. *Classification and Explanatory Role of Physical Principles*

In order to clarify the structure of mechanics, and to facilitate the following discussion, Tables I and II employ the following classification of physical principles according to their explanatory role[23]:

[23] The classification given here is extracted from [29]. The full classification described therein contains additional types of principle which are not required in the present discussion.

	I. Quantities of motion	II. Energetic framework	III. Force framework
Entities	Particles	Massive & Massless Components	Massive & Massless Components
Properties	mass (m); position ($\mathbf{r}$), velocity ($\mathbf{u}$); scalar quantity of motion, $f_m(u)$	*Particles:* mass; position, velocity; energy $f_m(u) = \beta m + mu^2/2$; momentum $m\mathbf{u}$ *Massless component:* energy, ΔE	*Particles:* mass; position, velocity; energy $f_m(u) = \beta m + mu^2/2$; momentum $m\mathbf{u}$ *Massless component:* energy, ΔE
Principles & Assumptions	**U1** Principle of inertia **EQ1** Relativity **EL1** Asymptotic conservation of total scalar quantity of motion in elastic collision **C1** Additivity of mass **C2** Additivity of scalar quantities of motion **S1** Specific elastic collision	**EQ1** Relativity **EL2&3** Continuous conservation of total energy and momentum **C3** Additivity of energies of massive and massless components	**U1** Principle of inertia **EQ1** Relativity **EL3** Continuous conservation of total momentum **C5** Composition of influences **S2** Abruptness model of motion-change & concept of influence
Results	Corpuscular energy: $\beta m + mu^2/2$ Corpuscular momentum: $m\mathbf{u}$	System mass is conserved ($\sum m_i = \sum \widetilde{m}_i$) Massless energy is frame-invariant ($\Delta E' = \Delta E$) (M, E, $\mathbf{P}$) is macrostate of system, where $E, \mathbf{P}$ are its *total* energy and momentum	For two bodies: (i) $m\mathbf{a_i} = \mathbf{F}_i(m_1, m_2; \mathbf{r}_1, \mathbf{r}_2; \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2; \dots)$, with force $\mathbf{F}_i$ frame-independent; (ii) $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$, with the $\mathbf{F}_i$ central if velocity-independent Composition of forces ($\mathbf{F}_i = \sum_{i \neq j} \mathbf{F}_{ij}$) Work-energy theorem ($dT = \mathbf{F} \cdot d\mathbf{x}$)
Remarks	Total corpuscular energy, $\sum \beta m_i + m_i u_i^2$, and momentum, $\sum m_i \mathbf{u}_i$, are asymptotically conserved in elastic processes.	Total system energy, $\Delta E + \sum \beta m_i + m_i u_i^2$, and momentum, $\sum m_i \mathbf{u}_i$, are continuously conserved in all processes.	
Motivations	That the sum total of a scalar quantity of motion be asymptotically conserved in an elastic collision.	That total system energy be *continuously* conserved in an elastic collision.	That a system of interacting bodies evolve *deterministically.*

TABLE I: *Structure of nonrelativistic mechanics.* The derivation occurs in three distinct steps. In each step, the table summarizes (a) entities and their properties; (b) the principles and assumptions employed; (c) the main results, and (d) the key motivation. Each principle or assumption is preceded by a label (U, EQ, EL, C, CR, S) indicating the category of principle (uniformity, equivalence, eliminative, compositional, correspondence, special) to which it belongs (see Sec. V A 1) followed by a number. The principles are numbered so as to emphasize the parallelism with the derivation of relativistic mechanics. As a consequence, one principle, namely C4 (additivity of momenta of the massive and massless components), is not used above. Note that the results of one step are incorporated into the following step (if one exists). The transition from Step I to Step II is driven by the desideratum that energy be *continuously*—not just asymptotically—conserved. The desire to treat *continuous* interactions between *separated* bodies drives the transition from Step II to Step III.

1. *Uniformity Principles (U).* A uniformity principle posits constancy of some property in a particularly simple case. As uniformity seems to demand little or no explanation (in comparison to non-uniformity), uniformity principles often have a grounding role in a theory. Examples of uniformity principles include the principle of inertia (describing the simple case of the motion of an isolated body), and the principle of indifference (uniform *a priori* probabilities) in Bayesian probability theory (which prescribes how to assignment a probability distribution when no specific information is available).

2. *Equivalence Principles (EQ).* An equivalence principle asserts that the same physical laws apply to physical phenomena observed from two or more different standpoints, or to a physical system placed in two or more different contexts. These principles enable one to explain what can happen by pointing to *something else* that can happen, Huygens' derivations of his laws of

	I. Quantities of motion	II. Energetic framework	III. Force framework
Entities	Particles	Massive & Massless Components	Massive & Massless Components
Properties	mass (m); position ($\mathbf{r}$), velocity ($\mathbf{u}$); scalar quantity of motion, $F_m(u)$	*Particles:* mass; position, velocity; energy $\gamma(u)mc^2$; momentum $\gamma(u)m\mathbf{u}$ *Massless component:* energy, ΔE; momentum, $\Delta\mathbf{P}$	*Particles:* mass; position, velocity; energy $\gamma(u)mc^2$; momentum $\gamma(u)m\mathbf{u}$ *Massless component:* energy, ΔE; momentum, $\Delta\mathbf{P}$
Principles & Assumptions	**U1** Principle of inertia **EQ1** Relativity **EL1** Asymptotic conservation of total scalar quantity of motion in elastic collision **C1** Additivity of mass **C2** Additivity of scalar quantities of motion **CR1** Correspondence of relativistic energy expression in the limit of small speeds **S1** Specific elastic collision **S3** Rest energy has no mass-independent contribution	**EQ1** Relativity **EL2&3** Continuous conservation of total energy and momentum **C3&4** Additivity of energies and momenta of massive and massless components	**U1** Principle of inertia **EQ1** Relativity **EL3** Continuous conservation of total momentum **C5** Composition of influences **S2** Abruptness model of motion-change & concept of influence
Results	Corpuscular energy: $\gamma(u)mc^2$ Corpuscular momentum: $\gamma(u)m\mathbf{u}$	Massless energy and momentum transform between frames in the same manner as corpuscular energy and momentum (E, $\mathbf{P}$) is macrostate of system, where $E, \mathbf{P}$ are its *total* energy and momentum	$d\mathbf{p}/dt = \mathbf{F}$, with force $\mathbf{F}$ frame-dependent; Composition of forces ($\mathbf{F}_i = \sum_{i\neq j}\mathbf{F}_{ij}$) holds in body's instantaneous rest frame Work-energy theorem ($dT = \mathbf{F}\cdot d\mathbf{x}$)
Remarks	Total corpuscular energy, $\sum\gamma(u_i)m_ic^2$, and momentum, $\sum\gamma(u_i)m_i\mathbf{u}_i$, are asymptotically conserved in elastic processes.	Total system energy, $\Delta E + \sum\gamma(u_i)m_ic^2$, and momentum, $\Delta\mathbf{P} + \sum\gamma(u_i)m_i\mathbf{u}_i$, are continuously conserved in all processes.	
Motivations	That the sum total of a scalar quantity of motion be asymptotically conserved in an elastic collision.	That total system energy and momentum be *continuously* conserved in an elastic collision.	That one be able to treat continuous interactions between separated bodies.

TABLE II: *Structure of relativistic mechanics.* The derivation occurs in three distinct steps. In each step, the table summarizes (a) entities and their properties; (b) the principles and assumptions employed; (c) the main results, and (d) the key motivation. Each principle or assumption is preceded by a label (U, EQ, EL, C, CR, S) indicating the category of principle (uniformity, equivalence, eliminative, compositional, correspondence, special) to which it belongs—see Sec. V A 1—followed by a number. Note that the results of one step are incorporated into the following step (if one exists). The transition from Step I to Step II is driven by the desideratum that energy and momentum be *continuously*—not just asymptotically—conserved. The desire to treat *continuous* interactions between *separated* bodies drives the transition from Step II to Step III.

collision being an exemplar[24]. Examples of equivalence principles include Galileo's principle of relativity and Einstein's equivalence principle.

3. *Eliminative Principles (EL).* An eliminative principle asserts that not all *conceivable* physical states, pairs of states (at two different times), or processes are possible, and specifies a constraint that *realizable* states, pairs of states, or processes must satisfy. Examples of eliminative principles include the principles of conser-

[24] For example, using Galilean relativity, one can explain what happens in an elastic head-on collision of equal bodies moving at unequal speeds u_1, u_2 in terms of what happens when those some bodies collide at equal speeds $(u_1 + u_2)/2$. See also footnote 31.

vation of energy and momentum, the principle of least action, and Pauli's exclusion principle[25].

4. *Compositional Principles (C).* A compositional principle asserts that, at some level of description, the description of a larger entity is determined by the corresponding description of its components. Various entities can be referred to, such as systems, trajectories, and quantities (such as energy or action) associated therewith. Examples of compositional principles include the additivity of mass, the vector additivity of forces, and the quantum mechanical tensor product rule for composite systems.

5. *Correspondence Principles (CR).* A correspondence principle asserts that there exists some quantitative agreement between two theoretical models of the 'same' physical system, often in some limit or other special case. Examples of correspondence principles include the quantum mechanical average-value correspondence principle [30, 31], which posits that the expected value of certain quantum mechanical operator relations agree with the corresponding classical mechanical relations.

6. *Special Principles (S).* Miscellaneous special assumptions or principles that do not fall under any of the other categories. Examples of special principles include the specific collision assumed in Step I, and the specific model of motion-change posited in Step III.

In the case of mechanics, the principle of relativity (EQ1) posits how a given situation will appear to different observers without constraining what dynamical processes are possible, and is thus part of the *kinematics*. The principle of inertia (U1) and the specific collision (S1) assumed in Step I both assume the possibility of specific kinds of motion, the former concerning a single isolated body, the latter concerning two bodies interacting via a collision; and both are the basis for the *dynamics*. The conservation principles (EL1–3) and the explicit model of motion-change (S2) are both integral parts of the dynamics, but each has a different explanatory role: the first explains why certain conceivable (or describable) motions do not in fact occur (because they do not conserve certain quantities of motion), while the second goes further and explains why a system in given initial state unfolds in a specific manner given the influence (or force) functions.

The compositional principles (C1–5) enable the analysis of a system composed of many entities; or, conversely, the building-up of a larger system from subsystems.. For example, the composition of influences (C5) allows the instantaneous behaviour of a given particle in a system of N particles to be explained in terms of the instantaneous behaviour of that particle when it is one component of $(N-1)$ two-particle systems.

Finally, in the derivation of relativistic mechanics, a special assumption (S3) concerning the rest energy of a body (which, in the present derivation, appears to be ultimately grounded by appeal to experiment) and a simple correspondence assumption (CR1), are additionally employed.

B. Grounding Mechanics in Symmetry Principles

In his 'Unreasonable effectiveness of mathematics in the natural sciences' [1], Wigner posits a three-fold hierarchy in physics: events, laws of nature, and symmetry principles. In particular, just as the laws of physics express regularities in events, symmetry principles express regularities in laws of physics—in short, symmetry principles are meta-laws. From this perspective, the laws posited in a physical theory are more secure to the extent to which they can be traced to symmetry principles.

As mentioned in the Introduction, the early development of mechanics was based on the key ideas of conservation and relativity, which are both symmetry principles (of type EL and EQ, respectively)[26]. However, in the process of their formalization and refinement, they acquired additions that were not obviously or clearly traced to symmetry principles.

Consider, for example, the formal principle of asymptotic conservation of energy, namely that $\sum m_i u_i^2$ is conserved under dynamical evolution of an isolated system of masses undergoing elastic contact interactions. The core of this principle—that a certain total 'quantity of motion' is conserved under a system's time evolution—is what one could regard as a *pure* symmetry principle. However, the *quantitative* part of this principle posits a *specific* quantity of motion, namely $\sum m_i u_i^2$. One can,

[25] The first principle acts as a constraint on which start- and end-states can be dynamically connected; the second as a constraint on allowable paths connecting given initial and final configurations; and the third as a constraint on allowable quantum numbers ('old' quantum theory) or on possible multiparticle states ('new' quantum theory).

[26] For example, in the case of conservation of energy, the transformation under consideration is time evolution of the system; the 'object' transformed is the physical state of the system; and the equivalence relation between states is that they 'possess' the same total energy. The conservation principle thus posits that time evolution is a symmetry transformation of physical states with respect to this equivalence relation.

in turn, split this quantitative assertion into two distinct statements:

1. Each body possesses a scalar quantity of motion $m_i u_i^2$.

2. The *total* quantity of motion is the sum of those of the individual bodies.

The first is a *specific* assertion. It is not a symmetry principle or obviously related to one. The second is a compositional principle, and also not explicitly traced back to a symmetry principle. This does not preclude these elements being separately derived from symmetry principles. But, *taken in isolation,* the principle is a hybrid of two parts: a symmetry-based part, and a quantitative part ($\sum m_i u_i^2$) that is not grounded on symmetry principles.

However, in Step I, it has been shown that the above conservation principle can—using relativity—be derived starting from a more austere basis, namely the asymptotic conservation of the sum total scalar quantity of motion, $\sum_i f_m(u_i)$, of a system of masses undergoing elastic collisions, provided that one assume that a specific collision is possible, and provided that one assume the additivity of mass and energy. In this manner, the above quantitative conservation principle is brought into closer contact with symmetry principles. The asymptotic conservation of momentum then follows immediately via a second application of relativity.

More generally, then, the process of grounding an existing physical theory, such as classical mechanics, on symmetry principles requires a careful re-examination of its mathematical principles, including those that might appear to be 'symmetry principles' but in fact contain elements that are not obviously grounded in symmetry.

As summarized in Tables I and II, it is possible to build up classical mechanics, guided by symmetry principles, in a fairly systematic fashion. The key symmetry principles employed are conservation (EL1–3) and relativity (EQ1), together with the principle of inertia (U1). However, in addition, the derivation employs the following special and compositional assumptions:

1. Three special assumptions, namely (i) a specific collision (S1) (as depicted in Fig. 1); (ii) a specific model of motion-change (S2) (Sec. II C); and (iii) in the relativistic case, the assumption that a particle's rest energy has no mass-independent contribution (S3) (Sec. III A 1).

2. Five compositional assumptions, namely (i) the additivity of mass (C1), (ii) the additivity of scalar quantities of motion (C2), (iii, iv) the additivity of energies of massive and massless components (C3&4), and (v) the composition of influences (C5).

First, as mentioned in footnote 7, the specific collision can be largely justified on the basis of symmetry considerations. In contrast, the specific model of motion change is not based on a symmetry principle, but rather on the idea that continuous motion can be approximated by impulsive motion. Finally, in the relativistic case, it appears that assumption S3 is needed in order to rule out the possibility that a particle's rest energy has a contribution other than mc^2.

Second, compositional assumptions or principles lie in a separate category to symmetry principles, and yet seem to play as fundamental a role as symmetry principles in the building up of physical theories. Nevertheless, the *mathematical form* of certain compositional principles can be derived from symmetry considerations. For example, although the additivity of mass and the additivity of a scalar quantity of motion have been assumed (C1, C2), this additivity can, in fact, be *derived* from the symmetry of associativity. For example, if one assumes that the total mass of two bodies of mass m_1, m_2 is given by an unknown function $h(m_1, m_2)$, and one further requires that the mass of a system of *three* bodies can be determined by iteratively applying h in a pairwise fashion, one notices that this composition can occur in either of two ways, either as $h(m_1, h(m_2, m_3))$ or as $h(h(m_1, m_2), m_3)$. The requirement of *associativity* is that these two compositional pathways agree:

$$h(m_1, h(m_2, m_3)) = h(h(m_1, m_2), m_3). \qquad (75)$$

This functional equation, known as the associativity equation, implies that, without loss of generality, one can take h to be the *sum* of its arguments[27]. Similarly, the vector addition of directed quantities of motion (as needed in C4), can be derived from elementary axioms following an argument originally due to d'Alembert[28]. In that derivation, basic symmetries, such as rotational co-

[27] More precisely, on the assumption that h is differentiable at a point, one can show [32] that $h(a,b) = f^{-1}(f(a)+f(b))$, where f is a continuous, monotonic function. Hence, if one *regraduates* the masses m_i via f, so that $\mu_i \equiv f(m_i)$, then $\mu = \mu_1 + \mu_2$ is the total *regraduated* mass of the system of two bodies. However, since f is monotonic, one can just as well quantify the 'amount of matter' via the μ_i rather than the m_i. Hence, without loss of generality, one can say that mass is additive. The same line of argument applies to any scalar quantity, such as kinetic energy, associated with the bodies, provided that one has clear physical ground for believing that the total quantity for a system of bodies is a function of the quantities associated with each of the bodies.

[28] See, for instance, Ref. [33], Chapter 1. The core assumptions here are: (i) the resultant of two parallel forces has magnitude equal to the sum of the magnitudes of these forces, and points

variance and commutativity & associativity, play a leading role. Finally, as we have shown, it is possible to argue using relativity that, given the qualitative requirement that the total influence on a body is determined by the influence on that body due to each of the other bodies *separately* (C5), these influences (quantified as velocity changes) combine additively.

C. Role of Conservation and Relativity in Mechanics

The twin concepts of conservation and relativity played a vital role in the early development of mechanics. Of these two concepts, the notion of conservation proved to be the most difficult to formalize in a way that was consistent with other physical considerations of similar intuitive force. In the following, we first summarize how relativity and conservation guided the historical development of mechanics, and then show how the symmetry-based derivation of mechanics given in the previous sections illuminates many of the issues that were faced during this developmental process.

1. Historical Role of Conservation and Relativity

The notion of conservation was first formalized by Descartes through the principle that a system of colliding bodies conserves its total scalar 'quantity of motion'[29], a principle that guided the formulation of his laws of collision[30]. Galileo's principle of relativity enabled his derivation of parabolic motion from vertical free fall, and later enabled Huygens' deduction of the behaviour of equal bodies in head-on elastic collision. Huygens went even further, showing that one could *combine* conservation (in the form of a generic principle of the conservation of a scalar quantity of motion) and relativity to derive a new conservation law, namely the conservation of relative speed, applicable to *unequal* bodies in head-on collision[31].

However, this early development was also marked by a striking conceptual tension between Descartes' conservation principle and other physical considerations of similar intuitive force. In particular, in the process of being formalized and applied to the task of formulating laws of collision, Descartes' conservation principle was confronted with a number of challenges which brought into question not only its mathematical form, but also its range of applicability and the validity of its conceptual justification:

1. *Mathematical form of the quantity of motion.* Descartes' choice of the conserved quantity of motion, namely, mv, was dictated by mathematical simplicity, not by a physical principle. This fact was brought into focus by Huygens, who showed that Descartes' mv was incompatible with relativity, and further showed that, granted other established physical laws and principles, mv^2 (known after Leibniz [37] as *vis viva*) was the correct scalar quantity of motion[32].

2. *Elastic collisions as continuous processes.* Descartes viewed matter as pure extension, and collisions ac-

---

in the same direction; (ii) the resultant of a number of forces is commutative and associative; (iii) the resultant of two forces is rotationally covariant; (iv) the resultant of two equal forces varies continuously with the angle between these forces.

[29] In Principles II 36 [34], Descartes asserts: "there is a fixed and determined quantity of [motion] ... always the same in the universe as a whole even though there may at times be more or less motion in certain of its individual parts", and that "when one part of matter moves twice as fast as another twice as large, there is as much motion in the smaller as in the larger", roughly interpreted as the assertion that $\sum_i m_i u_i$ is the conserved quantity, where m is a measure of the 'size' of a body.

[30] Descartes' conservation principle was insufficient to account for collisional behaviour. Lacking another principle of similar scope capable of rectifying this insufficiency, Descartes introduced other considerations in a rather *ad hoc* manner. The defects of the resultant laws of collision were readily apparent. For example, Leibniz showed these laws to be inconsistent with the requirement of continuity [35, pp. 290–291]. Nevertheless, these laws were a spur to development of the correct laws.

[31] Huygens' laws of collisions can be be divided into two cases: (i) For equal bodies in head-on collision (whether elastic or not), all collisions involving bodies with unequal incident speeds follows via relativity from the case of equal incident speeds, the behaviour in this latter case being taken as axiomatic. (ii) For unequal bodies in head-on elastic collision, the additional assumption of the asymptotic conservation of total scalar quantity of motion, where the quantity of motion is a function of speed, and an auxiliary assumption (if one mass undergoes a change, so must the other) implies that the relative speed of the two masses is the same before and after the collision. *Proof sketch*: in any such a collision, there is a Galilean frame of reference in which the speed of one body does not change (comparing the initial and final states), only its direction of motion. Hence, its quantity of motion does not change. But, asymptotic conservation of total quantity of motion then implies that the speed of the *other* body also does not change. But if the direction of one mass changes, so must the other (by the auxiliary assumption). Hence, relative speed in this chosen frame is same before and after. But relative speed is frame-independent. Therefore, irrespective of the (inertial) frame in which the collision is viewed, the relative speed is unchanged. For details, see [23, pp. 313–317] and also [36, §9.4].

[32] Huygens' law of conservation of relative speed of two bodies in head-on elastic collision (see footnote 31) implies that the conserved quantity of motion cannot be mv (as can be seen by considering a body of mass $m < M$ striking a body of mass M initially at rest). Furthermore, appealing to Galileo's law of free fall and Torricelli's principle (that the centre of gravity of a system of interacting bodies cannot rise), Huygens showed that the conserved quantity of motion is, in fact, mv^2.

cordingly as instantaneous events between rigid geometric figures. In contrast, Newton and Leibniz insisted of the continuity of most natural processes, and accordingly viewed an elastic collision as a finite process involving deformation. But, in such a process, the bodies would be momentarily stilled in some reference frame. Thus, any principle positing the conservation of a total *scalar* quantity of motion could only apply to the collision's asymptotic states.

3. *Dissipation of motion in atomic collisions.* Newton (amongst others) believed that atoms in head-on collision would *lose* their motion[33], an idea that conflicted with the intuition underpinning Descartes' conservation principle (see footnote 29).

The hypothesized dissipation of motion during the collision of hard atoms, and the requirement of continuity for elastic collisions, resulted in a marginalization of scalar conservation principles[34], and lead—via a vectorialization of Descartes' conservation principle (due to Wren [39], Huygens [40, 41], Wallis [42], and Newton)—to a new conservation principle, namely the principle of conservation of momentum. However, this vectorialization severed the connection between the mathematical principle and Descartes' intuitive motivation for his principle; and a novel justification for the new principle was not readily forthcoming[35].

In addition, the challenge of expanding mechanics beyond collisional phenomena to encompass bodies continuously interacting at a distance made clear that new ideas or principles, beyond relativity and conservation, were required. Newton's framework, organized around the concept of force, provided the key new idea, namely a specific law—Newton's second law—relating a body's acceleration with the force acting upon it. Conservation of momentum was recast as a constraint (antiparallelism) on the forces exerted by two bodies upon one another, which also thereby provided some kind of intuitive justification for momentum conservation. Meanwhile relativity was incorporated by the requirement that force be independent of inertial frame. Thus, conservation and relativity were subsumed within the framework, with a specific law placed at its centre.

However, a number of developments in the nineteenth and early twentieth century brought the general principles of conservation and relativity once again firmly into the foreground:

1. *Interconversion phenomena.* Following the discovery of new interconversion phenomena in the first third of the nineteenth century, a scalar conservation principle, the conservation of energy, arose to fill the need for a quantitive means to coordinate these diverse (electrical, magnetic, thermal, mechanical, and chemical) phenomena [43]. During this period, mechanics was regarded as a *component* of a larger *energetic framework*, which allowed for the interconversion of energy of motion—quantified by *vis viva*—and non-motive forms of energy.

2. *Principled derivation of mechanics.* During the nineteenth century, there were numerous attempts to derive key features of Newtonian mechanics using general physical principles, such as relativity. For example, Laplace and Bélanger offered novel derivations of Newton's second law[36], while Schütz used relativity to derive momentum conservation from energy conservation [3].

3. *Interpretation of Maxwell's equations.* In the last third of the nineteenth century, the interpretation of Maxwell's equations in terms of a privileged frame of reference brought the validity of the principle of relativity (and hence Newtonian mechanics) into question. Einstein's special relativity not only rescued Galileo's principle of relativity from this doubt, but, through the derivation of a new kinematics and dynamics, demonstrated anew its fecundity.

By the close of the foregoing developments, the energetic framework (extended to include massless momentum), with its conservation laws, had become established as an indispensable means to coordinate the distinct physical theories of mechanical, electromagnetic, and thermal phenomena which had been formulated. Finally, Einstein's theory of relativity showed that energy and momentum conservation were, in fact, two sides of a single conservation law. In particular, Laue's theorem showed

[33] Newton (amongst others) asserted that atoms were hard bodies that collide completely *inelastically* [38, pp. 4–5]. Hence the fundamental importance of formulating laws applicable to inelastic collisions.

[34] Although Leibniz championed the conservation of *vis viva*, a compelling account of the 'missing' quantity of motion at the stillpoint of an elastic collision, or at the end-point of an inelastic collision, was lacking. As a consequence, scalar conservation principles were marginalized. For example, in textbooks through to the end of the eighteenth century, elastic collisions were handled by using a situation-specific law (Huygens' conservation of the masses' relative speed—see Footnote 31), rather than the asymptotic conservation of *vis viva*—see [7] (Appendix) and [8].

[35] Some attempts were made to justify the mathematical principle of momentum conservation in terms of the law of the lever. See, for example, [39], and [35, pp. 203–206].

[36] See Ref. [2] for a detailed historical investigation into these derivations.

that, in a physical system describable by a stress-energy-momentum tensor, the total energy and momentum of the system transform in the same manner, namely as a four-vector.

2. *Role of Conservation and Relativity: An Analysis*

We now show how our symmetry-based derivation illuminates many of the issues that arose in relation to conservation and relativity in the historical development of mechanics.

In the nonrelativistic case:

1. *Principled derivation of scalar conserved quantity of motion.* As shown in Step I, one can posit Descartes' notion of conservation for the asymptotic states of an elastic collision and then, by appealing to the principle of relativity, derive the conserved scalar quantity of motion, $mu^2/2$. Hence, the tension between Descartes' original hypothesis (that mu is the scalar conserved quantity) and relativity—a tension recognized by Huygens—can be directly resolved, and leads to $mu^2/2$ without recourse to extraneous physical laws or principles (such as Galileo's laws of freefall—see footnote 32).

2. *Relationship between scalar and vector conservation.* Another application of relativity (via Schütz's argument) then leads from asymptotic energy conservation to asymptotic momentum conservation for elastic collisions. Thus, *in this special case*, these two principles, which were historically given such strikingly different intuitive justifications, are, in fact, intimately related, the former—when combined with relativity—*implying* the latter. Moreover, we see that, as long as the principle of relativity is presupposed, asymptotic scalar conservation *must* be accompanied by asymptotic vector conservation. The converse, however, does not hold—given the principle of relativity, asymptotic vector conservation can exist without asymptotic scalar conservation.

3. *Continuous energy conservation.* In Step II, the introduction of an energetic framework—with its notion of a massless form of energy—makes it possible to then posit that energy conservation holds *continuously*. Momentum conservation can also be posited to hold continuously, but without any evident *need* to introduce massless momentum. Thus, the energetic framework resolves the tension between scalar conservation and the requirement of continuity—as initially envisaged by Leibniz, but not embraced until the 1830s and 1840s.

4. *Co-existence of scalar and vector conservation, and their consequences.* These two conservation principles then yield nontrivial consequences, namely (i) total mass conservation (which implies no interconversion of rest energy to other forms of energy) and (ii) the frame-invariance of massless energy. Thus:
   (a) Once generalized within an energetic framework, momentum conservation no longer follows from energy conservation. Instead, the two conservation laws independently co-exist, each yielding important consequences.
   (b) One of those consequences is that total mass is conserved, a fact that therefore does not need to be independently assumed (as was the case historically).
   (c) The other consequence is that massless energy is *fundamentally different* from kinetic energy, and hence cannot (as Leibniz envisaged) be assumed to be due to the motion of microscopic particles in a nonrelativistic framework[37].

5. *Possibility of momentum-based dissipative mechanical theory.* Since momentum conservation holds continuously in the energetic framework without the need to posit a massless form of momentum, it is possible use continuous momentum conservation as a basis for a mechanical theory which allows for inelastic collisions, but which only explicitly tracks massive bodies. Such a theory is constructed in Step III by introducing a staccato model of motion change.

6. *Possibility of non-dissipative energy-based mechanical theory.* Since continuous energy conservation requires a massless form of energy, a dissipative mechanical theory which only tracks massive bodies cannot be build around continuous energy conservation. However, a non-dissipative (conservative) theory of such a type is possible.

In the relativistic case:

1. *Principled derivation of relativistic energy and momentum.* Step I generalizes fairly straightforwardly from the nonrelativistic case, yielding the corresponding relativistic expressions for energy and momentum. Noteworthy here is the fact that:

[37] We note that this implies that the (nonrelativistic) kinetic theory of gases is inconsistent—insofar as 'heat' is regarded as a form of massless energy, it is frame-invariant, and so cannot be represented by the kinetic energy of a set of particles, which is not frame-invariant.

(a) the expression for relativistic energy includes a rest energy component, mc^2; and

(b) one must assume that there is no contribution to a particle's rest energy other than mc^2.

2. *Massless energy and momentum.* In Step 2, the introduction of a massless form of momentum is essential in order to allow the conversion of kinetic energy to massless energy. Thus, the energetic framework must posit both massless energy *and* momentum. This stands against the historical development, in which it took the discovery of electromagnetic momentum to trigger the realization that momentum could be carried by something other than corpuscles.

3. *Nature of massless energy-momentum.* The generalized principles of conservation of energy and momentum then jointly imply that massless energy-momentum transforms as a four-vector, and hence transforms in the *same way* as massive energy-momentum. Thus, unlike the historical development, where the energy-momentum transformations laws were derived by consideration of the stress-energy-momentum tensor of an electromagnetic system [4] (see also [5, 6]), we see that the conservation laws directly imply that massless and massive energy-momentum have the same transformation laws; and they do so very generally since there is no need to specify any particular model of the massless component.

4. *Interconversion of energy-momentum.* As the transformation laws for energy and momentum are the same for the massless and massive components, the exchange of energy-momentum between these components is possible. In this connection, we note that, in contrast to the nonrelativistic case, a kinetic theory of gases is thereby rendered consistent.

VI. DISCUSSION

A. The relationship between symmetry transformations and conservation laws.

Noether's theorems establish a connection between symmetry transformations and conservation laws. This is typically taken to be the ground for such assertions as 'invariance under temporal displacement underlies the conservation of energy'. However, as pointed out in [44], the connections between *specific* symmetry transformations and *specific* conservation laws (say, between temporal displacement and conservation of energy) presume the *specific form* of the action for a mechanical system[38]. As this form is conventionally obtained by requiring that the Euler–Lagrange equations of motion agree with those of Newtonian mechanics[39], such assertions presume the latter.

The approach given here provides another way of seeing the connection between symmetry principles (which we take to include both the principle of relativity and conservation principles) and the quantities of motion (and their relation with their corresponding massless forms). Specifically, one can see how the two symmetry principles interweave to *produce* the quantities of motion, and then shape the equations of motion. This is a rather different connection from that suggested by an application of Noether's theorem, but are perhaps more fundamental in the sense that the considerations given here precede the equations of motion (whereas an application of Noether's theorems to classical mechanics presupposes them).

B. Pedagogical significance.

Classical mechanics is generally the first major physical theory to which a student is exposed, and serves as a linchpin in their subsequent physics education. As

---

[38] We give here the some of the relevant quotes from [44]: "The conserved quantities of classical mechanics are Noether charges only because the classical equations of motion are what they are. But whether or not the classical equations of motion hold is something that needs to be established...". And: "Given what the equations of motion are, and that they hold where they do, it is indeed necessary that the conservation laws hold, but that's just a conditional necessity. The connection between the symmetries of the equations of motion and conservation laws is shown by Noether's theorem. That these are the correct equations of motion, however, is a completely different matter."

[39] The assumptions underlying the least-action approach to non-relativistic particle mechanics can be broken down as follows: (i) the (configuration-space) trajectory, $x(t)$, of a particle system between times t_1, t_2 has an associated action $S[x(t)]$; (ii) the actual trajectory between given configurations at times t_1, t_2 is one that extremizes $S[x(t)]$; (iii) the action is given by the time integral of a function, L, of $x(t)$ and a finite number of temporal derivatives thereof; (iv) the function L has the form $L = T - V$, where T, V are the kinetic and potential energies of the system. Of these assumptions, the first *three* can be posited independently from Newton's equations of motion. However, the common view is that the *fourth*—$L = T - V$—arises through a transformation of Newton's equations of motion via d'Alembert's principle (a more direct approach is given in [45]). Although it is possible to use fundamental symmetries (homogeneity of space and time, isotropy of space, and Galilean invariance) to show that L is proportional to T for a single isolated particle [46, §4]; and, further, to use compositional symmetries to show that $L = \sum_i T_i$ for a set of noninteracting particles, we are not aware of a derivation of $L = T - V$ that avoids presuming Newton's equations of motion.

such, the manner of its presentation implicitly conveys the values and priorities of physics as it is currently practised, and significantly influences the degree to which the student will be prepared to comprehend further elaborations and developments of mechanics (such as relativistic dynamics, and Lagrangian and Hamiltonian mechanics) and the degree to which they will be able to integrate their understanding of mechanics with that of other physical theories (such as electromagnetism).

The standard approach to nonrelativistic classical mechanics is based around Newton's laws of motion and energy conservation. As we have described in Sec. II D, such an approach raises many questions, such as why the quantities of motion have the mathematical form that they do, but these questions cannot be adequately addressed within the context of the standard approach. Furthermore, from the standpoint of the standard approach, the transition to relativistic dynamics is rather opaque.

The approach given here paves the way for an alternative presentation of mechanics wherein the formalism of mechanics is derived systematically, guided primarily by symmetry principles. The emphasis on symmetry principles reflects the immense importance of such principles in modern physics, while the step-by-step derivation of mechanics on the basis of these principles reflects the growing trend in recent decades of better understanding our existing theories by *reconstructing* them systematically from physically well-motivated principles rather than taking their mathematical structure as a given. Indeed, the present approach to mechanics employs mathematical and methodological techniques borrowed from work on the reconstruction of quantum theory (see [47, 48], for instance).

Such a presentation would address many questions (summarized in Sec. II D) which are difficult to compellingly answer within the standard approach, but which frequently arise in the teaching of mechanics [49, 50]. In addition, as described in Sec. II D, such a presentation would enable a smooth transition to relativistic dynamics, and enable the student to clearly understand why and how the introduction of the Lorentz transformations to implement relativity brings about a cascade of changes in the dynamics—why, for instance, new expressions for corpuscular energy and momentum are required, why mass is no longer conserved, why massless momentum emerges, and why the energy and momentum of the massless component of an energetic system transform in the same way as the energy and momentum of the massive component. In contrast, in the usual presentations of relativistic dynamics, the connection between the dynamics and kinematics is obscured by a number of specific considerations, such as the use of the energy-frequency relationship of a photon.

Finally, as indicated in Sec. II D, many of the above questions reflect often decades-long debates in the history of mechanics. A symmetry-based presentation of mechanics puts these debates within touching distance, which would help to reveal the creative *process* by which mechanics was constructed (rather than presenting it as a finished product), and help cultivate an appreciation for some of the intellectual struggles which underpinned its development.

Although the details of how the approach developed here could be adapted for a first presentation of mechanics are beyond the scope of this paper, there would seem to be no obstacle to basing such a presentation on the twin notions of conservation and relativity. Indeed, some existing unconventional presentations approach mechanics via conservation principles.

For example, the Karlsruhe mechanics course [51] introduces particle momentum and the principle of conservation of momentum axiomatically, introduces Newton's second law axiomatically (with force interpreted as a 'momentum current strength'), introduces the notion of kinetic energy via a postulated relationship between 'energy current' and 'momentum current', and then shows that momentum conservation is consistent with Galileo's principle of relativity.

By incorporating the principle of relativity at the outset, however, it would be possible to *derive* the conserved quantities of motion (kinetic energy and momentum) rather than postulating them, and thereby reveal the intimate relationship between the quantities of motion and the notion of conservation. The Galilean invariance of the conservation laws would also thereby be made transparent. The issue of there being *two* distinct quantities of motion would thereby be encountered at the outset, and one could then introduce the distinction between asymptotic conservation and continuous conservation, which would provide clear motivation both for using momentum (rather than kinetic energy) as a basis for particle mechanics *and* for introducing noncorpuscular energy (such as heat) as a way of ensuring continuous energy conservation. Such an approach would have the added benefit of introducing students to the kind of symmetry-based thinking that characterises not only Huygens' approach to particle mechanics but also Einstein's special relativistic thought experiments, thereby paving the way for the transition to relativistic mechanics.

[1] E. P. Wigner, Comm. in Pure and Applied Math **13**, 1 (1960).
[2] O. Darrigol, Archive for History of Exact Sciences (2019).
[3] J. R. Schütz, Nachr. v. d. Gesellschaft der Wissenschaften zu Göttingen, Math.-Phy. Kl. **2**, 110 (1897).
[4] M. Laue, Ann. d. Phys **35**, 524 (1911).
[5] D. Giulini, International Journal of Geometric Methods in Modern Physics **15**, 1850182 (2018).
[6] H. C. Ohanian, Stud. Hist. Phil. Sci. **40**, 167 (2009).
[7] C. M. Iltis, Ph.D. thesis, University of Wisconsin, Madison (1967).
[8] C. M. Iltis, Studies in History and Philosophy of Science **1**, 135 (1970).
[9] C. M. Iltis, Isis **62**, 21 (1971).
[10] H. Simon, The Philosophical Magazine **38**, 888 (1947).
[11] E. A. Desloge, *Classical Mechanics.*, vol. 1 (John Wiley and Sons, Inc., 1982).
[12] R. Dugas, *History of Mechanics* (Dover Publications, 2011).
[13] A. Einstein, Bull. Amer. Math. Soc. **41**, 223 (1935).
[14] A. Einstein, Ann. Phys. **18**, 639 (1905).
[15] J. Ehlers, W. Rindler, and R. Penrose, Am. J. Phys. **33**, 995 (1965).
[16] M. V. Lalan, Comptes rendus de l'Académie des Sciences **198**, 1211 (1934).
[17] A. Einstein, Ann. Phys. **17** (1905).
[18] M. Planck, Verhandlungen Deutsche Physikalische Gesellschaft **8**, 136 (1906), English translation available at https://en.wikisource.org/wiki/Translation:The_Principle_of_Relativity_and_the_Fundamental_Equations_of_Mechanics (accessed 22 Nov 2019).
[19] R. C. Tolman, Phil. Mag. **23**, 375 (1912).
[20] C. M. Graney, The Physics Teacher **51**, 411 (2013).
[21] M. Clagett, *The Science of Mechanics in the Middle Ages* (University of Wisconsin Press, 1961).
[22] E. Grant, ed., *A source book in Medieval science* (Harvard University Press, 1974).
[23] E. Mach, *The Science of Mechanics: A Critical and Historical Account of Its Development* (Open Court Pub. Co., 1919), 4th ed.
[24] E. A. Desloge, Int. J. Theor. Phys. **15**, 349 (1976).
[25] E. A. Desloge, Int. J. Theor. Phys. **15**, 357 (1976).
[26] P. K. Sahoo and P. Kannappan, *Introduction to Functional Equations* (CRC Press, 2011).
[27] H. Arzeliès, *Relativistic Point Dynamics* (Pergamon Press, 1971).
[28] S. Sonego and M. Pin, Eur. J. Phys. **26**, 33 (2005).
[29] P. Goyal, in preparation.
[30] P. Goyal, Phys. Rev. A **78**, 052120 (2008).
[31] P. Goyal, arXiv:0910.2444 (2010).
[32] J. Aczél, *Lectures on Functional Equations and their Application* (Academic Press, 1966).
[33] J. Aczél and J. Dhombres, *Functional equations in several variables* (Cambridge Univ. Press, 1989).
[34] R. Descartes, *Principles of Philosophy* (Kluwer, 1982).
[35] R. S. Westfall, *Force in Newton's Physics: The science of dynamics in the seventeenth century* (Science History Publications, 1971).
[36] J. B. Barbour, *The Discovery of Dynamics* (Oxford University Press, 2001).
[37] G. W. Leibniz, Acta eruditorum pp. 161–163 (1686), English translation, with supplements, in *Philosophical papers and letters*, 2nd edition, ed. L. E. Loemker, pp. 296–302.
[38] W. L. Scott, *The conflict between atomism and conservation theory 1644 to 1960* (Macdonald (London), Elsevier (NY), 1970).
[39] C. Wren, Phil. Trans. Roy. Soc. **3**, 867 (1669).
[40] C. Huygens, Phil. Trans. Roy. Soc. **4**, 925 (1669).
[41] C. Huygens, J. Scavans pp. 19–24 (1669).
[42] J. Wallis, Phil. Trans. Roy. Soc. **3**, 864 (1669).
[43] T. S. Kuhn, in *The essential tension*, edited by T. S. Kuhn (University of Chicago Press, 1977).
[44] J. Wolff, Phil. Sci. **8**, 898 (2013).
[45] B. L. Beers, Am. J. Phys. **40**, 1636 (1972).
[46] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth Heinemann, 2000), 3rd ed.
[47] P. Goyal, K. H. Knuth, and J. Skilling, Phys. Rev. A **81**, 022109 (2010).
[48] P. Goyal, New J. Phys. **17**, 013043 (2015).
[49] C. Khiari, Lat. Am. J. Phys. Educ. **5** (2011).
[50] C. J. Papachristou, Nausivios Chora **4**, 153 (2012).
[51] F. Herrmann, M. Pohlig, and H. Schwarze, *The Karlsruhe Physics Course: Mechanics* (2019).

Appendix A: Solution of Functional Equations

In this appendix, the functional equations needed in the derivation of the energy and momentum of bodies are solved. If possible, we transform the functional equation of interest into a standard functional equation. For interest, we sometimes provide more than one possible method of solution. In each case, certain mathematical conditions must be satisfied by the unknown function in order for a solution to be obtained.

1. Solution of $f(u+v)+f(u-v)=2f(\sqrt{u^2+v^2})$.

We present two different solution methods for Eq. (1), one that transforms it into Jensen's functional equation, the other a direct solution by removing one degree of freedom.

a. *Solution by transformation into Jensen's functional equation*

Using the substitution $k(w^2)=f(w)$, Eq. (1) becomes

$$k\left([u^2+v^2]+2uv\right)+k\left([u^2+v^2]-2uv\right)=2k(u^2+v^2). \tag{A1}$$

Setting $x=u^2+v^2,\;\; y=2uv$, we obtain

$$k(x+y)+k(x-y)=2k(x), \tag{A2}$$

which is Jensen's equation, with x, y independently variable within $x>0, y>0$. If k is continuous, this equation, under the stated conditions, has general solution $k(z)=az+b$. As k is continuous whenever f is continuous,

$$f(v)=av^2+b \tag{A3}$$

is the general solution of Eq. (1) under the condition that f is continuous.

b. *Direct solution by removal of one degree of freedom*

Alternatively, one can directly solve Eq. (1) by removing one degree of freedom, albeit at the cost of the stronger regularity condition that f is analytic. Setting $u=v$ in Eq. (1) yields

$$f(2u)+f(0)=2f(\sqrt{2}u). \tag{A4}$$

If f is differentiable, then, for $n\geq 1$,

$$2^n f^{(n)}(2u)=2^{1+n/2}\, f(\sqrt{2}u). \tag{A5}$$

This yields $f^{(n)}(0)=0$ whenever $n\neq 2$. Hence, if f is analytic,

$$f(x)=av^2+b. \tag{A6}$$

2. Solution of $g(v+u)-g(v-u)=2g(\sqrt{u^2+v^2})\cdot v/\sqrt{u^2+v^2}$.

Solution of Eq. (6) is most readily obtained by removing one degree of freedom by setting $v=u$. Thence,

$$g(2u)-g(0)=\sqrt{2}\,g(\sqrt{2}u). \tag{A7}$$

Setting $u = 0$ fixes $g(0) = 0$. If g is differentiable, then, for $n \geq 1$,

$$2^n g^{(n)}(2u) = 2^{(n+1)/2}\, g^{(n)}(\sqrt{2}u). \tag{A8}$$

For $n \geq 2$, this yields $g^{(n)}(0) = 0$. Thus, if g is analytic,

$$g(u) = au. \tag{A9}$$

3. Solution of $\widetilde{F}(x) + \widetilde{F}(y) = 2\widetilde{F}((x+y)/2)$.

Equation (34), with $x = \gamma(u \oplus -v)$ and $y = \gamma(u \oplus v)$, has the form of Jensen's equation, but it is not immediately apparent that x, y are independent in some region. To see that this is so, it is helpful to express u, v in terms of rapidities:

$$\begin{aligned} u &= c \tanh \phi_1 \\ v &= c \tanh \phi_2. \end{aligned} \tag{A10}$$

Then $u \oplus v = c \tanh(\phi_1 + \phi_2)$, so that

$$\begin{aligned} \gamma(u \oplus v) &= \tilde{\gamma}(\phi_1 + \phi_2) \\ \gamma(u \oplus -v) &= \tilde{\gamma}(\phi_1 - \phi_2), \end{aligned} \tag{A11}$$

where $\tilde{\gamma}(\phi) \equiv (1 - \tanh^2 \phi)^{-1/2}$.

Now, $u > 0$ and $|v| < c$, so that $\phi_1 > 0$ and ϕ_2 is free. Consequently, $(\phi_1 + \phi_2)$ and $(\phi_1 - \phi_2)$ can be independently chosen. Further, since $\tilde{\gamma}$ is monotonic, $x = \tilde{\gamma}(\phi_1 + \phi_2)$ and $y = \tilde{\gamma}(\phi_1 - \phi_2)$ are independent in some region. Therefore, Eq. (34) has the solution $\widetilde{F}(x) = a + bx$.