

Three noncontextual hidden variable models for the Peres-Mermin square

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Abstract

I will argue that the Peres-Mermin square does not necessarily rule out a value-definite (deterministic) noncontextual hidden variable model if the operators are not given a physical interpretation satisfying the following two requirements: (i) each operator is uniquely realized by a single physical measurement; (ii) commuting operators are realized by simultaneous measurements. To underpin this claim, I will construct three hidden variable models for three different physical realizations of the Peres-Mermin square: one violating (i), another violating (ii), and a third one violating both (i) and (ii).

Keywords: Kochen-Specker theorem, contextuality, simultaneous measurability

1 Introduction

A value-definite (deterministic) hidden variable model for quantum mechanics is *noncontextual* if the outcome of no measurement represented by an operator in the formalism depends in any hidden state on which other simultaneous measurements are performed. The general aim of this paper is to show that the Kochen-Specker theorems cannot be successful in ruling out value-definite noncontextual hidden variable models for quantum mechanics until they are given a physical interpretation (realization) satisfying the following two requirements:

- (i) each operator is uniquely realized by a single physical measurement;
- (ii) commuting operators are realized by simultaneous measurements.

However strange it may seem, it is not at all trivial for a physical interpretation to satisfy both (i) and (ii). If one realizes each operator by a different measurement, then some commuting operators will turn out to be not simultaneously measurable. If one realizes each set of commuting operators by simultaneous measurements, then some operators will be realized by multiple measurements. After studying various interpretations of numerous Kochen-Specker theorems, I have to admit that I found no such interpretation for any Kochen-Specker theorem which would satisfy both (i) and (ii). In this paper, however, I will not argue for the claim that there is no such interpretation and consequently for the strong claim that there is no valid (algebraic) argument proving quantum contextuality. (For that see Hofer-Szabó, 2021a, b). Here I will only

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argue that if an interpretation of a Kochen-Specker theorem violates either (i) or (ii), then it can be given a value-definite noncontextual hidden variable model. To make my point, I will pick a specific Kochen-Specker theorem, the Peres-Mermin square (Peres, 1990; Mermin, 1992) and construct three different noncontextual hidden variable models for three different physical realizations thereof: the first violating (i), the second violating (ii), and the third violating both (i) and (ii). These specific models will simply highlight the truism that no mathematical no-go theorem *per se* can prove anything about the physical world without an appropriate physical interpretation.

In Section 2, I introduce the Peres-Mermin square; in Section 3, I give three different physical realizations for it; and in Section 4, I construct the three value-definite noncontextual hidden variable models. I discuss the results in Section 5.

2 The Peres-Mermin square

The Peres-Mermin square is the following 3×3 matrix of self-adjoint operators:

$$\begin{array}{ccc} \sigma_z \otimes I & I \otimes \sigma_z & \sigma_z \otimes \sigma_z \\ I \otimes \sigma_x & \sigma_x \otimes I & \sigma_x \otimes \sigma_x \\ \sigma_z \otimes \sigma_x & \sigma_x \otimes \sigma_z & \sigma_y \otimes \sigma_y \end{array}$$

where σ_x , σ_y , and σ_z are the Pauli operators and I is the unit operator on the two dimensional complex Hilbert space. Each operator in the matrix has two eigenvalues, ± 1 , and are arranged in such a way that two operators are commuting if and only if they are in the same row or in the same column. For example,

$$\begin{aligned} [\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y] &= \sigma_x \otimes \sigma_x \cdot \sigma_y \otimes \sigma_y - \sigma_y \otimes \sigma_y \cdot \sigma_x \otimes \sigma_x \\ &= \sigma_x \sigma_y \otimes \sigma_x \sigma_y - \sigma_y \sigma_x \otimes \sigma_y \sigma_x \\ &= i\sigma_z \otimes i\sigma_z - (-i)\sigma_z \otimes (-i)\sigma_z \\ &= -\sigma_z \otimes \sigma_z + \sigma_z \otimes \sigma_z \\ &= 0 \end{aligned}$$

Commuting operators have common eigenstates. Expressed in the computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and using the notation

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

the common eigenstates and the associated eigenvalues of the three operators in the three subsequent *rows* of the Peres-Mermin square are the following:

First row	$\sigma_z \otimes I$	$I \otimes \sigma_z$	$\sigma_z \otimes \sigma_z$
$ \Psi_1\rangle = 00\rangle$	+1	+1	+1
$ \Psi_2\rangle = 01\rangle$	+1	-1	-1
$ \Psi_3\rangle = 10\rangle$	-1	+1	-1
$ \Psi_4\rangle = 11\rangle$	-1	-1	+1

Second row	$I \otimes \sigma_x$	$\sigma_x \otimes I$	$\sigma_x \otimes \sigma_x$
$ \Psi'_1\rangle = ++\rangle$	+1	+1	+1
$ \Psi'_2\rangle = -\rangle$	+1	-1	-1
$ \Psi'_3\rangle = +-\rangle$	-1	+1	-1
$ \Psi'_4\rangle = --\rangle$	-1	-1	+1

Third row	$\sigma_z \otimes \sigma_x$	$\sigma_x \otimes \sigma_z$	$\sigma_y \otimes \sigma_y$
$ \Psi''_1\rangle = \frac{1}{\sqrt{2}}(0+\rangle + 1-\rangle)$	+1	+1	+1
$ \Psi''_2\rangle = \frac{1}{\sqrt{2}}(0+\rangle - 1-\rangle)$	+1	-1	-1
$ \Psi''_3\rangle = \frac{1}{\sqrt{2}}(1+\rangle + 0-\rangle)$	-1	+1	-1
$ \Psi''_4\rangle = \frac{1}{\sqrt{2}}(1+\rangle - 0-\rangle)$	-1	-1	+1

Table 1: Common eigenstates and eigenvalues of the operators in the three rows of the Peres-Mermin square

Similarly, the common eigenstates and eigenvalues of the three operators in the three *columns* are the following:

First column	$\sigma_z \otimes I$	$I \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$ \Phi_1\rangle = 0+\rangle$	+1	+1	+1
$ \Phi_2\rangle = 0-\rangle$	+1	-1	-1
$ \Phi_3\rangle = 1+\rangle$	-1	+1	-1
$ \Phi_4\rangle = 1-\rangle$	-1	-1	+1

Second column	$I \otimes \sigma_z$	$\sigma_x \otimes I$	$\sigma_x \otimes \sigma_z$
$ \Phi'_1\rangle = +0\rangle$	+1	+1	+1
$ \Phi'_2\rangle = -0\rangle$	+1	-1	-1
$ \Phi'_3\rangle = +1\rangle$	-1	+1	-1
$ \Phi'_4\rangle = -1\rangle$	-1	-1	+1

Third column	$\sigma_z \otimes \sigma_z$	$\sigma_x \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$
$ \Phi''_1\rangle = \frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	+1	+1	-1
$ \Phi''_2\rangle = \frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	+1	-1	+1
$ \Phi''_3\rangle = \frac{1}{\sqrt{2}}(01\rangle + 10\rangle)$	-1	+1	+1
$ \Phi''_4\rangle = \frac{1}{\sqrt{2}}(01\rangle - 10\rangle)$	-1	-1	-1

Table 2: Common eigenstates and eigenvalues of the operators in the three columns of the Peres-Mermin square

Observe that the common eigenstates of the operators in the first two rows and columns are product states, whereas those in the third row and column are entangled states. The vectors in the third column form the so-called Bell basis with $|\Phi''_4\rangle$ as the singlet state. However, I will also refer to the vectors in the third row as Bell states.

Also observe that the product of the eigenvalues of the three operators in all the three rows and in the first two columns is +1, whereas it is -1 for the operators in the third column. As a mathematical consequence, one cannot fill in a 3×3 matrix with numbers ± 1 such that these numbers conform in each row and column to one of the four triples of eigenvalues. Call this mathematical fact the *Peres-Mermin contradiction*. The Peres-Mermin contradiction is often interpreted as the impossibility to provide a value-definite noncontextual hidden variable model for quantum mechanics. Namely, such a model should consist of such hidden states which assign values to the operators in such a way that any triple of operators in a row or column have values according to one of the four triples of eigenvalues in that row or column.

Note, however, that without a physical realization of the operators such a conclusion is without any physical justification. And indeed, in the next Section we will provide three different physical realizations of the Peres-Mermin square for which there *is* a value-definite noncontextual hidden variable model. The models for these realizations will avoid the Peres-Mermin contradiction in the following two ways. First, if the realization violates requirement (i)—that is some operators are realized by multiple measurements, then the model will not assign the same value for these measurements in some of the hidden states. (Note that the fact that physically different measurements are represented by the same operator in quantum mechanics does not mean that

they need to have the same outcome in every hidden state. All that is required is that they have the same *distribution* of outcomes in every quantum state.) Second, if the realization violates requirement (ii)—that is some commuting operators are not realized by simultaneous measurements, then the model will not respect the constraint coming from the corresponding triples of eigenvalues since this constraint is not empirically justified. (Note that for the empirical justification of the constraint that in a certain row or column the admissible triples of eigenvalues are just those four which are in Table 1 or 2, we need to perform the three measurements realizing the corresponding three operators *simultaneously* and check which combinations of outcomes come up and which do not.) Thus, one can avoid the Peres-Mermin contradiction and construct a value-definite noncontextual hidden variable model for a given realization of the Peres-Mermin square if either (i) or (ii) is violated. In the first case one avoids the contradiction by writing more than one number in some entries of the matrix; in the second case by filling in the matrix such that it conforms only to those constraints in Table 1 or 2 which are backed by simultaneous measurements.

Now, let us turn to the concrete realizations.

3 Three realizations of the Peres-Mermin square

A realization (interpretation) of the Peres-Mermin square is an association of operators in the matrix with real-world measurements. In all three realizations to be provided, the operators will represent measurements performed on pairs of photons prepared in a given quantum state. The realizations will differ in what these measurements exactly are.

First realization. The schematic picture of the first realization is portrayed in Figure 1:

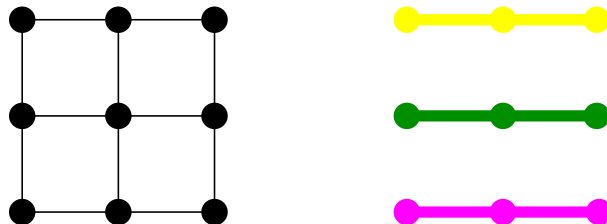


Figure 1: First realization of the Peres-Mermin square

On the left graph, the black vertices represent the 9 operators in the Peres-Mermin square and the (hyper)edges connecting the vertices represent the commutativity relation between the operators. On the right graph, the coloured vertices represent the 9 measurements uniquely associated with the operators and the edges represent the simultaneous measurability relation.

What are these 9 measurements?

Well, actually there are only three physical measurement procedures, to the result of which one applies three different functions. (This is why you find only three different colors on the graph each representing a different physical measurement. The thickened edges express the fact that the measurements in each triple are *trivially* simultaneously measurable: they are just

different functions of the same physical measurement.) The three physical measurements are the following:

L_{zz} : measure the *linear polarization* of both photons along a given transverse axis z (with outcome $+1$ if the photon passes the polarizer and -1 if not);

L_{xx} : measure the *linear polarization* of both photons along an axis x at 45° from the axis z ;

B : perform a *Bell state measurement* on the photon pair with four outcomes corresponding to the Bell states $|\Psi_1''\rangle$, $|\Psi_2''\rangle$, $|\Psi_3''\rangle$, and $|\Psi_4''\rangle$ in Table 1.

The 9 measurements arise from the 3 physical measurements by applying certain functions on the measurement results. After performing the linear polarization measurement L_{zz} and obtaining four different combinations of outcomes (± 1 on the left and ± 1 on the right), one can do three different things: either one registers only the outcome on the *left* wing ($l(L_{zz})$); or the outcome on the *right* wing ($r(L_{zz})$); or registers the *product* of the two outcomes ($t(L_{zz})$). Similarly, one can define $l(L_{xx})$, $r(L_{xx})$, and $t(L_{xx})$ from applying the same functions to the result of L_{xx} . Finally, one can apply three functions f , g , or h to the results of the Bell state measurement B . These functions assign to each outcome of B the corresponding eigenvalue of the operators $\sigma_z \otimes \sigma_x$, $\sigma_x \otimes \sigma_z$, and $\sigma_y \otimes \sigma_y$ according to the third row of Table 1. For example, if the outcome of the Bell state measurement B is $|\Psi_1''\rangle$, then $f(B) = +1$, $g(B) = +1$ and $h(B) = +1$.

The operators and the corresponding measurements realizing the operators are portrayed in Table 3:

Operators			Measurements		
$\sigma_z \otimes I$	$I \otimes \sigma_z$	$\sigma_z \otimes \sigma_z$	$l(L_{zz})$	$r(L_{zz})$	$t(L_{zz})$
$I \otimes \sigma_x$	$\sigma_x \otimes I$	$\sigma_x \otimes \sigma_x$	$r(L_{xx})$	$l(L_{xx})$	$t(L_{xx})$
$\sigma_z \otimes \sigma_x$	$\sigma_x \otimes \sigma_z$	$\sigma_y \otimes \sigma_y$	$f(B)$	$g(B)$	$h(B)$

Table 3: First realization of the Peres-Mermin square

Now, the realization satisfies requirement (i) since each operator is uniquely realized by a measurement. But it violates requirement (ii) since the measurements, for example, realizing the commuting operators in the third column $t(L_{zz})$, $t(L_{xx})$, and $h(B)$ are *not simultaneously measurable* for the simple reason that no two of the three measurements L_{zz} , L_{xx} and B can be performed on the same pair of photons at the same time. Note that the measurement L_{zz} requires two polarization beam splitters each oriented along direction z on the opposite wings; the measurement L_{xx} requires two polarization beam splitters each oriented along direction x on the opposite wings; and the Bell state measurement B requires a complicated arrangement of beam splitters and polarization beam splitters in the way of the two photons (Lütkenhaus et al., 1999). These measurement arrangements are incompatible, they cannot be performed simultaneously on the same pair of photons. Consequently, one cannot experimentally verify via these measurements whether the outcomes of $t(L_{zz})$, $t(L_{xx})$, and $h(B)$ *in a given run of the*

experiment conform to one of the rows of Table 2 or not. But then neither the hidden variable models need to respect these constraints. We came back to this point in Section 4.

Now, one might wonder whether the lack of simultaneous measurability between the measurement represented by the vertical triples of operators can be cured by introducing new measurements. This leads us to the second realization of the Peres-Mermin square.

Second realization. The second realization is portrayed in Figure 2:

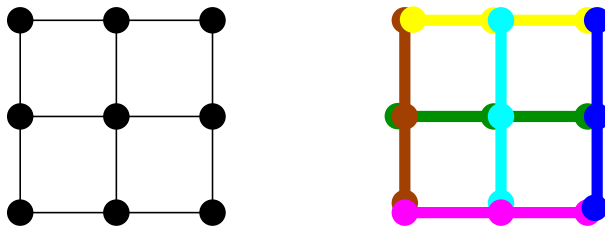


Figure 2: Second realization of the Peres-Mermin square

As we see on the right graph, we have now 6 measurements: the three old measurements: L_{zz} , L_{xx} and B plus three new ones realizing the three vertical triples:

L_{zy} : measure the *linear polarization* of the left photon along the axis z and the linear polarization of the right photon along the axis x ;

L_{xz} : measure the *linear polarization* of the left photon along the axis x and the linear polarization of the right photon along the axis z ;

B' : perform another *Bell state measurement* on the photon pair with four outcomes corresponding to the Bell states $|\Phi_1''\rangle$, $|\Phi_2''\rangle$, $|\Phi_3''\rangle$, and $|\Phi_4''\rangle$ in Table 2.

Just as above, there are 9 measurements arising from the 3 new physical measurements: $l(L_{zx})$, $r(L_{zx})$, $t(L_{zx})$; and $l(L_{xz})$, $r(L_{xz})$, $t(L_{xz})$; and finally $f'(B')$, $g'(B')$, $h'(B')$. (In these latter measurement, we again apply three functions f' , g' and h' assigning to each outcome of the Bell state measurement B' the corresponding eigenvalue of the operators $\sigma_z \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$, and $\sigma_y \otimes \sigma_y$ according to the third column of Table 2.) The operators and corresponding measurements realizing the operators are portrayed in Table 4:

Operators			Measurements		
$\sigma_z \otimes I$	$I \otimes \sigma_z$	$\sigma_z \otimes \sigma_z$	$l(L_{zz}) / l(L_{zx})$	$r(L_{zz}) / r(L_{xz})$	$t(L_{zz}) / f'(B')$
$I \otimes \sigma_x$	$\sigma_x \otimes I$	$\sigma_x \otimes \sigma_x$	$r(L_{xx}) / r(L_{zx})$	$l(L_{xx}) / l(L_{xz})$	$t(L_{xx}) / g'(B')$
$\sigma_z \otimes \sigma_x$	$\sigma_x \otimes \sigma_z$	$\sigma_y \otimes \sigma_y$	$f(B) / t(L_{zx})$	$g(B) / t(L_{xz})$	$h(B) / h'(B')$

Table 4: Second realization of the Peres-Mermin square

Now, this realization of the Peres-Mermin square, contrary to the first realization, satisfies requirement (ii) since commuting triples of operators are all realized by simultaneous measurements. However, requirement (i) is now violated since every operator is realized by two different measurements. Four of the nine realizations are unproblematic: due to locality, the measurements $l(L_{zz})$ and $l(L_{zx})$ can be considered the same measurements since in both measurements one does the same thing in the left wing of the experiment. Similarly, one can argue that

$$r(L_{zz}) = r(L_{xz}) \quad r(L_{xx}) = r(L_{zx}) \quad l(L_{xx}) = l(L_{xz})$$

However, the other five pairs of measurement are not the same measurements:

$$t(L_{zz}) \neq f'(B') \quad t(L_{xx}) \neq g'(B') \quad f(B) \neq t(L_{zx}) \quad g(B) \neq t(L_{xz}) \quad h(B) \neq h'(B')$$

For example, $t(L_{\{zz\}})$ and $f'(B')$, both realizing the operator $\sigma_z \otimes \sigma_z$ in the upper right entry, are two physically different measurement arrangements; they cannot be performed at the same time on the same pair of photons. Thus, requirement (i) is violated.

Now, let us go over to the third realization where neither (i) nor (ii) holds.

Third realization. The third realization is portrayed in Figure 3:

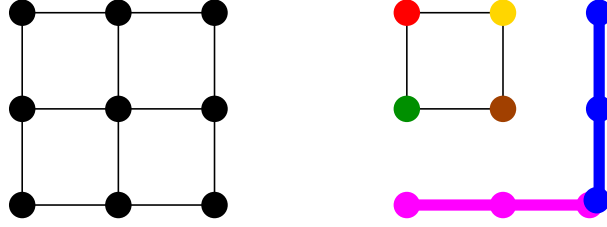


Figure 3: Third realization of the Peres-Mermin square

As we can see on the right graph, we have now again 6 measurements:

$L_{l(z)}$: measure the linear polarization of the left photon along the axis z ;

$L_{r(z)}$: measure the linear polarization of the right photon along the axis z ;

$L_{l(x)}$: measure the linear polarization of the left photon along the axis x ;

$L_{r(x)}$: measure the linear polarization of the right photon along the axis x ;

B : perform a Bell state measurement with four outcomes corresponding to the Bell states $|\Psi_1''\rangle$, $|\Psi_2''\rangle$, $|\Psi_3''\rangle$, and $|\Psi_4''\rangle$;

B' : perform a Bell state measurement with four outcomes corresponding to the Bell states $|\Phi_1''\rangle$, $|\Phi_2''\rangle$, $|\Phi_3''\rangle$, and $|\Phi_4''\rangle$.

From the two Bell state measurements we again obtain the measurements $f(B)$, $g(B)$, $h(B)$ and $f'(B')$, $g'(B')$, $h'(B')$. The operators and corresponding measurements realizing the operators are portrayed in Table 5:

Operators			Measurements		
$\sigma_z \otimes I$	$I \otimes \sigma_z$	$\sigma_z \otimes \sigma_z$	$L_{l(z)}$	$L_{r(z)}$	$f'(B')$
$I \otimes \sigma_x$	$\sigma_x \otimes I$	$\sigma_x \otimes \sigma_x$	$L_{r(x)}$	$L_{l(x)}$	$g'(B')$
$\sigma_z \otimes \sigma_x$	$\sigma_x \otimes \sigma_z$	$\sigma_y \otimes \sigma_y$	$f(B)$	$g(B)$	$h(B) / h'(B')$

Table 5: Third realization of the Peres-Mermin square

Now, in this third realization of the Peres-Mermin square both requirement (i) and (ii) are violated: the operator $\sigma_y \otimes \sigma_y$ in the bottom right corner is realized by two different measurements $h(B)$ and $h'(B')$; and the measurements $L_{r(z)}$ and $f'(B')$ are not simultaneously measurable (since the linear polarization measurement and the Bell state measurement require different experimental arrangements; see again Lütkenhaus et al., 1999).

To sum up, we have three different realizations of the Peres-Mermin square: one violating requirement (i), one violating requirement (ii), and a third one violating both (i) and (ii). Next, we construct a value-definite noncontextual hidden variable model for each realization.

4 Three value-definite noncontextual hidden variable models for the Peres-Mermin square

First realization. Let $|\Psi\rangle$ denote the quantum state of the beam of the photon pairs. A value-definite noncontextual hidden variable model for the first realization of the Peres-Mermin square is the following. We have three physical measurements, L_{zz} , L_{xx} , and B , each with four outcomes. Let $i, j, k = 1 \dots 4$ denote the indices running through these outcomes. The hidden variable model will then consist of 4^3 deterministic hidden states

$$\lambda^{ijk} \quad i, j, k = 1 \dots 4$$

providing the outcomes i , j and k for the measurements L_{zz} , L_{xx} , and B , respectively. The probability of these hidden states is

$$p(\lambda^{ijk}) = |\langle \Psi_i, \Psi \rangle|^2 \cdot |\langle \Psi'_j, \Psi \rangle|^2 \cdot |\langle \Psi''_k, \Psi \rangle|^2$$

Clearly, these probabilities add up to 1 and the model recovers the statistical predictions of quantum mechanics with respect to the above three measurements.

Note that this hidden variable model does not respect Table 2 in Section 2. In other words, it is possible that some hidden states (with nonzero probability) will provide such outcomes for a triple of measurements in a given *column* which do not conform to any triple of eigenvalues in Table 2. For example, for the measurements $t(L)$, $t(L)$, and $h(B)$ represented by the operators $\sigma_z \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$, and $\sigma_y \otimes \sigma_y$ in the third column, the hidden state λ^{111} will provide the outcomes

$$t(L_{zz}) = +1 \quad t(L_{xx}) = +1 \quad h(B) = +1$$

The triple $(+1, +1, +1)$, however, is not an admissible triple of eigenvalues according to the third column of Table 2. But this is no problem since the measurements L_{zz} , L_{xx} , and B , and consequently $t(L)$, $t(L)$, and $h(B)$ cannot be performed simultaneously in a given run of the experiment. Thus, the hidden variable model need not respect Table 2. (Note that Table 2 should be respected by a model only if it contains simultaneous measurements represented by operators in the *columns* of the Peres-Mermin square. Otherwise Table 2 does not put any constraint on the model.)

Second and third realization. For the second and third realization of the Peres-Mermin square we construct the same value-definite noncontextual hidden variable model. In both cases we have six physical measurements. In the second realization:

$$L_{zz} \quad L_{xx} \quad L_{zx} \quad L_{xz} \quad B \quad B' \quad (1)$$

with

$$l(L_{zz}) = l(L_{zx}) \quad r(L_{zz}) = r(L_{xz}) \quad r(L_{xx}) = r(L_{zx}) \quad l(L_{xx}) = l(L_{xz}) \quad (2)$$

In the third realization:

$$L_{l(z)} \quad L_{r(z)} \quad L_{l(x)} \quad L_{r(x)} \quad B \quad B' \quad (3)$$

The measurements (1)-(2) and (3), however, can easily be translated into one another via Table 1 and 2 in a one-to-one way. For example,

$$L_{zz} = 1 \quad L_{xx} = 4 \quad L_{zx} = 2 \quad L_{xz} = 2$$

if and only if

$$L_{l(z)} = +1 \quad L_{r(z)} = +1 \quad L_{l(x)} = -1 \quad L_{r(x)} = -1$$

and similarly for the other 15 measurement outcomes. So it is enough to provide a hidden variable model for the third realization.

Our hidden variable model will consist of $2^4 \cdot 4^2$ deterministic hidden states

$$\lambda^{ijklmn} \quad i, j, k, l = \pm 1, \quad m, n = 1 \dots 4$$

providing the outcomes $i \dots n$ for the six measurements in (3). The probability of these hidden states is

$$p(\lambda^{ijklmn}) = p^{ijkl} \cdot |\langle \Psi''_m, \Psi \rangle|^2 \cdot |\langle \Phi''_n, \Psi \rangle|^2$$

where the probabilities p^{ijkl} can be straightforwardly constructed using Arthur Fine's method developed in detail in Proposition 2 in (Fine, 1982). Here we do not repeat the construction; we only note that since the eight eigenprojections

$$P_{\sigma_z \otimes I}^\pm \quad P_{I \otimes \sigma_z}^\pm \quad P_{\sigma_x \otimes I}^\pm \quad P_{I \otimes \sigma_x}^\pm \quad (4)$$

of the four operators in the 2×2 top left submatrix of the Peres-Mermin square do not (in any combination) violate the Clauser-Horne inequality, there exists a value-definite noncontextual hidden variable model for any quantum state $|\Psi\rangle$. Fine explicitly constructs the hidden variable model with the 16 probabilities p^{ijkl} . (One just substitutes the projections with $+1$ eigenvalues in (4) for Fine's A, B, A', B' and the projections with -1 eigenvalues for $\bar{A}, \bar{B}, \bar{A}', \bar{B}'$, respectively, to get the probabilities p^{ijkl} .) For example, if $|\Psi\rangle = |\Psi_1\rangle$, we obtain

$$p^{+1+1kl} = \frac{1}{4} \quad k, l = \pm 1$$

and all the other probabilities are 0.

Note again that this hidden variable model is also not needed to conform to those constraints coming from Table 1 or 2 which are not backed by simultaneous measurability. Consider, for example, the operator $\sigma_z \otimes \sigma_z$. On the second realization, $\sigma_z \otimes \sigma_z$ is doubly realized: it can be measured either as $t(L_{zz})$ or as $f'(B')$. Consider now the hidden state $\lambda^{+1+1klm^4}$. In this hidden state $t(L_{zz}) = 1$ assigning the eigenvalue $+1$ to $\sigma_z \otimes \sigma_z$ (according to the first row in Table 1) and $f'(B') = 4$ assigning the eigenvalue -1 to $\sigma_z \otimes \sigma_z$ (according to the third column in Table 2). But since $t(L)$ and $f'(B')$ are different measurements, the two eigenvalues (measurement outcomes) need not match. (Still, one can easily find a $|\Psi\rangle$ for which $p(\lambda^{+1+1klm^4}) \neq 0$.)

Similarly, on the third interpretation, the measurements $P_{r(z)}$ and $f'(B')$ are not simultaneously measurable. Therefore, the hidden states need not conform to first row of Table 1.

Let me note, finally, that all the above models are noncontextual since the response of the system to a measurement in a given hidden state does *not* depend on which other measurements are simultaneously performed with it. The model also is nonconspiratorial since the distribution of the hidden states does not depend on the measurement choices (which are not even mentioned).

5 Discussion

In the paper it was shown that the Peres-Mermin square does not necessarily rule out a non-contextual hidden variable model if the physical realization of the operators does not satisfy the following two requirements:

- (i) each of the 9 operators is uniquely realized by a single measurement;
- (ii) commuting operators are realized by simultaneous measurements.

To make my point, I constructed three hidden variable models for three different physical realizations of the Peres-Mermin square: one violating (i), another violating (ii), and a third one violating both (i) and (ii). These models were not to suggest that quantum mechanics might admit a noncontextual hidden variable model. Far from it, we know from the violation of the Bell inequalities that this is impossible. Rather, by the above constructions I intended to show that in order to interpret the Kochen-Specker theorems as proving quantum contextuality, one needs to carefully interpret the operators in the theorems. More concretely, the Kochen-Specker theorems prove quantum contextuality only if the operators featuring in the theorem can be given a physical interpretation satisfying requirements (i) and (ii).

Now, requirement (i) is often ignored because it is implicitly required from a hidden variable model that any two measurements which are represented by the same operator should give the same outcome in every hidden state. Sometimes this constraint is also called noncontextuality (Spekkens, 2005). In (Hofer-Szabó 2021a, b) I argued against calling this constraint noncontextuality by showing that the two constraints are logically independent. Note that this constraint is violated in the second and third realization of the Peres-Mermin square while both models are noncontextual in the sense defined above.

As for requirement (ii), it is often tacitly assumed that commutativity and simultaneous measurability are synonyms. But they are not. In (Hofer-Szabó 2021a, b) again, I argued that from the pure fact that two operators are commuting it does not follow that any two measurements realizing the operators can be performed at the same time on the same system. The two measurement procedures may well be physically incompatible, just like the polarization measurement and Bell state measurements in the second and third realization. But realizing commuting operators by simultaneous measurements is necessary if we want to physically justify the mathematical no-go theorems.

In this paper, however, I did not intend to argue for requirement (i) and (ii). I simply wanted to show that they are *sine qua non* for taking the Kochen-Specker theorems to prove quantum contextuality.

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