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Bases for Structures and Theories I

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Abstract. Sometimes structures or theories are formulated with different sets of primitives and yet are definitionally equivalent. In a sense, the transformations between such equivalent formulations are rather like basis transformations in linear algebra or co-ordinate transformations in geometry. Here an analogous idea is investigated. Let a relational signature $P = \{P_i\}_{i \in I_P}$ be given. For a set $\Phi = \{\phi_i\}_{i \in I_\Phi}$ of L_P-formulas, we introduce a corresponding set $Q = \{Q_i\}_{i \in I_\Phi}$ of new relation symbols and a set of explicit definitions of the Q_i in terms of the ϕ_i. This is called a definition system, denoted d_Φ. A definition system d_Φ determines a *translation function* $\tau_\Phi : L_Q \to L_P$. Any L_P-structure A can be uniquely definitionally expanded to a model $A^+ \models d_\Phi$, called $A + d_\Phi$. The reduct $A + d_\Phi$ to the Q-symbols is called the *definitional image* $D_\Phi A$ of A. Likewise, a theory T in L_P may be extended a definitional extension $T + d_\Phi$; the restriction of this extension $T + d_\Phi$ to L_Q is called the *definitional image* $D_\Phi T$ of T. If T_1 and T_2 are in disjoint signatures and $T_1 + d_\Phi \equiv T_2 + d_\Theta$, we say that T_1 and T_2 are *definitionally equivalent* (wrt the definition systems d_Φ and d_Θ). Some results relating these notions are given, culminating in two characterization theorems for the definitional equivalence of structures and theories.

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1. Introduction

Sometimes theories are formulated with different sets of primitives and yet are definitionally equivalent. The *non-logical primitives* of a formalized language L are called its *signature*. There are many examples of theories (often involving formalized systems of arithmetic and set theory) formulated in very different signatures, which are nonetheless "equivalent". To take a simple example, consider the theory T_1 of a reflexive relation:

$$T_1 : \qquad \forall x P(x, x),$$

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expressed with a binary relation symbol P. Suppose we introduce a new binary relation symbol, Q, and give an explicit definition of it—call this definition d_1—in terms of P as follows:

$$d_1 : \qquad \forall x \forall y (Q(x, y) \leftrightarrow (P(x, y) \wedge x \neq y)).$$

Call the extended theory $T_1 + d_1$. Then $T_1 + d_1 \vdash \forall x \neg Q(x, x)$.

Let T_2 be the theory saying that Q is an irreflexive relation:

$$T_2 : \qquad \forall x \neg Q(x, x).$$

Consider following "*inverse definition*" of P in terms of Q, call it d_2:

$$d_2 : \qquad \forall x \forall y (P(x, y) \leftrightarrow (Q(x, y) \vee x = y)).$$

Call the extended theory $T_2 + d_2$. Then $T_2 + d_2 \vdash \forall x P(x, x)$. Moreover, we have:

$$T_1 + d_1 \vdash d_2 \ \text{ and } \ T_2 + d_2 \vdash d_1.$$

These two theories, T_1 and T_2, are in fact *equivalent* with respect to these definitions. That is, $T_1 + d_1$ and $T_2 + d_2$ are *logically equivalent*:

$$\boxed{T_1 + d_1 \equiv T_2 + d_2.}$$

We say that T_1 and T_2 have a "common definitional extension" and are therefore "definitionally equivalent" (see Definition 27 below).

As the reader may have guessed, this example derives from the standard mathematical method of passing between a non-strict preorder $\leq$ and its corresponding strict preorder $<$: they may be defined in terms of each other. To put the above in the more standard notation:

$$\{\forall x (x \leq x), \forall x \forall y (x < y \leftrightarrow (x \leq y \wedge x \neq y))\}$$
$$\equiv \{\forall x \neg (x < x), \forall x \forall y (x \leq y \leftrightarrow (x < y \vee x = y))\}.$$

In effect, the explicit definitions d_1 and d_2 are "mutual inverses". This notion will play a major role later.

Moving on to increasingly sophisticated examples, there are equivalent reformulations of Peano arithmetic PA with different primitives from the usual signature $\{0, S, +, \times\}$. For example, one might take exponentiation as the single basic notion, along with certain axioms for exponentiation, along with explicit definitions for $0, S, +$ and $\times$ (along with induction). The result is equivalent to PA.

As is well-known, one can interpret a modification we shall call ZF_0 of ZF set theory, obtained by removing the axiom of infinity, adding its negation, along with an additional axiom of "transitive containment", into PA. Moreover, the inverse of this interpretation interprets PA into ZF_0.[1] So PA is *definitionally equivalent* to the theory of *finite sets* ZF_0.

[1] A detailed analysis and proof of this "mathematical folklore" result is given in Kaye and Wong [13], who show that there exist translation functions between PA and ZF_0 which are mutual inverses (Theorem 20 of their paper). A fairly detailed discussion of the equivalence of Peano arithmetic and finite set theory ZF_0 may also be found in the lecture notes Andréka and Németi [2], pp. 41–45.

A similar relationship holds between the theory of *formalized syntax* and PA. Suppose S_A is the theory of concatenation for strings from alphabet A, with $|A| \geq 2$, and with the appropriate induction principle. Then S_A is *definitionally equivalent* to PA.[2]

There are other examples—from mathematics, logic and philosophy of science.[3] To return to the broader point, the transformations between such equivalent formulations are rather like "basis transformations" in linear algebra and other parts of mathematics. In this paper and the follow-up, an analogous idea is investigated.[4]

2. Syntax, Structures and Theories

Throughout, everything we consider is 1-sorted, relational and first-order.

Definition 1. Let $P = \{P_i\}_{i \in I}$ be a set and let $a : P \to \mathbb{N}$. The pair $S = (P, a)$ is called a one-sorted *signature*, and a is called the *arity* function for S. The multiset $t = (a(P_i) \mid P_i \in P)$ is called the *similarity type* of the signature S. If $a(P_i) = 0$, then P_i is called a *sentence letter* (or a *propositional atom*). The *alphabet* of S is P.

Definition 2. (P^c, a^c) is a *copy* of (P, a) iff the similarity types of (P, a) and (P^c, a^c) are the same. (P^c, a^c) is a disjoint copy of (P, a) when, in addition, $P \cap P^c = \varnothing$.

Definition 3. L_P is the first-order language over the signature $S = (P, a)$, where each symbol P_i is a primitive relation symbol of arity $a(P_i)$. We will sometimes call L_P "the P-language".

[2] A weak theory of concatenation without the induction principle is now usually called TC (Grzegorczyk [8]), and the precise interpretability relationship of TC and Robinson arithmetic Q has recently been clarified. Although similar systems had been studied before (Quine [15]; Tarski et al. [17]), the undecidability of TC is demonstrated in Grzegorczyk [8]; in Grzegorczyk and Zdanowski [9], the essential undecidability of TC is demonstrated through an interpretation of TC into Robinson arithmetic Q. Subsequently, Visser and Sterken [18], Ganea [7] and Švejdar [16] demonstrated the interpretability of Q into TC.

[3] As another example, there is the theory CEM, of "classical extensional mereology", in a language with basic binary relation symbol $\preceq$ ($x \preceq y$ means "x is part of y"). There is a definitionally equivalent theory I shall call F (for "fusions") with a basic binary operation symbol $\oplus$ (where $x \oplus y$ can be read "the fusion of x and y"). The detailed formulation and proof of definitional equivalence are given in Ketland and Schindler [12].

[4] Often, we have translations/interpretations $T_1 \to T_2$ and $T_2 \to T_1$, for theories which appear, on the surface, to be quite different. But it is not automatically true that T_1 and T_2 are definitionally equivalent: the translations involved must be *mutual inverses* of each other. (See our Theorem 2 below.) A valuable discussion of this point, and a criterion for it to hold, is Friedman and Visser [6], who show that when two theories are "bi-interpretable via identity-preserving interpretations", then they are definitionally equivalent (Sect. 5 of their paper). Moreover, they give an example of two finitely axiomatized sequential theories that are bi-interpretable but not definitionally equivalent (Sect. 7 of their paper). An example of a pair of theories which "define each others' models" but which are not definitionally equivalent is given in Andréka et al. [1].

Definition 4. Let

$$\Phi = \{\phi_1, \dots\} = \{\phi_i\}_{i \in I_\Phi}$$

be a set of L_P-formulas, which will be called *defining formulas*. Given the ϕ_i, a corresponding set

$$Q = \{Q_i\}_{i \in I_\Phi}$$

of new relation symbols Q_i is introduced, such that the arity of each Q_i matches the arity of its corresponding ϕ_i. The new language L_Q will sometimes be called "the Q-language". The combined language is then called $L_{P,Q}$.

A theory T in L is a set of L-sentences. When we require deductive closure, we say so. We use a deductive system such that, if $\Delta \vdash \alpha$, then $\Delta \vdash \forall x \alpha$, so long as x doesn't appear free in any formulas in Δ.[5] $T \vdash \alpha$ means: there exists a derivation of α from the axioms/rules of T. An L-structure A will always interpret all *variables*. So, we can always write $A \models \alpha$, even where α has free variables, since A will assign a value x^A to each variable $x \in \mathrm{FV}(\alpha)$.[6] The Completeness Theorem holds in the usual form: $\Delta \vdash \alpha$ iff, for any $A \models \Delta$, $A \models \alpha$. A theory T in L_P is said to be *deductively closed* iff, for all $\alpha \in \mathrm{Sent}(L_P)$, if $T \vdash \alpha$, then $\alpha \in T$. The *deductive closure* of T (written $\mathrm{DedCl}(T)$) is $\{\alpha \in \mathrm{Sent}(L_P) \mid T \vdash \alpha\}$.

So far as I can tell, nothing in this paper uses either proof theoretic methods or model theoretic methods beyond what is taught at intermediate logic.[7] We do introduce specific new terminology for the following notions:

Definition systems	d_Φ.
Definitional expansions/extensions	$A \mapsto A + d_\Phi$ and $T \mapsto T + d_\Phi$.
Definitional image operator	$A \mapsto D_\Phi A$ and $T \mapsto D_\Phi T$.
Definitional equivalence	$A \xleftrightarrow[\Theta]{\Phi} B$ and $T_1 \xleftrightarrow[\Theta]{\Phi} T_2$.

Definition 5. A *structure* A for the language L_P specifies a non-empty domain, $\mathrm{dom}(A)$; and interprets each variable x of L_P as an element $x^A \in \mathrm{dom}(A)$; and interprets each n-ary relation symbol P_i as a n-ary relation $(P_i)^A \subseteq (\mathrm{dom}(A))^n$.

Definition 6. Given signature P, let P^c be a disjoint copy of P. Let A be an L_P-structure. Then the *disjoint copy* A^c of A in L_{P^c} is defined by setting $\mathrm{dom}(A^c) = \mathrm{dom}(A)$ and, for each P_i, setting $(P_i^c)^{A^c} = (P_i)^A$. Let T be an L_P-theory. Then the *disjoint copy* T^c of T in L_{P^c} is defined by replacing every occurrence of P_i, in any sentence in T by the new symbol P_i^c.

[5] The deductive system I usually have in mind is the Hilbert system set out in Machover [14] or in Enderton [5].

[6] For any variable x, the denotation of x in A is x^A. We define A^x_a to be the structure just like A, except that, for the variable x, we have $x^{A^x_a} = a$.

[7] The terminology and definitions we use largely follow those of Machover [14], Hodges [10], or Enderton [5].

We may next give inductive definitions of the *denotation* function $t \mapsto t^A$ (specifying what any term t in L_P refers to in A) and the *satisfaction* relation $\models$ between A and L_P-formulas.

Definition 7. If A_1 and A_2 are L_P-structures, then an *isomorphism*

$$f : A_1 \to A_2$$

is a bijection from $\mathrm{dom}(A_1)$ to $\mathrm{dom}(A_2)$ satisfying the preservation condition that $f[(P_i)^{A_1}] = (P_i)^{A_2}$, for each relation symbol P_i in the signature P. This is written

$$A_1 \overset{f}{\cong} A_2$$

(or simply $A_1 \cong A_2$ if the isomorphism is left implicit).

Definition 8. If A is an L_P-structure and T, T_1, T_2 are sets of L_P-sentences:

(1)	$\mathrm{Th}_{L_P}(A)$	:=	the set of L_P-sentences true in A.
(2)	$A \models T$	:=	for all $\alpha \in T$, $A \models \alpha$.
(3)	$\mathrm{Mod}_{L_P}(T)$	:=	the class of L_P-structures $A \models T$
(4)	$T_1 \equiv T_2$	:=	$\mathrm{Mod}(T_1) = \mathrm{Mod}(T_2)$.
(5)	$T_1 \vdash T_2$	:=	for all $\alpha \in T_2$, $T_1 \vdash \alpha$.

$\mathrm{Th}_{L_P}(A)$ is called $\mathrm{Th}(A)$ if it's clear from context what language is involved. Likewise, $\mathrm{Mod}_{L_P}(T)$ is called $\mathrm{Mod}(T)$ if it's clear from context what language is involved.

If T_1 and T_2 are deductively closed theories, then $T_1 \vdash T_2$ iff $T_2 \subseteq T_1$. The Completeness Theorem tells us that $T_1 \equiv T_2$ iff $T_1 \vdash T_2$ and $T_2 \vdash T_1$.

Definition 9. Let A be an L_P-structure and A^+ be an $L_{P,Q}$-structure. Then A^+ is an *expansion* of A iff for all P_i, $(P_i)^{A^+} = (P_i)^A$. This is equivalent to saying that A is a *reduct* of A^+. If A^+ is an $L_{P,Q}$-structure, its reduct to P is denoted $A^+ \restriction_P$ (an L_P-structure) and its reduct to Q is denoted $A^+ \restriction_Q$ (an L_Q-structure) and we have: $(P_i)^{A^+ \restriction_P} = (P_i)^{A^+}$ and $(Q_i)^{A^+ \restriction_Q} = (Q_i)^{A^+}$.

The central property of expansions is that the truth value of a formula in the smaller language L_P remains invariant as we pass from an L_P-structure to an expanded structure for $L_{P,Q}$: if $\alpha \in L_P$ and an $L_{P,Q}$-structure A^+ is an expansion of an L_P-structure A, then $A^+ \models \alpha$ iff $A \models \alpha$.

Definition 10. A theory T^+ is an *extension* of T iff T is a subset of T^+. Let signatures P, Q, and corresponding languages L_P, L_Q and $L_{P,Q}$ be given. An extension T^+ in $L_{P,Q}$ of T in L_P is called a *conservative* extension of T with respect to L_P-formulas iff, for any L_P-formula α,

$$T^+ \vdash \alpha \Rightarrow T \vdash \alpha$$

3. Definition

Definition 11. Given the set $\Phi = \{\phi_i\}_{i \in I}$ of L_P-formulas, we introduce a disjoint set $Q = \{Q_i\}_{i \in I}$ of new relation symbols, with card Q = card Φ, and with the arity of Q_i matching the arity of ϕ_i, and let n_i be $a(\phi_i)$. The *definition system* over Φ, which we write as,

$$d_\Phi$$

is the set of explicit definitions,

$$\forall x_1 \dots x_{n_i}(Q_i(x_1, \dots, x_{n_i}) \leftrightarrow \phi_i)$$

where $\{x_1, \dots x_{n_i}\} = \mathrm{FV}(\phi_i)$. These define the new symbols Q_i in terms of the L_P-formulas ϕ_i. We shall sometimes write $\forall \overline{x}(Q_i(\overline{x}) \leftrightarrow \phi_i)$ instead of $\forall x_1 \dots x_n(Q_i(x_1, \dots, x_n) \leftrightarrow \phi_i)$.[8]

Definition 12. If A is an L_P-structure and $\phi(x_1, \dots, x_n)$ is an L_P-formula, then

$$\phi(A) := \{(a_1, \dots, a_n) \in (\mathrm{dom}(A))^n \mid A \models \phi[a_1, \dots, a_n]\}$$

is the relation that ϕ defines in A.

Definition 13. An $L_{P,Q}$-structure A^+ is a *definitional expansion* of an L_P-structure A with respect to the definition system d_Φ just if A^+ is an expansion of A interpreting each Q_i, and $A^+ \models d_\Phi$. This ensures that $(Q_i)^{A^+} = \phi_i(A^+)$, for each Q_i.

Given any L_P-structure A, it is clear that there is a unique definitional expansion $A^+ \models d_\Phi$. We introduce the following notation for this expansion:

Definition 14. $A + d_\Phi :=$ the unique definitional expansion $A^+ \models d_\Phi$.

We are going to treat the definitional expansion map

$$A \mapsto A + d_\Phi$$

as a unary operator $+d_\Phi$ (indexed by Φ), taking us from L_P-structures to $L_{P,Q}$-structures. It is clear that it is well-defined (i.e., unique, given Φ). It also satisfies the following useful "right cancellation" law (this amounts, in essence, to taking a reduct):

Lemma 1. *If* $A_1 + d_\Phi \cong A_2 + d_\Phi$ *then* $A_1 \cong A_2$.

Definition 15. A relation symbol P_i in the signature P is *explicitly definable* in T just if

$$T \vdash P_i(\overline{x}) \leftrightarrow \theta$$

for some formula θ in the language of the subsignature $P \backslash P_i$. We say that θ is a *defining formula* for P_i in T.

[8] As explained below, d_Θ will be an abbreviation for the set of definitions of the form $\forall \overline{x}(P_i(\overline{x}) \leftrightarrow \theta_i)$, where the θ_i are L_Q-formulas.

Definition 16. A relation symbol P_i in the signature P is *implicitly definable* in T in L_P just in case, given any pair of models $A, B \models T$, with $\text{dom}(A) = \text{dom}(B)$ and which assign the same extension to all P_j *except* P_i, we have $(P_i)^A = (P_i)^B$.

Beth's Theorem states that a relation symbol P_i is implicitly definable in T iff P_i is explicitly definable in T.[9]

Definition 17. Given a definition system d_Φ, *the definitional extension* of T wrt Φ is $T + d_\Phi$. We say that T^+ in $L_{P,Q}$ is a *definitional extension* of T in L_P just if

$$T^+ \equiv T + d_\Phi,$$

for some definition system d_Φ, where Φ is some set of L_P-formulas.

So, each new relation symbol Q_i is explicitly defined in $T + d_\Phi$ and the defining formula for $Q_i(\overline{x})$ is simply ϕ_i. Analogous to what we did with structures, we are going to treat the definitional extension map

$$T \mapsto T + d_\Phi$$

for theories as a unary operator $+d_\Phi$ (indexed by Φ), taking us from L_P-theories to $L_{P,Q}$-theories. Again, it is well-defined (i.e., unique, given Φ) and satisfies an analogous "right cancellation" law for L_P-theories:

Lemma 2. *The following are straightforward consequences of the definitions:*

(1) *If* $T_1 + d_\Phi \equiv T_2 + d_\Phi$ *then* $T_1 \equiv T_2$.

(2) $A \models T$ *iff* $A + d_\Phi \models T + d_\Phi$.

(3) *If* $B \models T + d_\Phi$, *then* $B = A + d_\Phi$, *for some* $A \models T$.

Before moving on to translations, we give three standard lemmas about definitional and conservative extensions (the converse of Lemma 3 is far from being true):

Lemma 3. *If* T^+ *in* $L_{P,Q}$ *is a definitional extension of* T *in* L_P, *then* T^+ *is a conservative extension of* T *wrt* L_P*-formulas.*

Lemma 4. $T + d_\Phi$ *is a conservative extension of* T *for* L_P*-formulas.*

Lemma 5. *Let* T *in* L_P *and* T^+ *in* $L_{P,Q}$ *be such that* $T \subseteq T^+$. *Suppose that, for any model* $A \models T$, *there is an expansion* $A^+ \models T^+$. *Then* T^+ *is a conservative extension of* T *for* L_P*-formulas.*

[9]For a proof, based on Craig's interpolation lemma, see Boolos and Jeffrey [3], Ch. 25.

4. Translation

Definition 18. Let a definition system d_Φ be given. Define the *translation*, induced by Φ

$$\tau_\Phi : L_Q \to L_P$$

as follows. For symbols Q_i, variables $x, y, \overline{x}$, and for L_Q-formulas $\alpha, \alpha_1, \alpha_2$: #

(1)	$\tau_\Phi(Q_i(\overline{x}))$	$:=$	$(\phi_i)'$
(2)	$\tau_\Phi(x = y)$	$:=$	$(x = y)$
(3)	$\tau_\Phi(\neg\alpha)$	$:=$	$\neg\tau_\Phi(\alpha)$
(4)	$\tau_\Phi(\alpha_1 \# \alpha_2)$	$:=$	$\tau_\Phi(\alpha_1)\#\tau_\Phi(\alpha_2)$
(5)	$\tau_\Phi(\mathbf{q}x\alpha)$	$:=$	$\mathbf{q}x\tau_\Phi(\alpha)$.

is any binary connective, $\mathbf{q}$ is a quantifier and $(\phi_i)'$ is the result of ensuring that the free variables appearing ϕ_i are relabelled, to match those of $Q_i(\overline{x})$. We call τ_Φ the *translation* induced by Φ. It maps from the new language L_Q back to the original language L_P.[10]

Lemma 6. *For any $\alpha, \beta \in L_Q$, if $\alpha \vdash \beta$ then $\tau_\Phi(\alpha) \vdash \tau_\Phi(\beta)$.*

Proof. We will prove this using a lemma below. Suppose $\tau_\Phi(\alpha) \nvdash \tau_\Phi(\beta)$. This gives us a model $A \models \tau_\Phi(\alpha)$ and $A \not\models \tau_\Phi(\beta)$. By Lemma 15(1) below, $D_\Phi A \models \alpha$ and $D_\Phi A \not\models \beta$. So, $\alpha \nvdash \beta$. □

Lemma 6 is a general property of translations, but its converse is not true in general.

Corresponding to a translation $\tau_\Phi : L_Q \to L_P$ is its "lift" $\tau_\Phi^+ : L_{P,Q} \to L_P$ from the combined language $L_{P,Q}$ down to L_P:

Definition 19. Let Φ be given, along with definition system d_Φ. Define the *lifted translation* τ_Φ^+ induced by Φ

$$\tau_\Phi^+ : L_{P,Q} \to L_P$$

as follows. For symbols Q_i, P_i, variables $x, y, \overline{x}$: Along with the requirement

(1)	$\tau_\Phi^+(Q_i(\overline{x}))$	$:=$	$(\phi_i)'$
(2)	$\tau_\Phi^+(x = y)$	$:=$	$(x = y)$
(3)	$\tau_\Phi^+(P_i(\overline{x}))$	$:=$	$P_i(\overline{x})$

that τ_Φ^+ commutes with the logical operators on the full language $L_{P,Q}$.

Thus, the translation τ_Φ is the *restriction* to L_Q of its lift, τ_Φ^+.

Note that because the translations we are interested in always act as the identity on equations, it is always the case that $\vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$ if α is an equation. Thus, in inductive proofs establishing biconditionals of the form $\alpha \leftrightarrow \tau_\Phi(\alpha)$, we only need to check the condition holds for atomic formulas which are not identity formulas.

[10] The clauses (3)–(5) are usually read as saying "τ_Φ commutes with the logical operators".

Lemma 7. *We have:*

(1)	$A + d_\Phi \models \alpha \leftrightarrow \tau_\Phi(\alpha)$	*for any* $\alpha \in L_Q$.
(2)	$A + d_\Phi \models \alpha \leftrightarrow \tau^+_\Phi(\alpha)$	*for any* $\alpha \in L_{P,Q}$.
(3)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$	*for any* $\alpha \in L_Q$.
(4)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau^+_\Phi(\alpha)$	*for any* $\alpha \in L_{P,Q}$.

Proof. For (2), we reason by induction. Let α be an atomic $L_{P,Q}$-formula. As noted above, for equations $(x = y)$, the translation $\tau_\Phi(x = y)$ is trivially $(x = y)$: so $\alpha \leftrightarrow \tau_\Phi(\alpha)$ always holds for equations. Suppose α is an atomic formula of the form $Q_i(\overline{x})$. Then its translation $\tau^+_\Phi(\alpha)$ is ϕ_i. Since we have $A + d_\Phi \models \forall\overline{x}(Q_i(\overline{x}) \leftrightarrow \phi_i)$, we have: $A + d_\Phi \models Q_i(\overline{x}) \leftrightarrow \tau_\Phi(Q_i(\overline{x}))$. Instead let α be $P_i(\overline{x})$. Then its translation $\tau^+_\Phi(\alpha)$ is simply α. So, $A + d_\Phi \models \alpha \leftrightarrow \tau^+_\Phi(\alpha)$. The other cases are shown by induction on the construction of α.

For (1), the result follows from (2), by restricting to L_Q-formulas (since $\tau^+_\Phi(\alpha) = \tau_\Phi(\alpha)$ for $\alpha \in L_Q$).

For (4), reasoning by induction, let α be an atomic L_P, Q-formula. If α is atomic, then the condition is trivial. Suppose α has the form $P_i(\overline{x})$. Again, he condition is trivial, since $\tau^+_\Phi(P_i(\overline{x}))$ is $P_i(\overline{x})$. Instead, suppose α has the form $Q_i(\overline{x})$. Then its translation $\tau^+_\Phi(\alpha)$ is ϕ_i. $Q_i(\overline{x}) \leftrightarrow \phi_i$ is a theorem of $T + d_\Phi$, by construction. The other cases are shown by induction on the construction of α.

For (3), the result follows from (4), by restricting to L_Q-formulas. □

Definition 20. Let $\tau_\Phi : L_Q \to L_P$ be the translation induced by d_Φ. If T_2 is a theory in L_Q, then the *image* of T_2 under τ_Φ is the set of L_P-sentences:

$$\tau_\Phi[T_2] := \{\tau_\Phi(\alpha) \in L_P \mid \alpha \in T_2\}$$

If T_1 is a theory in L_P, then the *pre-image* of T_1 under τ_Φ is the set of L_Q-sentences:

$$(\tau_\Phi)^{-1}[T_1] := \{\beta \in L_Q \mid \tau_\Phi(\beta) \in T_1\}$$

Similarly, if $\Theta = \{\theta_i\}_{i \in I_P}$ is a set of L_Q-formulas and d_Θ is the corresponding definition system over Θ (for the primitives P_i of L_P), we can define a translation

$$\tau_\Theta : L_P \to L_Q$$

by requiring that τ_Θ commute with the logical operators and, for atomic L_P-formulas: Likewise, we can also define the lifted translation $\tau^+_\Theta : L_{P,Q} \to L_Q$.

(i)	$\tau_\Theta(P_i(\overline{x})) :=$	$(\theta_i)'$
(ii)	$\tau_\Theta(x = y) :=$	$(x = y)$

Definition 21. Let $\tau : L_Q \to L_P$ be a translation. Let T_1 be a theory in L_P and T_2 be a theory in L_Q. Then we say:

(1) τ *interprets* T_2 into T_1 iff $T_1 \vdash \tau[T_2]$.

(2) τ *faithfully interprets* T_2 into T_1 iff $T_1 \equiv \tau[T_2]$.

One may compare the condition $T_1 \equiv \tau[T_2]$ with Visser's definition of faithful interpretability:

> We write $K : U \lhd_{\text{faith}} V$ for: K is a faithful interpretation of U in V. This means that: for all U-sentences A, we have: $U \vdash A$ iff $V \vdash A^{\tau_K}$. (Visser [19], p. 6).

Thus $\tau_\Phi : T_2 \lhd_{\text{faith}} T_1$ holds iff, for all $\alpha \in L_{T_2}$, we have: $T_2 \vdash \alpha$ iff $T_1 \vdash \tau_\Phi(\alpha)$. Thus, $\tau_\Phi : T_2 \lhd_{\text{faith}} T_1$ iff $T_1 \equiv \tau_\Phi[T_2]$. This establishes:

Lemma 8. *τ_Φ faithfully interprets T_2 into T_1 iff, for all $\alpha \in L_Q$, we have: $T_2 \vdash \alpha$ iff $T_1 \vdash \tau_\Phi(\alpha)$.*

Definition 22. Let $\tau_\Phi : L_Q \to L_P$ and $\tau_\Theta : L_P \to L_Q$ be translations induced by d_Φ and d_Θ. Let T_1 be an L_P theory. Let T_2 be an L_Q theory. Then τ_Θ is an *right inverse* of τ_Φ in T_1 iff, for any $\alpha \in L_P$,

$$T_1 \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$$

We write this more suggestively as:

$$(\tau_\Phi \tau_\Theta = 1)_{T_1}$$

And τ_Θ is an *left inverse* of τ_Φ in T_2 iff, for any $\beta \in L_Q$,

$$T_2 \vdash \beta \leftrightarrow \tau_\Theta(\tau_\Phi(\beta))$$

Likewise, we write this more suggestively as:

$$(\tau_\Theta \tau_\Phi = 1)_{T_2}$$

The following two lemmas are easy to prove, and yet hold to the key to much that follows. Both lemmas use "invertibility conditions", of the form:

$$A + d_\Phi \models d_\Theta$$
$$T + d_\Phi \vdash d_\Theta.$$

As we see later, these conditions express a very strong constraint on the set Φ of defining L_P-formulas involved—the property of being a "*representation basis*" for A (or T) with inverse Θ.

Lemma 9. *Suppose that $A + d_\Phi \models d_\Theta$. Then:*

(1)	$A + d_\Phi \models \alpha \leftrightarrow \tau_\Phi(\alpha)$	*for* $\alpha \in L_Q$.
(2)	$A + d_\Phi \models \alpha \leftrightarrow \tau^+_\Phi(\alpha)$	*for* $\alpha \in L_{P,Q}$.
(3)	$A + d_\Phi \models \alpha \leftrightarrow \tau_\Theta(\alpha)$	*for* $\alpha \in L_P$.
(4)	$A + d_\Phi \models \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$	*for* $\alpha \in L_Q$.
(5)	$A + d_\Phi \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$	*for* $\alpha \in L_P$.
(6)	$A + d_\Phi \models \tau_\Phi(\theta_i) \leftrightarrow \theta_i$.	
(7)	$A + d_\Phi \models \tau_\Theta(\phi_i) \leftrightarrow \phi_i$.	
(8)	$A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$	*for* $\alpha \in L_P$.
(9)	$A + d_\Phi \models \alpha \leftrightarrow \tau^+_\Theta(\alpha)$	*for* $\alpha \in L_{P,Q}$.

Proof. Claims (1) and (2) are already established in Lemma 7(1,2) and do not need the side condition. (They are included for convenience of reference.)

For (3), the proof is analogous to the proof of Lemma 7(1), but using the fact that $A+d_\Phi \models d_\Theta$. Reasoning by induction, let α be an atomic L_P-sentence, say $P_i(\overline{x})$. Then its translation $\tau_\Theta(\alpha)$ is θ_i. And $A + d_\Phi \models \forall\overline{x}(P_i(\overline{x}) \leftrightarrow \theta_i)$. So $A + d_\Phi \models P_i(\overline{x}) \leftrightarrow \theta_i$. That is, $A + d_\Phi \models P_i(\overline{x}) \leftrightarrow \tau_\Theta(P_i(\overline{x}))$. The other cases are shown by induction on the construction of α.

For (4), we already have that $A + d_\Phi \models \alpha \leftrightarrow \tau_\Phi(\alpha)$, for any $\alpha \in L_Q$. But $\tau_\Phi(\alpha)$ is an L_P-formula. So, by (3), we have: $A + d_\Phi \models \tau_\Phi(\alpha) \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$. So, $A + d_\Phi \models \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$, as required.

For (5), the reasoning is analogous to that for (4). And (6) and (7) are merely applications of (1) and (3).

For (8), using (5), we have $A + d_\Phi \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$, for any $\alpha \in L_P$. But $A + d_\Phi$ is an expansion of the L_P-structure A, and $\alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$ is an L_P-formula. Thus, $A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$.

For (9), the proof is analogous to the proof of (2), but is applied to the "lift" $\tau^+_\Theta : L_{P,Q} \to L_Q$ of τ_Θ. Reasoning by induction, let α be an atomic L_P-sentence, say $P_i(\overline{x})$. Then its translation $\tau^+_\Theta(\alpha)$ is θ_i. And $A+d_\Phi \models P_i(\overline{x}) \leftrightarrow \theta_i$, since $A + d_\Phi \models d_\Theta$. Instead, let α be an atomic L_Q-sentence, say $Q_i(\overline{x})$. Then its translation $\tau^+_\Theta(\alpha)$ is $Q_i(\overline{x})$. Trivially, $A + d_\Phi \models Q_i(\overline{x}) \leftrightarrow Q_i(\overline{x})$. The other cases are shown by induction on the construction of α. □

The following lemma, and the corresponding proofs, is a near repetition of the previous one, except that it deals with theories:

Lemma 10. *Suppose that* $T + d_\Phi \vdash d_\Theta$. *Then:*

(1)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$	*for* $\alpha \in L_Q$.
(2)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau^+_\Phi(\alpha)$	*for* $\alpha \in L_{P,Q}$.
(3)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Theta(\alpha)$	*for* $\alpha \in L_P$.
(4)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$	*for* $\alpha \in L_Q$.
(5)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$	*for* $\alpha \in L_P$.
(6)	$T + d_\Phi \vdash \tau_\Phi(\theta_i) \leftrightarrow \theta_i$.	
(7)	$T + d_\Phi \vdash \tau_\Theta(\phi_i) \leftrightarrow \phi_i$.	
(8)	$T \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$	*for* $\alpha \in L_P$.
(9)	$T + d_\Phi \vdash \alpha \leftrightarrow \tau^+_\Theta(\alpha)$	*for* $\alpha \in L_{P,Q}$.

Proof. Essentially, a repetition of the proofs for Lemma 9. □

Lemma 11. *Suppose* T_1 *is an* L_P*-theory and* T_2 *is an* L_Q*-theory. Then:*

(1)	*If* $(\tau_\Phi\tau_\Theta = 1)_{T_1}$ *then* $T_1 + d_\Phi \vdash d_\Theta$.
(2)	*If* $(\tau_\Theta\tau_\Phi = 1)_{T_2}$ *then* $T_2 + d_\Theta \vdash d_\Phi$.

Proof. For (1), we assume τ_Θ is a right inverse of τ_Φ in T_1. I.e., for any $\alpha \in L_P$, $T_1 \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$. Thus, $T_1 \vdash P_i(\overline{x}) \leftrightarrow \tau_\Phi(\tau_\Theta(P_i(\overline{x})))$. Since $\tau_\Theta(P_i(\overline{x}))$ is θ_i, we have $T_1 \vdash P_i(\overline{x}) \leftrightarrow \tau_\Phi(\theta_i)$. Taking the definitional extension, $T_1 + d_\Phi \vdash P_i(\overline{x}) \leftrightarrow \tau_\Phi(\theta_i)$. Now, for any L_Q-formula β, we have $T_1 + d_\Phi \vdash \beta \leftrightarrow \tau_\Phi(\beta)$ from Lemma 7(2). So, since $\theta_i \in L_Q$, $T_1 + d_\Phi \vdash \theta_i \leftrightarrow \tau_\Phi(\theta_i)$. Thus, $T_1 + d_\Phi \vdash P_i(\overline{x}) \leftrightarrow \theta_i$, as required.

We obtain (2) by relabelling everything (T_2 is now a theory in L_Q). □

Lemma 11 says that if τ_Θ is a right-inverse of τ_Φ, relative to T, then every relation symbol P_i from the original language can be explicitly defined from the θ_i. In a sense, the original definition system, d_Φ is a kind of *inverse* of d_Θ.

5. Definitional Images

Definition 23. Let A be an L_P-structure. Then the L_Q-structure $D_\Phi A$ is defined by:

$$D_\Phi A := (A + d_\Phi) \restriction_{L_Q}$$

$D_\Phi A$ is called the *definitional image* of A with respect to Φ.

Immediately, we see that the following three conditions provide an equivalent characterization of $D_\Phi A$:

(1)	$\text{dom}(D_\Phi A) = \text{dom}(A)$
(2)	$x^{D_\Phi A} = x^A$, for any variable x
(3)	$(Q_i)^{D_\Phi A} = \phi_i(A)$, for each new symbol Q_i.

Lemma 12. *For any* $\alpha \in L_Q$: $\alpha(D_\Phi A) = \alpha(A + d_\Phi)$.

Although $A + d_\Phi$ is by construction a definitional expansion of A, it is by no means automatically true that $A + d_\Phi$ is a definitional expansion of $D_\Phi A$. The requirement for this to hold is that each primitive relation $(P_i)^A$ be definable in $D_\Phi A$, by some formula, say θ_i.

Turning to theories, we introduce analogous concepts:

Definition 24. The *definitional image* of T, with respect to Φ, is the restriction of the deductive closure of $T + d_\Phi$ to the new language L_Q. The definitional image of T with respect to Φ is denoted $D_\Phi T$. That is,

$$D_\Phi T := \text{DedCl}(T + d_\Phi) \restriction_{L_Q} = \{\beta \in L_Q \mid T + d_\Phi \vdash \beta\}$$

The definitional image $D_\Phi T$ of a theory T in L_P is, essentially, the *pre-image* $(\tau_\Phi)^{-1}[T]$ of T under the translation τ_Φ induced by Φ:

Lemma 13. $(\tau_\Phi)^{-1}[T] \equiv D_\Phi T$.

Proof. Suppose $D_\Phi T \vdash \beta$, for $\beta \in L_Q$. Thus, $T + d_\Phi \vdash \beta$. But $T + d_\Phi \vdash \beta \leftrightarrow \tau_\Phi(\beta)$, by Lemma 10(1). So, $T + d_\Phi \vdash \tau_\Phi(\beta)$. Since $T + d_\Phi$ conservatively extends T for L_P-formulas, $T \vdash \tau_\Phi(\beta)$. Hence, $\beta \in (\tau_\Phi)^{-1}[T]$. And so $(\tau_\Phi)^{-1}[T_1] \vdash \beta$. Conversely, suppose $\beta \in (\tau_\Phi)^{-1}[T]$. So, $T \vdash \tau_\Phi(\beta)$. So, $T + d_\Phi \vdash \tau_\Phi(\beta)$. But $T + d_\Phi \vdash \beta \leftrightarrow \tau_\Phi(\beta)$, by Lemma 10(1) again. Thus, $D_\Phi T \vdash \beta$. □

Definition 25. Let A be an L_P-structure, B an L_Q-structure, T_1 an L_P-theory and T_2 an L_Q-theory. Then we say:

(1)	Φ *defines* B in A iff $B \cong D_\Phi A$.
(2)	Φ *proof-theoretically defines* T_2 in T_1 iff $T_2 \equiv D_\Phi T_1$.
(3)	Φ *model-theoretically defines* T_2 in T_1 iff $\text{Mod}(T_2) = D_\Phi[\text{Mod}(T_1)]$.

The second of these, (2), amounts to saying that $T_2 \vdash D_\Phi T_1$ and $D_\Phi T_1 \vdash T_2$. The third is equivalent to saying that

$$\text{Mod}(T_2) = (\text{Mod}(T_1 + d_\Phi)) \restriction_{L_Q} .$$

Note that the restriction $\restriction_{L_Q}$ is taken *after* the models are extracted from the definitional extension $T_1 + d_\Phi$. If the restriction is taken first, we get the rather different set $\text{Mod}((\text{DedCl}(T_1 + d_\Phi)) \restriction_{L_Q})$ of models: i.e., $\text{Mod}(D_\Phi T_1)$. Indeed, this is generally a superset of $D_\Phi[\text{Mod}(T)]$. As Lemma 15(3) will show, we have: $D_\Phi[\text{Mod}(T)] \subseteq \text{Mod}(D_\Phi T)$.

6. Some Book-Keeping Lemmas

We next provide several groups of “book-keeping” lemmas about translations and definitional images. The first, Lemma 14, primarily concerns theories. The second group, Lemma 15, concerns semantics and models. The third group (in particular, Lemma 16) establishes five calculationally useful equivalences for the "definition invertibility condition" on structures:

$$\boxed{A + d_\Phi \models d_\Theta.}$$

The fourth group establishes some analogous results for theories, including the main equivalence (Lemma 19) for the "definition invertibility condition" on theories:

$$\boxed{T + d_\Phi \models d_\Theta.}$$

Lemma 14. *Each of the following is true:*

(1)	$T + d_\Phi$ *is a conservative extension of* $D_\Phi T$ *for* L_Q*-formulas.*
(2)	$D_\Phi T_1 \vdash T_2$ *iff* $T_1 \vdash \tau_\Phi[T_2]$.
(3)	τ_Φ *interprets* T_2 *into* T_1 *iff* $D_\Phi T_1 \vdash T_2$.
(4)	τ_Φ *interprets* $D_\Phi T$ *into* T.
(5)	$D_\Phi T = \{\beta \in L_Q \mid T \vdash \tau_\Phi(\beta)\}$.
(6)	$D_\Phi T \vdash \beta$ *iff* $T \vdash \tau_\Phi(\beta)$ *(for any* $\beta \in L_Q$*).*
(7)	$T_2 \equiv D_\Phi T_1$ *iff, for all* $\beta \in L_Q$*:* $T_2 \vdash \beta$ *iff* $T_1 \vdash \tau_\Phi(\beta)$.
(8)	$T_2 \equiv D_\Phi T_1$ *iff* τ_Φ *faithfully interprets* T_2 *into* T_1.

Proof. To establish (1), note that $D_\Phi T$ is simply the restriction of the deductive closure of $T + d_\Phi$ to L_Q-sentences. So if $T + d_\Phi \vdash \alpha$, where $\alpha \in L_Q$, then $D_\Phi T \vdash \alpha$, as required. Statement (2) simply expresses the relationship between images and pre-images. Statement (3) is an immediate corollary of (2). Statement (4) is an immediate corollary of (3).

For (5), note that $D_\Phi T = (\tau_\Phi)^{-1}[T]$, from Lemma 13 above. Thus, $D_\Phi T = \{\beta \in L_Q \mid T \vdash \tau_\Phi(\beta)\}$. For (6), let $\beta \in L_Q$. Then, from (5), we have $D_\Phi T \vdash \beta$ iff $T \vdash \tau_\Phi(\beta)$, as required.

For (7), suppose first that $T_2 \equiv D_\Phi T_1$. So, for all $\beta \in L_Q$, $T_2 \vdash \beta$ iff $D_\Phi T_1 \vdash \beta$. But from (6), $D_\Phi T_1 \vdash \beta$ iff $T_1 \vdash \tau_\Phi(\beta)$. So, for all $\beta \in L_Q$, $T_2 \vdash \beta$ iff $T_1 \vdash \tau_\Phi(\beta)$, as required.

Instead suppose that for all $\beta \in L_Q$, we have: $T_2 \vdash \beta$ iff $T_1 \vdash \tau_\Phi(\beta)$. Then reason as follows: So, $T_2 \equiv D_\Phi T_1$, as required.

$T_2 \vdash \beta$	iff $T_1 \vdash \tau_\Phi(\beta)$ (from the hypothesis)
	iff $T_1 + d_\Phi \vdash \beta$ (as $T_1 + d_\Phi \vdash \beta \leftrightarrow \tau_\Phi(\beta)$; Lemma 10(1))
	iff $D_\Phi T_1 \vdash \beta$ (from the definition of $D_\Phi T_1$; i.e., Definition 24).

(8) is an immediate corollary of (7) and Lemma 8. □

Lemma 15. *Each of the following holds:*

(1)	*For any* $\alpha \in L_Q$*:* $D_\Phi A \models \alpha$ *iff* $A \models \tau_\Phi(\alpha)$.
(2)	*If* $A \models T$*, then* $D_\Phi A \models D_\Phi T$.
(3)	$D_\Phi[Mod(T)] \subseteq Mod(D_\Phi T)$.
(4)	*Let* T_2 *be a theory in* L_Q*. If* $A + d_\Phi \models T_2$ *then* $D_\Phi A \models T_2$.

Proof. (1): By induction. Suppose $\alpha = Q_i(\overline{x})$. Thus,

$$\begin{aligned} D_\Phi A \models \alpha &\Leftrightarrow ((x_1)^{D_\Phi A}, \ldots, (x_n)^{D_\Phi A}) \in (Q_i)^{D_\Phi A} \\ &\Leftrightarrow ((x_1)^A, \ldots, (x_n)^A) \in \phi_i(A) \\ &\Leftrightarrow A \models \phi_i \\ &\Leftrightarrow A \models \tau_\Phi(Q_i(\overline{x})) \\ &\Leftrightarrow A \models \tau_\Phi(\alpha). \end{aligned}$$

The case of equality, and the induction steps are routine.

(2): Let $A \models T$ and $D_\Phi T \vdash \alpha$, for some $\alpha \in L_Q$. We want to show $D_\Phi A \models \alpha$. We do this using the translation τ_Φ. Since $D_\Phi T \vdash \alpha$, it follows, by Lemma 14(4), that $T \vdash \tau_\Phi(\alpha)$. And so, $A \models \tau_\Phi(\alpha)$. From (1), $D_\Phi A \models \alpha$ iff $A \models \tau_\Phi(\alpha)$. So, $D_\Phi A \models \alpha$, as required.

(3): Let $B \in D_\Phi[\text{Mod}(T)]$. So, there is some $A \models T$ with $B \cong D_\Phi A$. Since $A \models T$, we have $D_\Phi A \models D_\Phi T$, by (2). So, $B \in \text{Mod}(D_\Phi T)$.

(4): Suppose that for any β such that $T_2 \vdash \beta$, we have $A + d_\Phi \models \beta$. We want to show that for any β such that $T_2 \vdash \beta$, we have $D_\Phi A \models \beta$. So, let $T_2 \vdash \beta$ and $A + d_\Phi \models \beta$. By Lemma 7(1), $A + d_\Phi \models \tau_\Phi(\beta)$. So, $A \models \tau_\Phi(\beta)$. By (1), $D_\Phi A \models \beta$. ☐

Neither the converse of Lemma 15(2) nor the converse of the inclusion in Lemma 15(3) is true.

Lemma 16. *The following are equivalent:*

(1)	$A + d_\Phi \models d_\Theta$.
(2)	*For all* P_i*, for all* Q_j*:* $(P_i)^{A+d_\Phi} = \theta_i(A + d_\Phi)$ *&* $(Q_j)^{A+d_\Phi} = \phi_j(A + d_\Phi)$.
(3)	$A + d_\Phi \cong D_\Phi A + d_\Theta$.
(4)	$D_\Theta D_\Phi A = A$.
(5)	$A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$ *(with* $\alpha \in L_P$*)*

Proof. For (1) ⇔ (2). This is simply unwinding the definitions.

For (1) ⇒ (3), suppose $A + d_\Phi \models d_\Theta$. Then, from (2), we have, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = \theta_i(A + d_\Phi) \ \& \ (Q_j)^{A+d_\Phi} = \phi_j(A + d_\Phi).$$

So, since $\theta_i(A + d_\Phi) = \theta_i(D_\Phi A + d_\Theta) = (P_i)^{D_\Phi A + d_\Theta}$, we have, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = (P_i)^{D_\Phi A+d_\Theta} \ \& \ (Q_j)^{A+d_\Phi} = \phi_j(A+d_\Phi).$$

So, since $\phi_j(A+d_\Phi) = (Q_j)^{D_\Phi A+d_\Theta}$, we have, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = (P_i)^{D_\Phi A+d_\Theta} \ \& \ (Q_j)^{A+d_\Phi} = (Q_j)^{D_\Phi A+d_\Theta}.$$

So, $A + d_\Phi \cong D_\Phi A + d_\Theta$, as required.

For (3) $\Rightarrow$ (1). This reverses the reasoning (1) $\Rightarrow$ (3). Suppose $A + d_\Phi \cong D_\Phi A + d_\Theta$. That is, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = (P_i)^{D_\Phi A+d_\Theta} \ \& \ (Q_j)^{A+d_\Phi} = (Q_j)^{D_\Phi A+d_\Theta}.$$

Now $(Q_j)^{D_\Phi A+d_\Theta} = (Q_j)^{D_\Phi A} = \phi_j(A) = \phi_j(A+d_\Phi)$. So, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = (P_i)^{D_\Phi A+d_\Theta} \ \& \ (Q_j)^{A+d_\Phi} = \phi_j(A+d_\Theta).$$

But trivially, $D_\Phi A+d_\Theta \models d_\Theta$. So, $D_\Phi A+d_\Theta \models P_i(\overline{x}) \leftrightarrow \theta_i$, for each P_i. Thus,

$$(P_i)^{D_\Phi A+d_\Theta} = \theta_i(D_\Phi A + d_\Theta).$$

But we also have

$$D_\Phi A + d_\Theta \cong A + d_\Theta$$

Therefore,

$$(P_i)^{D_\Phi A+d_\Theta} = \theta_i(A + d_\Theta)$$

Thus, for all P_i, for all Q_j,

$$(P_i)^{A+d_\Phi} = \theta_i(A+d_\Theta) \ \& \ (Q_j)^{A+d_\Phi} = \phi_j(A+d_\Theta).$$

And therefore, $A + d_\Phi \models d_\Theta$, as required.

For (1) $\Rightarrow$ (4). Suppose $A + d_\Phi \models d_\Theta$. So, from (3) above, we have:

$$A + d_\Phi \cong D_\Phi A + d_\Theta$$

Therefore, $D_\Phi A + d_\Theta \models d_\Phi$. A proof analogous to that of (3) above gives us that if $B + d_\Theta \models d_\Phi$ then $B + d_\Theta \cong D_\Theta B + d_\Phi$, and so, if $D_\Phi A + d_\Theta \models d_\Phi$ then $D_\Phi A + d_\Theta \cong D_\Theta D_\Phi A + d_\Phi$. And so, we may conclude,

$$D_\Phi A + d_\Theta \cong D_\Theta D_\Phi A + d_\Phi$$

And thus,

$$A + d_\Phi \cong D_\Theta D_\Phi A + d_\Phi$$

And right cancellation gives,

$$A \cong D_\Theta D_\Phi A$$

But since the map D_Φ leaves the domain invariant, we have $A = D_\Theta D_\Phi A$.

For (4) $\Rightarrow$ (5), suppose $A = D_\Theta D_\Phi A$. We have, applying Lemma 15(1) twice, for any $\alpha \in L_P$, $\beta \in L_Q$:

$$A \models \tau_\Phi(\beta) \text{ iff } D_\Phi A \models \beta$$
$$D_\Phi A \models \tau_\Theta(\alpha) \text{ iff } D_\Theta D_\Phi A \models \alpha$$

So, for $\alpha \in L_P$,

$$A \models \tau_\Phi(\tau_\Theta(\alpha)) \text{ iff } D_\Theta D_\Phi A \models \alpha$$

But $A = D_\Theta D_\Phi A$, and so,

$$A \models \tau_\Phi(\tau_\Theta(\alpha)) \text{ iff } A \models \alpha$$

And thus,

$$A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$$

as required.

For (5) $\Rightarrow$ (1), assume $A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$. So,

$$A + d_\Phi \models P_i(\overline{x}) \leftrightarrow \tau_\Phi(\tau_\Theta(P_i(\overline{x})))$$

But $\tau_\Theta(P_i(\overline{x})) = \theta_i$. So,

$$A + d_\Phi \models P_i(\overline{x}) \leftrightarrow \tau_\Phi(\theta_i)$$

And $\theta_i \in L_Q$, and so, from Lemma 7(1),

$$A + d_\Phi \models P_i(\overline{x}) \leftrightarrow \theta_i$$

and thus,

$$A + d_\Phi \models d_\Theta$$

as required. □

Turning next to theories:

Lemma 17. *$D_\Phi T + d_\Theta \vdash \alpha \leftrightarrow \tau_\Theta(\alpha)$, for all $\alpha \in L_P$.*

Proof. We reason by induction on the construction of α. Let $\alpha = P_i(\overline{x})$. Then we have: $D_\Phi T + d_\Theta \vdash P_i(\overline{x}) \leftrightarrow \theta_i$. But $\tau_\Theta(P_i(\overline{x})) = \theta_i$. And thus, $D_\Phi T + d_\Theta \vdash P_i(\overline{x}) \leftrightarrow \tau_\Theta(P_i(\overline{x}))$. The equality case and compound cases proceed routinely. (Notice this is analogous to Lemma 10(1).) □

Lemma 18. *Suppose that $T + d_\Phi \vdash d_\Theta$. Then, for any $\alpha \in L_Q$:*

(1)	$D_\Phi T \vdash \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$
(2)	$D_\Phi T + d_\Theta \vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$.

Proof. For (1), we have, for any $\beta \in L_Q$, if $T + d_\Phi \vdash \beta$, then $D_\Phi T \vdash \beta$, since $D_\Phi T$ is simply the restriction of the set of theorems of $T + d_\Phi$ to L_Q. By Lemma 10(4), we have $T + d_\Phi \vdash \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$, for any $\alpha \in L_Q$. Now $\alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$ is also in L_Q. Thus, $D_\Phi T \vdash \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$.

For (2), from Lemma 17, we have, for any $\alpha \in L_Q$: $D_\Phi T + d_\Theta \vdash \tau_\Phi(\alpha) \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$, since $\tau_\Phi(\alpha) \in L_P$. By (1), we have $D_\Phi T + d_\Theta \vdash \alpha \leftrightarrow \tau_\Theta(\tau_\Phi(\alpha))$. Hence, $D_\Phi T + d_\Theta \vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$, as required. □

The next lemma is the most important result needed for Theorem 2 given in Sect. 8:

Lemma 19. *The following are equivalent*

(1)	$T + d_\Phi \vdash d_\Theta$.
(2)	$T + d_\Phi \equiv D_\Phi T + d_\Theta$.

Proof. (1) ⇒ (2). Suppose $T + d_\Phi \vdash d_\Theta$. We want to show

(a)	$T + d_\Phi \vdash D_\Phi T + d_\Theta$.
(b)	$D_\Phi T + d_\Theta \vdash T + d_\Phi$.

First, for (a), let $A + d_\Phi \models T + d_\Phi$. So, since $T + d_\Phi \vdash d_\Theta$, we have $A + d_\Phi \models d_\Theta$. So, by Lemma 16(3), we have $A + d_\Phi \cong D_\Phi A + d_\Theta$. Since $A \models T$, we have $D_\Phi A \models D_\Phi T$. So, $D_\Phi A + d_\Theta \models D_\Phi T + d_\Theta$. Thus, $A + d_\Phi \models D_\Phi T + d_\Theta$. And therefore, since A was arbitrary, $T + d_\Phi \vdash D_\Phi T + d_\Theta$, as required.

For (b), we want to show that $D_\Phi T + d_\Theta \vdash T + d_\Phi$. That is, for for any $L_{P,Q}$-formula α,

$$\text{If } T + d_\Phi \vdash \alpha \text{ then } D_\Phi T + d_\Theta \vdash \alpha.$$

First, note that we may relabel Lemma 9(2) in terms of some L_Q-structure B and definition system d_Θ, rather than A and d_Φ, to obtain: for any $\alpha \in L_{P,Q}$,

(i)	$B + d_\Theta \models \alpha \leftrightarrow \tau^+_\Theta(\alpha)$.

For a contradiction, suppose we have some $\alpha \in L_{P,Q}$ such that

(ii)	$T + d_\Phi \vdash \alpha$
(iii)	$D_\Phi T + d_\Theta \nvdash \alpha$.

Since $T + d_\Phi \vdash d_\Theta$, we have, from Lemma 10(9), that $T + d_\Phi \vdash \alpha \leftrightarrow \tau^+_\Theta(\alpha)$ and since $T + d_\Phi \vdash \alpha$, we have:

(iv)	$T + d_\Phi \vdash \tau^+_\Theta(\alpha)$.

From (iii), there exists an L_Q-structure $B \models D_\Phi T$ with $B + d_\Theta \not\models \alpha$. So, from (i), $B + d_\Theta \not\models \tau^+_\Theta(\alpha)$. And since $\tau^+_\Theta(\alpha) \in L_Q$, we have:

(v)	$B \not\models \tau^+_\Theta(\alpha)$.

Recall that $D_\Phi T = \{\beta \in L_Q \mid T + d_\Phi \vdash \beta\}$. So, since $B \models D_\Phi T$, it follows that, for any $\beta \in L_Q$, if $T + d_\Phi \vdash \beta$, then $B \models \beta$. Thus, if $T + d_\Phi \vdash \tau^+_\Theta(\alpha)$, then $B \models \tau^+_\Theta(\alpha)$. So, from (iv), we infer:

(vi)	$B \models \tau^+_\Theta(\alpha)$.

Contradiction.

(2) $\Rightarrow$ (1). Let $T + d_\Phi \equiv D_\Phi T + d_\Theta$. Then $T + d_\Phi \vdash D_\Phi T + d_\Theta$. Thus, $T + d_\Phi \vdash d_\Theta$, as required. □

Lemma 20. *$D_\Theta D_\Phi T \vdash \alpha$ iff $T \vdash \tau_\Phi(\tau_\Theta(\alpha))$, for $\alpha \in L_P$.*

Proof. If we examine the definitions of $D_\Phi T$ and $D_\Theta T_2$ (where T_2 is in L_Q), we get

$$D_\Phi T = \{\alpha \in L_Q \mid T \vdash \tau_\Phi(\alpha)\}$$
$$D_\Theta T_2 = \{\alpha \in L_P \mid T_2 \vdash \tau_\Theta(\alpha)\}$$

Together, these imply that $D_\Theta D_\Phi T \vdash \alpha$ iff $T \vdash \tau_\Phi(\tau_\Theta(\alpha))$. □

Lemma 21. *Suppose $T + d_\Phi \vdash d_\Theta$. Then*

(1)	$D_\Phi T + d_\Theta \vdash d_\Phi$.
(2)	$D_\Theta D_\Phi T \equiv T$.

Proof. For (1), suppose $T + d_\Phi \vdash d_\Theta$. By Lemma 18(2), for any $\alpha \in L_Q$,

$$D_\Phi T + d_\Theta \vdash \alpha \leftrightarrow \tau_\Phi(\alpha)$$

Thus,

$$D_\Phi T + d_\Theta \vdash Q_i(\overline{x}) \leftrightarrow \tau_\Phi(Q(\overline{x}))$$

Thus,

$$D_\Phi T + d_\Theta \vdash Q_i(\overline{x}) \leftrightarrow \phi_i$$

as required.

For (2), suppose $T + d_\Phi \vdash d_\Theta$. We want to show that, for any $\alpha \in L_P$, we have: $D_\Theta D_\Phi T \vdash \alpha$ iff $T \vdash \alpha$. By Lemma 20, we have

$$D_\Theta D_\Phi T \vdash \alpha \text{ iff } T \vdash \tau_\Phi(\tau_\Theta(\alpha))$$

Since $T + d_\Phi \vdash d_\Theta$, by Lemma 10(8), we have

$$T \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$$

Hence,

$$T \vdash \alpha \text{ iff } T \vdash \tau_\Phi(\tau_\Theta(\alpha))$$

And therefore,

$$D_\Theta D_\Phi T \vdash \alpha \text{ iff } T \vdash \alpha$$

as required. □

Lemma 22. *The following are equivalent:*

(1)	$T \equiv D_\Theta D_\Phi T.$
(2)	$T \vdash \alpha$ *iff* $T \vdash \tau_\Phi(\tau_\Theta(\alpha))$, *with* $\alpha \in L_P$.

Proof. (1) ⇒ (2). From the previous lemma, $D_\Theta D_\Phi T \vdash \alpha$ iff $T \vdash \tau_\Phi(\tau_\Theta(\alpha))$. So, if $T \equiv D_\Theta D_\Phi T$, we infer that $T \vdash \alpha$ iff $T \vdash \tau_\Phi(\tau_\Theta(\alpha))$, for $\alpha \in L_P$.

(2) ⇒ (1). Suppose, for $\alpha \in L_P$,

$$T \vdash \alpha \text{ iff } T \vdash \tau_\Phi(\tau_\Theta(\alpha))$$

We already have

$$D_\Theta D_\Phi T \vdash \alpha \text{ iff } T \vdash \tau_\Phi(\tau_\Theta(\alpha))$$

So, for $\alpha \in L_P$

$$D_\Theta D_\Phi T \vdash \alpha \text{ iff } T \vdash \alpha$$

Thus, $D_\Theta D_\Phi T \equiv T$. □

Notice that, in the case of structures, we have:

$$D_\Theta D_\Phi A = A \text{ then } A \models \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$$

However, unlike that case, for *theories*, we only have established:

$$\text{If } T \equiv D_\Theta D_\Phi T \text{ then } T \vdash \alpha \text{ iff } T \vdash \tau_\Phi(\tau_\Theta(\alpha))$$

So, we have not established the provability of the biconditional $\alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$ inside T itself.[11]

[11] That said, I do not have a counterexample. It is conceivable that the example given in Andréka et al. [1] might yield such a counter-example.

7. Definitional Equivalence

We next explain what it means for structures and theories to be definitionally equivalent.[12] Intuitively, a pair of structures are definitionally equivalent when they have a common definitional expansion. And a pair of theories are definitionally equivalent when they have a common definitional extension.

Throughout the next two definitions, A is an L_P-structure, $\Phi = \{\phi_i\}_{i \in I_1}$ is a set of L_P-formulas; Q is the new disjoint signature corresponding to Φ; and B is an L_Q-structure. Similarly, $\Theta = \{\theta_i\}_{i \in I_2}$ is a set of L_Q-formulas. d_Φ is the definition system of the Q_i primitives in terms of the ϕ_i, and d_Θ is the definition system the P_i primitives in terms of the θ_i. Similarly, T_1 is an L_P-theory and T_2 is an L_Q-theory. Please note that the definitions of these notions given below assume *disjoint signatures*.

Definition 26. Structures A and B are *definitionally equivalent* wrt d_Φ and d_Θ iff

$$A + d_\Phi \cong B + d_\Theta.$$

If this is so, we write:

$$A \overset{\Phi}{\underset{\Theta}{\longleftrightarrow}} B$$

Definition 27. Theories T_1 and T_2 are *definitionally equivalent* wrt d_Φ and d_Θ iff

$$T_1 + d_\Phi \equiv T_2 + d_\Theta.$$

To express this, we write:

$$T_1 \overset{\Phi}{\underset{\Theta}{\longleftrightarrow}} T_2$$

These definitions require that the signatures of A and B (or T_1 and T_2) be disjoint. But is not A obviously definitionally equivalent to itself? Is not a theory definitionally equivalent to itself? Well, one can always arrange for a pair of structures A and B in overlapping signatures to be reformulated as *copies* A^c and B^c in entirely disjoint signatures (see Definitions 2, 6 above). If A^c is a disjoint copy of A in L_{P^c}, then clearly A^c is definitionally equivalent to A with respect to the trivial definition systems: That is,

$d_P^{P^c}$:	$\forall \overline{x}(P_i^c(\overline{x}) \leftrightarrow P_i(\overline{x}))$
$d_{P^c}^P$:	$\forall \overline{x}(P_i(\overline{x}) \leftrightarrow P_i^c(\overline{x}))$.

$$A + d_P^{P^c} \cong A^c + d_{P^c}^P$$

A similar copying procedure can be adopted for theories too. If we have a theory T in L_P, and T^c is a disjoint copy of T in L_{P^c}, then clearly T^c is definitionally equivalent to T with respect to the definitions:

$$T + d_P^{P^c} \equiv T^c + d_{P^c}^P$$

[12]The concept appears to have first articulated by de Bouvère [4] and Kanger [11].

So, one can give a more general definition of definitional equivalence by first applying this copying procedure to both structures, and then applying the definition above.[13]

8. Main Results

We finally give two main results, which characterize definitional equivalence.

Theorem 1. *The following are equivalent:*

(1)	$A \underset{\Theta}{\overset{\Phi}{\longleftrightarrow}} B$
(2)	$D_\Phi A \cong B$ *and* $D_\Theta B \cong A$

Proof. (1) ⇒ (2). Let $A \underset{\Theta}{\overset{\Phi}{\longleftrightarrow}} B$. Thus, $A + d_\Phi \cong B + d_\Theta$. So,

$$A + d_\Phi \models d_\Theta$$
$$B + d_\Theta \models d_\Phi$$

Thus, by Lemma 16(3) (switching labels in the second case), we have:

$$A + d_\Phi \cong D_\Phi A + d_\Theta$$
$$B + d_\Theta \cong D_\Theta B + d_\Phi$$

Thus,

$$B + d_\Theta \cong D_\Phi A + d_\Theta$$
$$A + d_\Phi \cong D_\Theta B + d_\Phi$$

And by right cancellation,

$$B \cong D_\Phi A$$
$$A \cong D_\Theta B$$

(2) ⇒ (1). Let $D_\Phi A \cong B$ and $D_\Theta B \cong A$. Since we have two isomorphisms, we can arrange, without loss of generality, for $\text{dom}(D_\Phi A) = \text{dom}(B)$, and $\text{dom}(D_\Theta B) = \text{dom}(A)$. We want to show First, we work out the extensions

(a)	$(P_i)^{A+d_\Phi} = (P_i)^{B+d_\Theta}$
(b)	$(Q_i)^{A+d_\Phi} = (Q_i)^{B+d_\Theta}$

of P_i and Q_i in $A+d_\Phi$. For Q_i, we have $(Q_i)^{A+d_\Phi} = \phi_i(A)$. Using the assumption $D_\Theta B = A$, we have $(P_i)^{A+d_\Phi} = (P_i)^{D_\Theta B+d_\Phi}$. So, $(P_i)^{A+d_\Phi} = \theta_i(B)$. And using the assumption $D_\Phi A = B$, we have $(P_i)^{A+d_\Phi} = \theta_i(D_\Phi A)$. Summarizing:

Next working out the extensions in $B + d_\Theta$, and using the assumption $D_\Phi A = B$, we have $(P_i)^{B+d_\Theta} = (P_i)^{D_\Phi A+d_\Theta}$ and $(Q_i)^{B+d_\Theta} = (Q_i)^{D_\Phi A+d_\Theta}$. I.e., And (i) and (iii) imply (a), while (ii) and (iv) imply (b). □

[13] See Andréka and Németi [2] for a related but slightly different procedure.

(i)	$(P_i)^{A+d_\Phi} = \theta_i(D_\Phi A)$.
(ii)	$(Q_i)^{A+d_\Phi} = \phi_i(A)$.

(iii)	$(P_i)^{B+d_\Theta} = \theta_i(D_\Phi A)$.
(iv)	$(Q_i)^{B+d_\Theta} = \phi_i(A)$.

Theorem 2. *The following are equivalent:*

(1)	$T_1 \overset{\Phi}{\underset{\Theta}{\longleftrightarrow}} T_2$.
(2)	$(\tau_\Phi \tau_\Theta = 1)_{T_1}$ *and* $T_2 \equiv D_\Phi T_1$.

Proof. For (1) $\Rightarrow$ (2), suppose $T_1 \overset{\Phi}{\underset{\Theta}{\longleftrightarrow}} T_2$. Thus, $T_1 + d_\Phi \equiv T_2 + d_\Theta$. Thus, $T_1 + d_\Phi \vdash d_\Theta$.

Then, from Lemma 10(8), we have $T_1 \vdash \alpha \leftrightarrow \tau_\Phi(\tau_\Theta(\alpha))$, for $\alpha \in L_P$. I.e., $(\tau_\Phi \tau_\Theta = 1)_{T_1}$. And secondly, since $T_1 + d_\Phi \vdash d_\Theta$, we have $T_1 + d_\Phi \equiv D_\Phi T_1 + d_\Theta$, by Lemma 19. So, $T_2 + d_\Theta \equiv D_\Phi T_1 + d_\Theta$, and by right cancellation, $T_2 \equiv D_\Phi T_1$.

For (2) $\Rightarrow$ (1), suppose $(\tau_\Phi \tau_\Theta = 1)_{T_1}$ and $T_2 \equiv D_\Phi T_1$. From Lemma 11(1), we may conclude that $T_1 + d_\Phi \vdash d_\Theta$, and from this, we may conclude that $T_1 + d_\Phi \equiv D_\Phi T_1 + d_\Theta$, via Lemma 19. But since $T_2 \equiv D_\Phi T_1$, we may conclude that $T_1 + d_\Phi \equiv T_2 + d_\Theta$, as required. □

Indeed, from $T_1 + d_\Phi \equiv T_2 + d_\Theta$, we may also conclude $(\tau_\Phi \tau_\Theta = 1)_{T_1}$, $(\tau_\Theta \tau_\Phi = 1)_{T_2}$, $T_2 \equiv D_\Phi T_1$ and $T_1 \equiv D_\Theta T_2$. For example, if $T_2 + d_\Theta \vdash d_\Phi$, from Lemma 10(8), by relabelling, we have $T_2 \vdash \beta \leftrightarrow \tau_\Theta(\tau_\Phi(\beta))$, for $\beta \in L_Q$, and thus $(\tau_\Theta \tau_\Phi = 1)_{T_2}$.

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