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What types should not be

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Abstract In a series of papers Ladyman and Presnell raise an interesting challenge of providing a pre-mathematical justification for homotopy type theory. In response, they propose what they claim to be an informal semantics for homotopy type theory where types and terms are regarded as mathematical concepts. The aim of this paper is to raise some issues which need to be resolved for the successful development of their types-as-concepts interpretation.

1 Introduction

We say that a putative foundation for mathematics has justificatory autonomy if it is possible to justify its claims without first assuming some background mathematical theory, or rather, if we can give a pre-mathematical informal explanation of the meaning of the basic notions upon which the theory rests. According to this idea, to provide a justification for a mathematical theory is nothing but to develop an informal semantics that describes the intended interpretation of that theory without assuming previous mathematical knowledge. For example, the justificatory autonomy of a material set theory such as ZFC can be said to follow from the iterative conception of set (Boolos, 1971), a standard intuitive justification for the axioms of sets according to which the mathematical universe is divided into a well-ordered cumulative hierarchy of sets.

In a series of three long papers, Ladyman and Presnell (2015, 2016, 2017) raise an interesting challenge to a promising candidate for foundation of mathematics known as homotopy type theory: the standard justifications for the axioms and rules of inference of the framework appear to be largely based on sophisticated concepts from homotopy theory, whereas the primitive concepts of a supposedly autonomous foundation (in the sense being considered) should not be grounded on notions borrowed from other fields of mathematics.[1] In response to this observation, Ladyman and Presnell (2016) announced a program for developing an informal semantics for homotopy type theory based on an interpretation of types as mathematical concepts, which is used to claim that homotopy type theory can be an autonomous foundation for mathematics.

The aim of this paper is to raise some issues which need to be resolved for the successful development of Ladyman and Presnell's types-as-concepts interpretation—our primary purpose is, hence, negative. More specifically, our main criticism will be directed to the following points:

(i) It is no improvement to justify path induction in terms of two other unjustified principles;

(ii) The types-as-concepts interpretation is incompletely specified, in particular it does not explain what concepts are and how type formers are to be understood;

(iii) Under one reasonable such specification, the types-as-concepts interpretation justifies the law of excluded middle, in the unrestricted form that is incompatible with univalence;

(iv) The interpretation of universes makes it unclear, at best, as to when and whether the "concepts" in question are intensional or extensional;

(v) The proposed justification of univalence is incomplete, and there is no discussion of higher inductive types.

It is important to stress that our goal is not to completely demolish the program of Ladyman and Presnell, but to identify some crucial issues that they will have to overcome in order to make their program plausible. We believe that the concern of Ladyman and Presnell is legitimate, although, in our view, they fail to provide a clear answer as to whether or not homotopy type theory has autonomy as a foundation for mathematics.

[1] It is not the logical autonomy of homotopy type theory (whether it is possible to use it as a framework or language for mathematics without appealing to some prior theory) but its justificatory autonomy that has been questioned by Ladyman and Presnell—for briefness, 'autonomy' means 'justificatory autonomy' throughout this paper, unless stated otherwise. For a detailed discussion of various forms of autonomy (in the context of category theory), the reader is encouraged to consult Linnebo and Pettigrew (2011, §3,§5).

The remainder of this paper is structured as follows. Section 2 gives an overview of homotopy type theory and the standard informal semantics of constructive type theory known as the "meaning explanations". Section 3, the main section of the paper, examines Ladyman and Presnell's response to the challenge of autonomy and their types-as-concepts interpretation, an informal semantics which can be seen as an alternative to the meaning explanations. Section 4 briefly compares the meaning explanations and the types-as-concepts interpretation in terms of their pre-mathematical nature and which axioms and rules of inference they justify. Section 5 gives some closing remarks.

2 Homotopy type theory

In a broad sense, homotopy type theory (UFP, 2013) is a foundational language for mathematics in which equalities between terms of a certain type are explicitly treated as paths between points in a certain space. It extends the so-called "identity type" of constructive type theory (Martin-Löf, 1975) by allowing for new ways of obtaining equality proofs between terms.

The identity type of x and y in A, written $x =_A y$, is a type that can be formed for any terms x and y of type A. Traditionally, however, its introduction rule states that reflexivity refl_x is the only term of $x =_A x$ (UFP, 2013, §1.12):

$$\frac{x : A}{\mathsf{refl}_x : x =_A x}.$$

In the homotopy type theory book (UFP, 2013), the theory is presented as an augmentation of constructive type theory with two axioms for equality, namely, the univalence axiom and higher inductive types, which populate the identity type with new terms (other than refl_x) for obtaining equality proofs.

This conventional presentation of homotopy type theory is the one considered by Ladyman and Presnell (2015, 2016, 2017) in their series of papers.[2] To avoid confusion, I shall term it 'Book HoTT' and use 'homotopy type theory' in the broad sense described above.

[2]Except that Ladyman and Presnell (2015, 2016, 2017) have never discussed higher inductive types in their papers, so we omit the subject here for conciseness. The univalence axiom is introduced in Section 3.3.

2.1 The meaning explanations

While the status of homotopy type theory as an autonomous foundation is debatable, the matter is certainly less controversial for constructive type theory: the correctness of the rules of the theory is well-known to possess a simple but rigorous justification provided by the meaning explanations (Martin-Löf, 1982), a standard intuitive semantics that validates the rules of inference of a type theory via an explanation grounded on computation.

Starting with a primitive untyped notion of computation, a meaning explanation ascribes expressions to terms or types based on their computational behavior, and then explains the meaning of a judgment depending on the values the expressions involved in that judgment compute to (Constable et al., 1985; Nordström et al., 1990). The basic principle is that the correctness of a rule of inference is determined by the correctness of the judgments it is composed of. Computation is given by an operational semantics that prescribes a collection of valid programs as sequences of computational steps and consequently provides us with a precise notion of evaluation—a finite sequence of one-step computations from closed terms (terms with no occurrence of free variables) to their canonical normal form (values). With a particular notion of evaluation in hands we endow each one of the basic forms of judgments of the type theory with meaning by saying what is to know a judgment of each form.

In constructive type theory we have four forms of judgments that assert that something is a type, two types are definitionally equal, a term is a member of a type, and two terms are definitionally equal at a type (Martin-Löf, 1984, p.3). Those judgments typically have their meaning explained as follows:

(i) To know that A is a type is to know that A evaluates to a canonical type;
To know a canonical type is to know
   (a) how to construct a canonical term of that canonical type;
   (b) how to show that two canonical terms of it are definitionally equal;

(ii) To know that A and B are definitionally equal types is to know that they have exactly the same terms;

(iii) To know that a is term of type A is to know that a evaluates to a canonical term of that type;

(iv) To know that a and b are definitionally equal terms at type A is to know that a and b evaluate to definitionally equal canonical terms of that type;

The schemes above are just a rough characterization and a detailed description of the meaning explanations is out of the scope of this paper (see Martin-Löf (1984)). Fortunately, enough has been said to suggest that types can be understood as program specifications and terms as programs.

It will come as no surprise to many that the meaning explanations of constructive type theory fail to justify Book HoTT: the use of axioms in a type theory is known to block computation in general, because axioms introduce new canonical terms to types without specifying how to compute with them (UFP, 2013, §0); so when one extends a type theory with an axiom one might break the process of evaluation of a term to its canonical form, thus undermining any attempt to endow the theory with a meaning explanation since the theory no longer posses a well-behaved notion of computation. In sum, Book HoTT adds new canonical terms to the identity type without explaining how the type theory is supposed to compute with them.

Even worse, since the reflexivity term is the only canonical term of the identity type in constructive type theory, and because $\mathsf{refl}_x : x =_A y$ only holds if x and y are definitionally equal terms, the meaning explanations validate a "reflection rule" that collapses any equality $x =_A y$ into a definitional equality of x and y at A (Dybjer, 2012). But this rule implies a principle known as uniqueness of identity proofs (UIP), which states that any two equality proofs are equal up to a higher equality proof:

$$\frac{a, b : A \quad p, q : a =_A b}{\mathsf{uip} : p =_{a =_A b} q}. \tag{UIP}$$

Since UIP is inconsistent with homotopy type theory (UFP, 2013, §2.14,§3.1.9) (otherwise it would not be possible to keep track of different equality proofs between two terms), the meaning explanations of constructive type theory will fail to justify any particular presentation of homotopy type theory.

3 The program of Ladyman and Presnell

3.1 Path induction

From the discussion in the preceding section it can be seen that the constant concern of Ladyman and Presnell (2015, 2016, 2017) about the status of homotopy type theory as an autonomous foundation for mathematics is well founded. Homotopy type theory is just a particular form of type theory, and while type theories are usually thought of having a justification in the meaning explanations, that is not the case for Book HoTT.[3] Nevertheless, the main objection of Ladyman and Presnell is not that the autonomy of homotopy type theory is dubious because it does not admit a meaning explanation. Rather, their point is that what they see as the informal semantics offered in the homotopy type theory book (UFP, 2013) is not pre-mathematical.

They start their series of papers arguing that the justification for the elimination rule of the identity type offered in the homotopy type theory book (UFP, 2013) is largely based on advanced concepts from homotopy theory while it is supposed to be pre-mathematical. This is the central topic of their first paper (Ladyman and Presnell, 2015) and a more general statement of the problem along the same lines is found in their second paper (Ladyman and Presnell, 2016, §§3.4.1–4).[4] The elimination rule for the identity type roughly states that given two terms $x, y : A$, an equality proof $p : x =_A y$ and a type family P, it is enough to have a term $u : P(x, x, \mathsf{refl}_x)$ in order to define a term of the type $P(x, y, p)$ (UFP, 2013, §1.12.1),

$$\frac{x, y : A, \quad p : x =_A y, \quad u : P(x, x, \mathsf{refl}_x)}{\mathsf{pathind}_P(x, y, p, u) : P(x, y, p)}.$$

Inspired by the homotopy interpretation of type theory (UFP, 2013, §2) in which a type A is a space, a term $x : A$ is a point in the space A, and an equality proof $p : x =_A y$ is a path in the space A with initial point x and final point y, this elimination rule is often called "path induction" in homotopy type theory because on the intended interpretation it implies that to prove a property of a path $p : x =_A y$ it suffices to assume that p is the identity path $\mathsf{refl}_x : x =_A x$ since all properties expressible in the sytem are homotopy invariant, and so are preserved under homotopy of paths.

It is difficult to see why Ladyman and Presnell (2015) insist on focusing their attention on path induction specifically, since their motivation is not so clear in their first paper. But perhaps there is a reason for that: a standard way of justifying elimination rules of type formers—or propositions, according to the propositions-as-types paradigm, a well-know correspondence between types and propositions formulated by Howard (1980)—is to establish that the meaning of that type former is determined by its introduction rules. This general idea of "logical harmony" is very well explained in the words of Gentzen:

[3] It is not easy to say whether Ladyman and Presnell are aware of that because the meaning explanations are constantly omitted from the discussions in their series of papers (Ladyman and Presnell, 2015, 2016, 2017).

[4] The problem is not stated again in the third paper of the series; instead, Ladyman and Presnell (2017, §1) refer the reader to their second paper.

> The introductions represent, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only ‘in the sense afforded it by the introduction of that symbol’. An example may clarify what is meant: We were able to introduce the formula $A \supset B$ when there existed a derivation of B from the assumption formula A. If we then wished to use that formula by eliminating the $\supset$-symbol [...], we could do this precisely by inferring B directly, once A has been proved, for what $A \supset B$ attests is just the existence of a derivation of B from A. (Gentzen and Szabo, 1969, p.80-1)

This gives us a form of justification for elimination rules that is used by many different semantics, including, notably, the meaning explanations. Type-theoretically, harmony states that the introduction rules of a type specify the only way of generating terms of that type, so in order to define a function out of that type it is enough to determine what it should do on terms explicitly obtained by introduction rules. Unfortunately, harmony can only explain path-induction if we maintain that the only terms of the identity type are reflexivity, and this contradicts the introduction of univalence and higher inductive types as additional ways of generating new equalities. Since the most direct route for the justification of path induction is blocked for Book HoTT, it is possible that Ladyman and Presnell chose to focus at first on path induction with this scenario in mind. Without a proper clarification, however, it seems as if they are regarding the justification of path induction as an individual problem at this point of their program, instead of providing a single informal semantics that can simultaneously justify all the axioms and rules of inferences of Book HoTT, including path induction—they will only address this point in their second and third papers, which we discuss in Sections 3.2 and 3.3.

The first response of Ladyman and Presnell (2015, §6) to the challenge of autonomy is to simply articulate a supposedly pre-mathematical justification for path induction. This justification leaves much to be desired, however, since what they propose is to take two unjustified rules of type theory and justify path induction using them as primitive principles.

One of those principles that are taken for granted is a type-theoretic generalization of Leibniz’s law of indiscernibility of identicals that states that if two entities x and y are identical then every property P possessed by x is also possessed by y and vice versa (Ladyman and Presnell, 2015, §6.3),

$$\frac{x, y : A, \quad p : x =_A y, \quad u : P(x)}{\mathsf{transport}_P(p, u) : P(y)}.$$

The other one (referred to as the uniqueness principle for the identity type) basically states that refl_x determines all equalities $x =_A y$ up to the equality of x and y in A (Ladyman and Presnell, 2015, §6.2),

$$\frac{a : A, \quad w : \sum_{(x:A)} a =_A x}{\langle a, \mathsf{refl}_a \rangle =_{\sum_{(x:A)} a =_A x} w},$$

where the pair constructor $\langle x, y \rangle$, where $x : A$ and $y : B(x)$, is the canonical term of the associated type $\sum_{(x:A)} B(x)$.

Finally, Ladyman and Presnell (2015, §6.4) show that path induction can be derived from those two principles and conclude their paper by saying that 'path induction can be justified on the basis of pre-mathematical principles' (§7).

But no comprehensive justification for the correctness of these two principles is found in the paper. Instead they just take it as given that Leibniz's law of indiscernibility of identicals is 'a fundamental part of our pre-mathematical understanding of identity' (Ladyman and Presnell, 2015, §6.3) and appeal to the fact that each type should come with a uniqueness principle without ever properly indicating a means to justify them, at least other than saying that these 'formal statements capture more precisely the sense in which the token constructors for a type give us all the tokens of that type.' (Ladyman and Presnell, 2015, §6.1).[5]

3.2 The types-as-concepts interpretation

In their second paper, Ladyman and Presnell (2016) consider a general approach to the challenge of autonomy where instead of addressing a particular problem of justification as an isolated problem, as in their first work, they present a full informal semantics with the intention of justifying all the axioms and rules of inference of Book HoTT at once. Those new justifications comprises Ladyman and Presnell's previous argument for path induction (2015, §6) which happens to be given again in the second paper (2016, §6).

[5]See Klev (2017, §5) for a more detailed argumentation against Ladyman and Presnell's justification of path induction and a demonstration of the justification of this same rule via the meaning explanations of constructive type theory (Klev, 2017, §3).

Ladyman and Presnell (2016) start proposing their own ideas for the informal semantics with a short digression where they discuss an "initially appealing" interpretation of terms as mathematical objects and types as kinds of mathematical objects that is rejected under the allegation that, among other reasons, it subscribes to a strong kind of realism, and that it cannot explain why each term must belong to exactly one type (§5.1). Immediately after that failed attempt, Ladyman and Presnell (2016, §5.2) articulate an alternative account of terms and types as mathematical concepts: terms and types correspond to specific and general (mathematical) concepts, respectively, so the membership judgment $a : A$ means that the specific concept given by a falls under (or is an instance of) the general concept given by A. For example, we can understand the judgment $\mathtt{0} : \mathbb{N}$ as the assertion that the concept *being zero* is an instance of the general concept *being a natural number*. Ladyman and Presnell (2016) are fully aware that they must say a few things about concepts to get this interpretation off the ground, but instead of developing a complete theory of concepts they just highlight seven particular features of concepts that according to them are relevant to the discussion. They only mention that:

(i) Concepts may exist regardless of the existence of an instance of it;

(ii) Concepts can be concrete or abstract, and specific or general;

(iii) Concepts need not be mind-dependent;

(iv) Concepts have an intensional nature;

(v) Complex concepts can be formed from basic ones;

(vi) Features of concepts can be abstracted;

(vii) General concepts can be instantiated with specific concepts.

Although it is arguably pre-mathematical and it has no commitment to realism (it does not rely on the existence of mathematical objects), this interpretation still needs some refinement—Ladyman and Presnell (2016) quickly observe that conceiving terms as specific concepts does not allow them to explain why a term must belong to exactly one type. This fact forces them to slightly modify their account by viewing terms not as specific concepts, but as specific concepts *qua* instance of a general concept. Finally, given their subscription to the propositions-as-types paradigm, Ladyman and Presnell (2016, §7.2) seem to imply that a proposition is true when the concept that it represents has an instance.

It seems doubtful whether the types-as-concepts interpretation can make things better for homotopy type theory because an explanation of terms and types as concepts can only shed light on the matter if we take for granted that our understanding of the nature of concepts is not problematic. Later in the paper, Ladyman and Presnell (2016, §8.1) consider the objection that it is unclear what concepts are and what their metaphysical status is. In response they maintain that

> although we do invoke concepts in our interpretation of tokens and types, the features of concepts that we rely upon in Section 5.2 are only those straightforward features that follow from our intuitive understanding of concepts. Thus we do not depend upon any advanced or intricate theory of concepts, and therefore do not need to give a comprehensive detailed account of concepts. (Ladyman and Presnell, 2016, §8.1)

But without a "comprehensive detailed account of concepts" one is free to wonder what exactly are the features of concepts that should follow from our natural understanding of them since the list (i)-(vii) is obviously non-exhaustive.

With such a naive characterization of concepts there is nothing preventing us to take for granted other intuitively appealing features commonly attributed to them such as the principle that a (mathematical) concept must have clear delimitations to the extent that it should be determined, with regard to every particular thing in a mathematical domain of discourse, whether it falls under that concept or not.[6] When taken together with the intensionality of concepts this principle simply means that a concept must be well-defined as far as mathematics is concerned. Concepts must have sharp boundaries and a real number such as π should not be confused with a topological space, nor a degree such as $45°$ with a group, for

[6] Frege (1903, §57) makes a strong case that in an exact science such as mathematics the extension of all concepts must be completely determined, otherwise a mathematical symbol would not refer to exactly one object (and consequently lack a reference). In his *Foundations of Arithmetic*, for instance, Frege (1884) writes:

> All that can be demanded of a concept from the point of view of logic and with an eye to rigor of proof is only that the limits to its application should be sharp, that we should be able to decide definitively about every object whether it falls under that concept or not. (Frege, 1884, §74)

example. This principle does not appear to be less appealing than the properties (i)-(vii) conveniently mentioned by Ladyman and Presnell (2016, §5.2) and their naive account of concepts naturally admits it as a fundamental part of our intuitive understanding as well. It turns out that this principle allows the validation of an integral part of classical reasoning in the types-as-concepts interpretation, namely, the unrestricted law of excluded middle, which states that for any proposition A, either A or its negation, $\neg A$, is the case. As we shall see next, the problem is that instantiation of (mathematical) concepts becomes a bivalent property: given any concept with sharp boundaries it must be determined for every particular mathematical entity whether it instantiates that concept or not, so it must be the case that in the whole totality of mathematical entities either there is something that falls under that concept or nothing falls under it.[7]

What does the law of excluded middle say from the perspective of the types-as-concepts interpretation? Ladyman and Presnell (2016) mention that the 'rules governing the formation of types are understood as ways of composing concepts to form more complex concepts' (§7.2) but they never extended the types-as-concepts interpretation to give a full account of all the type formers of Book HoTT in their series of papers.[8] In other words, for our question

[7]Notice that this inference requires the addition of a reasoning principle called the "limited principle of omniscience" (restricted to the domain of all mathematical entities) to our meta-theory. This principle states that the if a property is decidable for all objects then the existence of an object with that property is also decidable, that is, $\forall x(P(x) \vee \neg P(x)) \to (\exists x P(x) \vee \neg \exists x P(x))$. Although the limited principle of omniscience is weaker than the unrestricted law of excluded middle, it is important to be clear that it is non-constructive. More importantly, our argument for the potential validation of the law of excluded middle will not depend on the limited principle of omniscience, because the key point is that the type-as-concepts interpretation is unable to refute that 'in the whole totality of mathematical entities either there is something that falls under that concept or nothing falls under it'. Another implicit assumption in the argument is what exactly it means in the proposed concept-explanation of function-types for a concept to "result in" another concept. But since this interpretation is the one offered by Ladyman and Presnell (2017, §2.3) themselves we shall not bother with it here. As we shall see in the next section, univalence cannot be formulated without function-types, so a vague semantic account of functions will inevitably lead to a semantic indeterminacy of univalence as far as the types-as-concepts interpretation is concerned. Thus, all those considerations do not detract from our main objection: the types-as-concepts interpretation is very incomplete.

[8]Except for a brief explanation of the function type: 'a function is something that takes an instance of one concept and produces an instance of the other' (Ladyman and Presnell, 2017, §2.3). In fact, Ladyman and Presnell do not cover all the basic forms of judgments of Book HoTT (which are the same as constructive type theory): they provide no explanation of what it means for two types to be definitionally equal and two terms to be definitionally equal at a

to be a well-formulated one we need to close this gap by ourselves. In doing that, we are adopting for the sake of argument what seems to be the most natural and straightforward possible extension of the types-as-concepts interpretation—and, coincidentally, given the propositions-and-types correspondence, the following can be seen as a propositions-as-concepts alternative account of the traditional Brouwer–Heyting–Kolmogorov informal semantics of the logical constants of propositional constructive logic (Troelstra and van Dalen, 1988) that deals with conceptual subsumption instead of the usual notion of provability.

Write $[\![A]\!]$ for the general concept that the type A represents. We have that:

($\times$) A concept falling under $[\![A \times B]\!]$ is a pair containing a concept falling under $[\![A]\!]$ and a concept falling under $[\![B]\!]$;

($+$) A concept falling under $[\![A + B]\!]$ is either a concept falling under $[\![A]\!]$ or a concept falling under $[\![B]\!]$ plus the information of which concept has been instantiated;

($\rightarrow$) A concept falling under $[\![A \rightarrow B]\!]$ is a concept that, given any concept falling under $[\![A]\!]$, results in a concept falling under $[\![B]\!]$;

($\neg$) A concept falling under $[\![\neg A]\!]$ is a concept falling under $[\![A \rightarrow \bot]\!]$;
No concept falls under the empty concept $[\![\bot]\!]$.

Now suppose that A is a type that stands for the general concept $[\![A]\!]$. From the assumption that any concept either has something falling under it or not it follows that either $[\![A]\!]$ has an instance or not. If the former is the case, we have a concept falling under $[\![A]\!]$ and therefore $[\![A + \neg A]\!]$ has an instance. If the latter holds then nothing falls under $[\![A]\!]$ and $[\![A \rightarrow \bot]\!]$ must have an instance, for if something falls under $[\![A]\!]$ the concept *a concept falls under* $[\![A]\!]$ *and no concept falls under* $[\![A]\!]$, which is coextensional with $[\![\bot]\!]$, is instantiated. Thus, the assumption that nothing falls under $[\![A]\!]$ leads to an instance of $[\![A + \neg A]\!]$ and, since A is arbitrary, we have that $A + \neg A$ obtains for any A.

Why should we worry about whether or not the types-as-concepts interpretation validates the law of excluded middle or

type from the perspective of the types-as-concepts interpretation. Given that Ladyman and Presnell (2015, 2016, 2017) still have much unfinished business left, it is hard to tell what makes their types-as-concepts interpretation into a serious informal semantics even for basic constructive type theory at this point.

any one of its equivalents? The key problem is that the types-as-concepts interpretation may not serve as an informal semantics for homotopy type theory because the law of excluded middle, in its general form that holds for all propositions, is known to be inconsistent with the univalence axiom (UFP, 2013, cor. 3.2.7). Put differently, the types-as-concepts interpretation may be logically incapable of justifying the univalence axiom (to be discussed in the next section). Although this does not conclusively undermines the types-as-concepts interpretation the argument presented here indicates that Ladyman and Presnell's program cannot rest upon a naive account of concepts as they wish: a theory of concepts that is rigorous to a certain extent is absolutely necessary in order to make the types-as-concepts interpretation sufficiently plausible. While it is not our intention here to defend the view that any concept either has something falling under it or not, we understand that it is Ladyman and Presnell's responsibility to argue against it and provide a comprehensive theory of concepts that leaves no margin for interpretation.

3.3 The univalence axiom

In a nutshell, the univalence axiom is a claim about universe types (a hierarchy of types that contain other types as their canonical terms) that implies that two types are equal when they are equivalent in a technical sense (UFP, 2013, §2.10,§4.2-4),

$$\mathsf{ua} : A \simeq B \to A =_{\mathcal{U}_i} B.$$

To be precise, the full univalence axiom asserts that the canonical function in the other direction, which is defined by path induction as a map from reflexivity to the canonical automorphism of a type, is an equivalence—a statement that, very interestingly, may be pronounced as "equality is equivalent to equivalence". The function ua is thus obtained as the inverse of that map.

Universes allow for greater flexibility in a type theory. The homotopy type theory book (UFP, 2013), for instance, uses a particular formulation where the claim that A is a type is often treated as a claim of type membership in some universe: A is a term of a universe $\mathcal{U}_i$ for some universe level i (and having left the indices implicit we could simply write $A : \mathcal{U}$).[9] This is the sense in which we can say that univalence speaks of equality of types rather than equality of terms at a universe type.

[9]Representing universes and their corresponding types with the same notation is a characteristic of a postulation of universes a la Russell (Martin-Löf, 1984).

In their third paper, Ladyman and Presnell (2017) extend their types-as-concepts interpretation to universes and offer an account of the univalence axiom in such terms. To that end they must explain how we may understand universes as concepts and how we may justify univalence as a claim about concepts. Ladyman and Presnell (2017) interpret universes "by understanding them as domains of discourse, where a domain of discourse consists of the concepts and propositions that are understood and defined in a given discussion" (§2.3) and observe that this characterization can justify the usual rules for universes (UFP, 2013, § A.1.1-8,A.2.3-10). Such a detailed discussion would take us too far from our subject, but the crucial point here is the extensional character that is attributed to domains of discourse (it is essential that types have an extensional criterion of identity for the justification of univalence, otherwise we could not identify them when they are equivalent). According to Ladyman and Presnell, '[t]here is nothing to a domain of discourse beyond what particular concepts it contains' (2017, §2.3) and this is what allows domains of discourse to be conceived extensionally rather than intensionally.

Ladyman and Presnell appear to contradict themselves here. On the one hand, universes are domains of discourse which are extensional; on the other hand, universes are types and stand for general concepts—but in their previous paper Ladyman and Presnell (2016) argued extensively that one of the fundamental features of concepts is their intrinsic intensional nature:

> Concepts are intensional: they correspond (roughly) to descriptions rather than to extensional collections. Hence (to use two famous examples), 'the morning star' and 'the evening star' are two distinct specific intensions, although they have the same extension; and 'human' and 'featherless biped' are distinct general intensions although they have the same extension. We can have empty concepts, even necessarily empty concepts, and indeed multiple distinct empty concepts. (Ladyman and Presnell, 2016, §5.2.4)

Extensional entities should always be compatible with intensional criteria of identity, but as far as intensional entities and extensional equality are concerned this statement is clearly false. It is natural to expect an explanation of why domains of discourse (which are general concepts) contradict the inherent intensionally of concepts, but no such clarification is given in their account (Ladyman and Presnell, 2017).

On the basis of their representation of universes as domains of discourse, Ladyman and Presnell (2017) support a description of univalence as the claim that all domains of discourse are univalent and that of equivalent types as a single mathematical concept under multiple different presentations (§5.1). However, as Ladyman and Presnell (2017, §5.1) admit, to justify such a view there is more to be done.

Inspired by an informal motivation for the univalence axiom given by Awodey (2014), who argues that univalence captures a common principle of reasoning embodied in everyday mathematical practice, Ladyman and Presnell (2017, §7) question if a similar justification which only appeals to elementary pre-mathematical considerations is available to us. Awodey's main argument is that if one wishes for a foundational system that can reflect the ordinary mathematical practice where isomorphic objects are identified, then we have good reasons to choose type-theoretical foundations and, in particular, univalent ones. Ladyman and Presnell (2017, §7.4) identify two problems with Awodey's proposal. The first one is a gap between "isomorphism" and equivalence: two objects A and B are typically said to be isomorphic if there are structure-preserving maps

$$f : A \to B \qquad \text{and} \qquad g : B \to A$$

such that

$$g \circ f = 1_A \qquad \text{and} \qquad f \circ g = 1_B.$$

But, type-theoretically, this relation, which in the homotopy type book is called "quasi-invertibility" (UFP, 2013, §2.4.6), contains too little data and has a poor behavior: if we have an isomorphism between A and B witnessed by the functions $f : A \to B$ and $g : B \to A$, the isomorphism induced by f may have multiple proofs that g is a quasi-inverse of f, meaning that "the" isomorphism induced by a function need not be unique (UFP, 2013, §4.1). While one might say that quasi-invertibility is a natural expression of isomorphism from traditional mathematics, the statement of full univalence in terms of quasi-inverses leads to a contradiction. For if the canonical function that maps equalities to isomorphisms has a quasi-inverse ua', it would have to send the collection of isomorphisms induced by a function to the same equality, thus identifying them and contradicting the fact that "the" isomorphisms induced by a function are not necessarily unique.[10] Although quasi-invertibility is logically equivalent to equivalence (in that a function is an equivalence if and only if it is quasi-invertible) (UFP, 2013, §4.2), the inference is certainly

[10] Readers may refer to Ladyman and Presnell (2017, §6.5) for more detailed discussion of the contradiction.

not pre-mathematical and Ladyman and Presnell's point is how to modify Awodey's argument in a way that replaces 'isomorphism' by 'equivalence' without validating the inconsistent formulation of univalence in terms of isomorphism. But it is possible that Awodey is only using the term 'isomorphism' informally as a stand-in for the technical notion of equivalence, in which case their objection may be dismissed.

The second problem with Awodey's argument, as pointed out by Ladyman and Presnell (2017, §7.4), is that it only seems to be able to justify part of the univalence axiom, namely, the function $\mathsf{ua} : A \simeq B \to A =_{\mathcal{U}_i} B$. That is to say, there is a canonical function $\mathsf{idtoeqv} : A =_{\mathcal{U}_i} B \to A \simeq B$ (UFP, 2013, §2.10.1) in the opposite direction, defined by path induction by mapping reflexivity paths to auto-equivalences, but there is no indication that the consideration of ua as an inverse to $\mathsf{idtoeqv}$ in an equivalence (which is something the full univalence axiom requires) is a common practice of working mathematicians.

Ladyman and Presnell (2017) consider possible arguments in response to those issues but they ultimately conclude that in order to develop Awodey's argument into a complete justification for univalence they still need to answer a few questions, for instance—whether their account of universes as domains of discourse is well justified, why we should take a structuralist view of mathematics, and whether their proposed amendments to Awodey's argument are legitimate.[11]

4 Types as concepts or programs?

As was discussed in the previous section, it appears that the types-as-concepts interpretation can only serve as an incompletely specified informal semantics for homotopy type theory, at least in its current stage of development. Ladyman and Presnell (2017) do not provide a clear interpretation of universes and type formers, and, besides never addressing higher inductive types, and they do not show that the types-as-concepts interpretation can validate univalence.

What about the meaning explanations? As seen in Section 2.1, the meaning explanations of constructive type theory cannot justify homotopy type theory either, but the reason is different: the existence of equality proofs other than the reflexivity term has no

[11] Ladyman and Presnell (2017) discuss an alternative justification for univalence in §8.1, but since they conclude that '[w]hat would be sacrificed on this approach, then, is the idea that Univalence is true' it is not so clear what is gained with the proposed idea. Thus, we shall not discuss it here.

justification from the viewpoint of this interpretation. Nevertheless, this does not show that there are no meaning explanations for homotopy type theory. The key observation here is that a meaning explanation (a description of how terms and types are assigned to untyped computations) could be offered to an alternative computational presentation of homotopy type theory, that is to say, a stronger type theory that implements univalence and higher inductive types not as axioms but theorems.

Cubical type theory is a positive step in this direction. Bezem et al. (2014) have given a constructive model of type theory that validates the univalence axiom using cubical sets. This is a variant of Voevodsky's simplicial model (Kapulkin and Lumsdaine, forthcoming), using a mathematical concept of cubes due to Kan (1955) which is more amenable to constructive methods. Cubical type theories (type-theoretic paraphrases of models in cubical sets) have been proposed since then. In particular, Cohen et al. (2018) have developed a cubical type theory which proves univalence and has possible extensions with some higher inductive types, as made explicit in Coquand et al. (2018). Recently, Angiuli et al. (2017) and Cavallo and Harper (2018) have built a realizability model that can be seen as a higher-dimensional generalization of the meaning explanations of Martin-Löf for a cubical type theory with a cumulative hierarchy of univalent universes, full univalence and many higher-inductive types. Their realizability model may indeed be seen as a computational justification for homotopy type theory, but, as it relies on the mathematical concept of cubical set, further work would be required to investigate whether it could also constitute a legitimate pre-mathematical justification for homotopy type theory.

The current state of affairs regarding the justification of homotopy type theory via the meaning explanations and type-as-concepts interpretation is summarized in Table 1.

Finally, another possibility to be considered is that the homotopy interpretation criticized by Ladyman and Presnell (2015, 2016, 2017) can be perhaps made pre-mathematical through the use of spatial intuitions in the sense of Tsementzis (forthcoming), where an intuitive semantics for homotopy type theory is given based on notions such as "shapes", "points", "symmetries" etc. But a thorough analysis of the pre-mathematical nature of the primitive assumptions the interpretation relies on and what rules it justifies (Tsementzis, forthcoming, §3.1–2) would exceed the scope of this paper and is thus seen as future work.

Table 1: A comparison between some informal semantics for type theory

Meaning explanations (Constructive type theory)	
Does it justify path induction?	yes
Does it justify the reflection rule/UIP?	yes
Does it justify univalence?	no
Does it justify higher inductive types?	no
Does it justify the law of excluded middle?	no
Is it pre-mathematical?	yes
Type-as-concepts	
Does it justify path induction?	unknown
Does it justify the reflection rule/UIP?	unknown
Does it justify univalence?	unknown
Does it justify higher inductive types?	unknown
Does it justify the law of excluded middle?	unknown
Is it pre-mathematical?	yes
Meaning explanations (Cubical type theory)	
Does it justify path induction?	yes
Does it justify the reflection rule/UIP?	no
Does it justify univalence?	yes
Does it justify higher inductive types?	yes
Does it justify the law of excluded middle?	no
Is it pre-mathematical?	unknown

5 Conclusion

In conclusion, Ladyman and Presnell (2017) fail to provide a decisive answer as to whether homotopy type theory can serve as an autonomous foundation. Their types-as-concepts interpretation certainly cannot be seen as a pre-mathematical semantics for homotopy type theory yet, as some crucial issues will have to be overcome to make that possible.

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