A Problem with the Dependence of Informal Proofs on Formal Proofs

Fenner Tanswell

Abstract

In this paper I examine the phenomenon of informal proofs as found in mathematical practice and the difficulties these face concerning rigour and correctness. I focus on one particular type of response, which I call *derivationist*, which seeks to explain these in terms of underlying formal derivations. I proceed to set out five desiderata that the derivationist approach should aim to satisfy. With particular emphasis on Azzouni's derivation-indicator account, I raise a dilemma for the type of link that must be posited from informal proofs to formal derivations: that it must either be agent-independent or else agent-dependent. I show that derivationist theories want to take the first horn, but that considerations of proof identity, uniqueness and informal content determining formal structure are serious obstacles in that direction. I further argue that the other horn is incompatible with the original motivations of the derivationists. Thus I conclude that the desiderata for a derivationist theory cannot be satisfied.

1 Introduction

We can distinguish two types of proof: *informal proofs* and *formal proofs* (or *proofs* and *derivations*). On the one hand, formal proofs are given an explicit definition in a formal language: proofs in which all steps are either axioms or are obtained from the axioms by the applications of fully-stated inference rules. On the other hand, informal proofs are proofs as they are written and produced in mathematical practice. They may make assumptions about the intended audience's background knowledge and ability to follow lines of reasoning, skip over tedious or routine steps and make reference to semantic properties and properties of mathematical objects[1] without stating these fully. They also are not confined to formal languages: though mathematical symbolism may be used, natural language, diagrams and mixed-mode explanations are freely employed too.

While formal proofs, in our sense, may be defined mathematically in any

[1] As seen through the 'Plato-tinted spectacles' described in (Buldt, Löwe & Müller, 2008).

number of ways[2], informal proofs are much harder to pin down precisely. We may give a general description of what they are like (as I began above), or as others have similarly done: "...what we do to make each other believe our theorems...[an] argument which convinces the qualified, skeptical expert." (Hersh, 1997, p. 153); "...a kind of meaningful narrative... more like a story, or even a drama, conveyed to us in language calling on our semantic and intuitive understanding." (Robinson, 1991, p. 269); "...a conceptual proof of customary mathematical discourse, having an irreducible semantic content..." (Rav, 1999, p. 11) or "a sequence of thoughts convincing a sound mind" (Gödel, 1953, p. 341). However, the real problem is not giving such a general description of what informal proofs are like, but it is rather to sort those informal proofs which are correct and rigorous from those which are not.

While we may associate deductive reasoning and logicality with formal proofs in formal systems, actual mathematics is regularly presented informally using informal proofs.[3] This challenges any proponent of an account of philosophy of mathematics to also give an account of how proving, as it is practiced, relates to the idealised notion of formal proofs. There are many directions to take for such an account, from Lakatosian dialectics (see Lakatos, 1976) all the way to denying that any mathematics took place before Frege. In this paper I want to focus on just one family of responses, wherein the rigour and correctness of informal proofs is taken to be dependent (in some sense) on associated formal proofs. Call this family of views the *derivationist* approach.[4] There are a number of different connections that informal proofs can be argued to have to their formal counterparts: reductions, logical forms, explications, abbreviations, sketches, formalisations, etc. In this paper I will look at one particular proposal by Azzouni: that informal proofs indicate underlying formal proofs.[5]

I will begin by laying out some desiderata that any successful derivationist account of informal proofs must meet. I will then explain Azzouni's account of informal proofs, focusing on the particular connection between formal and informal proofs that is posited and how well Azzouni's account would meet the given desiderata. In section 4 I present a dilemma, asking whether the link from informal proofs to underlying formal derivation is an agent-independent one or whether it is dependent on the agent who is presenting the proof. I take Azzouni to need the former in order to be successful in obtaining his brand of derivationism, but but in section 5 I will criticise this horn of the dilemma based

[2] Avoiding, for the purposes of this paper, the need to fully get to grips with what it means to be formal. For work towards this see (MacFarlane, 2000; Dutilh Novaes, 2011).

[3] What is 'actual mathematics'? The intended answer here is *mathematics as it is practiced* but this is only enlightening in that it points to further questions that need to be addressed, concerning which parts of mathematical practice are relevant. For the purposes of this paper I take actual mathematics to simply be that published in mathematics journals, presented at conferences and taught in mathematics classes. A number of interesting discussions of this question can be found in (Mancosu, 2008).

[4] I specifically avoid calling this formalism because the derivationist stance is broader and may encompass positions that would traditionally fall outside of the formalist school of thought.

[5] The bulk of this position is given in (Azzouni, 2004a) and (Azzouni, 2005a).

on a problem of *overgeneration*. Azzouni can avoid this problem if he adopts the second horn of the dilemma, but in section 6 I will argue that this is not compatible with Azzouni's theory. I shall therefore argue that the account is deficient in dealing with the various desiderata it is aiming to address. Finally, I will conclude that the fundamental diffculty that prevents Azzouni's account from being successful is one that is a general roadblock to successfully providing a derivationist account of informal proofs.

2 Minimal Desiderata of a Derivationist Account of Informal Proofs

In this section I shall lay down the minimal aims that a derivationist account should achieve in dealing with the problem of informal proofs. By making the these intentions clear from the outset, we will be able to see where conflicts arise.

We can begin with two desiderata that were already mentioned:

> **(Rigour)** To give an account of how informal proofs are (or can be said to be) rigorous through their connection to formal proofs.

> **(Correctness)** To distinguish correct informal proofs from incorrect ones i.e. the connection should only link the informal proofs that are correct to the justifying formal proofs.

The first of these is precisely the challenge the derivationist faces in arguing that informal proofs can be rigorous if they are connected to formal proofs in the right kind of way. The second adds to this the need to properly distinguish the correct informal proofs from incorrect ones. One could interpret this as the intention not to over-generate through the posited connection: it would be undesirable for the link matching informal proofs to formal proofs to also associate *flawed* informal proofs with justifying formal proofs.

Since informal proofs arise from mathematical practice and the way in which we engage with and do mathematics, another desideratum is the following:

> (**Agreement)** To explain how, in practice, mathematicians manage to consistently converge and agree on the correctness of informal proofs. (Additionally, to give an account of informal proofs that were conceived of long before we had a sufficiently strong account of formal proofs to support them.)

The main part of **(Agreement)** is to actually engage with informal proofs as a social phenomenon; to explain how and why the mathematical community has employed informal proofs, as well as how the underlying link the derivationist account argues for relates to this practice. The addendum presses the requirement further, asking for the account to also explain how the cumulative nature of mathematics fits with the fact that formal proofs are a rather recent discovery (in the strong sense that a derivationist account needs). A requirement like this is to avoid the immediate objection that might be raised: that formal proofs cannot underwrite informal ones, because historically we have been using the latter far longer.

Now I will impose a stronger demand on the derivationists, the demand that their account doesn't simply state what the link is between formal and informal proof (abbreviating, indicating, logical form etc.) but that instead it gives some substance to the link.

> **(Content)** To show how the content of an informal proof determines the structure of the formal proof(s) it maps to.

A reason that informal proofs do present a substantial difficulty is that, in many ways, they are and appear quite different to any formal proofs. In answering such a difficulty, then, saying that the relation between them is of a certain kind is the easy part; showing that it is so is much harder. What the account needs to provide is an explanation of how exactly the informal proof can be used to pick out some formal proof or proofs. The picking out must surely (and at least partially) follow the content of the informal proof, so the account needs to tell us about how this content determines the structure of the formal proof that is associated with it.

We may elaborate the above further, to require a response to the particular tricky cases:

> **(Techniques)** To provide an explanation of apparently inherently informal techniques.

A main example of what is required here is dealing with diagrams in mathematics. A legitimate response is to argue for some kind of eliminability thesis for diagrams: that all diagrams must be eliminable from proofs entirely. Of course, such an argument would need to be given to complete the account, and may bring additional commitments. Other examples are proofs using symmetry, or the 'untraversed gaps' described in (Fallis, 2003).

3 Azzouni's Derivation-Indicator View

Azzouni's derivation-indicator view of mathematical practice, as presented in (Azzouni, 2004a, 2005a), takes the link between informal and formal proofs to be that informal proofs indicate underlying formal proofs.[6] In his own words:

> I take a proof to indicate an 'underlying' derivation... Since (a) derivations are (in principle) mechanically checkable, and since (b) the algorithmic systems that codify which rules may be applied to

[6]Although it should be noted that Azzouni has largely dropped the 'indicating' terminology in later developments of the view in (Azzouni, 2005b) and (Azzouni, 2009) for reasons we will see in section 4.

> produce derivations in a given system are (implicitly or, often nowadays, explicitly) recognized by mathematicians, it follows that if proofs really are devices mathematicians use to convince one another of one or another mechanically-checkable derivation, this suffices to explain why mathematicians are so good at agreeing with one another on whether some proof convincingly establishes a theorem. (Azzouni, 2004a, p. 84)

The focus here is very much on answering **(Agreement)**, dealing with the general social conformity regarding good and bad proofs. However, it is clear that for Azzouni this is closely linked to **(Rigour)** and **(Correctness)** in that the link will explain the agreement in terms of informal proofs being correct or rigorous due to underlying formal proofs.

An interesting aspect of Azzouni's view is that the formal proofs are defined more liberally than usual. He takes them to be located within 'algorithmic systems', which are not restricted in the ways we generally take formal proofs to be:

> I've already stressed that 'algorithmic systems' are restricted neither to a particular logic, a particular subject-matter, nor even to an explicit language (as opposed to something diagrammatic or pictorial). What is required is that 'proofs', however these be understood, are (in principle) mechanically recognizable. (Azzouni, 2004a, p. 86)

This has already been criticised (see Rav, 2007), with a response from Azzouni in (Azzouni, 2009), so I shall not take up this discussion here. However, in the present context the motivation for this view should garner at least some sympathy, for Azzouni is explicitly trying to leave open a straightforward route to meeting the demands of **(Techniques)**, in particular those regarding diagrams as used in mathematics. This focus on diagramatic reasoning becomes clearer if we note Azzouni's reference to another of his papers analysing diagrammatic reasoning in Euclid's Elements (Azzouni, 2004b), suggesting that he believes diagrammatic proofs do not always need to be informal, so long as they are given a mechanically checkable structure.[7] More on Azzouni's views of diagrams in mathematics can also be found in (Azzouni, 2013).

Turning now to the question of how exactly it is that derivation-indication links informal proofs to formal ones (and thereby the question of **(Content)**), Azzouni does not argue that each informal proof is underwritten by some unique formal proof in one algorithmic system. That would, he claims, be implausible as an account of mathematical practice because in reality mathematicians are not held to one specific inference system. Furthermore, if an account did limit mathematicians to one specific formal system it would be open to objections based on incompleteness phenomena. Instead, in Azzouni's view each informal proof relates to a family of formal proofs which are located in a number of different algorithmic systems.

[7]Understanding formal proofs as mechanically checkable ones takes one of the stances on the debate over what it means to be formal found in (Dutilh Novaes, 2011).

> It doesn't much matter where in the family of algorithmic systems we take 'the' derivation indicated by a proof to be located... since algorithmic systems embedded in one another are so embedded to conserve derivational results, we can take the derivation indicated to be one located in any algorithmic system within which the result occurs and is surveyable. (Azzouni, 2004a, pp. 93–94)

The conservativity requirement holding between algorithmic systems in which 'the same' formal proof is located comes closely coupled with a translation of the ideas up and down systems:

> Indeed, provided one is very strict about concept-individuation conditions, what can be claimed is that the new systems come with all-new concepts—and the old ones have simply been stipulatively identified with (some of these) new concepts. Such a stipulative identification of concepts that proves valuable is innocuous solely because of the cumulative way that algorithmic systems are embedded in one another: none of the old results regarding the old set of concepts are jettisoned—new material has only been added. (Azzouni, 2004a, p. 98)

Azzouni rightly observes the need to deal with **(Techniques)** and, specifically, that many informal techniques do not seem to point directly to something formal. The particular example Azzouni gives is using symmetry, i.e. doing one part of a proof and then observing that another part is proved symmetrically. What is understood is that the part of the proof already given could be easily edited and adjusted to give the other part, though the exact details of such an adjustment are never given. The solution he offers is that in the course of informal proofs mathematicians may be using 'meta-level' reasoning, which means that the system(s) that the indicated derivation is located in will be 'larger':

> When formalized as a derivation, such a proof will necessarily contain metamathematical elements which naturally drive it into the form of a derivation in a system strictly larger than one about, say, the objects officially under study. Mathematicians automatically ascend to a discussion of what can be taken to be properties and relations of the relations and properties of the objects they are proving results about. (Azzouni, 2004a, p. 94)

Here, his discussion of how to deal with the case of symmetry additionally reveals some of the main evidence of what his view on **(Content)** is. It appears that aside from these tricky cases of meta-level reasoning and the like, the actual link from informal proofs to formal ones will usually be a straightforward 'filling in the gaps'-type process. However, in developing the view further in later work, Azzouni explicitly moves away from this 'filling in the gaps' account to a more sophisticated picture separating the way we come to understand informal proofs (through 'inference packages') from the way that corresponding formal proofs are determined (see Azzouni, 2005b, p. 40). Regardless, the worries I raise hereafter apply equally to both.

4 A Dilemma

It is now time to start exploring the relation of derivation-indication more thoroughly. A particularly weak understanding would be to see informal proofs as a kind of time-saving communicative device, allowing mathematicians to quickly transfer formal proofs by indicating them to one another using informal proofs. However, this is not Azzouni's intended meaning; he, in fact, explicitly rules out the idea that mathematicians need to be aware of the underlying derivations ("I should add that it isn't a requirement on 'indicating' that mathematicians, generally, be aware that their proofs indicate derivations." (Azzouni, 2005a, fn. 16) or similarly (Azzouni, 2009, fn. 17)). So if not this, what is meant by indicating? Since the general intention is to give an account of **(Agreement)**, **(Correctness)** and **(Rigour)**, it appears that what is required is that indication is some kind of dependence relation, but what properties it should have is just one of many questions that must be faced to complete the account.

The particular question I propose to press for this account is the following: is derivation-indication agent-dependent or agent-independent?[8] Since, in essence, it is a proof that indicates a derivation it is relevant to ask who the supposed agent in this dilemma is. The proposal is that, on the one hand, the dependence link could be argued to not involve any kind of agent (say mathematician, student, listener, reader or anyone else that is involved in the particular instance of the proof). On the other hand, the agent-dependent horn of the dilemma suggests that the link from informal to formal proof may not be fully present in the proof itself, but instead something over and above generated by the practice of proving i.e. something that is added by some involved agent.

In what follows I will examine the two horns of this dilemma, arguing that Azzouni is proposing an agent-independent link between formal and informal proof. However, I will contend that taking this horn will not be successful, based on a problem of the informal proofs corresponding to multiple, non-equivalent formal proofs. Taking the link to be agent-dependent, I argue, is not an escape option for the derivationist though, because doing so fails to satisfy the original motivations of the derivationist enterprise.

5 Agent-Independent Derivation-Indicators

In this section I will consider the agent-independent horn of the dilemma, investigating the correspondence it posits between informal and formal proofs in order to show the ways in which this correspondence cannot support the answers to the various desiderata set out above.

Let us consider the following question: does each informal proof relate to just one unique formal proof or to many of them? We have already seen that for Azzouni each informal proof relates to a whole family of derivations, due to the

[8] This question is very close to the question of whether formalisation is a process that varies with the agent performing it, like Carnap's notion of explication (Carnap, 1945) or whether it instead is a process of revealing the 'deep structure' of the target phenomenon.

fact that he believes that 'the' formal proof is located in a range of algorithmic systems and, strictly speaking, these are different proofs.[9] The question can be reissued in these terms, though: for some given informal proof, is there a unique formal proof relative to each algorithmic system it appears in? In all cases where Azzouni touches on the issue, he seems to want each informal proof to pick out one unique formal proof per algorithmic system, within the upper and lower bounds.[10]

Let us think about this kind of uniqueness, since proof identity conditions are central to the problem I raise this section.

We already saw in section 3 that on the derivationist picture the informal proofs depend on formal proofs to be able to answer the various desiderata laid out above. In the light of this, the identity of proofs is highly relevant because it affects which formal proof(s) an informal proof depends on, and consequently impacts how well the desiderata are met. For example, if an informal proof does not depend on a unique formal proof (per algorithmic system) but instead depends on multiple, non-identical or non-equivalent formal proofs, then this could lead to further difficulties, say, in satisfying **(Rigour)** and **(Correctness)**. For it is the underlying formal proofs that are meant to be ensuring the rigour and correctness of informal proofs, but if there are multiple different formal proofs simultaneously being depended upon this undermines the effectiveness of the explanation the derivation-indicator account gives. For example, what is there then to stop an informal proof from corresponding to both one correct and one incorrect formal proof? The point is that if it is the case that the informal proof does not uniquely determine which formal proof it depends on, then the dependence is far weaker than is required to actually satisfy the desiderata. Once it is conceded that there are multiple different, non-equivalent formal proofs underlying some informal proof, we can immediately ask why it is these particular ones that are selected and what ensures that it is only correct and rigorous formal proofs that are picked out. Now, if we need an extra step to clarify why the informal proof only corresponds to just to those formal proofs which do ensure rigour and correctness, then it is this additional step that is doing all of the philosophical work and the account given has failed to properly answer the questions posed.

If the underlying formal proof is unique in some sense, then it seems the structure of the formal proof could, perhaps, be related to the content of the informal proof and avoid this underdetermination. Such considerations are also clearly present in Azzouni's theory: the fact that he writes of the underlying formal proof in the singular[11], even when it is in fact located in different algorithmic systems with different languages, does not appear to be accidental. Of course, we did see that this required two extra components. Firstly, the moves

[9] This is because a formal proof is relative to a formal system and language.

[10] It should be noted that Azzouni, despite attempting to deal with some of the key issues of mathematical practice and informal proof, is never particularly explicit about the answers to these questions. Dealing with the various options for what he can and may want to mean is precisely the current undertaking.

[11] As evidenced by the quotations in section 3.

'upwards' had to be conservative of the derivational results, to make sure 'the' formal proof is still present as one extends the system. Secondly, we need to be able to identify proofs up and down systems to ensure they are still the same in this crucial sense. As we saw above, this is achieved by stipulatively identifying concepts between formal systems. I shall return to these moves once the worry has been further articulated.

In the remainder of this section I will show that the lack of a unique determination of the formal proofs an informal proof depends on does indeed occur and that as a result the problem just described applies to the derivation-indicator view on the agent-independent horn of the dilemma.[12] Another way of describing this problem is as an *overgeneration* problem. The idea is not that informal proofs are too resistant to formalisation but instead that they are not resistant enough. There are multiple, equally legitimate formal proofs corresponding to any given informal proof and it is this multitude which throws doubt on there being any deep philosophical significance to the correspondence at all.

When proposing this overgeneration worry for derivationist views, something that is often brought up in response is whether or not the difference between the various proofs is *substantial*. The thought is, presumably, that if the type of difference between the various formal proofs is only minor or insubstantial, then the proofs may be essentially the same and so the overgeneration problem loses its bite. However, I do not find this distinction particularly helpful in avoiding the problem for two reasons. Firstly, while being essentially the same may hold for two formal proofs with only some minor change, making lots of minor changes could add up to a substantial change quite easily. Secondly, I don't expect there is any robust way of separating the variations between formal proofs into substantial and insubstantial ones, but rather that the variations will come in degrees from very minor all the way to being totally different proofs. Nonetheless, I will accept the distinction for the sake of argument and proceed to why I think there will be both the smaller and the more substantial variations between the formal proofs that some given informal proof will depend on.

Given some informal proof, it is straightforward to see that there must be a selection of formal proofs that it corresponds to just from the minor and insubstantial variations that can be introduced. Examples I have in mind are variable-renaming; changing the order of independent lemmas; switching between inter-definable logical constants; changing between the order you prove bi-conditionals (i.e. starting right-to-left or left-to-right) etc. Of course, the kind of changes that are minor will depend on the particular proof, since at times these rather innocuous differences can be relevant (or even crucial) to the success of the proof. This not only supports my claim that the distinction between minor and substantial differences is not a robust one, but also the more general argument I am making that even the minor differences can potentially cause problems for the agent-independent take on derivationism.

Now Azzouni has essentially two options. He can stick to his guns, as it

[12] I believe the other horn does not suffer this same problem, as will be discussed in section 6.

were, and insist that for any given informal proof there is just one formal proof per algorithmic system, in which case he fails to capture basic intuitions about formal proof identity concerning these minor variations, say, as well as being exposed to a worry about the arbitrariness of the particular proof that underlies the informal proof. Alternatively, he can accept that there is instead some *equivalence class of formal proofs* in each system matching up to any informal proof. In this case, for some given informal proof and an appropriate algorithmic system, there is a class of formal proofs that the informal proof indicates. It seems obvious that Azzouni should take the latter option; given that he accepts inter-system identity of proof, intra-system identity does not appear to be any more problematic.

However innocuous intra-system identity may seem, it is in fact deeply problematic, even in cases of insubstantial variation. To begin, a concern is that even though we have seen some suggestions for the acceptable minor variations listed above, if the minor variations do still keep the given formal proof 'essentially the same', then we would certainly like a more complete description of the kind of variations that are acceptable. With this comes the further need to justify such choices and convince us that adding up the differences will not eventually amount to a more substantial change. Considering the huge variety of systems that we could be talking about here, these demands will not realistically be met. The rhetorical point, though, is that the granularity of the notion of proof identity in play will have a bearing on how well the theory holds up under scrutiny.

Even if there are answers to the questions of the previous paragraph, this does not settle the matter concerning proof identity. Azzouni's theory, for good reason, identifies proofs between different algorithmic systems via the stipulative identification between concepts and conservative translations between the systems. Again, if we were dealing with just a single formal proof in each algorithmic system then this process might work, but if there is an equivalence class of formal proofs underlying some informal proof in each algorithmic system, then once again there are technical issues that must be addressed. Even when the variations are minor relative to some particular algorithmic system, those differences could be exacerbated and enlarged by the translation between systems. Formal proofs that were essentially the same (in the sense of being in the same equivalence class) in one system could, for all we know, be translated to proofs that are no longer the same according to the equivalence conditions in that other algorithmic system. Suppose we have two formal proofs P and Q in algorithmic system A that are both in the equivalence class underlying some informal proof, then translate them in Azzouni's sense to some other algorithmic system B. There is no guarantee that the translations $t(P)$ and $t(Q)$ will be in the equivalence class for the informal proof in system B.

There are two ways that one might try to avoid this concern of identity and translation: by appeal to conservativity and stipulative identity. Conservativity ensures that no results are jettisoned when moving between algorithmic systems, so we are safe in the knowledge that whatever we have a proof for in the weaker system will also have a proof in the stronger one. Yet this is certainly not enough to avoid the problem, since the way it is posed does not require the result to disappear, rather that the translation may take minor differences and make them substantial in the translation process. This can certainly happen if the result is still present in the new system. The fact that the identification between systems is stipulative can also not do any work here, because as we saw above the stipulative identity is only argued to be innocuous thanks to the conservativity. Now I argue that when it comes to formal proof identity, the stipulation of identity might not be innocuous (in that substantial differences might creep into proofs during translation) and that conservativity does not allow a way out of this fact, therefore making use of stipulative identity would beg the question.

So much for minor variations; what of more substantial ones? Are there ways in which the underlying formal proofs can differ which amount to significant and sizeable differences? I believe that there certainly are and will now give an example where this can be seen. First, though, I want to give some thought as to what 'substantial' differences could be like. There is a sense in which the type of differences is constrained by the informal proof that the formal proofs all correspond to, yet this constraint does not, I argue, prevent substantial differences from appearing. The two most straightforward places to see this are in the treatment of mathematical objects and the mathematical dependencies a theorem has. Firstly, the treatment of the objects of an informal proof have to give some formal reconstruction of the objects in terms of relevant properties (at least those that are used in the proof). How the objects are represented in the formal system, then, will affect how the formal details of the proof go. Even for these details there may be multiple different ways to do things (totally ignored in the informal proof). Together we get different formal constructions (which will only have to overlap in some crucial properties) with different technical details. Of course, differences in the representation will have knock-on and snowballing effects the further through the proof we go, as the different representations and details of the formal proofs cascade along. After all, the exacting nature of formal proofs brings with it a delicate balance that must be maintained for the proof to be correct. Secondly, by representing the proof in different places, the mathematical dependencies that the proof has will be altered to support the type of specific inferences that may be made in that system. From all of these factors, the appearance of variations between the formal proofs that are substantial should be expected.

Let us flesh this out with a concrete example. The one I have in mind is that of the mutilated chessboard.[13] The statement and proof are the following:

> An ordinary chess board has had two squares—one at each end of a diagonal—removed. There is on hand a supply of 31 dominos, each of which is large enough to cover exactly two adjacent squares of the board. Is it possible to lay the dominos on the mutilated chess

[13] This example is central in (Robinson, 1991) and can also be found in (Black, 1946) and (Gardner, 1957).

> board in such a manner as to cover it completely? (Black, 1946, p. 157)
>
> It is impossible ... and the proof is easy. The two diagonally opposite corners are the same color. Therefore their removal leaves a board with two more squares of one color than of the other. Each domino covers two squares of opposite color, since only opposite colors are adjacent. After you have covered 60 squares with 30 dominos, you are left with two uncovered squares of the same color. These two cannot be adjacent, therefore they cannot be covered by the last domino. (Gardner, 1957)

This example has intentionally been chosen as one which is intuitively correct, rigorous and understandable but also has a great deal of freedom regarding the underlying formal derivations that Azzouni's theory is committed to.

One common response that the deriviationists usually have open to them is, interestingly, not available in this case. The response would be that Azzouni could give up the need for translations between systems and

As a final point against the agent-independence of underlying formal proofs, I add that the problems of the identity and uniqueness of formal proofs strike me as the *easier* ones to answer compared to questions about the *identity of informal proofs*. There are very strong intuitions concerning which informal proofs are the same and which are not (an issue that is, for example, important to properly crediting mathematicians for their new discoveries). Presumably, in trying to examine mathematical practice, at least some attention should be paid to ideas of informal identity. An even broader line of difficulties would emerge from this though, concerning whether informal proofs which are informally identical should indicate the same classes of formal proofs and if not, why not.

All of the above follows Azzouni along the agent-independent horn of the dilemma. Taking the other horn would make matters like this far easier to deal with, since which class of formal proofs (both inter- and intra-system) underlies the informal proof would depend on the particular agent and circumstances of the informal proof. Unfortunately, the second horn cannot be what Azzouni wants because it does not suffice to establish the derivationist claims, as I will argue in the next section.

6 Agent-Dependent Derivation-Indicators

So let us consider the other horn of the dilemma, which has it that the formal proof(s) underlying any given informal one are agent-dependent and supplied over and above what is already present in the proof itself. This horn would yield great benefits: there would be readily available practical evidence that proofs can be linked to formal derivations from the field of Formal Mathematics, in which there is an ever-growing collection of computer-checkable formal counterparts for well-known mathematical proofs.[14] At least on the surface, the success of this grand formalisation project should add great credence to the idea that informal proofs can be linked to derivations. However, we once again encounter the chasm that the dilemma opens up. Formal Mathematics is very clearly agent-dependent, with different mathematicians converting different informal proofs to equally different derivations. In this section I will make the case that the agent-dependent horn of the dilemma is not available to Azzouni or other derivationists.

Firstly, as we have seen, Azzouni insists that the agents need not be aware of the indicated derivation that underlies the informal proof they are communicating. This in itself seems to put a stop to agent-dependence for Azzouni, for if the formal proof depends on agents who have no access to the formal proof there is little hope of success in this direction. Furthermore, one of the main desiderata for Azzouni, that of **(Agreement)**, would be left in a far more precarious position. For the social agreement on what constitues a correct proof is explained in terms of the indicated derivations, but if the link is now agent-dependent then there is no given reason why any two people will have the *same* class of derivations underlying the informal proof. In this case, mathematics could then end up as a lot of talking past one another.

The original reasons for wanting to reject the notion that mathematicians are aware of the underlying formal derivations are good ones. Firstly, this simply does not match up to the reality of mathematical practice. Secondly, this would fail to answer the clause of **(Agreement)** which asks for an explanation of mathematics done long before there were formal proofs in mathematics. Finally, formal proofs for mathematics tend to be long and unwieldy therefore not the kind of thing that are 'easy' to know. In (Pelc, 2009), it is argued that the formal counterparts to informal proof of theorems that have already been proved may very well not just be currently inaccessible to us, but beyond the physical limits of our universe to ever potentially check.[15] By avoiding having the mathematicians aware of the underlying derivations, Azzouni will be sidestepping these three concerns. Except, if Azzouni were to now take the second horn of the dilemma then these worries would be back with a vengeance. For in that case he would need to explain how the link from informal to formal proofs can be agent-dependent while the agent may nonetheless have no access to the formal proof, which is precisely the type of worry he was attempting to sidestep.

Generalising somewhat to other derivationist arguments, there is an even more crucial reason that they should not want to accept that their link is agent-dependent. This is that whatever the posited link may be from informal proofs to their formal counterparts, this link is one of *dependence*. The entire project is aimed at explaining the utility of informal proofs in terms of formal proofs, with their philosophically more straightforward use of logic and deductive reasoning. The desiderata of **(Correctness)** and **(Rigour)** can be tackled by

[14] In mechanical proof-checkers such as *Coq*, *Mizar*, *Isabelle*, etc.

[15] See also (Boolos, 1987) for another unwieldy formal proof for a clear informal one.

taking advantage of the dependence of informal proofs on formal ones to import the story of rigour or correctness we have for the latter. However, if we make this link agent-dependent, then the clear waters are muddied once again by the complicated relationship that the posited link has with the mathematicians themselves. For the entire point of the undertaking is to resolve the tricky problem of the practical, real-life side of mathematics in the philosophically simpler terms of formal derivations. If the posited link is agent-dependent, the very difficulty that we were resolving simply re-emerges at another level. In short, the attempt to answer the problem of informal proofs in mathematical practice will find itself once again dealing with the practical difficulties of formalisation. This should be unacceptable to any derivationist account.

7 Conclusion

Having seen that the agent-dependent approach is not compatible with the derivationist aims, let me now return to the first horn of the dilemma and give a reason as to why an independent link from informal proofs to underlying formal ones is going to be particularly hard to establish.

The reason is embodied in the desideratum of **(Content)**. To successfully give the type of account that the derivationist is after, one has to go from the informal, implicit, gappy and oftentimes hidden structure of the informal proof to a fully explicit formal proof, which has picked out everything down to the smallest details. But one of the obvious reasons that formal proofs are rarely employed in practice is that these minutiae will get in the way of explanation, comprehension and communication of proofs. The result is, unsurprisingly, that they are often left out. What follows, then, is that the link is adding extra structure and detail in going from informal to formal proofs.

Positing an agent-independent link, the details that have to be filled in should be routine and fully determined. The thought behind this could be, for example, that getting from the informal to the formal just involves taking all the gaps in the informal proof and applying a mechanical process of filling in these gaps. However, while this might work in a few cases, say where the target proof is already close to being fully formal, it is certainly not a general procedure. Mainly this is because even rather small gaps can have multiple different ways of being filled out.

Although such multiple realisations of the formal proofs corresponding to informal ones don't pose such a problem to a weaker, agent-dependent notion of formalisation, if we want an independent link this is a serious problem because it compels us to go beyond the link to explaining which realisation is the correct one or how they can all be correct, and as we have seen neither option is particularly easy. The difficulty of filling in these gaps is further compounded by the fact that proofs have a great deal of structure, which means that how we fill in a gap at one point can and does affect the options for different gaps. Believing that the answers to these technicalities is somehow already present in the proof and objectively determined is entirely misguided.

The moral, then, is that satisfying **(Content)** is really quite a challenging problem. Interestingly, the problem is one that extends far beyond Azzouni's particular proposal to derivationist projects generally. Whether one wants to reduce all mathematics to formal derivations, claim that informal proofs reveal a complete logical form, or any other proposal in this direction, the hard problem of **(Content)** is a serious roadblock.

A weaker but maybe more acceptable proposal might be something like the following: that informal proofs should be theoretically formalisable, in the sense that the mathematician producing them should be confident that all of the moves involved in the proof are logical, deductive inferences which in theory could be made explicit in some way. Such an agent-dependent view does want to associate logicality and formality with explicit deductive steps, but doesn't rely on some dependence of the full informal proof on some particular class of formal ones. Of course, there are still many issues with such a proposal, which will have to be discussed elsewhere.

References

Azzouni, J. (2004a): "The Derivation-Indicator View Of Mathematical Practice", *Philosophia Mathematica (3)* 12, pp. 81–105.

Azzouni, J. (2004b): "Proof And Ontology In Euclidean Mathematics" in Kjeldsen, T. H., Pedersen, S. A. & Sonne-Hansen, L. M. (eds.) (2004): *New Trends in the History and Philosophy of Mathematics*, Denmark: University Press of Southern Denmark, pp. 117–133.

Azzouni, J. (2005a): "How To Nominalize Formalism", *Philosophia Mathematica (3)* 13, pp. 135–159.

Azzouni, J. (2005b): "Is There Still A Sense In Which Mathematics Can Have Foundations?" in Sica, G. (ed.) (2005): *Essays on the Foundations of Mathematics and Logic*, Milan: Polimetrica, pp. 9–47.

Azzouni, J. (2009): "Why Do Informal Proofs Conform To Formal Norms?", *Foundations of Science* 14, pp. 9–26.

Azzouni, J. (2013): "That We *See* That Some Diagrammatic Proofs Are Perfectly Rigorous", *Philosophia Mathematica (3)* 21, pp. 323–338.

Bancerek, G. (1995): "The Mutilated Chess Board Problem— checked by Mizar", in Matuszewski, R. (ed.): *The QED Workshop II, Technical Report No. L/1/95*, pp. 37-38.

Black, M. (1946): *Critical Thinking*, Prentice Hall.

Boolos, G. (1987): "A Curious Inference", *Journal of Philosophical Logic* 16, pp. 1–12.

Buldt, B.; Löwe, B. & Müller, T. (2008): "Towards A New Epistemology Of Mathematics", *Erkenntnis* 68, pp. 309-329.

Carnap, R. (1945): "The Two Concepts Of Probability", *Philosophy and Phenomenological Research* 5, pp. 513–532.

Dutilh Novaes, C. (2007): *Formalizing Medieval Logical Theories*, Springer.

Dutilh Novaes, C. (2011): "The Different Ways In Which Logic Is (Said To Be) Formal", *History and Philosophy of Logic* 32, pp. 303–332.

Došen, K. (2003): "Identity Of Proofs Based On Normalization And Generality", *Bulletin of Symbolic Logic* 9, pp. 477–503.

Fallis, D. (2003): "Intentional Gaps In Mathematical Proofs", *Synthese* 134, pp. 45–69.

Gardner, M. (1957): "Mathematical Games" in *Scientific American* Iss. March 1957.

Gödel, K. (1953): Is Mathematics Syntax Of Language? in Feferman S. et al. (eds.) (1995): *Kurt Gdel Collected Works, Vol. III*, Oxford: Oxford University Press, pp. 334–364

Hersh, R. (1997): "Prove— Once More And Again", *Philosophia Mathematica* 5, pp. 153–165.

Lakatos, I., (1976): *Proofs And Refutation*, Cambridge: Cambridge University Press.

MacFarlane, J. (2000): *What Does It Mean To Say That Logic Is Formal?*, *PhD Thesis*, University of Pittsburgh.

Mancosu, P. (ed.) (2008): *The Philosophy Of Mathematical Practice*, Chippenham: Oxford University Press.

Pelc, A. (2009): "Why Do We Believe Theorems?", *Philosophia Mathematica* 17, pp. 84–94.

Rav, Y. (1999): "Why Do We Prove Theorems?", *Philosophia Mathematica* 7, pp. 5–41.

Rav, Y. (2007): "A Critique Of A Formalist-Mechanist Version Of The Justification Of Arguments In Mathematicians' Proof Practices", *Philosophia Mathematica* 15, pp. 291-320.

Robinson, J. A. (1991): "Formal And Informal Proofs" in Boyer, R. S. (ed.) (1991): *Automated Reasoning*, Kluwer Academic Publishers, pp. 267-282.