Trans-statistical behavior of a multiparticle system in an ontology of properties

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Abstract

In the last years, the surprising bosonic behavior that a many-fermion system may acquire, has raised interest because of theoretical and practical reasons. This trans-statistical behavior is usually considered to be the result of approximation modeling methods generally employed by physicists when faced with complexity. In this paper, we take a tensor product structure and an ontology of properties approach and provide two versions (standard and algebraic) of a toy model in order to argue that trans-statistical behavior allows for a realistic interpretation.

Keywords composite bosons – non-individual bundle – ontology of properties – tensor product structure

Section 1 Introduction

1.1. Indistinguishability and statistics

In classical mechanics, a composite system of two or more identical particles rearranged because of a permutation between them is statistically considered a different microstate. This fact leads to Maxwell-Boltzmann statistics. In quantum mechanics (QM), an analogous permutation does not yield a statistically different possibility. For this reason, it is said that quantum identical particles are indistinguishable. That means that any permutation between them cannot yield any observable consequence. The indistinguishability postulate (IP) of QM may be formulated as follows (see Butterfield 1993):

> IP: If the vector $|\psi\rangle$ represents the state of a composite system whose components are indistinguishable particles, then the expectation value of any observable represented by an operator O must be the same for $|\psi\rangle$ and for any permutation $|\psi'\rangle$

$$|\psi'\rangle = P|\psi\rangle : \langle\psi'|O|\psi'\rangle = \langle\psi|O|\psi\rangle \qquad (1)$$

In order to satisfy IP, a restriction to states is usually introduced in QM: the symmetrization postulate (SP). IP is satisfied by symmetric $|\psi_S\rangle$ or antisymmetric $|\psi_A\rangle$ states with respect to permutation operator P. Both of them are eigenvectors of P with eigenvalues (1) and (-1)

$$\begin{aligned} P|\psi_S\rangle &= |\psi_S\rangle \\ P|\psi_A\rangle &= -|\psi_A\rangle \end{aligned} \tag{2}$$

So, a formulation for SP may be (see Fortin and Lombardi 2021)

> SP: Any system of many identical particles is represented by either a totally symmetric quantum state (bosons) or a totally antisymmetric quantum state (fermions), where symmetry and antisymmetry are defined in terms of permutations P.

$$|\psi'\rangle = P|\psi\rangle = \pm|\psi\rangle \tag{3}$$

In order to obtain a symmetric $|\psi_S\rangle$ or antisymmetric state $|\psi_A\rangle$ from a generic state $|\psi\rangle$, symmetrizer S and antisymmetrizer A operators should be applied to it

$$\begin{aligned} S|\psi\rangle &= |\psi_S\rangle \\ A|\psi\rangle &= |\psi_A\rangle \end{aligned} \tag{4}$$

It must be noted that fermions (half-integer spin) and bosons (integer spin) have very different behaviors. A many-fermion system is represented by an antisymmetric state, and, therefore, as the Pauli Exclusion Principle states, it is not possible to find two fermions in the same state. This gives rise to the Fermi-Dirac statistic for fermions. On the contrary, in accordance with Bose-Einstein statistics, two different bosons can be in the same state.

1.2. Trans-statistical behavior

Taking into account SP, it is quite clear that quantum particles must behave as fermions or bosons. However, under certain circumstances, physicists found surprising bosonic behavior in many-fermion systems. That is the well-documented phenomenon of composite bosons or simply co-bosons. In this paper we call it trans-statistical behavior of quantum particles. On the one hand, the issue has raised theoretical interest among many researchers. Law (2005) found that the degree of entanglement between constituent fermions in a multiparticle system determines how close it behaves as a system of composite bosons. As a result, interactions are not strictly needed for the phenomenon to arise. If there are interactions, they apparently only reinforce correlations which are the determinant factor for bosonic behavior. Chudzicki *et al*. (2010) obtained a generalization of Law's approach in terms of creation and annihilation operators. On the other hand, the issue is also relevant for practical reasons since it has connections with several applications such as quantum information processing (Gigena and Rossignoli 2015), Bose-

Einstein condensates (Avancini *et al*. 2003, Rombouts *et al*. 2003), excitons (Combescot *et al*. 2001) and Cooper pairs in superconductors (Belkhir el at. 1992). Recently, some of these studies have been applied to describe both fermionic and bosonic behavior of confined Wigner molecules (Cuestas el at. 2020).

1.3. A non-realistic approximation

It is a usual assumption that trans-statistical behavior is a phenomenon that should not be interpreted realistically, but simply as a result of approximation methods frequently employed in experimental physics (see Tichy *et al*. 2014 for an example). Most phenomena are so complex that they just cannot be modelled in a realistic manner. In turn, it is necessary to work with models that only provide an approximate description of the object under scrutiny. Physicists are well aware that in these circumstances, approximate models may predict behavior that cannot be expected for a physical real object. For the sake of clarity, it is not really expected that a real pendulum will exhibit perpetual motion. That is only predicted for an approximate model. Analogously, it is not believed that a many-fermion system really behaves as a system of bosons. Trans-statistical behavior of identical particles –it is believed– is only a suitable description for the observed phenomenon that arises from approximate models of many-fermion systems under specific conditions, in which entanglement is apparently a key factor.

1.4. A TPS approach

In this paper we tackle trans-statistical behavior from a different perspective. We take a tensor product structure (TPS) approach (see Harshman and Wickramasekara 2007). As it is well-known, a TPS is a particular way (among many) to factorize the Hilbert space into subspaces or, from an algebraic approach, decompose the algebra of observables into subalgebras in order to split a system into subsystems. We benefit from studies that defend the idea that the notion of separability between subsystems is not absolute but relative to a particular partition (Zanardi 2001). In this work we explore the possibility that the relativity of separability extends to quantum statistics. Such relativity of separability leads also to the question of which of the many mathematically possible TPSs should be endowed with physical significance. The very idea of what a system is has also been put into discussion (Dugić and Jeknić 2008). At this point, the matter demands that we also adopt a philosophical perspective. The fact that a multiparticle system may be factorized in many equally legitimate structures, poses the question of what is the ontological picture that trans-statistical behavior entails if it is realistically interpreted.

1.5. A realistic interpretation

We are proposing a toy model in which different TPSs give rise to both fermion-like or boson-like behavior. We aim to show that trans-statistical behavior is built into QM formalism, in a way that favors a realistic interpretation of that phenomenon. It is important to emphasize that we are not intending to create an approximate model to capture such systems empirically, as usually performed by experimental physicists. We just play mathematically with QM formalism to create a toy model. Before proceeding, it is also necessary to make clear in what sense we are talking about reality. We are not referring to it as a *noumenon* in a naïve manner. Our concept of reality is a relative one. It is reality as it is constituted by the theory, in our case QM. It is a categorical-conceptual framework endowed with ontological significance (see Lombardi 2021). In simple terms, we talk of reality as if QM were true.

1.6. Towards an ontological lesson

If trans-statistical behavior were indeed built into QM formalism, it would be possible to take it as a fundamental concept of the theory. And if, at the same time, we assume a realistic stance towards QM, we could benefit of trans-statistical phenomenon to learn a lesson about QM ontology. A topic of debate in QM ontology is what ontological concept is adequate to refer to a quantum system. There are traditional ontologies, which are favored by the familiar particle-picture in physics, in that properties are attributes of individuals. The ontology of individuals and properties suggests that fermions should retain their identity when merged in a composite and only in a merely descriptive manner could behave as bosons. There are also ontologies based exclusively on properties (see da Costa and Lombardi 2014). From this perspective, a system may be just a non-individual bundle of properties. If a system is a bundle, there is no need that it preserves its identity when it enters in a composite. This ontology of non-individual bundles would allow us to claim that statistical behavior does not depend upon any identity conditions previously possessed by quantum systems and so, construe trans-statistical behavior in a realistic manner.

1.7. Content of the next sections

In section 2, a first version of our toy model will be proposed. In this first version, we work with a standard Hilbert space formalism. States will have logical priority over observables. Consequently, systems will be identified from their vector state and standard indistinguishability (IP), and symmetrization (SP) postulates will be employed. In this version, we settle two specific TPSs to account for both fermionic and bosonic behavior. On the one hand, alpha-partition will entail fermionic behavior. On the other, beta-partition will induce a bosonic one. We work in this section exclusively at a mathematical level.

In section 3, we will not only work mathematically. The basic lines of an ontology of properties for QM will be exposed. This ontology was originally suggested by the algebraic formalism of QM, which grants priority to observables over states. So, a second version of our toy model will be proposed, in which the two partitions settled in section 2 are reconsidered from the algebraic approach. The main idea is to show that the same set of observables yields fermion-like or boson-like behavior with respect to different TPSs, both in a single system in the very same state. It will be concluded that trans-statistical behavior –so modeled– allows a realistic interpretation that strengthens a non-individual bundle ontological picture. We end up with some final remarks

Section 2 The toy model in Hilbert space

In this section we will present a toy model in which it is possible to study the bosonic behavior of particles composed of fermions. Although we deal with only four fermions in this model, it could be easily generalized to any even number of fermions. Our interest is to argue in favor of the relativity of quantum statistics by making clear that fermionic and bosonic behavior arise as a result of considering different partitions of a multiparticle system. The striking feature of our model is that alternative symmetrization or antisymmetrization of the state of the system will not be required to obtain alternative statistics. The different decompositions are performed in this section in terms of different tensor product structures of the multiparticle system Hilbert space.

2.1.1. Fermion-like decomposition (alpha-TPS)

The toy model is a system composed of 4 elementary systems of the following type.

The component systems: Let us consider a spin ½ quantum system S represented in its own Hilbert space $\mathcal{H}$. Its Hamiltonian $\hat{H}$ has eigenstates $|n\rangle$ with energy E_n, that is $\hat{H}|n\rangle = E_n|n\rangle$. Then, each state $|\varphi\rangle \in \mathcal{H}$ can be written as $|\varphi\rangle = \sum_n c_n |n\rangle$.

The composite system: Now we will consider a quantum system $U = S_1 \cup S_2 \cup S_3 \cup S_4$ with an associated Hamiltonian $\hat{H}_U = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$ whose eigenstates $|N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle$ generate the Hilbert space $\mathcal{H}_U = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$. Then, $\hat{H}|N\rangle = E_N|N\rangle$ where $E_N = E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4}$, and each state $|\psi\rangle \in \mathcal{H}_U$ can be written as

$$|\psi\rangle = \sum_N c_N |N\rangle = \sum_{n_1,n_2,n_3,n_4} c_{n_1,n_2,n_3,n_4} |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle \tag{5}$$

Since they are fermions (spin ½), the wave function is antisymmetric under the exchange of the labels of any pair of particles. So, if permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ and $P_{3\leftrightarrow 4}$ are defined as

$$\begin{aligned}
P_{1\leftrightarrow 2}|N\rangle &= |n_2\rangle \otimes |n_1\rangle \otimes |n_3\rangle \otimes |n_4\rangle, \; P_{1\leftrightarrow 3}|N\rangle = |n_3\rangle \otimes |n_2\rangle \otimes |n_1\rangle \otimes |n_4\rangle \\
P_{1\leftrightarrow 4}|N\rangle &= |n_4\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_1\rangle, \; P_{2\leftrightarrow 3}|N\rangle = |n_1\rangle \otimes |n_3\rangle \otimes |n_2\rangle \otimes |n_4\rangle \\
P_{2\leftrightarrow 4}|N\rangle &= |n_1\rangle \otimes |n_4\rangle \otimes |n_3\rangle \otimes |n_2\rangle, \; P_{3\leftrightarrow 4}|N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_4\rangle \otimes |n_3\rangle
\end{aligned} \tag{6}$$

This condition imposes a restriction over the possible states. Then the only possible coefficients c_N (or c_{n_1,n_2,n_3,n_4}) are those such that

$$P_{1\leftrightarrow 2}|\psi\rangle = P_{1\leftrightarrow 3}|\psi\rangle = P_{1\leftrightarrow 4}|\psi\rangle = P_{2\leftrightarrow 3}|\psi\rangle = P_{2\leftrightarrow 4}|\psi\rangle = P_{3\leftrightarrow 4}|\psi\rangle = -|\psi\rangle \tag{7}$$

Because of the very way it is constructed, the Hilbert space of the composite system can be trivially factorized into four equivalent subspaces. This is the alpha tensor product structure (TPS_A) that had to be considered. In summary, *U* is a composite system of fermions whose wave function is antisymmetric with respect to TPS_A.

2.1.2. Boson-like decomposition (beta-TPS)

The decomposition of the state $|N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle$ is not the only one that can be done on the complete system U . For example, it can be described as a system composed of components systems of the following type.

The component systems: Let us consider the system $S_i = S_1 \cup S_2$ represented in its own Hilbert space $\mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2$. Its Hamiltonian $\hat{H}_i = \hat{H}_1 + \hat{H}_2$ has eigenstates $|m_i\rangle = |n_1\rangle \otimes |n_2\rangle$ with energy $E_{m_i} = E_{n_1} + E_{n_2}$, that is $\hat{H}|m_i\rangle = E_{m_i}|m_i\rangle$. Then, each state $|\varphi^i\rangle \in \mathcal{H}_i$ can be written as $|\varphi^i\rangle = \sum_{m_i} c_{m_i}|m_i\rangle$. Let us also consider another system $S_{ii} = S_3 \cup S_4$ represented in its own Hilbert space $\mathcal{H}_{ii} = \mathcal{H}_3 \otimes \mathcal{H}_4$. Its Hamiltonian $\hat{H}_{ii} = \hat{H}_3 + \hat{H}_4$ has eigenstates $|m_{ii}\rangle = |n_3\rangle \otimes |n_4\rangle$ with energy $E_{m_{ii}} = E_{n_3} + E_{n_4}$, that is $\hat{H}|m_{ii}\rangle = E_{m_{ii}}|m_{ii}\rangle$. Then, each state $|\varphi^{ii}\rangle \in \mathcal{H}_{ii}$ can be written as $|\varphi^{ii}\rangle = \sum_{m_{ii}} c_{m_{ii}}|m_{ii}\rangle$.

If we consider these components, the toy model is a composed system of two elementary systems. This is beta tensor product structure (TPS_B) employed in our model, since the Hilbert space that defines system *U* can be factorized into two subspaces.

The composite system: Now the same system U can be described as $U = S_i \cup S_{ii}$ with an associated Hamiltonian $\hat{H}_U = \hat{H}_i + \hat{H}_{ii}$ whose eigenstates $|N\rangle = |m_i\rangle \otimes |m_{ii}\rangle$ generate the Hilbert space $\mathcal{H}_U = \mathcal{H}_i \otimes \mathcal{H}_{ii}$. Then, $\hat{H}|N\rangle = E_N|N\rangle$ where $E_N = E_{m_i} + E_{m_{ii}}$, and each state $|\psi\rangle \in \mathcal{H}_U$ can be written as

$$|\psi\rangle = \sum_N c_N |N\rangle = \sum_{m_i, m_{ii}} c_{m_i, m_{ii}} |m_i\rangle \otimes |m_{ii}\rangle \tag{8}$$

To study the statistical aspects, it is necessary to define new permutation operators. This is because, for example, the labels 1 and 2 from the old operator $P_{1\leftrightarrow 2}$, no longer refer to subsystems that are present in this partition. To be able to permute the new particles it is necessary to define the operator

$$P_{i\leftrightarrow ii}\left|N\right\rangle = P_{i\leftrightarrow ii}\left|m_i\right\rangle\otimes\left|m_{ii}\right\rangle = \left|m_{ii}\right\rangle\otimes\left|m_i\right\rangle \tag{9}$$

This is the only permutation operator that exists in this partition. Since the particles S_i and S_{ii} are linked with the particles S_1, S_2, S_3 and S_4 in a direct way, it is easy to see that there is a relation between the permutation operators

$$P_{i\leftrightarrow ii}\left|N\right\rangle = \left|m_{ii}\right\rangle\otimes\left|m_i\right\rangle = \left|n_3\right\rangle\otimes\left|n_4\right\rangle\otimes\left|n_1\right\rangle\otimes\left|n_2\right\rangle = P_{1\leftrightarrow 3}P_{2\leftrightarrow 4}\left|N\right\rangle \tag{10}$$

So, the relation between permutation operators from both TPS is

$$P_{i\leftrightarrow ii} = P_{1\leftrightarrow 3}P_{2\leftrightarrow 4} \tag{11}$$

It should be noted that so far we have not changed the state, we have only written it in a new way. Therefore, the coefficients c_N have the same restrictions as before. Then, it is possible to compute how $P_{i\leftrightarrow ii}$ operates on the state $\left|\psi\right\rangle$

$$P_{i\leftrightarrow ii}\left|\psi\right\rangle = P_{1\leftrightarrow 3}P_{2\leftrightarrow 4}\left|\psi\right\rangle = -P_{1\leftrightarrow 3}\left|\psi\right\rangle = \left|\psi\right\rangle \tag{12}$$

In summary, under this decomposition (TPS_B) *U* is a composite system of bosons whose wave function is symmetric.

2.1.3. The relativity of statistics with respect to partition

Having arrived at this result, it is important to note that the fact that a set of fermions happens to form a new non-fundamental particle with bosonic behavior is not new. Indeed, it has long been known that a group of protons and neutrons, all spin 1/2, can join together to form an atomic nucleus. For example, two protons together with two neutrons join together through nuclear forces to form a nucleus of Helium 4. These atomic nuclei are bosons that exhibit empirically testable bosonic behavior such as superfluidity (Brooks and Donnelly 1977). In this case, the strong nuclear force holds the particles of the nucleus together so tightly that it is possible to think that the nucleus is a new entity. However, in the toy model presented in this work, the particles do not interact with each other and this argument is not valid. There are also other more recent examples such as atomic Bose-Einstein condensates (Avancini *et al*. 2003, Rombouts *et al*. 2002), excitons (Combescot *et al*. 2008, Rombouts *et al*. 2002), and Cooper pairs in superconductors (Belkhir and Randeria 1992). However, the mathematical treatment of all these models includes important approximations that obscure the ontological question about this type of physical systems. Then, in the case of bosons composed of fermions, the question arises that a group of bosons can share the same quantum state, but the fermions that compose them cannot, due to the Pauli exclusion principle.

That question finds an answer in our toy model, since the same state which is symmetric under TPS_B is antisymmetric under TPS_A. That is, the state $|\psi\rangle$ is symmetric with respect to the permutation operator $P_{i\leftrightarrow ii}$ in the TPS_B perspective but antisymmetric with respect to the permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ y $P_{2\leftrightarrow 3}$ in the TPS_A perspective. The relativity of statistics with respect to partition has been clearly stated for our model. This suggests that trans-statistical behavior is built into QM formalism. But we still have two ways to define subsystems, i. e., alpha and beta partitions. This brings us to the question, is it a fermionic or a bosonic system? At this stage of the present work, our answer cannot be far away from the orthodox point of view, that is, that the fundamental level has ontological priority. In this case, the fundamental level is that of the TPS_A and therefore the system is composed of fermions. While the TPS_B corresponds to the collective behavior of the fermions taken in pairs. From this perspective, bosonic behavior becomes an appearance. It is the unreal behavior of particles that do not really exist except as pseudoparticles or virtual particles. However, a definitive answer will come out from the algebraic version of our toy model that will be provided in the next section, once the ontology of properties for QM has been introduced.

Section 3 Trans-statistical behavior in an ontology of properties

3.1. Classical and quantum particles as individual objects

From a philosophical perspective, an individual is an object that possess an identity that makes it distinguishable from other objects and is able to retain its identity over time. It is also believed to be the bearer of a set of properties, such as location in space and time. The individuality of such an object may be regarded to be granted for something over and above the properties that it possesses, such as substance. Alternatively, Leibniz Identity Principle (PII) establishes that individual identity depends only on the properties possessed by the object. Identification of the individual object over time is made possible by its spatiotemporal trajectory. The ontological category of individual fits properly when referring to classical particles. But it runs into trouble when applied to quantum particles. Quantum indistinguishability is known to prevent particles of the same kind to be re-identified once a permutation is performed between them. Moreover, contextuality prevents quantum particles to possess well-defined properties. As a consequence, omnimode determination principle that is expected to be satisfied by any individual object is violated by particles in the quantum domain. They do not even have well-defined spatiotemporal trajectories, which would have allowed to identify them over time and keep track of particles being permuted.

These features led some of the founding fathers of QM (Born and Heisenberg) to radically discard the category of individual to refer to quantum particles. They are simply not individuals. This idea was reflected in early discussions (see Weyl 1931). This constitutes the so-called Received View concerning this matter, which eventually entailed the development of non-standard formal systems to represent non-individual objects (Krause 1992). Recently, a variety of authors criticized the Received View claiming that the category of individual may hold if we drop PII or at least some of its strongest forms. In order to make this view consistent with quantum statistics, van Fraassen (1985) argued that it is not necessary to admit equiprobability for each possible configuration as usually assumed in statistical mechanics. Alternatively, French (1989) proposed that states that are neither symmetric nor antisymmetric are ontologically possible but physically inaccessible. From this perspective, quantum particles are considered individual objects that are contingently in states that make them indistinguishable. Muller and Saunders (2008) explored the possibility of weakly discern between quantum identical particles in relational terms.

3.2. An ontology of properties for quantum systems

In the context of modal interpretations of QM, some authors proposed a new quantum ontology of properties without individuals (see da Costa, Lombardi and Lastiri 2013; da Costa and Lombardi 2014; Lombardi and Dieks 2016). The choice for this ontology is strongly suggested by the aforementioned quantum features (contextuality and indistinguishability). Our guiding hypothesis is that also trans-statistical behavior matches the most with a non-individual ontology.

3.2.1. Ontology of properties and algebraic formalism

Usual presentations of QM employ Hilbert space formalism. It is mathematically built from a set of vectors, which in turn represent possible physical states of the system. System observables are represented by operators that act on already defined state vectors. The logical priority of system states over observables that characterizes Hilbert space formalism favors an ontology of individuals and properties. Systems are individuals identified by their state space and observables are properties that inhere in them (see Ballentine 1998, 234-235).

As it is known, it is also possible to employ an algebraic formalism in QM. Taking this option allows even a greater degree of generality, since mix states cannot be represented in Hilbert space formalism. In algebraic formalism, the set of physical observables are represented by an algebra of operators. System state is represented by a functional that act upon those already defined operators, in order to compute expected values. In this case, logical priority of observables over states suggests an ontology of properties, where there may be no individuals. Systems are defined exclusively by their algebra of observables. State functional is simply a device that codifies quantum probabilities (see Ballentine 1998, 48).

3.2.3. Ontology of properties. Semantic correspondences

To put it more formally, an ontology of properties without individuals is defined by the following semantic correspondences (see Fortin and Lombardi 2021):

- The algebra of *self-adjoint operators* represents the set of *physical observables* that define a quantum system, which in turn corresponds to the set of *instances of universal type-properties* in the ontological domain.

- *Eigenvalues* of self-adjoint operators represent possible *physical values*, which in turn corresponds to possible *case-properties* belonging to each type-property.

- *Probability functions* represent physical *probability distributions* for each physical observable, which in turn corresponds to *ontological propensities* of each possible case-property

- *Functionals* over algebra of observables represent *physical states*. This last item has no ontological counterpart, since physical states are just devices that assign a probability distribution for each observable.

It is important to notice that we do not talk about physical outcomes but about physical values because the ontology of properties was first developed in the context of modal interpretations of QM. In this family of interpretations, the observables may have determined values regardless of a measurement context. A preferred context is defined *a priori* and each modal interpretation postulates a particular actualization rule. Nonetheless, the ontology of properties is equally suitable for the standard interpretation or for others not belonging to the modal family.

3.2.4. Quantum systems as non-individual bundles of possible properties

The ontology of properties yields a picture of quantum systems in which they are just bundles of possible case-properties without any individual identity. The familiar particle-picture assumed in physical practice is generally discarded and could be retained only under peculiar circumstances. It is important to stress that it is not the traditional bundle of actual properties, designed in metaphysics to account for classical individual objects without the notion of substance. A quantum system could not be this latter kind of bundle because of contextuality restrictions definitively stated in Kochen-Specker theorem (1967). Even more important is to emphasize that bundles of possible case-properties are no longer object of PII. It is not a matter that PII is false. It simply just not applies to them. Bundles of possible case-properties do not retain any identity each time they merge into a composite bundle or split into them. These features of the ontology of properties make it adequate to overcome the difficulties that quantum contextuality and quantum indistinguishability impose upon the design of a QM ontology. As it will be soon formally stated, this ontology fits properly also with trans-statistical behavior, in which it is observed that somehow a set of fermions loses its identity and become a set of bosons under certain circumstances. It is rather obvious that an ontology based in individuals could not in any way construe that phenomenon in a realistic manner. The ontology of properties certainly does. Of course, a basic assumption that is previously needed to choose for this ontology is to endow modality with an ontological meaning.

3.2.5. Ontology of properties and indistinguishability

An additional result of the ontology of properties for QM is a re-statement of traditional indistinguishability postulate (IP see eq. 1 in section 1) that makes symmetrization postulate (SP see eq. 3 in section 1) a natural consequence of the ontology. When two or more indistinguishable bundles are combined, it is natural to expect that the instances of universal type-properties belonging to the composite bundles do not distinguish between those component bundles. More simply, when two indistinguishable bundles merge into a single whole, which component bundle is taken first and which second does not matter at all. Mathematically, the restriction that yields the observed statistics is no longer imposed over states (as in SP) but directly over observables. IP_{obs} is formulated as (see Lombardi and Castagnino 2008 and Fortin and Lombardi 2021 for a complete justification)

$$O' = P^{\dagger} O P : \langle \psi | O' | \psi \rangle = \langle \psi | O | \psi \rangle \tag{13}$$

Then, the observables that respect this condition will be symmetric, that is $O_{sym} = P^{\dagger} O_{sym} P$ and form the space $\mathcal{O}_{sym}$ (see Fortin and Lombardi 2021 for details). In contrast with standard IP, IP_{obs} is ontologically motivated, since a bundle is symmetric if its constituents are identical. Let us consider two bundles h^1 and h^2 defined by different instances of the same algebra of observables $\mathcal{O}_1 = \mathcal{O}_2$ such that $h^1 \triangleq h^2$. That means that these bundles are represented in the physical domain by systems or "particles" of the same kind and must be considered indistinguishable. Of course, different indices in this case does not mean physical distinguishability. These two bundles merge in a composite bundle h^U such that $h^U = h^1 * h^2$. Consequently, the algebra $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_2 \vee \mathcal{O}_1$ defines bundle h^U. Now the restriction over observables $O_U \in \mathcal{O}_U$ established in IP_{obs} (eq. 13) must be carried out. This requires that observables $O_U = \sum_{ij} k_{ij} \left(O_{1i} \otimes O_{2j} \right)$ are such that $O_{1i} \otimes O_{2j} = O_{2i} \otimes O_{1j}$. This means that observables O_U belonging to bundle h^U are symmetric with respect to permutation of bundles h^1 and h^2 (see Fortin and Lombardi 2021).

The restriction imposed by (eq. 13) includes both the case of fermions and bosons. This is because the permutation operator appears twice, then both in the case that the state ($|\psi_S\rangle$) is eigenstate of P with eigenvalue 1

$$\langle \psi_S | O_{sym} | \psi_S \rangle = \langle \psi_S | P^{\dagger} O_{sym} P | \psi_S \rangle = (1)^2 \langle \psi_S | O_{sym} | \psi_S \rangle = \langle \psi_S | O_{sym} | \psi_S \rangle \tag{14}$$

and in the case that it is -1 ($|\psi_A\rangle$)

$$\langle\psi_A|O_{sym}|\psi_A\rangle = \langle\psi_A|P^{\dagger}O_{sym}P|\psi_A\rangle = (-1)^2\langle\psi_A|O_{sym}|\psi_A\rangle = \langle\psi_A|O_{sym}|\psi_A\rangle \quad (15)$$

the eigenvalue appears squared. To account for bosons or fermions separately, it is necessary to further restrict the space of observables. Usually, to obtain the symmetric/antisymmetric state $|\psi_S\rangle / |\psi_A\rangle$ from a generic state $|\psi\rangle$, it is necessary to apply the operator *S/A* respectively $|\psi_S\rangle = S|\psi\rangle / |\psi_A\rangle = A|\psi\rangle$, then the expectation value of an observable O is

$$\langle O\rangle_{|\psi_S\rangle} = \langle\psi_S|O|\psi_S\rangle = \langle\psi|S^{\dagger}OS|\psi\rangle = \langle\psi|O_{\mathcal{S}}|\psi\rangle = \langle O_{\mathcal{S}}\rangle_{|\psi\rangle} \quad (16)$$

$$\langle O\rangle_{|\psi_A\rangle} = \langle\psi_A|O|\psi_A\rangle = \langle\psi|A^{\dagger}OA|\psi\rangle = \langle\psi|O_A|\psi\rangle = \langle O_A\rangle_{|\psi\rangle} \quad (17)$$

It is easy to see that observables of the type $O_S = S^{\dagger}OS$ form the subspace $\mathscr{O}_S \subset \mathscr{O}_{sym}$ and observables of the type $O_A = A^{\dagger}OA$ form the subspace $\mathscr{O}_A \subset \mathscr{O}_{sym}$. Therefore, the same empirical reason that imposes the restriction to symmetric-bosonic states $|\psi_S\rangle$ or antisymmetric-fermionic states $|\psi_A\rangle$ in the usual presentations, from the present perspective imposes the restriction to bosonic observables $O_S \in \mathscr{O}_S$ or fermionic observables $O_A \in \mathscr{O}_A$.

3.3. A semi-classical behavior

An ontology that focuses on properties has the advantage that it allows us to define a system from a set of observables. That is, if a quantum system is defined from a set of properties, and those properties are each linked to a mathematical operator, then it is possible to define the physical system from a set of observables. This allows us to build a coherent description of systems that exhibit classical and quantum behavior at the same time. This is the case, for example, of a tunnel-effect transistor. So let us consider it. On the one hand, at first glance it can be considered a classic object. It is possible to touch it, measure its position and speed simultaneously at all times, when it moves it follows a defined trajectory, etc. However, when we examine the internal processes of the transistor it becomes obvious that it is a quantum object. This is due to the fact that, in addition to endless quantum interactions between the atoms that make up the transistor, the tunnel effect occurs inside it. To describe this situation from the point of view of a state-centered ontology, it is necessary to define a global state ρ of the transistor and then define subsystems. Each subsystem *S* will have its reduced state ρ_S which is obtained by taking the partial trace over the other degrees of freedom (the environment *E*), then $\rho_S = Tr_E(\rho)$. According to the decoherence theory, to demonstrate that this subsystem has a classic behavior it is necessary to show that the reduced state becomes diagonal in a decoherence time and therefore it can be considered a classical-like state. In addition, it is necessary to show that there is another partition in which the reduced state is not diagonal and therefore retains its quantum character. This state-based description can lead to ontologically sensitive situations. For example, the existence of systems in which it is possible to make a partition such that all its subsystems have reduced diagonal states, and therefore classical, but the joint state is not diagonal (see Castagnino *et al*. 2010a and Lombardi *et al.* 2012).

From the point of view of a property-centered ontology, the description of this type of system does not pose any challenge. It is possible to define a classical system represented by the space of observables $\mathcal{O}_c$ that includes all the observables that behave classically. On the other hand, observables that behave quantically form the space of quantum observables $\mathcal{O}_q$. From this space it is possible to define another quantum system that behaves in a quantum way. With this approach, the classical / quantum character of an object depends on which are the relevant observables that are considered. For an exhaustive review of the decoherence and the classical limit from the point of view of the observables see Fortin and Lombardi 2019, 2016, 2014, Fortin 2014, Castagnino *et al*. 2008, 2010b, 2010c, Castagnino and Fortin 2011a, 2011b, 2012.

The example above aims to show the comparative advantages that a property-centered ontology generally has over a state-centered one. Nonetheless, it is important to stress that the example above is not exactly analogous with the algebraic version of the toy model that we present below. In the tunnel-effect transistor, classical behavior and quantum behavior could not be modeled exactly as if they arose from the very same system, since in order to account for different behaviors, it was necessary to define different sets of observables. In what follows, it will become clear that instead, fermionic and bosonic behavior arise from the very same set of observables. So, the ontological lesson we can draw from our algebraic toy model is even stronger.

3.4. Algebraic version of the toy model

3.4.1. Definition of the total system based in its observable space

Let us consider an aggregate h^U of indistinguishable bundles $h^1 \triangleq h^2 \triangleq h^3 \triangleq h^4$ such that $h^U = h^1 * h^2 * h^3 * h^4$. This aggregate of bundles h^U, which is itself a new bundle, is in the physical domain a composite system U of indistinguishable subsystems $S_1 = S_2 = S_3 = S_4$ each of them with spin ½ such that $U = S_1 \cup S_2 \cup S_3 \cup S_4$. We are adopting here an ontology of properties suggested by the algebraic approach of QM, so subsystems $S_1 = S_2 = S_3 = S_4$ are not defined in Hilbert space, but by the algebras of observables $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4$, where each algebra represents each subsystem type-properties. System U is defined in terms of an algebra $\mathcal{O}_U$ such that $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 \vee \mathcal{O}_3 \vee \mathcal{O}_4$ which is the minimal algebra generated by the subsystems algebras.

Since these subsystems are indistinguishable and consequently the bundle h^U is symmetrical with respect to any permutation of component bundles, the operators representing observables $O_U \in \mathcal{O}_U$ are symmetric in accordance with IP_{obs} (eq. 13)

$$O'_U = P_\alpha^\dagger O_U P_\alpha = O_U \tag{18}$$

Where P_α represents each element of the set $\{P_\alpha\}$ of all possible permutation operators relative to alpha-partition (TPS_A) of system U. In addition, the observables O_U are symmetric with respect to the only admissible permutation relative to beta-partition (TPS_B), since $P_\beta = P_{i\leftrightarrow ii}$ is equivalent to one of the elements of the set $\{P_\alpha\}$, i. e. the product of the permutation operators $P_{1\leftrightarrow 3}P_{2\leftrightarrow 4}$ (see eq. 11). Consequently, if observables O_U satisfy condition $O'_U = P_\alpha^\dagger O_U P_\alpha = O_U$ (eq. 18), they also satisfy

$$O'_U = P_\beta^\dagger O_U P_\beta = O_U \tag{19}$$

This means that every observable O_U belonging to $\mathcal{O}_U$ is permutation invariant with respect to both partitions

$$\forall O_U \in \mathcal{O}_U, O_U = P^\dagger O_U P \tag{20}$$

But this is not the whole story, since $\mathcal{O}_U$ includes both fermionic and bosonic observables (see eq. 14 and 15). It is necessary to introduce specifically the fermionic character of the TPS_A subsystems.

3.4.2. Fermionic subalgebra of observables

In section 2, because of the value of spin ½ of the component systems in TPS_A, we demanded that the state of the system U were antisymmetric with respect to permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ and $P_{3\leftrightarrow 4}$ (eq. 7). However, in this section we are adopting an ontology of properties. The system state will be considered just a device that assigns a probability at each possible event. It plays no roll in identifying the system. The fermionic character that our bundle may assume ought to be defined exclusively in terms of its properties. So, the empirical behavior of a fermionic system will be obtained by imposing a further restriction to its observables. Consider the antisymmetrizer projector corresponding to the TPS_A

$$A_A = \frac{1}{\sqrt{N!}} \sum_{i=1}^{\alpha} \pm P_\alpha \tag{21}$$

Notice that the projector A is alpha-indexed in correspondence with the permutations that define it. The operator P_α (also alpha-indexed) represents each possible permutation (including the identity I) belonging to TPS_A, $N!$=24 is the quantity of those permutations, and $(\pm)$ depends of the parity of P_α: $(+)$ if it is even or $(-)$ if it is odd. Usually in QM, the antisymmetrizer projector is applied to a generic state $A_A|\psi\rangle = |\psi_{A_\text{A}}\rangle$. Instead, we are applying it to our observables

$$A_\text{A}^\dagger O_U A_\text{A} = O'_U \tag{22}$$

That operation allows us to define a fermionic subalgebra $\mathcal{O}^F \subset \mathcal{O}_U$ such that

$$\forall O_F \in \mathcal{O}^F, O_F = A_\text{A}^\dagger O_U A_\text{A} \tag{23}$$

Which is the algebra in respect with any generic state $|\psi\rangle$ will behave as antisymmetric

$$\langle\psi|O_F|\psi\rangle = \langle\psi|A_\text{A}^\dagger O_U A_\text{A}|\psi\rangle = \langle\psi_{A_\text{A}}|O_U|\psi_{A_\text{A}}\rangle \tag{24}$$

3.4.3. Bosonic subalgebra of observables

In section 2, we found that the same coefficients c_N that make the system state antisymmetric with respect to the set $\{P_\alpha\}$ of permutations, turn it symmetric with respect to operator $P_{i\leftrightarrow ii}$. That is

$$P_\alpha|\psi_{A_\text{A}}\rangle = -|\psi_{A_\text{A}}\rangle \Rightarrow P_{i\leftrightarrow ii}|\psi_{A_\text{A}}\rangle = |\psi_{A_\text{A}}\rangle \tag{25}$$

Then, every antisymmetric state in the TPS$_\text{A}$ $|\psi_{A_\text{A}}\rangle$ is a symmetric state $|\psi_{S_\text{B}}\rangle$ in the TPS$_\text{B}$

$$\forall|\psi_{A_\text{A}}\rangle / |\psi_{A_\text{A}}\rangle = A_\text{A}|\psi\rangle \rightarrow |\psi_{A_\text{A}}\rangle = |\psi_{S_\text{B}}\rangle \tag{26}$$

However, the inverse relationship is not valid

$$P_{i\leftrightarrow ii}|\psi_{A_\text{A}}\rangle = |\psi_{A_\text{A}}\rangle \not\Rightarrow P_\alpha|\psi_{A_\text{A}}\rangle = -|\psi_{A_\text{A}}\rangle \tag{27}$$

This is easy to see in a trivial example. Let us consider the state

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}\left(|n_1\rangle\otimes|n_2\rangle\otimes|n_3\rangle\otimes|n_4\rangle + |n_3\rangle\otimes|n_4\rangle\otimes|n_1\rangle\otimes|n_2\rangle\right) \\ |\psi\rangle &= \frac{1}{\sqrt{2}}\left(|m_i\rangle\otimes|m_{ii}\rangle + |m_{ii}\rangle\otimes|m_i\rangle\right) \end{aligned} \tag{28}$$

If we apply the operator $P_{i\leftrightarrow ii}$ we obtain the same state

$$\begin{aligned} P_{i\leftrightarrow ii}|\psi\rangle &= \frac{1}{\sqrt{2}}\left(|m_{ii}\rangle\otimes|m_i\rangle + |m_i\rangle\otimes|m_{ii}\rangle\right) = |\psi\rangle \\ P_{i\leftrightarrow ii}|\psi\rangle &= \frac{1}{\sqrt{2}}\left(|n_3\rangle\otimes|n_4\rangle\otimes|n_1\rangle\otimes|n_2\rangle + |n_1\rangle\otimes|n_2\rangle\otimes|n_3\rangle\otimes|n_4\rangle\right) = |\psi\rangle \end{aligned} \tag{29}$$

But if we apply, for example $P_{1\leftrightarrow 2}|\psi\rangle$, we don't obtain the same state changed sign

$$P_{1\leftrightarrow 2}|\psi\rangle = \frac{1}{\sqrt{2}}\left(|n_2\rangle \otimes |n_1\rangle \otimes |n_3\rangle \otimes |n_4\rangle + |n_4\rangle \otimes |n_3\rangle \otimes |n_1\rangle \otimes |n_2\rangle\right) \neq -|\psi\rangle \tag{30}$$

This means that to build the bosonic subalgebra of this model we cannot simply apply the canonical symmetrizer operator to a generic state in $\mathrm{TPS_B}$. If we define this operator in the standard way, $\tilde{S}_{\mathrm{B}} = \frac{1}{2}(I + P_{i\leftrightarrow ii})$, the space of observables of the form $O_U = \tilde{S}_{\mathrm{B}}^{\dagger} O_U \tilde{S}_{\mathrm{B}}$ is bigger than $\mathcal{O}^F$. Then, we will try in other way, we can build the bosonic subspace in this model by means of a symmetrizer operator defined as

$$S_{\mathrm{B}} = A_{\mathrm{A}} \tag{31}$$

Oddly enough, this operator is a legitimate symmetrizer for the $\mathrm{TPS_B}$ because all antisymmetric states in the $\mathrm{TPS_A}$ are symmetric in the $\mathrm{TPS_B}$. Then, we are applying it to our observables

$$S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}} = O'_U \tag{32}$$

That operation allows us to define a bosonic subalgebra $\mathcal{O}^B \subset \mathcal{O}_U$ such that

$$\forall O_B \in \mathcal{O}^B, O_B = S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}} \tag{33}$$

The observables generated with the operator S_{B} are "less" than those generated by $\tilde{S}_{\mathrm{B}}$, however it generates all that are necessary to describe this model. Then, $\mathcal{O}^B$ is the algebra in respect with any generic state $|\psi\rangle$ will behave as symmetric

$$\langle\psi|O_B|\psi\rangle = \langle\psi|S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}}|\psi\rangle = \langle\psi_S|O_U|\psi_S\rangle \tag{34}$$

Since there is a direct relation between S_{B} and A_{A}, we have

$$\langle\psi|O_F|\psi\rangle = \langle\psi|A_A^{\dagger} O_U A_A|\psi\rangle = \langle\psi|S_B^{\dagger} O_U S_B|\psi\rangle = \langle\psi|O_B|\psi\rangle \tag{35}$$

Then, the same observables can be interpreted as fermionic observables from the $\mathrm{TPS_A}$ and as bosonic observables from the $\mathrm{TPS_B}$. This fact invites us to change the notation with which we call the algebra of observables, instead of $\mathcal{O}^F$ we will use $\mathcal{O}_{\mathrm{A}}^F$ and instead of $\mathcal{O}^B$ we will use $\mathcal{O}_{\mathrm{B}}^B$. In this way, we can say that both algebras are the same, that is

$$\mathcal{O}_{\mathrm{A}}^F = \mathcal{O}_{\mathrm{B}}^B = \mathcal{O} \tag{36}$$

The difference in the notation is that if $\mathcal{O}$ is considered from different partitions the system has fermionic or bosonic behavior.

3.4.4 Trans-statistical behavior and its full ontological lesson

The previous result is relevant and may allow us to affirm that the system itself is neither fermionic nor bosonic. In the Hilbert space version of our toy model, we were able to find that quantum statistics can be considered as TPS-relative. That result made possible to take trans-statistical behavior as a fundamental concept of QM, built into its formalism. But in that first version of the model, it was not yet possible to interpret trans-statistical behavior realistically and learn the full ontological lesson from it since there was a correspondence between a particular statistical behavior and a specific definition of subsystems. Let us recall the exact correspondences

Fermionic behavior to subsystems $S_1 \cup S_2 \cup S_3 \cup S_4 = U$ (see subsection 2.1.1)

Bosonic behavior to subsystems $S_i \cup S_{ii} = U$ (see subsection 2.1.2)

That is, statistical behavior seemed to be attached to peculiar identity conditions of the subsystems that entail it. It was necessary to endow one of the partitions (TPS_A) with ontological significance and the other with just a descriptive meaning. That criterion corresponds with the ontology of individuals suggested by the way systems are defined in Hilbert space formalism (through state vectors that may be factorized together with Hilbert space).

When the toy model was translated to algebraic formalism, we obtained a relevant result at the end of last subsection: it was possible to define the multiparticle system of our model by means of a single set of observables that yields both fermionic or bosonic behavior when considering different TPSs. Namely

Whole system U is defined by the algebra $\mathcal{O}_{\mathrm{A}}^{F} = \mathcal{O}_{\mathrm{B}}^{B} = \mathcal{O}$ (see eq. 36)

As a consequence, statistical behavior of the total system no longer depends on particular identity conditions of its possible subsystems. Algebraic approach makes possible to work with the system as a whole, which in itself is neither fermionic nor bosonic. That is, we obtained a single set of observables for the whole system that is compatible with both statistical behaviors.

Quantum statistics was attached to a specific definition of subsystems, which corresponds in the ontological domain to peculiar identity conditions. But when we work with the algebraic formalism it was possible to directly constrain the observables of the whole system in order to make them symmetric or antisymmetric with respect to particular partitions. In this case, a single set of observables represents the whole system, without establishing a priori preferences for one partition or other. That single set may yield statistical behavior when properly considered. Statistical behavior from this perspective is not linked anymore to peculiar identity conditions of the system subsystems. Only a physical criterion outside from the formalism can give ontological priority to any of the partitions.

Therefore, from the algebraic approach there is no ontological priority between both TPS. They only express equally legitimate statistical behavior that a multiparticle system may exhibit as a whole. This ontological flexibility corresponds with the relaxed identity conditions that are assigned to quantum systems when they are considered in the ontological domain as non-individual bundles of properties. They can merge or split without carrying with them restrictive identity conditions. That flexibility is required to realistically interpret trans-statistical behavior. In turn, trans-statistical behavior thus construed reinforces the ontology of properties for QM in a virtuous theoretical circularity.

Conclusions

In this work, we proposed a toy model of trans-statistical behavior taking a TPS approach in the standard Hilbert space formalism. This model suggests that quantum statistics is TPS-relative and that trans-statistical behavior is built into QM formalism. As a result, trans-statistical behavior may be considered a part of QM and not merely a matter of approximation modelling methods. From this approach the multiparticle system is ontologically fermionic and descriptively bosonic. Fermions taken in pairs may behave like bosons, but it is not an ontologically proper bosonic behavior. As a consequence, if we adopt an ontology of individuals for QM suggested by Hilbert space formalism, trans-statistical behavior cannot be realistically interpreted.

The algebraic version of the toy model shows that a fermionic or bosonic system may be defined by the same set of observables. Since each observable semantically corresponds to a property in an ontology of properties without individuals, it is possible to interpret trans-statistical behavior from an ontology in which quantum systems are just non-individual bundles of properties. In this ontology, bundles do not retain their identity when they merge into a single whole or split into subsystems. This feature of the ontology of properties allows to interpret trans-statistical behavior in a realistic manner. The argument ran as follows. The algebraic toy model suggests that a multiparticle system may not be regarded as fermionic or bosonic until a specific TPS is chosen. Since statistical behavior does not depend upon a peculiar physical identity of quantum systems, it is possible to claim that a multiparticle system really behaves both as fermions or bosons. As a result, trans-statistical behavior is realistically interpreted.

Last but not least, realistically-construed trans-statistical behavior reinforces the ontology of properties originally proposed to deal with other ontological challenges posed by QM, such as contextuality and indistinguishability.

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