

Quantum Conditional Probabilities and New Measures of Quantum Information

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Abstract

We use a novel form of quantum conditional probability to define new measures of quantum information in a dynamical context. We explore relationships between our new quantities and standard measures of quantum information such as von Neumann entropy. These quantities allow us to find new proofs of some standard results in quantum information theory, such as the concavity of von Neumann entropy and Holevo's theorem. The existence of an underlying probability distribution helps to shed some light on the conceptual underpinnings of these results.

1 Introduction

Quantum information is primarily understood in terms of von Neumann entropy and related quantities [1, 2]. Due to quintessentially quantum phenomena such as entanglement, quantum information measures—such as conditional von Neumann entropy and mutual von Neumann information—lack well-defined underlying probability distributions. Nevertheless, despite their own somewhat mysterious conceptual underpinnings, these quantities have proved useful for reframing and clarifying aspects of quantum information. Many of the relationships satisfied by classical information measures are mirrored by their quantum analogues [1–3], sometimes quite remarkably, as in the case of strong subadditivity [4].

In this paper, we define and study new forms of quantum information that complement the standard quantities. The key ingredients in our approach are conditional probability distributions, first studied in [5, 6], that provide an underlying picture for the type of information being described. In particular, we are able to provide a description of information flow in the context of open quantum systems whose dynamical evolution is well-approximated by linear, completely positive, trace-preserving (CPTP) maps, without any explicit appeal to larger Hilbert spaces or ancillary systems. We show that some standard results of quantum information theory emerge quite naturally from our perspective.

Section 2 provides some relevant background on classical and quantum information. In Section 3, we define new forms of quantum conditional entropy and quantum mutual information in terms of quantum conditional probabilities, and briefly describe a dynamical interpretation of these quantities. In Section 4, we use the results of the previous section to analyze processes under which there is growth in entropy (in the sense of Shannon) and to provide new proofs of the concavity of von Neumann entropy and quantum data processing. We demonstrate that our quantum data-processing inequality provides a natural interpretation

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of Holevo's theorem in a dynamical context, showing that Holevo's χ acts as an upper bound on the amount of information that can flow from a system's initial configuration to a later one. We conclude in Section 5 with some ideas for generalizing our quantum conditional probabilities, and other future directions.

2 Background

2.1 Shannon Entropy, Density Matrices, and von Neumann Entropy

Consider a random variable X whose set of outcomes $\{x\}_x$ occur according to a probability distribution $\{p(x)\}_x$. Using this data, we can compute expectation values, standard deviations, and so on. Assuming a discrete set of outcomes, the average information encoded in the probability distribution is given by its Shannon entropy,

$$H(X) \equiv - \sum_x p(x) \log p(x). \quad (1)$$

In quantum theory, observables are a non-commutative generalization of random variables, with their set of eigenvalues playing the role of the set of possible outcomes. A given density matrix $\hat{\rho}$ generalizes the role of a probability distribution, allowing us to compute statistical quantities such as the expectation value of an observable $\hat{\mathcal{O}}$:

$$\langle \mathcal{O} \rangle = \text{Tr}[\hat{\rho} \hat{\mathcal{O}}]. \quad (2)$$

The simplest density matrices correspond to pure states, and can be expressed as a projection operator of the form $|\Psi\rangle\langle\Psi|$. In this simple case, the formula (2) reduces to

$$\langle \mathcal{O} \rangle = \text{Tr}[|\Psi\rangle\langle\Psi| \hat{\mathcal{O}}] = \langle \Psi | \hat{\mathcal{O}} | \Psi \rangle. \quad (3)$$

In general, a density matrix has infinitely many possible decompositions over sets of projectors $\{\hat{\Pi}_\alpha\}_\alpha$,

$$\hat{\rho} = \sum_\alpha \lambda_\alpha \hat{\Pi}_\alpha, \quad \hat{\Pi}_\alpha = |\phi_\alpha\rangle\langle\phi_\alpha|, \quad (4)$$

where the set $\{\lambda_\alpha\}_\alpha$ consists of non-negative, real numbers that sum to unity, and where $\{|\phi_\alpha\rangle\}_\alpha$ is not necessarily an orthonormal set of states. Each such decomposition has a corresponding Shannon entropy:

$$H(\{\lambda_\alpha\}) = - \sum_\alpha \lambda_\alpha \log \lambda_\alpha. \quad (5)$$

The decomposition that *minimizes* [7] the Shannon entropy consists of the eigenvalues and corresponding eigenprojectors of $\hat{\rho}$,

$$\hat{\rho} = \sum_i p_i \hat{P}_i, \quad \hat{P}_i = |\Psi_i\rangle\langle\Psi_i|, \quad (6)$$

where $\{|\Psi_i\rangle\}_i$ is the set of eigenstates of $\hat{\rho}$. The von Neumann entropy of a density matrix $\hat{\rho}$ is this minimal Shannon entropy of $\hat{\rho}$,

$$S(\hat{\rho}) \equiv -\text{Tr}[\hat{\rho} \log \hat{\rho}] = - \sum_i p_i \log p_i, \quad (7)$$

and therefore represents the minimum amount of average information that can be encoded in a system described by $\hat{\rho}$.

2.2 Classical Conditional Entropy and its Quantum Counterpart

Classically, the conditional entropy of a random variable Y given another random variable X is defined in terms of a conditional probability distribution $p(y|x)$ that describes correlations between possible outcomes

of the two random variables Y and X . Specifically, the conditional entropy of a random variable Y given that X takes the value x is defined to be

$$H(Y|x) \equiv - \sum_y p(y|x) \log[p(y|x)]. \quad (8)$$

The full conditional entropy is then

$$H(Y|X) \equiv \sum_x H(Y|x)p(x) = - \sum_{x,y} p(y|x)p(x) \log[p(y|x)], \quad (9)$$

which can be thought of as the average information encoded in Y given a particular outcome of X , averaged across all the possible outcomes of X .

Conditional entropies satisfy the identity

$$H(Y|X) = H(Y, X) - H(X), \quad (10)$$

where $H(Y, X)$ is the Shannon entropy of the joint distribution in X and Y . The identity (10) captures the intuition that the conditional entropy measures the information about Y encoded in its correlation with X in excess of information encoded in X alone.

In the quantum case, the pair of random variables X and Y are replaced by a bipartite quantum system AB , with a corresponding density matrix $\hat{\rho}_{AB}$. The standard definition of conditional von Neumann entropy adopts the form of the classical relation (10), with $S(\hat{\rho}_{AB})$ in place of the classical joint entropy and $S(\hat{\rho}_B)$ substituted for $H(X)$, where $\hat{\rho}_B$ is the reduced density matrix for subsystem B , as defined by the partial trace over subsystem A . That is, the conditional von Neumann entropy is given by

$$S(A|B) \equiv S(\hat{\rho}_{AB}) - S(\hat{\rho}_B), \quad \hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}]. \quad (11)$$

Unlike classical conditional entropy, conditional von Neumann entropy defined by (11) lacks an underlying probability distribution, as can be seen from the fact that $S(A|B)$ can be negative [1] when subsystems A and B are entangled. In [8], the authors introduce a conditional amplitude operator $\hat{\rho}_{A|B}$ as one possible generalization of a conditional probability distribution, but the operator is *not* a density matrix, and thus lacks a clear interpretation itself. Operational approaches are quite fruitful (see [9] for example), but they do not always clarify the conceptual underpinnings of such quantities.

3 Quantum Conditional Probabilities and Information

3.1 Quantum Conditional Probabilities

The type of information measures studied in this paper are built from quantum conditional probabilities first explored in the context of the minimal modal interpretation of quantum theory [5, 6]. While the quantities we discuss here require nothing beyond standard quantum theory for their formulation, we adopt the language of the minimal modal interpretation, as it provides a useful way to describe what follows.

To start, imagine that at a given time, a quantum system is described by an ‘objective’ density matrix $\hat{\rho}_Q$ —objective in the sense that it is empirically optimal among all possible density matrices that an external observer could assign to the system.¹ Now suppose that from the initial time to a later time, the density matrix evolves from $\hat{\rho}_Q$ to a final density matrix $\hat{\rho}_R$ according to a linear CPTP map:

$$\hat{\rho}_R = \mathcal{E}_{R \leftarrow Q}\{\hat{\rho}_Q\}. \quad (12)$$

¹The paradigmatic example of a nontrivial objective density matrix is the reduced density matrix of a maximally entangled qubit.

The initial and final density matrices have respective spectral decompositions

$$\hat{\rho}_Q = \sum_q p_q \hat{P}_q, \quad \hat{P}_q = |\Psi_q\rangle\langle\Psi_q|, \quad (13)$$

$$\hat{\rho}_R = \sum_r p_r \hat{P}_r, \quad \hat{P}_r = |\Psi_r\rangle\langle\Psi_r|. \quad (14)$$

According to the minimal modal interpretation, every quantum system has an actual underlying state corresponding to one of the eigenstates of the system's density matrix, but that actual underlying state is hidden from external observers unless the system's density matrix is a projector. In our present example, the system's actual underlying state evolves from being one of the eigenstates of $\hat{\rho}_Q$ to being one of the eigenstates of $\hat{\rho}_R$. Collectively, the eigenstates of $\hat{\rho}_Q$ represent the initial possible underlying states of the system, and the eigenstates of $\hat{\rho}_R$ represent the final possible underlying states.

The evolution of the possible underlying states of the system is defined stochastically in terms of quantum conditional probabilities. For example, the probability that the system's later state is $|\Psi_r\rangle$ given that it was initially $|\Psi_q\rangle$ is defined to be

$$p_{\mathcal{E}}(r|q) \equiv \text{Tr}[\hat{P}_r \mathcal{E}_{R \leftarrow Q} \{\hat{P}_q\}] = \langle\Psi_r | \mathcal{E}_{R \leftarrow Q} \{\hat{P}_q\} | \Psi_r \rangle. \quad (15)$$

Note that throughout this paper, lower-case index labels q, q', \dots and r, r', \dots on states correspond respectively to upper-case system configuration labels Q and R . We adopt analogous conventions for other system configuration labels.

Regardless of the interpretation of quantum theory, the quantities defined by (15) exhibit almost all of the standard properties of conditional probabilities. In particular, they are non-negative real numbers that sum to unity and satisfy the law of total probability,

$$p_r = \sum_q p_{\mathcal{E}}(r|q) p_q, \quad (16)$$

which follows from substituting (15) into (12). We may associate the quantum conditional probabilities $p_{\mathcal{E}}(r|q)$ with a density matrix

$$\hat{\rho}_{R|q}^{\mathcal{E}} \equiv \sum_r p_{\mathcal{E}}(r|q) \hat{P}_r, \quad (17)$$

which satisfies

$$\hat{\rho}_R = \sum_q p_q \hat{\rho}_{R|q}^{\mathcal{E}}, \quad (18)$$

due to (16).

The crucial difference between classical and quantum conditional probabilities is that the latter fail to satisfy Bayes' theorem:

$$p_{\mathcal{E}}(r|q) p_q \neq p_{\mathcal{E}}(q|r) p_r. \quad (19)$$

The failure of Bayes' theorem reflects the non-commutativity of quantum observables, and therefore the inability to define a symmetric joint probability distribution. From a dynamical perspective, Bayes' theorem fails due to the generic irreversibility of $\mathcal{E}_{R \leftarrow Q}$, as is evident from the case in which $\mathcal{E}_{R \leftarrow Q}$ represents a projective measurement.

In general, linear CPTP evolution of an eigenprojector of the initial density matrix yields a nontrivial density matrix defined by

$$\hat{\rho}_q^R \equiv \mathcal{E}_{R \leftarrow Q} \{\hat{P}_q\}. \quad (20)$$

Introducing a new label r_q to distinguish the eigenprojectors $\{\hat{P}_{r_q}\}_{r_q}$ of this density matrix, we can write down the spectral decomposition of this density matrix:

$$\hat{\rho}_q^R = \sum_{r_q} p_{\mathcal{E}}(r_q|q) \hat{P}_{r_q}. \quad (21)$$

Note that for each fixed value of q , the basis of eigenprojectors $\{\hat{P}_{r_q}\}_{r_q}$ can be different, and will generically differ from $\{\hat{P}_r\}_r$.

Nevertheless, the set of these density matrices must combine to yield $\hat{\rho}_R$,

$$\hat{\rho}_R = \sum_q p_q \hat{\rho}_q^R. \quad (22)$$

It follows from a few lines of linear algebra that

$$\hat{\rho}_{R|q}^{\mathcal{E}} = \sum_r \hat{P}_r \hat{\rho}_q^R \hat{P}_r, \quad (23)$$

with

$$p_{\mathcal{E}}(r|q) = \sum_{r_q} \beta(r|r_q) p_{\mathcal{E}}(r_q|q), \quad \beta(r|r_q) = \text{Tr}[\hat{P}_r \hat{P}_{r_q}] = |\langle \Psi_r | \Psi_{r_q} \rangle|^2. \quad (24)$$

Note that the Born probability distribution $\beta(r|r_q)$ is doubly stochastic, implying that

$$S(\hat{\rho}_{R|q}^{\mathcal{E}}) \geq S(\hat{\rho}_q^R), \quad (25)$$

where we are using a standard result from probability theory. We provide an explicit proof of a more general result in the appendix.

So far, our description of the quantum conditional probabilities (15) has been dynamical, with $\mathcal{E}_{R \leftarrow Q}$ thought of as an evolution map. However, the same ideas can be applied to the quantum relationships between systems and their subsystems by noting that partial traces are an example of a linear CPTP map. We provide a more detailed sketch of these ideas in Section 5. In what follows, we will continue to focus on the dynamical picture, in which a single system evolves according to $\mathcal{E}_{R \leftarrow Q}$.

3.2 New Measures of Quantum Information

Combining the quantum conditional probabilities of (15) with Shannon's entropy formula yields a new type of quantum conditional entropy. Using the initial and final density matrices defined in (13) and (14), respectively, we let

$$J_{\mathcal{E}}(R|q) \equiv - \sum_r p_{\mathcal{E}}(r|q) \log[p_{\mathcal{E}}(r|q)] = S(\hat{\rho}_{R|q}^{\mathcal{E}}) \quad (26)$$

be the quantum conditional entropy of our system given that the system's initial underlying state corresponded to the eigenstate $|\Psi_q\rangle$ of $\hat{\rho}_Q$. We will argue that we can interpret this quantity as the entropy added to the system during its evolution given the initial underlying state of the system. The full quantum conditional entropy is the average over all possible initial eigenstates of $\hat{\rho}_Q$,

$$J_{\mathcal{E}}(R|Q) \equiv \sum_q J_{\mathcal{E}}(R|q) p_q = - \sum_{q,r} p_{\mathcal{E}}(r|q) p_q \log[p_{\mathcal{E}}(r|q)]. \quad (27)$$

We also define a new type of quantum mutual information,

$$I_{\mathcal{E}}(R : Q) \equiv \sum_{q,r} p_{\mathcal{E}}(r|q) p_q \log \left[\frac{p_{\mathcal{E}}(r|q)}{p_r} \right]. \quad (28)$$

The relation

$$I_{\mathcal{E}}(R : Q) = S(\hat{\rho}_R) - J_{\mathcal{E}}(R|Q) \quad (29)$$

follows directly from the definitions of quantum conditional entropy (27) and quantum mutual information (28), mirroring the classical identity

$$I(Y : X) = H(Y) - H(Y|X). \quad (30)$$

In a dynamical context, mutual information can be thought of as measuring the information that is shared between the initial and final system configurations.

3.2.1 Evolution from a Pure State

To illustrate the interpretations of the quantities (26) and (28), we examine two special cases. To start, consider a system that is initially in a known pure state $|\Psi\rangle$. Suppose that it evolves according to a linear CPTP map \mathcal{E} , so that we lose track of its initially pure state:

$$\hat{\rho}_R = \mathcal{E}\{\hat{P}_{\Psi}\}, \quad \hat{P}_{\Psi} = |\Psi\rangle\langle\Psi|. \quad (31)$$

In this situation, we have conditional probabilities

$$p_{\mathcal{E}}(r|\Psi) = p_r, \quad (32)$$

and hence we have the quantum conditional entropy

$$J_{\mathcal{E}}(R|Q) = J_{\mathcal{E}}(R|\Psi) = -\sum_r p_r \log p_r = S(\hat{\rho}_R). \quad (33)$$

In words, the increase in the system's entropy arises solely from the evolution of the system. We can also characterize this statement in terms of the mutual information, which vanishes,

$$I_{\mathcal{E}}(R; \Psi) = S(\hat{\rho}_R) - J_{\mathcal{E}}(R|\Psi) = 0, \quad (34)$$

thereby showing that no information is carried over from the system's initial state to its final configuration.

This linear CPTP map can be thought of as modeling a process in which the system becomes more entangled with its surrounding environment.² From this perspective, the quantum conditional entropy measures the growth of entanglement between a system and its environment.

3.2.2 Unitary Evolution

Now consider a system whose initial and final density matrices are $\hat{\rho}_Q$ and $\hat{\rho}_R$ as expressed in (13) and (14), respectively. Suppose that the evolution is unitary, so that for some unitary operator \hat{U} , we have

$$\hat{\rho}_R = \mathcal{U}_{R \leftarrow Q}\{\hat{\rho}_Q\} = \hat{U}\hat{\rho}_Q\hat{U}^\dagger, \quad \hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{I}, \quad (35)$$

where \mathbb{I} is the identity. Under such evolution, the eigenvalues of $\hat{\rho}_Q$ are unchanged and the eigenstates rotate into the set of eigenstates of $\hat{\rho}_R$,

$$\hat{P}_q^R = \hat{U}\hat{P}_q^Q\hat{U}^\dagger, \quad (36)$$

²This interpretation assumes that the map is faithful to the underlying physics, rather than capturing measurement or modeling errors.

where the upper label emphasizes that the evolution carries us from the initial configuration Q to the final configuration R . In this situation, the conditional probabilities (15) are trivial,

$$p_{\mathcal{U}}(r|q) = \text{Tr}[\hat{P}_r^R \hat{U} \hat{P}_q^Q \hat{U}^\dagger] = \text{Tr}[\hat{P}_r^R \hat{P}_q^R] = \delta_{rq}. \quad (37)$$

The quantum conditional entropy of this process is therefore zero and the quantum mutual information is equal to the von Neumann entropy of the system, showing that the uncertainty in the state of the system before the evolution is the sole source of uncertainty in the state afterward.

3.3 Some Identities and Inequalities

Due to the existence of an underlying probability distribution, the quantum conditional entropy (26) and mutual information (28) satisfy various relationships familiar from classical information theory.

- Conditional entropy and mutual information are always non-negative:

$$J_{\mathcal{E}}(R|Q) \geq 0, \quad I_{\mathcal{E}}(R : Q) \geq 0. \quad (38)$$

- A system's mutual information cannot be greater than the system's initial entropy:

$$I_{\mathcal{E}}(R : Q) \leq S(\hat{\rho}_Q). \quad (39)$$

- A system's conditional entropy cannot be greater than the system's final entropy:

$$J_{\mathcal{E}}(R|Q) \leq S(\hat{\rho}_R). \quad (40)$$

The inequalities (38), (39), and (40) can be proved following similar steps to those from classical information theory. We provide details in the appendix.

4 Entropy Growth and Data Processing

4.1 Unital Evolution and Projective Measurement

A unital linear CPTP map satisfies

$$\mathcal{E}_{R \leftarrow Q}\{\mathbb{I}\} = \mathbb{I}. \quad (41)$$

The conditional probabilities (15) for a unital linear CPTP map are doubly stochastic:

$$\begin{aligned} \sum_q p_{\mathcal{E}}(r|q) &= \text{Tr}\left[\hat{P}_r \mathcal{E}_{R \leftarrow Q}\left\{\sum_q \hat{P}_q\right\}\right] \\ &= \text{Tr}\left[\hat{P}_r \mathcal{E}_{R \leftarrow Q}\{\mathbb{I}\}\right] \\ &= \text{Tr}\left[\hat{P}_r\right] \\ &= 1. \end{aligned} \quad (42)$$

Thus, when the evolution of a system is unital linear CPTP, then the von Neumann entropy grows,

$$S(\hat{\rho}_Q) \leq S(\hat{\rho}_R), \quad (43)$$

which follows from the law of total probability (16) relating p_r and p_q and the double-stochasticity of $p_{\mathcal{E}}(r|q)$ in this case, as proved in the appendix.

A projective measurement is an example of a unital process. Suppose that we measure an observable with eigenstates $\{|\Psi_m\rangle\}_m$. If we isolate the measurement device and refrain from learning the outcome, then the post-measurement density matrix is well approximated by

$$\hat{\rho}_M = \mathcal{M}\{\hat{\rho}_Q\} = \sum_m \hat{P}_m \hat{\rho}_Q \hat{P}_m, \quad (44)$$

which is clearly unital. As a result, we see that measurements increase the entropy of a system.

4.2 Concavity of von Neumann Entropy

The quantities described earlier allow us to demonstrate certain standard properties of quantum information. Consider the concavity of von Neumann entropy

$$\sum_i p_i S(\hat{\rho}_i) \leq S(\hat{\rho}), \quad \hat{\rho} = \sum_i p_i \hat{\rho}_i, \quad (45)$$

where $\hat{\rho}$ is an arbitrary density matrix, the set $\{\hat{\rho}_i\}_i$ is a collection of density matrices that can exceed the dimension of the Hilbert space, and the set of values $\{p_i\}_i$ are non-negative and sum to unity.

To prove (45), recall from Section 3.1 that a linear CPTP map can be used to produce just such a decomposition of a density matrix into a set of other density matrices,

$$\hat{\rho}_R = \mathcal{E}\{\hat{\rho}_Q\} = \sum_q p_q \mathcal{E}\{\hat{P}_q\} = \sum_q p_q \hat{\rho}_q^R = \sum_q p_q \hat{\rho}_{R|q}, \quad (46)$$

where we have simplified the notation by suppressing some labels. Recall, too, that the density matrices $\hat{\rho}_{R|q}$ and $\hat{\rho}_q^R$ are related via a projective measurement map,

$$\hat{\rho}_{R|q} = \sum_r \hat{P}_r \hat{\rho}_q^R \hat{P}_r, \quad (47)$$

implying that

$$S(\hat{\rho}_q^R) \leq S(\hat{\rho}_{R|q}). \quad (48)$$

The quantum conditional entropy (27) can be expressed as the sum

$$J_{\mathcal{E}}(R|Q) = \sum_q p_q S(\hat{\rho}_{R|q}),$$

and satisfies the inequality (40). Thus,

$$\sum_q p_q S(\hat{\rho}_q^R) \leq \sum_q p_q S(\hat{\rho}_{R|q}) \leq S(\hat{\rho}_R),$$

which, given the arbitrariness of \mathcal{E} , demonstrates the concavity of von Neumann entropy.³

4.3 Quantum Markovianity and Data Processing

Consider a system that evolves from $\hat{\rho}_Q$ to $\hat{\rho}_R$, and then to $\hat{\rho}_S$, as described by the linear CPTP maps $\mathcal{E}_{R \leftarrow Q}$ and $\mathcal{E}_{S \leftarrow R}$, so that we have

$$\hat{\rho}_R = \mathcal{E}_{R \leftarrow Q}\{\hat{\rho}_Q\}, \quad \hat{\rho}_S = \mathcal{E}_{S \leftarrow R}\{\hat{\rho}_R\} = \mathcal{E}_{S \leftarrow R} \circ \mathcal{E}_{R \leftarrow Q}\{\hat{\rho}_Q\} = \mathcal{E}_{S \leftarrow Q}\{\hat{\rho}_Q\}. \quad (49)$$

³Note that we implicitly allow \mathcal{E} to involve a partial trace operation so that the Hilbert space dimension associated with the final density matrix $\hat{\rho}_R$ can be smaller than that of $\hat{\rho}_Q$.

Observe that

$$\hat{\rho}_R = \mathcal{E}_{R \leftarrow Q} \left\{ \sum_q p_q \hat{P}_q \right\} = \sum_q p_q \mathcal{E}_{R \leftarrow Q} \{ \hat{P}_q \}, \quad (50)$$

with corresponding conditional probabilities

$$p(r|q) = \text{Tr}[\hat{P}_r \mathcal{E}_{R \leftarrow Q} \{ \hat{P}_q \}], \quad (51)$$

where we suppress the map label as the mapping will be clear from the state indices.

Similarly, we have

$$\hat{\rho}_S = \mathcal{E}_{S \leftarrow R} \left\{ \sum_r p_r \hat{P}_r \right\} = \sum_r p_r \mathcal{E}_{S \leftarrow R} \{ \hat{P}_r \}, \quad p(s|r) = \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R} \{ \hat{P}_r \}], \quad (52)$$

as well as

$$\hat{\rho}_S = \mathcal{E}_{S \leftarrow Q} \left\{ \sum_q p_q \hat{P}_q \right\} = \sum_q p_q \mathcal{E}_{S \leftarrow Q} \{ \hat{P}_q \}, \quad p(s|q) = \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow Q} \{ \hat{P}_q \}]. \quad (53)$$

There are some subtle constraints required for the consistency of these processes. Using the law of total probability and (24), we have

$$\begin{aligned} p_s &= \sum_r p(s|r)p_r \\ &= \sum_{r,q} p(s|r)p(r|q)p_q. \\ &= \sum_{r,q,r_q} p(s|r)\beta(r|r_q)p(r_q|q)p_q. \end{aligned} \quad (54)$$

Similarly, we have

$$p_s = \sum_q p(s|q)p_q. \quad (55)$$

However, recall from (20) that

$$\mathcal{E}_{R \leftarrow Q} \{ \hat{P}_q \} = \sum_{r_q} p(r_q|q) \hat{P}_{r_q}. \quad (56)$$

So expanding out the definition of $p(s|q)$ and using $\mathcal{E}_{S \leftarrow Q} = \mathcal{E}_{S \leftarrow R} \circ \mathcal{E}_{R \leftarrow Q}$ gives

$$\begin{aligned} p_s &= \sum_q \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R} \{ \mathcal{E}_{R \leftarrow Q} \{ \hat{P}_q \} \}] p_q \\ &= \sum_q \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R} \left\{ \sum_{r_q} p(r_q|q) \hat{P}_{r_q} \right\}] p_q \\ &= \sum_q \sum_{r_q} \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R} \{ \hat{P}_{r_q} \}] p(r_q|q) p_q \\ &= \sum_q \sum_{r_q} p(s|r_q) p(r_q|q) p_q. \end{aligned} \quad (57)$$

Comparing (54) and (57), we find that a natural-looking consistency condition to impose would be

$$p(s|r_q) = \sum_r p(s|r)\beta(r|r_q). \quad (58)$$

Expanding out the definitions of the conditional probabilities in (58), we have on the right-hand side that

$$\begin{aligned}
\sum_r p(s|r)\beta(r|r_q) &= \sum_r \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R}\{\hat{P}_r\}] \langle \Psi_r | \hat{P}_{r_q} | \Psi_r \rangle \\
&= \sum_r \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R}\{ |\Psi_r\rangle \langle \Psi_r | \hat{P}_{r_q} | \Psi_r\rangle \langle \Psi_r | \}] \\
&= \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R}\{ \sum_r \hat{P}_r \hat{P}_{r_q} \hat{P}_r \}],
\end{aligned} \tag{59}$$

while the left-hand side of (58) is

$$p(s|r_q) = \text{Tr}[\hat{P}_s \mathcal{E}_{S \leftarrow R}\{\hat{P}_{r_q}\}]. \tag{60}$$

We conclude that

$$\mathcal{E}_{S \leftarrow R}\{ \sum_r \hat{P}_r \hat{P}_{r_q} \hat{P}_r \} = \mathcal{E}_{S \leftarrow R}\{\hat{P}_{r_q}\} \tag{61}$$

solves the constraint and that the map $\mathcal{E}_{S \leftarrow R}$ incorporates a projective measurement along the $\{\hat{P}_r\}_r$ basis in its definition. Conceptually, this projective measurement ensures that the intermediate composite state of the system and its environment re-factorize, thus leading to Markov-like evolution. We therefore have the relation

$$p(s|q) = \sum_r p(s|r)p(r|q). \tag{62}$$

The mutual information shared between the initial and final configurations is

$$I(S : Q) = \sum_{s,q} p(s|q)p_q \log \left[\frac{p(s|q)}{p_s} \right]. \tag{63}$$

The mutual information between the initial and intermediate configurations is

$$I(R : Q) = \sum_{r,q} p(r|q)p_q \log \left[\frac{p(r|q)}{p_r} \right]. \tag{64}$$

Using (62), the difference between these two quantities can be written as

$$\begin{aligned}
I(S : Q) - I(R : Q) &= \sum_{s,q,r} p(s|r)p(r|q)p_q \left(\log \left[\frac{p(s|q)}{p_s} \right] - \log \left[\frac{p(r|q)}{p_r} \right] \right) \\
&= \sum_{s,q,r} p(s|r)p(r|q)p_q \log \left[\frac{p(s|q)p_r}{p_s p(r|q)} \right].
\end{aligned}$$

Using Jensen's inequality,⁴ we have

$$\begin{aligned}
I(S : Q) - I(R : Q) &\leq \log \left[\sum_{s,q,r} p(s|r)p(r|q)p_q \frac{p(s|q)p_r}{p_s p(r|q)} \right] \\
&= \log \left[\sum_{s,q,r} p(s|r)p_r \frac{p(s|q)p_q}{p_s} \right] \\
&= \log \left[\sum_s p_s \frac{p_s}{p_s} \right] \\
&= 0.
\end{aligned}$$

We therefore arrive at a quantum version of the data-processing inequality,

$$I(S : Q) \leq I(R : Q), \tag{65}$$

capturing the idea that the information encoded in the system's initial configuration is increasingly diluted as the system is "processed."

4.4 A Holevo-Type Bound

Let us recall the statement of Holevo's bound. Consider a quantum system and let X be a classical random variable with possible outcomes $\{x\}_x$ and corresponding probability distribution p_x . Suppose that $\{\hat{\rho}_x\}_x$ is a collection of density matrices indexed by the possible outcomes x of X , and let $\hat{\rho}$ be the correspondingly averaged density matrix:

$$\hat{\rho} \equiv \sum_x p_x \hat{\rho}_x. \tag{66}$$

If we now measure a POVM $\{E_Y\}_y$ whose possible outcomes y form another classical random variable Y , then Holevo's bound states that the classical mutual information between X and Y is bounded from above by the quantity

$$\chi \equiv S(\hat{\rho}) - \sum_x p_x S(\hat{\rho}_x). \tag{67}$$

That is,

$$I(X : Y) \leq \chi. \tag{68}$$

In the two-step process described in Section 4.3, the mutual information between the initial configuration $\hat{\rho}_Q$ and the intermediate configuration $\hat{\rho}_R$ can be expressed as

$$I(R : Q) = S(\hat{\rho}_R) - J(R|Q) = S(\hat{\rho}_R) - \sum_q p_q S(\hat{\rho}_{R|q}), \tag{69}$$

where

$$\hat{\rho}_{R|q} = \sum_r \hat{P}_r \mathcal{E}_{R \leftarrow Q} \{ \hat{P}_q \} \hat{P}_r. \tag{70}$$

The quantity on the right-hand side of (69) is clearly an example of Holevo's χ quantity. We see that it emerges quite naturally as a form of mutual information in our formalism, and that Holevo's bound (68) arises as a manifestation of our quantum data-processing inequality (65). The Holevo bound's interpretation as a quantum version of the data-processing inequality has been discussed before (see for example [10]). Our dynamical interpretation of the bound provides another perspective that avoids any explicit embedding of the

⁴Jensen's inequality states that if $f(x)$ is a convex function of its argument x , then the average of $f(x)$ provides an upper bound for the original function applied to the average of its argument. Here we apply Jensen's inequality to $-\log x$.

system of interest into a larger composite system. Instead, we capture the role of the broader environment through the formalism of linear CPTP maps.

5 Conclusion and Future Directions

5.1 Systems and Subsystems

Our focus in this paper has been on a dynamical interpretation of quantum information in a system whose evolution is described by a linear CPTP map. However, as mentioned in Section 3.1, the formalism is general enough to capture structural relationships between composite quantum systems and their subsystems. To begin, consider the parent system AB , formed from a pair of quantum subsystems A and B and described by the density matrix

$$\hat{\rho}_{AB} = \sum_m p_m^{AB} \hat{P}_m^{AB}, \quad \hat{P}_m^{AB} = |\Psi_m^{AB}\rangle\langle\Psi_m^{AB}|, \quad (71)$$

where we include the parent system's label AB on the system's eigenprojectors \hat{P}_m^{AB} and the corresponding probabilities p_m^{AB} . The subsystem density matrices are related to $\hat{\rho}_{AB}$ via the appropriate partial traces,

$$\hat{\rho}_A = \text{Tr}_B[\hat{\rho}_{AB}] = \sum_a p_a^A \hat{P}_a^A, \quad \hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}] = \sum_b p_b^B \hat{P}_b^B, \quad (72)$$

where the sets of eigenprojectors for subsystems A and B are $\{\hat{P}_a^A\}_a$ and $\{\hat{P}_b^B\}_b$, respectively.

Quantum probabilities that conditionally link subsystem eigenstates to a given eigenstate of the parent system are again defined using (15), substituting the relevant partial trace for the linear CPTP map in the formula. For instance, the conditional probability that $|\Psi_a^A\rangle$ is the actual underlying state of subsystem A given that the underlying state of AB is $|\Psi_m^{AB}\rangle$ is⁵

$$p(a|m) = \text{Tr}_A[\hat{P}_a^A \text{Tr}_B\{\hat{P}_m^{AB}\}]. \quad (73)$$

As in Section 3.1, the partial trace applied to system AB 's eigenprojector yields a density matrix

$$\hat{\rho}_m^A = \text{Tr}_B[\hat{P}_m^{AB}] = \sum_{a_m} p(a_m|m) \hat{P}_{a_m}^A. \quad (74)$$

We have

$$\hat{\rho}_A = \sum_m p_m \hat{\rho}_m^A = \sum_m p_m \hat{\rho}_{A|m}, \quad \hat{\rho}_{A|m} = \sum_a \hat{P}_a^A \hat{\rho}_m^A \hat{P}_a^A. \quad (75)$$

These relationships imply that the quantum entropy conditioned on the parent state $|\Psi_m^{AB}\rangle$ satisfies the inequality

$$S(\hat{\rho}_m^A) \leq J(A|m) = - \sum_a p(a|m) \log p(a|m) = S(\hat{\rho}_{A|m}), \quad (76)$$

due to the quantities $p(a|m)$ and $p(a_m|m)$ being related via the doubly stochastic distribution

$$\beta(a|a_m) = |\langle\Psi_a^A|\Psi_{a_m}^A\rangle|^2. \quad (77)$$

It is interesting to examine the von Neumann entropy of $\hat{\rho}_m^A$,

$$S(\hat{\rho}_m^A) = - \sum_{a_m} p(a_m|m) \log p(a_m|m), \quad (78)$$

⁵We again adopt the language of the minimal modal interpretation, though the mathematical content involves only textbook quantum theory.

and to note that it is naturally interpreted as the entanglement entropy of subsystem A conditioned on the parent system AB actually occupying the pure state $|\Psi_m^{AB}\rangle$. Note that when the parent system is in a pure state, then $\hat{\rho}_m^A = \hat{\rho}_{A|m}$ and $J(A|m)$ is the entanglement entropy of subsystem A .

The full quantum conditional entropy is defined as

$$J(A|AB) = \sum_m p_m J(A|m). \quad (79)$$

Therefore (76) implies

$$\sum_m p_m S(\hat{\rho}_m^A) \leq J(A|AB). \quad (80)$$

There are also intriguing relationships between our quantum conditional entropy (26,27) and conditional von Neumann entropy (11). Observe that the inequality satisfied by our version of quantum mutual information can be re-expressed as

$$S(\hat{\rho}_A) - J(A|AB) \leq S(\hat{\rho}_{AB}), \quad (81)$$

where the initial density matrix is taken to be $\hat{\rho}_{AB}$ and the final density matrix is $\hat{\rho}_A$. Rearranging terms and applying the definition of conditional von Neumann entropy yields

$$-S(B|A) \leq J(A|AB). \quad (82)$$

In the presence of entanglement, $S(B|A)$ may take on negative values, leading to a *positive* lower bound on $J(A|AB)$. The result naturally captures the idea that when subsystems are entangled, there is a non-zero minimal uncertainty about their states even given information about the parent system.

The above sketch outlines intriguing connections between our quantum conditional probabilities and standard quantum information-theoretic concepts that arise from the rich structure of system-subsystem relationships in quantum theory. In future work, we will continue to explore these connections, along with related concepts such as quantum discord [11].

5.2 Generalizations of Quantum Conditional Probabilities

Our definition of quantum conditional probability (15) involves the eigenprojectors of initial and final density matrices (13) and (14), respectively. However, as we describe in Section 2, there are infinitely many decompositions of a nontrivial density matrix. Thus, we may consider quantities of the form

$$\mathcal{P}_{\mathcal{E}}(\rho|\kappa) = \text{Tr}[\hat{\Pi}_{\rho}^R \mathcal{E}_{R \leftarrow Q} \{\hat{\Pi}_{\kappa}^Q\}], \quad (83)$$

where

$$\hat{\rho}_Q = \sum_{\kappa} \lambda_{\kappa}^Q \hat{\Pi}_{\kappa}^Q, \quad \hat{\rho}_R = \sum_{\rho} \lambda_{\rho}^R \hat{\Pi}_{\rho}^R \quad (84)$$

are general convex decompositions of the system's initial and final density matrices, respectively, with generic projection operators

$$\hat{\Pi}_{\kappa}^Q = |\Phi_{\kappa}^Q\rangle\langle\Phi_{\kappa}^Q|, \quad \hat{\Pi}_{\rho}^R = |\Phi_{\rho}^R\rangle\langle\Phi_{\rho}^R|. \quad (85)$$

Note that such sets of projectors need not be orthogonal. However, if we demand that the quantities (83) behave as probabilities, then the set $\{\hat{\Pi}_{\rho}^R\}_{\rho}$ must resolve the identity,

$$\sum_{\rho} \hat{\Pi}_{\rho}^R = \mathbb{I}. \quad (86)$$

Nevertheless, such quantities fail to act as fully satisfactory conditional probabilities, as they do not obey a straightforward version of the law of total probability. Instead we have

$$\begin{aligned}\Lambda_\rho^R &= \text{Tr}[\hat{\Pi}_\rho^R \hat{\rho}_R] \\ &= \text{Tr}[\hat{\Pi}_\rho^R \mathcal{E}_{R \leftarrow Q} \{\hat{\rho}_Q\}] \\ &= \sum_\kappa \text{Tr}[\hat{\Pi}_\rho^R \mathcal{E}_{R \leftarrow Q} \{\hat{\Pi}_\kappa^Q\}] \lambda_\kappa^Q,\end{aligned}\tag{87}$$

and thus

$$\Lambda_\rho^R = \sum_\kappa \mathcal{P}_{\mathcal{E}}(\rho|\kappa) \lambda_\kappa^Q,\tag{88}$$

where we generically have $\Lambda_\rho^R \neq \lambda_\rho^R$ due to the possible nonorthogonality of the projectors.

Despite their failure to reproduce the law of total probability, the quantities (83) do satisfy the Kolmogorov axioms for a basic probability distribution. They are also examples of more general quantities of the form

$$F_p(\hat{A}, \hat{B}; \hat{K}) = \text{Tr}[\hat{A}^p \hat{K} \hat{B}^{1-p} \hat{K}^\dagger],\tag{89}$$

where \hat{A} and \hat{B} are positive semi-definite $N \times N$ matrices, \hat{K} is a fixed $N \times N$ matrix, and $0 \leq p \leq 1$. Lieb proved in [12] that trace quantities of the above type are non-negative concave maps. Observe that when \hat{A} and \hat{B} are taken to be projection operators with $p = 1/2$, and if \hat{K} is one of the operators in a Kraus representation of $\mathcal{E}_{R \leftarrow Q}$, then each term in the Kraus decomposition of (83) is of the form (89). Quantities such as (89) have been central to the understanding of generalized entropies, particularly the properties of quantum relative entropy, but their implications for the existence of probability distributions in quantum theory seem worth exploring further.

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Appendix: Proofs of Basic Information Inequalities

Properties of Doubly Stochastic Distributions

A conditional probability distribution $p(y|x)$ is called doubly stochastic if

$$\sum_x p(y|x) = 1.\tag{90}$$

If $p(y)$ and $p(x)$ are related via a doubly stochastic distribution,

$$p(y) = \sum_x p(y|x)p(x),\tag{91}$$

then the Shannon entropy of $p(y)$ is greater than or equal to that of $p(x)$. To see why, consider their difference:

$$H(X) - H(Y) = \sum_{x,y} p(y|x)p(x) \log\left(\frac{p(y)}{p(x)}\right).\tag{92}$$

Using Jensen's inequality, we have

$$\begin{aligned} H(X) - H(Y) &\leq \log \left(\sum_{x,y} p(y|x)p(x) \frac{p(y)}{p(x)} \right) \\ &= \log \left(\sum_{x,y} p(y|x)p(y) \right). \end{aligned} \quad (93)$$

At this stage, we can use the double stochasticity of $p(y|x)$ to obtain

$$H(X) - H(Y) \leq \log \left(\sum_y p(y) \right) = 0, \quad (94)$$

and hence

$$H(X) \leq H(Y), \quad (95)$$

as claimed.

While we have explicitly proved this result using classical notation, the proof applies to von Neumann entropies linked via the quantum conditional probabilities (15) defined in Section 3.1.

Non-Negativity

The non-negativity of quantum conditional entropy follows directly from its construction from non-negative conditional probabilities that cannot be greater than one. Non-negativity of our form of quantum mutual information arises by applying Jensen inequality to the definition (28):

$$I_{\mathcal{E}}(R : Q) = - \sum_{q,r} p(r|q)p_q \log \left[\frac{p_r}{p(r|q)} \right] \geq - \log \left[\sum_{q,r} p(r|q)p_q \frac{p_r}{p(r|q)} \right] = - \log(1) = 0. \quad (96)$$

These arguments thus prove (38).

Linear CPTP Evolution Cannot Increase Mutual Information

The difference between the quantum mutual information shared by the initial and final configurations, on the one hand, and the von Neumann entropy of the initial density matrix (13), on the other hand, is

$$\begin{aligned} I_{\mathcal{E}}(R : Q) - S(\hat{\rho}_Q) &= - \sum_{q,r} p(r|q)p_q \log \left[\frac{p_r}{p(r|q)} \right] + \sum_q p_q \log p_q \\ &= \sum_{q,r} p(r|q)p_q \log \left[\frac{p(r|q)p_q}{p_r} \right]. \end{aligned} \quad (97)$$

The law of total probability (16) gives us

$$p_r \geq p_{\mathcal{E}}(r|q)p_q. \quad (98)$$

Thus, the monotonicity of the logarithm implies that

$$I_{\mathcal{E}}(R : Q) - S(\hat{\rho}_Q) \leq \sum_{q,r} p(r|q)p_q \log \left[\frac{p_r}{p_r} \right] = 0. \quad (99)$$

We have thus proved (39).

Conditional Entropy Cannot Exceed Final Entropy

The identity (29) can be rewritten as

$$J_{\mathcal{E}}(R|Q) = S(\hat{\rho}_R) - I_{\mathcal{E}}(R : Q). \quad (100)$$

Due to the positivity of mutual information, we immediately have that conditional entropy cannot exceed the final entropy of a system after a linear CPTP process, (40).

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