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Indeterminacy, Coincidence, and "Sourcing Newness" in Mathematical Research

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Abstract Far from being unwelcome or impossible in a mathematical setting, indeterminacy in various forms can be seen as playing an important role in driving mathematical research forward by providing "sources of newness" in the sense of [Hutter and Farías(2017)]. I argue here that mathematical coincidences, phenomena recently under discussion in the philosophy of mathematics, are usefully seen as inducers of indeterminacy and as put to work in guiding research directions. I suggest that to call a pair of mathematical facts (merely) a coincidence is roughly to suggest that the investigation of connections between these facts isn't worthwhile. To say of this pair, "That's no coincidence!" is to suggest just the opposite. I further argue that this perspective on mathematical coincidence, which pays special attention to what mathematical coincidences *do*, may provide us with a better view of what mathematical coincidences *are* than extant accounts. I close by reflecting on how understanding mathematical coincidences as generating indeterminacy accords with a conception of mathematical research as ultimately aiming to reduce indeterminacy and complexity to triviality as proposed in [Rota(1997)].

Keywords indeterminacy · mathematical coincidence · mathematical practice

1 Introduction

Cantor's paradise[1] (and the world of mathematics more generally) has seemed to many to mirror Dante's Paradise, where "law eternal" ordains that "no trace of chance can find a place."[2] However, mathematical coincidences or "accidents," phenomena recently under discussion in the philosophy of mathematics, threaten to disturb the idyllic bliss by injecting traces of the unexplained

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[1] [Hilbert(1926/1983), 191]
[2] [Dante, *Paradiso*, Canto XXXII, 52-56]

and unordained into paradise.[3] This emergence of apparent disorder needn't be seen as wholly unwelcome, however. Like Achilles in the Underworld,[4] some may find the unchanging, fully-determined nature of so-called paradise uninspiring and so pine for any source of something new. In what follows, I'll argue for taking this Achillean point of view on mathematical coincidences and the way they can—by drawing attention to a kind of indeterminacy—be "sources of newness" and motivators of mathematical investigation.[5]

More specifically, the plan for the paper is as follows. I'll begin in Section 2 by discussing some of the roles indeterminacy can play in inquiry in general and of the mathematical variety in particular. This discussion will be forced to find a sense of 'indeterminacy' that doesn't clash with the presumed exact and unchanging nature of mathematical reality. Next, in Section 3, I'll present a few examples that will introduce the concept of a mathematical coincidence. These examples will then be used to argue for an understanding of mathematical coincidences as sources of indeterminacy and newness that can motivate mathematical research in Section 4. When we look at what we *do* when we call something a mathematical coincidence rather than ask what it *is* for something to be a mathematical coincidence, an alternative to the standard account of the phenomenon ([Lange(2017), Ch. 8]), which will come up for discussion in Section 5, suggests itself. According to this alternative view, to call a pair of mathematical facts (merely) a coincidence is roughly to suggest that attention to how these facts are related would not be fruitful; to say, "That's no coincidence!" is to suggest just the opposite. Expanding on this basic picture and responding to some of the most obvious objections to it will be the project of the latter portion of Section 4 and Section 6 respectively.

2 A Sense of 'Indeterminacy'

John Dewey famously characterizes inquiry as the transformation of an indeterminate situation into one "*that is so determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole.*"[6] Here, a "situation" is deemed to be indeterminate when its constituent parts don't "hang together"; when it presents itself as open for questioning or as being uncertain; when the future of the situation can't be predicted or clearly made out; and when it "tends to evoke discordant responses" from those encountering it.[7] The full details of Dewey's view aren't important for present purposes, but what is important for the project of the

[3] On mathematical coincidence, see, e.g., [Baker(2009)], [Lange(2010)], and [Lange(2017), Ch. 8]. See also [Davis(1981)].

[4] [Homer, *Odyssey* 11.487-503]

[5] The idea of "sourcing newness" is drawn from [Hutter and Farías(2017)].

[6] [Dewey(1938), 104-105, emphasis in the original]

[7] See [Dewey(1938), 105-106]. Dewey's use of 'indeterminate situation' in this semi-technical sense helps block Russell's "counterexample" that, according to Dewey, a bricklayer's dealings with a pile of bricks is a form of inquiry [Russell(1946/1961), 823]. See [Gale(1959)] for more on Russell on Dewey on inquiry.

paper is the implication in Dewey's thinking that inquiry is premised on the existence of the appropriate sort of indeterminacy. If this line of thought is on the right track, we might expect, as Michael Hutter and Ignacio Farías have found, that "inducing" indeterminacy can serve as a way to "source newness" and advance an art, an industry, or a science by generating new sites for inquiry.[8]

Some examples of induced indeterminacy from the work of Hutter and Farías include attempts to suspend particular social norms or expectations within the artist's studio;[9] forcing the "transcription" of aspects of a star like Joan Crawford into the new, animated form of *Snow White*'s Evil Queen; and aiming to disrupt accepted valuations through various forms of critique.[10] In each case, there is an aim to create conditions of uncertainty and indeterminacy out of which something new may be hoped to emerge via the subsequent engagement and inquiry.

Mathematical uses of 'determinacy' and 'indeterminacy' are familiar enough: $y = x^2$ determines a value of y given a value for x, but $y \neq x^2$ doesn't;[11] $\lim_{x\to 0} \sin x/x$ is an indeterminate form since $\lim_{x\to 0} \sin x = \lim_{x\to 0} x = 0$;[12] $R[X]$ is a ring of polynomials in indeterminate X;[13] etc. But it's also common enough for the kinds of indeterminacy discussed by Hutter and Farías to find their place in mathematical settings. For example, although this reading doesn't capture his own intentions, the investigations of Saccheri into Euclidean geometry without the parallel postulate can be seen as relaxing a framework of rules to increase indeterminacy and allow something new—non-Euclidean geometry—to emerge.[14] *Adding* an axiom to an existing base system can also be seen as allowing an experiment in indeterminacy in the other direction: e.g., what happens if we restrict our set theories by adopting new rules in the form of the Axiom of Constructibility or the Axiom of Determinacy?[15] Part of the fruitfulness of category theory comes from the ability of functors to "transcribe" objects from one category to another; e.g., a topological object can be transported into an algebraic category by the fundamental group functor.[16] And criticizing and critiquing, say, foundations or logics, often serves to make room for reevaluation of how determinate our fundamental assumptions

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8 [Hutter and Farías(2017)]

9 Something like this induced indeterminacy may also be familiar as what the character Paul aims to produce in his rented apartment in *Last Tango in Paris*.

10 [Hutter and Farías(2017), 438-440, 441-442, 444]

11 See [Wittgenstein(1953/2009), §189] for discussion of this sense of 'determinacy.'

12 See, e.g., [Bartle and Sherbert(2011), Section 6.3].

13 See, e.g., [Mac Lane and Birkhoff(1999), Ch. III.6].

14 See [Saccheri(1733/2014), Book I].

15 See [Corfield(2003), 152] for more on the role of axioms in experimentation and creativity within mathematical research.

16 *Cf.* [Hatcher(2002), 21]: "Algebraic topology can be roughly defined as the study of techniques for forming algebraic images of topological spaces." See also the notion of "transport of structure" from [Bourbaki(1968), Ch. IV, §1].

and frameworks ultimately are.[17] The reevaluation and critique of accepted valuations and conceptualizations as a stimulus for mathematical research has also been noted at least since the work of Lakatos.[18] The importance of and role played by this Lakatosian variety of indeterminacy and "open texture" in the sense of [Waismann(1945), 123f.] has reemerged in recent discussions of formal vs. informal mathematics,[19] as well as in attempts to tie the ordinary practice of refining and homing in on mathematical concepts to the contemporary interest in "conceptual engineering."[20] Both of these natural connections further place the sense of 'indeterminacy' especially focused on by Hutter and Farías squarely within traditional bounds of research in the philosophy of mathematics.[21]

The instances of indeterminacy in mathematical settings above are all analogues of examples given in [Hutter and Farías(2017)], but the typology can readily be expanded as well. Consider the following reminiscence of Vaughan Jones (of Jones polynomial fame) discussing an experience which he used to "dislike intensely," but has...

> ...come to appreciate *and even search for*. It is the situation where one has two watertight, well-designed arguments that lead inexorably to opposite conclusions. Remember that research in mathematics involves a foray into the unknown. We may not know which of the two conclusions is correct or even have any feeling or guess. [...] The search for a chink in the armour often involve[s] many tricks including elaborate thought experiments and perhaps computer calculations. Much structural understanding is created, which is why I now so value this process [Jones(1998), 208-209, emphasis added].[22]

Jones describes here a situation where the vacillation of trust in each of his "watertight" arguments and in the consistency of mathematics as a whole creates an uncomfortable indeterminacy that leads to new understanding and knowledge.[23] The example, therefore, fits in nicely with those presented so far, which are more closely tied to those originally given in [Hutter and Farías(2017)].

[17] This is one way to view the resurgence of interest in type-theory and constructive logic initiated by the so-called univalent foundations program. See [Univalent Foundations Program(2013)].

[18] See, e.g., [Lakatos(1976)].

[19] See [Shapiro and Roberts(2021)] for discussion of open texture and mathematics more generally.

[20] See [Tanswell(2018)] for a compelling case that the project of conceptual engineering has much to learn from mathematical practice. See, e.g., [Burgess et al.(2020)] for a general introduction to the conceptual engineering project.

[21] Of course, none of this indeterminacy quite suggests the kind of "ontological indeterminacy" investigated in more metaphysically-focused literature: see, e.g., [Rosen and Smith(2004)] or [Barnes and Williams(2011)] for more on this purported type of indeterminacy. (See, e.g., [Lewis(1986), 212] for the 'purported' qualification.)

[22] In addition to indeterminacy, Jones suggests that this kind of situation also "induces obsessive and anti-social behaviour."

[23] See [Aberdein(2010)] for an attempt to mine the mistaken half of this kind of argument pair for a deeper understanding of mathematical error, mathematical fallacies, and the role of and justification for informal reasoning in mathematics.

Clearly, this is a fairly brief account of how indeterminacy in the sense of Dewey and Hutter and Farías finds its place in the setting of mathematics. Despite the brevity of these remarks, however, the examples do suggest that there's reason to think that inducing indeterminacy in at least this sense can be an important engine driving new mathematical research forward. And from there, it's clear enough how mathematical coincidences might usefully be seen as presenting us with indeterminacy in Dewey's sense as well: when two (or more) mathematical facts are thought to be coincidental, they don't "hang together" in any obvious way; there's a kind of unpredictability about how the facts might be related—if they ever are; and different responses to the juxtaposition of these facts are characteristically evoked. After presenting a few examples that illustrate the concept of a mathematical coincidence in the next section, I'll expand on this initial characterization of the indeterminacy induced by mathematical coincidences in Section 4.

3 A budget of coincidences

As mentioned in Section 1, mathematicians and philosophers have noticed the surprising fact that within mathematical practice facts or collections of facts are regularly classified as being "coincidental" (or not).[24] The following examples are just a small sampling of this phenomenon.[25]

Example 1. The thirteenth decimal digit of both π and e is 9.[26]

$$\begin{aligned} \pi &= 3.\ 1\ 4\ 1\ 5\ 9\ 2\ 6\ 5\ 3\ 5\ 8 \left| 9 \right| 7 \ldots \\ e &= 2.\ 7\ 1\ 8\ 2\ 8\ 1\ 8\ 2\ 8\ 4\ 5 \left| 9 \right| 0 \ldots \end{aligned}$$

Is this a coincidence? Presumably, yes.

Example 2. $1!, 2!, \ldots, 400!$ are all "Niven" (or "harshad") numbers. That is, each is divisible by the sum of its decimal digits.[27] Are all natural numbers Niven numbers or is it just a coincidence that the first $\sim$400 turn out to be?

All natural numbers are not Niven numbers. 432! is the first counterexample. The sum of the digits of 432! is $3^2 \cdot 433$. Since $432! = 432 \cdot 431 \cdot \ldots \cdot 2 \cdot 1$, and 433 is prime, 432! doesn't have 433 as a factor.

[24] Another way talk of coincidence can arise in mathematics is when a bad argument "coincidentally" reaches a true conclusion. As [Aberdein(2010)] shows, this kind of coincidence can be usefully investigated, but it is not of the type central to recent discussions of mathematical coincidence, so I'll set it aside in what follows.

[25] So-called "monstrous moonshine" [Conway and Norton(1979)] is another more famous and much more complicated source of examples of surprising (non)coincidences relating facts about finite groups and modular functions. See, e.g., [Gannon(2006)].

[26] [Davis(1981), 312]

[27] [Guy(1990), 10]

Example 3. Each of the following is a prime number: 31, 331, 3331, 33331, 333331, 3333331, 33333331.[28]

Are all numbers of the form $3\ldots31$ prime? Again, the answer is no. It's only a coincidence that the first several instances of numbers of this form are prime. $333333331 = 17 \cdot 19607843$, for example.

Example 2 and Example 3 naturally prompt questions about whether we can determine if an n will be a Niven number or if

$$\underbrace{3\ldots3}_{n}1$$

is prime just by looking at n. Example 1, however, doesn't obviously raise questions that might be usefully investigated, which may lead us to classify it as not merely a coincidence, but as a "mere" coincidence.

Example 4. Choose a "normal" three-digit number (i.e., aside from 111, 222, ...), say, 713. Subtract the number that results from arranging its digits in ascending order from the number that results from arranging its digits in descending order: $731 - 137 = 594$. Repeat this process again with the result: $954 - 459 = 495$. And again with 495: $954 - 459 = 495$. Choosing a few other initial numbers also yields a repeating 495 after several iterations of this operation—a so-called Kaprekar operation.[29] Will we end up with 495 regardless of the three-digit number we start with or are these results just a matter of coincidence?

It can be shown that 495 is the unique fixed point of the Kaprekar operation applied to normal three-digit numbers and that all normal three-digit numbers eventually reach the fixed point (in six steps or fewer), so this is no coincidence.[30] To find the fixed point, consider the following. Each iteration of the Kaprekar operation has the following form, where $0 \leq c \leq b \leq a \leq 9$.

$$\begin{array}{rccc} & a & b & c \\ - & c & b & a \\ \hline & A & B & C \end{array}$$

And the relations between a, b, c and A, B, C can be seen to be the following.

$$\begin{aligned} C &= 10 + c - a \\ B &= 10 + b - 1 - b = 9 \\ A &= a - 1 - c \end{aligned}$$

[28] [Guy(1988), 699]

[29] This operation is named after Dattathreya Kaprekar, who studied it and discovered a number of facts about it. See, e.g., [Kaprekar(1955)]. Kaprekar also gave the name 'harshad' to the harshad numbers in Example 2.

[30] For four-digit numbers, the operation always ends with 6174. Neither two-digit nor five-digit numbers have a repeating value (or "kernel"). See [Nishiyama(2012), 370].

Since we're looking for a fixed point of this operation, we need A, B, C to be some permutation of a, b, c. There are six possible permutations, but only $(A, B, C) = (c, a, b)$ solves the equations. And for this permutation, $(a, b, c) = (9, 5, 4)$. That is, applying the Kaprekar operation in the form of $954 - 459$ results in the fixed point of 495.[31] [Eldridge and Sagong(1988)] further shows that for any three-digit number in base r, the Kaprekar operation must reach the fixed point

$$\left(\frac{r-2}{2}, r-1, \frac{r}{2}\right)$$

just in case r is even.

Example 4 leads to at least the following natural question: For which n does the Kaprekar operation applied to length-n numbers terminate in a single value? And is there a reason that certain length numbers behave this way under the operation or is it merely coincidental that, e.g., three- and four-digit numbers have a fixed point but five-digit numbers do not?

Philosophers of mathematics have been interested in mathematical coincidences like these largely because they seem to present us with a puzzle: *everything* in mathematics is necessary, so it's not clear what sense could be made of there being mathematical *coincidences* or anything *accidental* in mathematics at all.[32] I hope to show in what follows, however, that if we pay more attention to the role the concept of a coincidence plays in mathematical practice—in particular to the ways in which mathematical coincidences can be used to "induce indeterminacy"—coincidences in mathematics needn't be seen as being inherently puzzling or problematic.

4 Mathematical Coincidences as Inducing Indeterminacy

As I suggested at the end of Section 2, it's natural to see the examples of mathematical coincidence offered in Section 3 as presenting us with indeterminacy in Dewey's sense; e.g., facts about the decimal representations of π and e don't seem to hang together in any sense; there's no way to predict how these facts could ultimately be connected; and noting that some of the decimal digits of π and e coincide naturally enough evokes different responses (e.g., some might say there's nothing more to be said here; some may think it points to something further to be discovered—perhaps that these digits coincide roughly every ten digits or something similar[33]). But I've also claimed

[31] A similar procedure can be used to show that 6174 is the fixed point of the Kaprekar operation applied to normal four-digit numbers. See [Nishiyama(2012), 364-365].

[32] This may also suggest, as [Fine(1994)] has argued regarding essence, that necessity is too "coarse-grained" of a notion to capture the phenomena and so an account of coincidences as not stemming from the "essences" of the objects involved is needed instead. Taking this line would produce a rather different view than the one to be investigated here, and will have to wait for another opportunity to be explored in any case.

[33] See [Davis(1981), 312-313].

the mathematical coincidences can serve as "sources of newness" and "*inducers* of indeterminacy" to use the phraseology from [Hutter and Farías(2017)]. In what way might this be the case?

When a mathematician first asks whether two or more mathematical facts are coincidental, attention is drawn to the fact that we don't have a way of connecting these facts to one another: there's no path from the one to the other(s) in a sense. Making this disconnect clear also makes clear that what might be said to get us from the one fact to the other(s) is not something that's yet settled, and so may be properly thought of as generating a new indeterminacy to be investigated. If we take mathematical coincidences to indicate a kind of indeterminacy in connections between mathematical facts that fail to "hang together" and so on in this way, they can also be seen as sources of the indeterminacy that pushes creativity and research forward as discussed in Section 2. It should be noted right away, however, that this highlighted indeterminacy and the attempts at its resolution aren't to be understood as a kind of anything-goes chaos. Rather they should be seen as placing the mathematician in a familiar constrained, but unsettled kind of situation familiar across various forms of inquiry and research. Consider the following comparison. In his *Confessions*, Augustine recounts the story of being part of a competition as a boy that asked a group of students to rewrite the angry speech of Juno from Book I of *The Aeneid* in prose. "The contest was to be won by the boy who found the best words to suit the meaning and best expressed the feelings of the sorrow and anger appropriate to the majesty of" Juno.[34] If we take a question such as, "What is the best way to render Juno's speech into prose?" a determinate answer wouldn't be expected, and it would be odd to suspect that there is a single best answer waiting to be discovered. Surely the judges would at least have some appropriate wordings in mind and would have had *some* sense of what a winning submission might look like, but the contest would not be the type of contest it was if they had fully specified the conditions under which an entry would win in advance. Part of what makes such a competition interesting for all those involved must be the learning about what good answers look like through pursuing an answer and through judging both good and bad attempts at providing one. Similarly, if mathematical coincidences induce indeterminacy in the sense of Dewey, we may see them as posing the question, "What is the best way to get from fact A to fact B?" By the nature of the case, we won't be able to predict how A and B end up connected (if they are ever connected), but that doesn't mean that anything at all can be put in place to do the connecting.

This understanding of connecting unconnected mathematical facts comes close to committing to the view that mathematics is created not discovered, but the view under consideration here needn't go that far globally.[35] Even if

[34] See [Augustine(1961), I.xvii]. *Cf.* the fairy tale discussed by Wittgenstein and cited in [Säätelä(2011), 173], where the prince asks a smith to bring him a "hubbub."

[35] See [Muntersbjorn(2003)] for an argument that this is a false dilemma anyway. Muntersbjorn argues that mathematics is best thought of as being "cultivated" rather than invented or discovered.

it did though, the problems involved with taking such a position often seem to be exaggerated. From the perspective of trying to take the facts of mathematical practice adequately into account, mathematicians themselves may be more inclined to describe certain developments in terms of creations rather than as discoveries.[36] Insisting that everything in mathematics must count as being discovered also commonly fails to distinguish between that which is eventually seen as being there from that which is somehow already there waiting to be found. These general remarks could be justified from a perspective on mathematics along the lines of Wittgenstein's,[37] but they can also be seen as simple descriptions of the realities of mathematical practice.[38] Since capturing mathematical practice in relation to mathematical coincidence is my primary goal here, I only want to claim that the understanding of how the pieces of a mathematical coincidence may come to be connected discussed so far is not in conflict with that practice: there's no doubt that mathematicians do regularly forge connections between different areas of research and disparate facts.

Noting these facts about the role of coincidence in mathematical practice may even help us understand the phenomenon more than a direct attempt at an analysis or definition would. In fact, the examples considered so far suggest a view along the following lines. In one of his so-called minor works, *The Art of Persuasion*, Pascal suggests that God wants divine truths to enter our lives by proceeding from the heart to the mind, not the other way around.

> [T]he saints [. . .] say, in speaking of things divine, that we must love them in order to know them.[39]

The beginning of an account that will be presented below might be called "Pascalian" in the sense that it suggests that our talk of mathematical coincidence is most appropriately seen as being involved in making us love mathematics in a way that leads us to mathematical knowledge and understanding. Even in this abstract domain, we need a loving guide to lead us along the path to knowledge.[40]

Other sciences can rely more on human needs and necessities to push forward investigation and research: new diseases come along; the climate changes; people demand faster computers; etc. Some of the science involved in these pursuits will push mathematics forward as well obviously: in order to build a

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36 *Cf.* [Gowers(2011)] and the commentary in [Rosen(2011)].

37 See, e.g., [Wittgenstein(1930/1975), §158] and [Wittgenstein(1956/1983), I §168].

38 See, e.g., [Martin(2020), §5.1].

39 "[*L*]*es saints* [. . .] *disent en parlant des choses divines qu'il faut les aimer pour les connaître.*"

40 Note that [Guy(1988), 698], e.g., disagrees with this kind of view. He suggests that early coincidences like the ones seen in Example 2 and 3 above are actually "the enemy of mathematical discovery" since they tend to send us on wild goose chases for proofs of theorems that are simply false. Of course, I'm not suggesting that there is some kind of foolproof path to successful mathematical discovery. In any given case, the mathematician will have to rely on background knowledge, experience, intuition, and so on to determine whether a direction of research suggested by a mathematical coincidence is worth the time and effort. My point is that these coincidences are useful ways of inducing indeterminacy that can spur research, not that they're the only guide available.

better bridge, you need to understand, say, some geometry, differential equations, and so on. Even the purest number theory can sometimes be driven forward by real-world applications such as the modern need for encryption online as well. But there's plenty of mathematics that certainly does not now and may never have real-world applications or be used to solve practical human problems. If we think that the general practice of mathematics is a worthwhile and significant human endeavor that needn't be justified in terms of applications, we need to continue finding ways to generate particular interest in it and even love for it.[41] One way of creating this kind of care, which we find in ordinary mathematical discourse already, is putting forward a given mathematical fact or group of facts as being no coincidence.[42] To make a claim like this is a way of both helping to generate an indeterminate situation in Dewey's sense that is open for inquiry and suggesting that there might be something worthy of interest there—something that's of interest independently of anything else that might make a piece of mathematics worth caring about. Similarly, saying that something is *just* (or *merely*) a coincidence is a way of directing attention away from the indeterminacy felt between the facts and suggesting that this is not something worth spending time on.[43] (The examples presented in Section 3 elicited these very sorts of reactions and judgments.) These kinds of evaluative judgments should be seen in a real sense as essential to the practice of mathematics. We don't have unlimited time or resources, so we need to focus attention on one area or problem rather than another now and then; researchers need to get other people interested in their problems to sustain their work, to get grants and postdocs; and so on. Talking in terms of mathematical coincidence is one way of contributing to the doing of all this work.

To see this kind of interest-generating role for mathematical coincidence in action in another simple case, consider the following pair of "worm-eaten arithmetic" problems.[44]

$$\begin{array}{lr} & \square\,\square\,\square\,\square\,\square \\ \times & \square\,\square\,\square\,\square\,\square \\ \hline & 1\,2\,3\,4\,5\,6\,7\,8\,9 \end{array}$$

$$\begin{array}{lr} & \square\,\square\,\square\,\square\,\square \\ \times & \square\,\square\,\square\,\square\,\square \\ \hline & 1\,2\,3\,4\,5\,6\,7\,8\,4 \end{array}$$

41 Seeing the ways in which "[m]athematics is *for* human flourishing" [Su(2020), 10, emphasis added] is another route to this same end.

42 Although he wouldn't agree with the details of the Pascalian view presented here, Marc Lange, e.g., agrees that coincidence talk can sometimes make it easier to recognize interesting issues. See [Lange(2010), 331].

43 Lange very briefly discusses and dismisses a view like this which holds that a coincidence "does not repay further study, it is not fruitful, it leads to no further interesting mathematics" [Lange(2017), 286]. (He also suggests that Roy Sorensen makes a proposal like this in an unpublished manuscript, "Mathematical Coincidences.") I'll discuss Lange's view further in Section 5 and comment on how the proposal under consideration differs from this dismissed one there as well.

44 [Nishiyama(2012), 372]

Each of these problems is solvable by the same method,[45] but due to the coincidence that the product arrived at in the first is 123456789, we can expect there to be a natural desire to find the answer, while 123456784 is somehow less exciting. I don't want to claim that the difference in interest is very great here at all or that the problem is itself of any great significance. However, I do claim that such a difference in interest is there nonetheless (as attested to in [Nishiyama(2012), 372]), and that fact is worth trying to understand.

There are a few other related roles that talk of coincidence in mathematics often plays and which are worth mentioning here as well before reaching a general summary of the view. Although these roles are clearly important, they are the sort of roles that philosophers of mathematics can be slow to pay attention to. The first is the fact relevant to mathematical pedagogy that presenting a new piece of mathematics as a case for which there's a question about whether or not a coincidence is involved can motivate students to learn the fact better as they do the work required to find out the answer on their own. There are numerous studies in the cognitive science literature and the literature on mathematics education that bear out the effectiveness of this approach.[46] One important part of mathematical practice is the training of new generations of mathematicians, and noticing and making use of the effect that this style of presentation can have on students seems to be a wise thing to do. Generally, seeing mathematical coincidence talk as essentially aimed at creating or dissolving a certain kind of care can play an important role in taking this kind of fact about pedagogy into proper consideration.

The idea that talk of mathematical coincidence can play an effective role in mathematical education is supported by the fact that in mathematical textbooks the most common use of 'coincidence' is in the context of saying that something is "no coincidence."[47] This is a way of saying that the facts presented so far may seem to be merely coincidental, but that one ought to "stay tuned" for a more satisfying story. The use of coincidence talk plays a different, but still central, role in research-level mathematics. In contrast to textbook writing, a search for the word 'coincidence' on MathOverflow, "a question and answer site for professional mathematicians,"[48] reveals that the term occurs most often by far in the context of questions of the form, "Is it a coincidence that X?" Discussions of "eventual counterexamples" on the site are also ger-

[45] $123456789 = 3^2 \cdot 3607 \cdot 3803 = 10821 \cdot 11409$ and $123456784 = 2^4 \cdot 11^2 \cdot 43 \cdot 1483 = 10406 \cdot 11864$.

[46] See, for example, [Buchbinder and Zaslavsky(2011)] and [Diaconis and Mosteller(1989), 859].

[47] For examples of this in a few popular textbooks across a variety of disciplines consider, e.g., [Artin(1991), 233], [Axler(1997), 177], [Dummit and Foote(2004), 55], and [Spivak(1994), 199, 528], where this kind of language only turns up in "no coincidnce" (or "no accident") contexts.

[48] <https://mathoverflow.net/tour> Accessed 25 March 2021. See [Martin and Pease(2013)] for some reflections on MathOverflow as a resource and the light it sheds on the production of mathematics.

mane here.[49] An ‘eventual counterexample’ is a counterexample to a general principle that holds long enough for one to conjecture that it holds in general. E.g., the factors of $x^n - 1$ for $n \leq 100$ all have coefficients in $\{-1, 0, 1\}$. It was conjectured that this would be the case for all n.[50] The theorem “[i]f n has at most two odd prime divisors, then the coefficients of $\Phi_n(x)$ [the n^{th} cyclotomic polynomial] are in $\{-1, 0, 1\}$,”[51] and the fact the d^{th} cylcotomic polynomials where d is a divisor of n are the factors of $x^n - 1$ shows that it’s no accident that the factors’ coefficients lie in $\{-1, 0, 1\}$—no number ≤ 100 has more than two distinct prime factors. The conjecture doesn’t hold for all n, however, since, e.g., $x^{105} - 1$ has $\Phi_{105} = \Phi_{3 \cdot 5 \cdot 7}$ as a factor and this polynomial has a coefficient of 2 in it. (It’s worth adding this example to those given in Section 3 because it highlights the way aspects of both coincidence and noncoincidence often show up in these sorts of cases. The fact that the conjecture doesn’t hold for all n makes it a coincidence that it holds for the first hundred or so cases, but it’s no coincidence that it held for each of those.) Asking whether or not something is a coincidence on a forum like MathOverflow appears to be one useful way of gauging whether other mathematicians find the particular fact worth investigating and of generating the kind of attention needed for progress to be made. (E.g., noting the facts mentioned above about the factors of $x^n - 1$ did lead to further interest and investigation on MathOverflow subsequently.[52]) This kind of sparked-interest can play the role of prompting other researchers to get involved in the assembly and correlation of answers to other non-*why*-questions from which answers to coincidence-related *why*-questions tend to emerge.[53,54]

Finally, and I hope not too fancifully, it’s easy when trying to engage in the sometimes cold, objective observation of a science to forget that people study mathematics and pursue mathematical research for the unique kinds of pleasures that can be obtained in such pursuits. There’s beauty to be appreciated in certain proofs and there’s enjoyment to be had in observing certain kinds of cleverness in mathematics that are specific to the practice and worth pursuing for their own sake. One further kind of enjoyment that one can experience when engaging in this kind of work is wondering whether something is a coincidence or not and then having the question settled.[55] This kind of

[49] <https://mathoverflow.net/questions/15444/examples-of-eventual-counterexamples> Accessed 29 August 2021.

[50] <https://mathoverflow.net/a/15506> Accessed 29 August 2021.

[51] See, e.g., [Brookfield(2016), 186].

[52] <https://mathoverflow.net/questions/109149/cyclotomic-polynomials-with-coefficients-0-pm1> Accessed 29 August 2021.

[53] *Cf.* Sylvain Bromberger’s advice to someone seeking an answer to a *why*-question: “My guess is that the rational thing for him to do is to forget about the *why*-question and to turn to other questions instead, remembering that answers to *why*-questions usually emerge from work on questions with more reliable credentials” [Bromberger(1992), 169].

[54] See [Martin and Pease(2013)] for more on this “fact-gathering”-role played by MathOverflow.

[55] Note that this kind of question can be settled without coming to the conclusion that it’s *true or false* that X is a coincidence. One way of settling the question would be to come to the opinion that the fact isn’t interesting and so just a coincidence or that it is interesting and so is no mere coincidence.

resolution can come about through new research or can be put before you by someone writing an article or textbook. This kind of pleasure is similar to one that one finds in reading certain works of fiction. Many people have had the experience of reading Dickens, e.g., *Martin Chuzzlewit*, and having the thought, "Ah, another very convenient coincidence, Charles!" But of course one needn't respond to the book that way. One might also wonder whether it really *is* a coincidence that such-and-such person just happens to show up at just the right moment, and this wonder can add to the enjoyment of the novel.[56] No doubt, some people will dislike this kind of story and experience, but not taking note of the phenomenology would be to miss an important part of what reading and writing novels is all about. Similarly, we would be missing some of the true multicolored mixture of mathematics and its practice if we did not pay attention to this role for mathematical coincidence within the practice.[57]

Clearly, I have not offered a *theory* of mathematical coincidence here. The beginnings of an account derived from the observations made so far, however, might take the following form. The point of talk of coincidence in mathematics from a "Pascalian" perspective is to marshall attention and interest in a group of mathematical facts by drawing attention to their indeterminacy (i.e., as a site for inquiry). That being the case, a deflationary/non-representational story about coincidence-talk suggests itself.[58] To call a collection of facts a coincidence can be taken to do the work of suggesting that these facts constitute a location of inquiry that's not worth investigating. To say that the facts are an *interesting* coincidence would be to indicate that although they're not worth further investigation, the facts seen together are nevertheless remarkable. To say, instead, that some facts are no coincidence would be to say that the facts are a site of indeterminacy that is (or—in the case of established connections—was) worthy of further investigation. It's being true or false that some facts are a coincidence would then likely be given a deflationary story derived from these facts about usage. An account of this kind would further argue that facts about the use of 'coincidence' talk in mathematics and what coincidences *are* aren't separable and that the former facts provide us with all there is to know about the later.[59] Although this "account" is quite sketchy, what it does do is capture and highlight the important roles the apparently strange phenomenon of mathematical coincidence plays within

[56] *Cf.* [Dannenberg(2008), 155]: "It would probably be difficult to discover a novelist more consummate in the art of coincidence than Dickens."

[57] See [Wittgenstein(1956/1983), III §46, emphasis in the original]: "[M]athematics is a MULTICOLOURED *mixture* of techniques of proof."

[58] See, e.g., [Williams(2010)] for a general framework for offering an explanation of meaning in terms of use along these lines. See [Pérez Carballo(2016)] on a non-representational view of mathematics as a whole.

[59] A full theory might proceed by providing more explicit rules for "introducing" and "eliminating" coincidence-talk; explaining how coincidence-claims embed in non-asserted contexts; etc. There seem to be ample tools and methods available for filling in some of these details if one were so inclined. *Cf.* [Thomasson(2020), Ch. 3] for a similar approach to handling talk of necessity and possibility especially in the area of metaphysics.

the practice of mathematics, in particular the role these coincidences play in inducing indeterminacy and as sources of newness in inquiry. My hope is that through the presentation of these roles and facts the phenomenon needn't feel so strange anymore—that is, that some of our philosophical puzzlement has been dissolved—and that some insight into mathematics and its practice has been achieved at the same time.

Let me emphasize one key characteristic of mathematical coincidence as I have described the phenomenon here though before moving on to discuss the most obvious objections to the view, which arise in relation to this characteristic. Calling a collection of facts a mathematical coincidence or not, according to the Pascalian view presented here, primarily engages one in evaluating it as being uninteresting or worthy of attention, respectively. This being the case, its being *true* that some mathematical fact, say, is a coincidence shouldn't be understood in terms of the world of mathematics providing a truthmaker for this claim.[60] Rather, if we're to talk about truth at all here (it will be of secondary importance to this sort of account in any case), to say that it's true that X is just a coincidence is to do no more than agree that X isn't worthy of special attention.[61] To say that it's true that X is no coincidence is similarly merely to put it forward as something worthy of our mathematical attention.

5 Comparison with Lange's Account

The understanding of mathematical coincidence discussed so far contrasts most directly with the account given by Marc Lange, which is the most developed in the literature.[62] Probably the most important contrast with the Pascalian approach outlined above is that Lange's analysis takes talk of coincidence in mathematics to be essentially descriptive.[63] I'll consider Lange's view in this section and another objection to the general Pascalian viewpoint in the next in order to further elaborate the approach on offer and respond to the most obvious objections to it.

Just as we had to find an understanding of 'indeterminacy' that could make sense in the setting of mathematics in Section 2, Lange's account can be seen as first finding a mathematical sense of 'coincidence' by modifying a more general definition of the term. The account moves from *The Oxford English Dictionary*'s definition of 'coincidence' as "[a] notable concurrence of events or circumstances having no apparent causal connection,"[64] to the considered view by analyzing 'notable' as sharing a noteworthy, mathematically

[60] *Cf.* [Floyd(2012), 232] for a similar claim about surprises in mathematics from a Wittgenstein-inspired perspective.
[61] X may be worthy of attention for other reasons, of course.
[62] See [Lange(2017)], Part III and especially Chapter 8.
[63] *Cf.* [Lange(2017), 304]: That some theorem is no coincidence "is a fact about mathematics no less than that the theorem holds." Whether or not this thought can only be captured by an account like Lange's depends on how robustly we want to understand the concept of a "fact." See, e.g., [Price(2011), §4].
[64] "coincidence, *n.*" *OED Online*. Oxford University Press. Accessed 25 March 2021.

"salient" or "natural" property,[65] substituting "a collection of mathematical facts" in for "concurrent events or circumstances," and suggesting that the analogue of "having no apparent causal connection" is "lacking a common, unified explanation."[66] After these adjustments we arrive at the following.

> For a collection of mathematical facts, it is a coincidence that they are all true iff the facts share a common, salient property but there is no single, unified explanation why.[67]

This definition makes use of the concept of a single, unified mathematical explanation,[68] and so can't be fully understood without some basic idea of what such a thing is supposed to be. However, given that the worries I'll raise in this section don't depend in any crucial way on the fine details of a theory of mathematical explanation, the simple idea that explanatory proofs—proofs demonstrating *why* a theorem holds rather than *that* it does—provide us with mathematical explanations should suffice.[69]

Before objecting to this account, it must be admitted that it seems like it's getting at something correct and interesting, and that it can accommodate many of the observations covered in the previous sections as well. E.g., what else makes it merely a coincidence that the thirteenth digit of π and e is 9 if not the fact that there's no unified explanation of that fact? And isn't it the belief that there is a unified explanation to be found that drives one to investigate apparent coincidences in the hopes of discovering fruitful mathematics? Despite this appeal and success, however, the following reasons suggest that there is still room for alternative views.

The first concern stems from the fact that Lange's understanding of mathematical coincidence seems not to mesh well with at least some of the things mathematicians themselves have said about coincidence in mathematics. For example, some of the examples in Philip Davis's original article on mathematical coincidence do not seem to fit Lange's account without a bit of forcing. Davis shows, e.g., that it's no coincidence that the first time $\sqrt{1141y^2+1}$ is an integer is when y is greater than 10^{25}. For this to count as a non-coincidence on Lange's account, we have to identify a collection of mathematical facts and then some salient feature shared by all of them. A unified explanatory proof that shows *why* all of these facts have this property then shows why bearing it is no coincidence. The best candidates for the mathematical facts making up

65 *Cf.* [Lange(2017), 277]

66 See [Lange(2010), 316-322] and [Baker(2009), 141].

67 Given an account like this, an alternative to the line I'm pursuing here, but that would be congenial to the general outlook, might be to follow [Locke(2020)] in his account of "metaphysical explanation for modal normativists" and use Lange's definition of 'coincidence' along with a non-descriptive story about non-causal explanation.

68 The qualification, "single, unified," aims to prevent one from claiming that by putting together an explanatory proof of A and an explanatory proof of B one has given an explanation for A and B.

69 For more detailed accounts, see, e.g., [Steiner(1978)], [Kitcher(1989)], or [Lange(2017)]. For worries about having mathematical explanation focus only on explanatory *proofs*, see [Lange(2018)] and [D'Alessandro(2020a)].

this non-coincidence seem to be the following: (1) when $y = 1$, $\sqrt{1141y^2 + 1}$ isn't an integer; (2) when $y = 2$, $\sqrt{1141y^2 + 1}$ isn't an integer; (3) when $y = 3$, $\sqrt{1141y^2 + 1}$ isn't an integer; .... The question is then whether or not to include the fact that when $y = 30693385322764657197397208$, $\sqrt{1141y^2 + 1}$ *is* an integer in this list. If it is included, then the shared property, which seems to be "y's not being an integer when substituted into $\sqrt{1141y^2 + 1}$" is no longer shared by all the facts. (And it's maybe a stretch to consider this property as being mathematically natural in any case.) If, on the other hand, this last fact is not included, then our explanation just shows that $\sqrt{1141y^2 + 1}$ isn't an integer for very many small values of y. But the non-coincidence given by Davis seems to want to account for *both* when $\sqrt{1141y^2 + 1}$ isn't an integer and when it is one once y gets to 30693385322764657197397208. Perhaps, instead, the shared property of each of these facts is that they all involve the expression $\sqrt{1141y^2 + 1}$—Davis's explanation does appeal to the continued fraction expansion of $\sqrt{1141}$ after all. This allows the 30693385322764657197397208$^{\text{th}}$ fact to be included in the list seamlessly, but, again, counting this as a mathematically natural property and analyzing the case in these terms feels to me as if it's fitting the example to the analysis rather than the other way around. That is, the route from noting that y must grow so large before we find an integral value of $\sqrt{1141y^2 + 1}$ to the reason why doesn't look like it really requires taking special note of this massive collection of other facts or how they're grouped together.

William D'Alessandro raises a similar sort of concern about salience in his investigation of proofs of quadratic reciprocity in relation to Lange's account of explanation, which is central to his account of mathematical coincidence.

> Mathematicians clearly view [quadratic reciprocity] as mysterious and in need of explanation, but what is its relevant outstanding [i.e., salient] quality? I don't think this is very obvious. Although one could probably shoehorn the case into any or all of Lange's three main categories, none seems particularly natural, and none leads to the right verdicts about proofs [D'Alessandro(2020b), 39].

Lange himself, being one of the excellent practitioners of "practice-first" philosophy of mathematics, of course wouldn't want to do any shoehorning to fit a case to his account, and there may be a way to make these examples fit the theory more snugly, but if mathematical coincidences can be understood at least largely apart from a theory of explanation as with a Pascalian-type account, perhaps some of these difficulties can be preemptively avoided. And given that mathematicians' assessments of the explanatory value of proofs don't always turn out as we'd expect them to, doing without the theory of explanation here might actually be the safest bet. Consider, e.g., the recent study [Mejía-Ramos et al.(2021)].[70] The authors asked a group of mathematicians to evaluate various proofs of the proposition "[i]f $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$" for explanatory value, which they did without trouble. As the au-

[70] See also [Inglis and Aberdein(2015)] for a similar kind of study.

thors note in §5 of their paper, however, this gives us at least some reason to question the adequacy of an account like Lange's since his view "predicts that if [a] result exhibits no noteworthy feature, then to demand an explanation of why it holds, not merely a proof that it holds, makes no sense" [Lange(2017), 255].[71]

Not only do some purported examples of mathematical coincidence not seem to fit naturally with Lange's account, but also both Davis and the mathematician E. H. Moore (surprisingly) suggest what is essentially the exact opposite of this view. They both claim that, "The existence of [a] coincidence implies the existence of an explanation"[72]; that is, once you see something interesting or unexpected going on, you're going to be able to find an explanation for it in some way. Lange's account says that coincidence implies that there is *no* explanation though, so clearly both can't be correct. The way Davis and Moore speak of coincidences as having explanations may seem unusual, but it is paralleled to some extent in ordinary discourse as well. If coincidences by definition have no explanation, "There's no explanation for this coincidence," would be tautologous, and something of the form, "X explains the coincidence," would be contradictory. I take it that neither statement is tautologous or contradictory, however. Certainly, one might suggest that it would be more careful to say, "X explains the *apparent* coincidence," in the latter case, but even if this were so, the fact that "X explains the coincidence" isn't nonsense, in combination with the claims of Davis and Moore from before, ought to provide some evidence against Lange's view and is more readily accounted for by an account like the Pascalian one discussed above.[73]

Finally, given all the roles mathematical coincidences were seen to play in Section 4, we should ask an account of mathematical coincidence to provide us with some explanation of the appropriateness or inappropriateness of various attitudes towards the facts involved in purported coincidences. For example, if a mathematician thinks that some collection of facts is merely coincidental and then spends the next five years investigating these same facts, something seems to have gone wrong. Similarly, if a mathematician thinks, "This is no coincidence!" and then feels no motivation at all to investigate, we might feel as if there's some kind of internal discord as well. According to a Pascalian view, these attitudes and motivations are accounted for by the simple fact that to call something a coincidence just is to have the attitude that its investigation wouldn't be worthwhile, and similarly for the no-coincidence case. While Lange's account has an explanation for these attitudes as well, we might (1) aim for an even tighter connection between the judgments and the motivation and (2) worry about whether his account gets the order of explanation right.

[71] It's also possible, as already suggested in n.67, that substituting a different account of explanation or modifying Lange's could do the required work just as well.

[72] See [Davis(1981), 320] and [Moore(1909)], which is cited in [Krieger(2003), 214]. Lange makes note of this part of the Davis article in a footnote, but thinks that it would be better interpreted in a less extreme form. See [Lange(2017), 445n.8].

[73] There remain questions about how literally to take and how heavily to weigh the opinions of mathematicians on these sorts of issues though of course. *Cf.* [Martin(2020), §5.2].

First, consider the following passage where Lange responds to a view similar to the one on offer here and also provides the material for explaining the facts about motivation under discussion.

> But it is not the case that [some facts are non-coincidental] *because* they suggest further interesting mathematics.[74] Rather, they suggest further interesting mathematics because they are non-coincidences. It is fruitful to think further about a non-coincidence because we may thereby uncover the facts that make it no coincidence. On my account, the reason why a genuine mathematical coincidence leads nowhere is that there is nowhere interesting for it to lead. In particular, a coincidence has no mathematical explanation; its components lack a common proof. The reason that a coincidence is not mathematically fruitful is that there is no explanation of it to be found [Lange(2017), 286-287].[75]

So, Lange's idea here is that a mathematician who deems some facts to be a coincidence won't be motivated to investigate them further because if they really are a coincidence, there's nothing further to be found to explain them. Similarly, someone who thinks, "That's no coincidence," thinks that because they think there is an explanation to be found and so may be motivated to try to find it by that belief.

The contrasting views of motivation in this area seem to be somewhat analogous to the difference between the realist and the expressivist in ethics. It has seemed to many to be obviously correct that, if you sincerely think that you ought to give money to the ACLU each month, then you ought to be motivated to do so in at least *some* way.[76] Expressivism has seemed appealing in light of this apparent connection between moral beliefs and motivation since it suggests that the way our ethical beliefs are related to the world is different from the way in which our other beliefs are. If one holds that moral beliefs do something like express a person's view of how the world ought to be rather than having something to do with the world as it is, a ready-made story is available about why a moral belief would automatically imply some kind of motivation.[77] On a view according to which we aim to have our beliefs about

[74] This is one important way in which the Pascalian account isn't just the account discussed in [Lange(2017), 286]. Lange commits the view there to the claim that non-coincidences are non-coincidental *because* they suggest further interesting mathematics. On a Pascalian view, there isn't any because playing a significant role. Further, the view in Lange is primarily a story about what mathematical coincidences are—according to the view, they're the facts that don't repay further study. The Pascalian viewpoint is, instead, primarily a story about what we do with coincidence talk that then takes mathematical coincidences themselves to be roughly the shadows of this talk.

[75] Note that this potentially commits the view to what might be called "explanation chauvinism": Is it really the case that explanations are all that are sought when a coincidence is investigated in mathematics? Compare with "proof chauvinism" as discussed in [D'Alessandro(2020a)] and [Lange(2018)].

[76] See [Smith(1994), 71-76] for an influential argument for this claim from someone not moved to expressivism by this "motivation problem." [Shafer-Landau(2000)] and [Railton(1986)], on the other hand, object to this sort of motivational internalism.

[77] See, e.g., [Schroeder(2010), Ch. 1.4].

morality accurately represent the world in the same way we want our beliefs about, say, crustaceans to do, this connection between motivation and belief is not so easily accounted for. The connection can be accounted for, however, in at least some way in the style of Lange’s account of coincidence. If I believe supporting the arts is important, then it’s no surprise that these beliefs then lead me to action. Given that mathematical coincidence talk doesn’t play a role in our lives anywhere near as important as our moral discourse, it may be thought that accounting for these facts about motivation in relation to our understanding of the phenomenon is not so pressing. And that’s true enough, but the considerations on each side of the realism/expressivism debate seem to be applicable to the evaluation of a Lange-style vs. a Pascalian account of mathematical coincidence as well. So, I claim that whatever weight one gives to the expressivist case against realism, that same force can reasonably be applied proportionally to the contrasting views being discussed here as well.

Second, it might be argued that the motivational account derived from the passage from Lange above gets the order of explanation wrong. Suppose I think that some mathematical facts are a coincidence. According to Lange’s view, this either comes to the same thing as my thinking that there’s no unified explanation connecting the facts or I think this because I believe there’s no unified explanation of the facts. This belief that there’s no unified explanation to be found is what then explains why I won’t spend time investigating the facts. But, surely, it isn’t the fact that there’s no unified explanation connecting these facts *so far* that leads me to call some collection of facts coincidental. That there’s no unified explanation so far must at least further be combined with a belief that one won’t ever be found. And the belief that a unifying explanation won’t ever be found seems to be playing the starring role in explaining my conclusion that the facts are coincidental as well as my lack of motivation to explore them. This belief that a unifying explanation won’t ever be found seems to be not much more than another way of describing an evaluation that the investigation of these facts won’t be worthwhile. That being the case, an account that takes this evaluation to come first, providing the motivation and grounds for the judgment seems to put the star of the show in the more appropriate role. Again, I don’t think this consideration is decisive, but it arguably captures the phenomenology we’re after more accurately and directly.

6 Does Anything Go?

I naturally expect the suggestion that whether a collection of mathematical facts is coincidental is not determined just by what the world of mathematics is like to meet with some resistance. Perhaps it makes sense of how coincidence talk is used in practice; perhaps it nicely squares with judgments that coincidence comes in degrees in mathematics[78]; and perhaps it even helps us understand otherwise strange claims made by mathematicians (e.g., Davis claims

[78] For the desirability of such a feature, see [Baker(2009), 148].

that the mathematician to some extent brings mathematical coincidences into existence—if coincidence talk is in the business of focusing attention here or there, this seems like it might be a reasonable enough claim; if not, perhaps not[79]); but *surely* there are facts of the matter here. Marc Lange gets at something like this thought when he claims that "[e]ven from a mathematically omniscient perspective, there are some mathematical coincidences."[80] Isn't someone who takes the Pascalian view described above committed to a kind of extreme relativism here where anything goes when it comes to talking about coincidence in mathematics?

In responding to this worry and the surrounding cluster of issues it raises in more detail than I did in Section 4, it should first be noted that, as a matter of fact, mathematical practice does tend to converge on whether or not something is appropriately called a coincidence. So, evidently, not just anything does go here. Part of what mathematical practice does—like all other sufficiently developed practices—is establish norms for the usage of the terms of art, like 'mathematical coincidence,' within the practice.[81] 'Coincidence' is a term we're all familiar with prior to any exposure to its use within mathematics, but it's important to see clearly both the similarities and differences between our ordinary usage and the uses the word finds within mathematics. Given the fact that to be part of the practice of mathematics is to be governed by the professional and informal norms of that practice, to be at least largely in line with other mathematicians' judgments about coincidence will in part be constitutive of being engaged in that practice. This being the case, there will be better or worse opinions about whether particular facts are coincidences or not, and these opinions will be judged by the standards of other practitioners. This fact alone should go a long way towards reducing the sting of this relativist worry.[82] Nevertheless, the further question that ought to be asked when deciding how seriously to take the objection that anything like the Pascalian view of Section 4 implies that there's no fact of the matter about whether something is a mathematical coincidence is, "What best explains these facts about mathematical practice?"[83]

Gilbert Harman and David Wiggins have each suggested that realism (in the metaphysical sense) is the view one should opt for when the best explana-

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[79] See [Davis(1981), 320]: "To some extent, [the mathematician] even brings [mathematical coincidence] about."

[80] See [Lange(2010), 316]. According to an alternative account of explanation for which explanations are intimately related to the answering of *why*-questions, this claim of Lange's would likely be judged to be mistaken though. E.g., [van Fraassen(1980), 130] suggests that an omniscient being wouldn't be in the business of explanation at all, so the distinction between coincidence and non-coincidence in Lange's terms would disappear.

[81] See, e.g., [MacIntyre(1981), Ch. 14] on this role of practice. See [Martin(2020)] for a general MacIntyrean pespective on mathematical practice.

[82] See [Field(2001)] for a similar view in relation to his "evaluativist" account of apriority. See also [Rosen(1994)] on the general difficulty of saying what exactly objectivity comes to in the first place.

[83] A simpler way out of the worry may be by adopting a deflationary account of what it is for there to be "facts of the matter" in this domain. See, e.g., [Thomasson(2020), Ch. 6.1].

tion for convergence like the one just described is the existence of a relevant fact.[84] In Harman's examples, the convergence of a group of scientists on the thought, "There goes a proton," is best explained by the existence of the proton, but convergence on the thought, "Setting cats on fire is wrong," is not best explained by the fact that this action is wrong. Rather, this convergence can be better explained by other psychological and sociological facts about human beings in certain places and times. The relevant question for present purposes is whether convergence within mathematical practice on a judgment that, say, X is coincidence is best explained by the fact that X *really is* a coincidence, whatever we ultimately take that claim to mean. My suggestion is that the real existence of something coincidental is not what leads to the convergence here. Instead, this convergence is explained, as suggested above, by the shared practice governing the interests and appropriateness of certain judgments of those making the relevant judgments. In fact, even an analysis based on Lange's account would seem to be committed to this view of the explanation of convergent views about mathematical coincidence: convergence on the view that X is no coincidence tends to happen prior to any explanation being found, and since there's generally no way to prove that A and B can't be given a single, unified explanation, convergence on the view that A and B is merely coincidental *can't* be reached on the basis of the fact that no explanation of this kind will be found.

Another familiar test for whether or not a domain of discourse should be thought of as being (metaphysically) realist is whether it exhibits what Crispin Wright calls "cognitive command." A domain of discourse exhibits cognitive command, roughly, if it's *a priori* that when there's a dispute, one side or the other is mistaken.[85] According to the Pascalian view, talk about coincidence fails to be a domain exhibiting cognitive command, but this fact should be considered a point in its favor. For many pairs or larger collections of mathematical facts, it doesn't seem as if it really makes sense to ask whether they're coincidental or not.[86] In these cases, one *could* insist that the facts are either coincidental or not or one could find a way to raise the question of whether they're coincidental meaningfully, but there's nothing like an *a priori* certainty that one side or the other must be mistaken given this imagined insistence or that an answer about the raised question must be reachable.

These responses to the worry that anything goes with mathematical coincidence talk as far as the Pascalian view is concerned are brief, and the back-and-forth could obviously be extended (indefinitely). However, I take it that the general line of response presented so far is enough to both show the Pascalian view not to have clearly unacceptable consequences and to indicate how this further conversation could be continued for those interested in carrying on.

84 See, e.g., [Harman(1977), Ch. 1.3] and [Wiggins(1987), 147, 149-151].

85 [Wright(1992), 92-93]

86 This fact about arbitrary collections of mathematical facts is also noted by Lange and accounted for by his view. See, e.g., [Lange(2017), 280].

7 Conclusion

One's view of the world and coincidences in it appears to be determined to a large degree by one's intuitive ideas about how the world works. If I think that good things generally come to those who wait, I might not find it coincidental that my patience is so often rewarded. If I have no idea about how the world and occurrences in it are governed, nearly anything could be deemed a coincidence: the sun's rising two days in a row might be a shock; the milk's going bad after being left out in the sun might be baffling; etc. The world of mathematics is one that we're often in the position of not knowing our way about very well and of not having an intuitive understanding that allows us to see certain things as expected or not. This is especially true when we're trying to break into unknown and not yet determinately settled parts of the subject while new research is underway. It's no surprise, then, that we seem to find coincidences so often in this domain. As we come to have hunches or insights or intuitions about how a particular corner of the world of mathematics works and is governed, we aim to remove our feelings of uncertainty, indeterminacy, and confusion by transforming the coincidences we find there from surprises into trivialities.[87] By focusing on how talk of mathematical coincidence functions within normal mathematical practice, I hope to have shed some light on how this talk and these judgments push us forward in the quest for understanding and the (temporary) conquering of indeterminacy in the domain mathematics.

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