# Implementing David Lewis' Principal Principle: A Program for Investigating the Relation between Credence and Chance

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Abstract: In place of the just-so stories and intuition mongering of analytical metaphysicians, I offer a program for understanding the relationship between credence and chance in quantum physics and show how a version of the program can be implemented with the help of some representation theorems.

## 1 Introduction: a short, depressing history of the Principal Principle

Forty some years ago David Lewis (1980) proposed a principle, dubbed the Principal Principle (PP), connecting rational credence and chance. A crude example that requires much refining is nevertheless helpful in conveying the intuitive idea. Imagine that you are observing a coin flipping experiment. Suppose that you learn—for the nonce never mind how—that the objective chance of Heads on the next flip is 1/2. The PP asserts that rationality demands that when you update your credence function on said information your degree of belief in Heads-on-the-next-flip should equal 1/2, and this is so regardless of other information you may have about the coin, such as that, of the 100 flips you have observed so far, 72 of the outcomes were Tails.

The large and ever expanding philosophical literature that has grown up around the PP exhibits a number of curious, disturbing, and sometimes jaw-dropping features.[1] To begin, there is a failure to engage with the threshold issue of whether there is a legitimate subject matter to be investigated. Bruno de Finetti's (1990, p. x) bombastic pronouncement that "THERE

[1] Here is a sample that conveys the flavor the literature: Arntzenius and Hall (2003); Bigelow, Collins, and Pargeter (1993); Black (1998); Haddock (2011); Hall (1994, 2004); Ismael (2008); Meacham (2010); Pettigrew (2012); Roberts (2001, 2013); Strevens (1995); Thau (1994); Vranus (2002, 2004).

IS NO PROBABILITY" was his way of asserting that there is no objective chance, only subjective or personal degrees of belief, and hence there is no need to try to build a bridge connecting credence to a mythical entity. Leaving doctrinaire subjectivism aside for the moment and assuming there is objective chance brings us to the next curious feature of the literature: the failure to engage with substantive theories of chance, despite the fact that various fundamental theories of modern physics—in particular, quantum theory—ostensibly speak of objective chance. Of course, as soon as one utters this complaint the de Finetti issue resurfaces since interpretive principles are needed to tease a theory of chance from a textbook on a theory of physics, and de Finetti's heirs—the self-styled quantum Bayesians (QBians)—maintain that the probability statements that the quantum theory provide are to be given a personalistic interpretation.[2] And this leads to the next curious and disturbing feature of the literature. The bulk of the philosophical discussion is couched in terms of classical probability theory, without any apparent recognition of the facts that quantum probability theory is not classical probability theory and that the way in which quantum probabilities are generated offers ways of linking credence and chance that have a bearing on the PP. The lack of engagement with substantive theories of chance also enables the avoidance of issue of whether and how it is possible learn what the objective chances are. Lacking an account of how the learning of chances is possible, the PP has the airy-fairy quality of a Principle of Revelation that requires that when a rational agent acquires knowledge of El Qanna's favorability ranking—never mind how she acquires such knowledge—she aligns her credence in X with how favorably El Qanna looks upon X.

If this parade of horribles were not already long enough there is another overriding concern: a lack of clarity about what the PP is. I refer not just to the fact that there are several competing formulations of the PP but to the more fundamental fact that there is an ambiguity in what is being claimed by any given formulation. To lay out this complaint in more detail a little more groundwork is helpful.

The vicissitudes of PP are best discussed in the setting of what I will call normative Bayesianism. There are two forms—classical and quantum—that take into account the differences in the event structures of classical and quantum settings. There are two matching tenets from the two forms involv-

[2]For an accessible introduction to QBism see von Baeyer (2016).

ing, respectively, synchronic and diachronic constraints on rational degrees of belief. The synchronic constraint requires that rational degrees of belief conform to the probability axioms—classical or quantum as the case may be. The diachronic constraint requires that updating proceeds in the classical case by Bayes conditionalization and in the quantum case by Lüders conditonalization. In both cases the diachronic constraint is silent about how an agent should update on zero-probabilty events, an issue that will be set aside here.[3]

The first way of construing the PP is that it is proposing an additional constraint on rational credence. It contemplates that there are credence functions that satisfy the tenets of normative Bayesianism but fail to align with objective chance when knowledge of its values is obtained, and it would label such credence functions as irrational despite their Bayesian pedigree. Alleged norms require justification. There are multiple justifications for the synchronic norm of Bayesianism: Dutch book arguments, scoring rule arguments, decision theoretic arguments, and more. The justification for the diachronic constraint is much thinner. As far as I am aware the only half-way convincing argument involves a diachronic Dutch book construction that has been subjected to much criticism. But for present purposes the diachronic constraint can be taken on board since what is at issue is whether there needs to be a further constraint linking credence and chance. If the PP is to serve as an additional normative constraint it requires justification. The only serious attempt to this effect I am aware of is a scoring rule argument (see Pettigrew 2012). It gets credit for ingenuity, but it is question begging. The scoring is given in terms of a measure of how well personal degree of belief tracks objective chance. But precisely what is at issue is how and why rational credence should track objective chance.

The second way to construe the PP is to take it not as proposing a new principle of rationality but rather as providing a kind of functional characterization of chance: whatever chance turns out to be, it is that which has the power to command credence in the way the PP contemplates. This point of view can serve as a useful heuristic when trying to interpret a physical theory so as to yield verdicts about objective chance. But adamantly refusing to apply the label 'chance' unless it fits the functional characterization provided by the PP turns it into prophecy that will either be empty or self-fulfilling.

Time to pack it in? Not yet. Another line of investigation is worth pur-

[3]See Earman (2020) for a discussion of this issue in classical and quantum probability.

suing: it seeks to give a more constructive content to the idea that objective chance is that which commands rational credence.

## 2 The program

The program starts from the sentiment that questions about the relation of credence and chance should be relativized to a substantive theory of chance, and it is open to the possibility that the answers may vary from theory to theory. It is up front in admitting that a theory of chance cannot be read off a textbook theory of physics but requires interpretational principles. Such principles are bound to be controversial—what did you expect! But the controversies will be of a form that is part and parcel of the philosophy of physics, and reducing issues about the PP to such a form is to be counted as progress. The goal of the program is to prove theorems about how rational credence is related to chance as embodied in the considered theory of chance. Taking to heart the pessimism about justifying the PP as a new principle of rationality, the program takes "rational credence" to mean simply credence satisfying the two norms of normative Bayesianism—no additional norms are to be appealed to. Then the chips are left to fall where they may. If some of the theorems can be plausibly construed as fulfilling the intuition that David Lewis had, score one—a big one—for David. But don't celebrate too much unless the considered theory of chance lends itself to an account of how agents can come to learn what the chances are. On the other hand, if no appropriate theorems are forthcoming then conclude that, as far as the considered theory of chance is concerned, the PP is to be put in the litter bin of untenable philosophical conceits. But continue the program by investigating how the PP fares in alternative substantive theories of chance.

What kind of theorem should we hope to prove if such a program is to lead to a vindication of the PP? Here I take my inspiration from two sources. The first is the theory of rational decision making under uncertainty, where the goal is to prove a representation theorem of the form: If an agent's preferences satisfy such-and-such rationality constraints then they can be represented as if she has a utility function and a probability function such that her decisions conform to the rule of maximizing expected utility. Second, I take to heart Jenann Ismael's (2008) insight that behind squabbles about specific formulations of the PP there is the more general principle that rational credence is, or as I would prefer to say, can be represented as subjective uncertainty about what the objective chances are. This generalized PP explains why achieving subjective certainty about the objective chances brings credences into line with (what the agent takes to be) chance, as the special PP requires.

With these guidelines in mind, the desired theorem should take the form of a representation theorem: Rational credence in an event $E$ can be represented as a weighted average of the possible objective chances, where the weight given to a chance value is the agent's personal probability of the proposition that the chance takes the specified value. Again "rational credence" is to mean simply credence satisfying the two norms of normative Bayesianism, and the representation theorem can be either an object level theorem or a meta-theorem of the considered theory of chance. The theorem should entail as a corollary that when updated on $F$ (by Bayes conditionalization in the case of classical probability or by Lüders conditonalization in the case of quantum probability) the agent's new rational credence in $E$ can again be represented as a weighted average of the possible objective chances, where now the weight given to a chance value is agent's $F$-updated personal probability assigned to the proposition that the chance takes the specified value. And from this it should follow that updating on the proposition that the chances are so-and-so brings the agent's credence of an event $E$ into alignment with said chance of $E$. Obviously, in order to enable the sought after representation theorem the considered theory of chance must contain propositions, in the domain of both credence functions and chance functions, that can play the contemplated role in the weighting of chances by the subjective credences assigned to the corresponding propositions. This is something that needs to be demonstrated, not assumed.

One potential glitch in implementing this program is an ambiguity in the synchronic norm of normative Bayesianism that went unremarked above. The additivity axiom for probabilities, classical or quantum, comes in different strengths—finite, countable, and complete additivity—and the different justifications for the synchronic norm mentioned above support different strengths of additivity. My tactic here is admittedly self-serving: use whatever strength of additivity is needed to get a representation theorem and then afterwards revisit the issue of how the needed form of additivity can be justified as a constraint on rational credence.

As an illustration of how the program can be implemented in the quantum context I will propose an account of how chance works in ordinary QM and show how it enables a representation theorem of the desired form. Before turning to this account it will be helpful consider in more detail what the philosophical literature would lead us to expect by way of a representation theorem.

## 3 The PP of philosophers' dreams

Philosophers who write about credence and chance in the classical setting use expressions like $ch(\bullet)$ to stand for a "chance function" that assigns chances to propositions in its domain, and expressions like $C_{ch}$ to stand for the proposition that the chances are given by the chance function $ch(\bullet)$. Presumably $C_{ch}$ and $C_{ch'}$ are to be regarded as logically incompatible when $ch \neq ch'$. In this notation the special PP would require that

> If $Cr$ is a rational credence function then for any chance function $ch$ such that $Cr(C_{ch}) \neq 0$
>
> $$Cr(E/C_{ch}) = ch(E) \tag{SPP}$$
>
> where $E$ is any proposition in the domain of both the credence function $Cr$ and the chance functions, and '/' denotes Bayes conditionalization (i.e. $Cr(E/F) = \dfrac{Cr(EF)}{Cr(F)}$), provided that $Cr(F) \neq 0$.)

Note that (SPP) requires that rational credence functions have the same additivity profile as chance functions. So if some chance functions are merely finitely additive while others are countably additive, or some are merely countably additive while others are completely additive, then no rational agent can satisfy (SPP) unless she denies chance by setting $Cr(C_{ch}) = 0$ for all $ch$, thereby satisfying (SPP) vacuously.[4] So unless some explanation is forthcoming as to why all chance functions have the same additivity profile, (SPP) is a non-starter.

Using the same notation, Ismael's general PP for representing credence as epistemic uncertainty about objective chance would take the form

[4] And even here there is a problem. If the disjunction of the $C_{ch}$ over all chance functions is a proposition in the domain of $Cr$ then (SPP) implies that a rational credence function cannot be completely additive.

If $Cr$ is a rational credence function then

$$\begin{aligned} Cr(E) &= \sum_{\{ch\}} ch(E)Cr(C_{ch}) && \text{(GPP)} \\ \sum_{\{ch\}} Cr(C_{ch}) &= 1 \text{ and } Cr(C_{ch}C_{ch'}) = 0 \text{ when } ch \neq ch' \end{aligned}$$

where the sum is taken over the class $\{ch\}$ of all chance functions.

Since presumably the Bayes updated credence function $Cr(\bullet/F)$, where $Cr(F) \neq 0$, is rational if $Cr$ is, we should have from (GPP) that

$$Cr(E/F) = \sum_{\{ch\}} ch(E)Cr(C_{ch}/F).$$

And for $F = C_{ch^*}$ for some particular chance function $ch^*$ we get back the (SPP): $Cr(E/C_{ch^*}) = ch^*(E)$ when $Cr(C_{ch^*}) \neq 0$.

I emphasize that (GPP) is simply hypothesized or imposed as a rationality constraint, and as far as I am aware there is no attempt to prove a representation theorem of the form (GPP) from a theory of classical chance.

The optimistic agenda is now set for understanding the relation between credence and quantum chance: Produce an account of quantum chance that yields quantum analogs of (SPP) and (GPP) as theorems for credence functions satisfying the norms of normative Bayesianism. The account should explain how it is possible to learn what the chances are, and should also explain why all chances satisfy the same form of additivity. Due to the difference in the event structure for classical and quantum events it would be surprising if this agenda could be attained in a completely straightforward manner. What we will find is that while there is a straightforward quantum analog of (SPP), the quantum analog of (GPP) has to be more nuanced due to the non-commutative nature of quantum events; specifically, there are many ways to parse quantum chances, and while there is always *a* way that yields a direct analog of the classical (GPP), in general the representation of credence as epistemic uncertainty about quantum chance contains an additional non-classical term that embodies quantum interference effects.

# 4 Proof of concept

The account I will offer of how chance works in ordinary QM is, I think, plausible. I will not attempt to defend it because it is on offer only as an illustration of how it is at least plausible that the program outlined above can be carried to fruition.

### 4.1 Sketch of the account of quantum chance for ordinary QM.

The algebra of observables is $\mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on the Hilbert space $\mathcal{H}$. The events or propositions to which probabilities are assigned are members of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$, the projection lattice of $\mathfrak{B}(\mathcal{H})$.[5] Quantum probability theory is then viewed as the study of quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where such a measure is a map $\Pr : \mathcal{P}(\mathfrak{B}(\mathcal{H})) \to [0,1]$ such that $\Pr(I) = 1$ and $\Pr(E_1 \vee E_2) = \Pr(E_1) + \Pr(E_2)$ when $E_1, E_2 \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are mutually orthogonal.[6]

A quantum state $\omega$ are normed positive linear functional $\omega : \mathfrak{B}(\mathcal{H}) \to \mathbb{C}$. Any such state $\omega$ induces on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ a quantum probability measure $\Pr_\omega(E) := \omega(E)$ for any $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. I adopt the widely shared attitude that the physically realizable states are the normal states, where $\omega$ is normal means that $\omega$ is completely additive on any family of mutually orthogonal projections or, equivalently, $\omega$ admits a density operator representation.[7] A vector state is a state $\psi$ such that there is a unit vector $|\psi\rangle \in \mathcal{H}$ with $\omega(A) = \langle\psi|A|\psi\rangle$ for all $A \in \mathfrak{B}(\mathcal{H})$. That $\omega$ is a mixed (or impure) state means that it can be expressed as a convex linear combination of other states, viz. $\omega = \lambda_1\xi_1 + \lambda_2\xi_2$ with $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$, and $\xi_1 \neq \xi_2$. A pure state is a non-mixed state, and for $\mathfrak{B}(\mathcal{H})$ the normal pure states coincide with the vector states.

The key interpretational principle used here is that the probability measure induced by a normal pure state gives the chances for events in $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$

[5] An element $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is a self-adjoint operator such that $E^2 = I$ (the identity operator). In the literature projections are referred to as Yes-No questions as well as events or propositions.

[6] For details see Hamhalter (2003). When $E_1, E_2 \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are mutually orthogonal $E_1 \vee E_2 = E_1 + E_2$.

[7] The complete additivity of a quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ means that $\Pr(\sum\limits_a E_a) = \sum\limits_a \Pr(E_a)$ for any family $\{E_a\}$ of mutually orthogonal projections. When the sum $\sum\limits_a \Pr(E_a)$ is over an uncountable index set $\mathcal{I}$, as can occur when the Hilbert space is non-separable, it is understood as $\lim_{\mathcal{F}} \sum_{a\in\mathcal{F}} \Pr(E_a)$ where the $\mathcal{F}$ are finite subsets of $\mathcal{I}$, and $\lim_{\mathcal{F}} \sum_{a\in\mathcal{F}} \Pr(E_a) = L$ means that for any $\epsilon > 0$ there is a finite $\mathcal{F}_0 \subset \mathcal{I}$ such that for any finite $\mathcal{F}$ with $\mathcal{I} \supset \mathcal{F} \supset \mathcal{F}_0$, $|\sum_{a\in\mathcal{F}} \Pr(E_a) - L| < \epsilon$.

when the system at issue is in said state. The support projection for a normal pure state $S_\psi \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ serves as the proposition that the chances are those induced by $\psi$.[8]

Updating is done by Lüders conditionalization, denoted by '//' to distinguish it from Bayes conditionalization. It is defined for a quantum probability measure Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that extends uniquely to a normal state $\omega$ on $\mathfrak{B}(\mathcal{H})$, in which case $\Pr(E//F) := \dfrac{\omega(FEF)}{\omega(F)} = \dfrac{\omega(FEF)}{\Pr(F)}$ for all $E, F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\omega(F) \neq 0$. When $E$ and $F$ commute Lüders conditionalization reduced to Bayes conditionalization since then $FEF = EF^2 = EF$ and $\Pr(E//F) = \dfrac{\Pr(E \wedge F)}{\Pr(F)} = \dfrac{\Pr(EF)}{\Pr(F)}$. The case for Lüders conditionalization as the proper analog of Bayes' conditionalization for a non-abelian algebra is strong and will not be reviewed here (see Bub 1977 and Cassinelli and Zanghi 1983). For $\dim(\mathcal{H}) > 2$ the generalized Gleason theorem implies that any completely additive probability measure Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ extends uniquely to a normal state $\omega$ on $\mathfrak{B}(\mathcal{H})$.[9] The exceptional case of $\dim(\mathcal{H}) = 2$ will be discussed below in Section 5.4

There is a fairly strong case for the assumption that physically realizable states must be normal (see Ruetsche 2011 and Earman and Ruetsche 2020). If this assumption is taken on board it explains why all chances have the same additivity profile; for any normal state on $\mathfrak{B}(\mathcal{H})$ induces on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ a probability measure that is completely additive. Complete additivity reduces to finite additivity when $\dim(\mathcal{H})$ is finite and to countably additivity if $\dim(\mathcal{H}) = \infty$ and $\mathcal{H}$ is separable. The further assumption that only *pure* normal states induce chances will be addressed below in Section 5.

All the ingredients needed to give a proof of concept in the form of a representation theorem for quantum chances are in place. In fact I will offer two.

[8]The support projection $S_\psi$ of a normal state $\psi$ is the smallest projection to which $\psi$ assigns probability 1. For a normal pure state (= vector state for the algebra $\mathfrak{B}(\mathcal{H})$) the support projection is the projection onto the ray spanned by the unit vector corresponding to $\psi$.

[9]Gleason's theorem was originally proved for separable $\mathcal{H}$ and countably additive Pr. It has been extended to: If $\mathcal{H}$ is a Hilbert space, separable or non-separable, $\dim(\mathcal{H}) \geq 3$, and Pr is a quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ then Pr extends to a unique state on $\mathfrak{B}(\mathcal{H})$; if Pr is completely additive it extends to a unique normal state on $\mathfrak{B}(\mathcal{H})$. And the theorem has been further extended to include more general von Neumann algebras.

## 4.2 Representation theorems.

For the classical (GPP) the sum that represents an agent's credence $Cr(E)$ in $E$ as an epistemically weighted average $\sum\limits_{\{ch\}} ch(E)Cr(C_{ch})$ of possible chances of $E$ is taken over the entire class $\{ch\}$ of classical chance functions since the $C_{ch}$ are logically incompatible and all of these mutually exclusive possibilities have to be taken into account for an accurate representation.

In the quantum case it would make no sense to sum over all possible chance making states (= normal pure states). The closest analog would be a sum over a set $\{\psi_a\}$ of chance-inducing normal pure states whose support projections $\{S_{\psi_a}\}$ form a partition of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$; specifically, $S_{\psi_a}S_{\psi_{a'}} = O$ when $a \neq a'$ (orthogonality of the different support projections) and $\sum\limits_a S_{\psi_a} = I$ (completeness). The orthogonality of the $S_{\psi_a}$ is the closest quantum analog of the logical incompatibility of the $C_{ch}$. There is a different sense of incompatibility in QM that has no classical analog and, as will be discussed below in Section 5.3, this is responsible for the fact that there is a non-trivial sense of transition probability quantum probability for non-orthogonal states that has no counterpart in classical probability.

The first representation theorem shows that, with a mild restriction on the Hilbert space, there always exists a privileged partition $\{S_{\psi_a}\}$ that apes as closely as possible the (GPP) of classical probability.

> *Theorem 1.* Let $Pr$ be a completely additive quantum probability measure on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\dim(\mathcal{H}) \geq 3$. Then there exists a countable set $\{\psi_a\}$ of mutually orthogonal normal pure states on $\mathfrak{B}(\mathcal{H})$ such that for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$
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> $$\begin{aligned} \Pr(E) &= \sum_a \psi_a(E)\Pr(S_{\psi_a}) \qquad \text{(QGPP)} \\ \sum_a \Pr(S_{\psi_a}) &= 1, \quad \Pr(S_{\psi_a}S_{\psi_{a'}}) = 0 \text{ for } a \neq a', \quad \sum_a S_{\psi_a} = I \end{aligned}$$
>
> where $S_{\psi_a} \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is the support projection for $\psi_a$.

Proof: The proof of the theorem is an easy consequence of combining Gleason's theorem with Theorem 7.1.12 of Kadison and Ringrose (1997). For a Pr satisfying the conditions of the Theorem, Gleason's theorem shows that Pr extends uniquely to a normal state $\omega$ on $\mathfrak{B}(\mathcal{H})$. For $\mathfrak{B}(\mathcal{H})$ the normal pure states are vector states, and the Kadison and Ringrose theorem shows that for any normal state—and, thus, for the state $\omega$ in question—there is a countable family $\{\psi_a\}$ of normal pure states (= vector states on $\mathfrak{B}(\mathcal{H})$) whose corresponding unit vectors $\{|\psi_a\rangle\}$ are mutually orthogonal such that $\omega = \sum_a \lambda_a \psi_a$, where the limit is understood in the sense of norm convergence and where $\sum_a \lambda_a = 1$ and $0 < \lambda_a < 1$. Since $\psi_{a'}(S_{\psi_a}) = 0$ for $a' \neq a$, we have $\omega(S_{\psi_{a'}}) = \lambda_{a'}$ and, thus, $\omega(A) = \sum_a \omega(S_{\psi_a})\psi_a(A)$ for $A \in \mathfrak{B}(\mathcal{H})$. Since $\omega(S_{\psi_a}) = \Pr(S_{\psi_a})$ and $\omega(E) = \Pr(E)$ for $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ we have $\Pr(E) = \sum_a \psi_a(E) \Pr(S_{\psi_a})$. Finally, $\sum_a \Pr(S_{\psi_a}) = \sum_a \lambda_a = 1$; $\Pr(S_{\psi_a} S_{\psi_{a'}}) = 0$ for $a \neq a'$ since $S_{\psi_a} S_{\psi_{a'}}$ is the null projection; and without loss of generality the $\{|\psi_a\rangle\}$ can be expanded if necessary to form a complete ON basis so that $\sum_a S_{\psi_a} = I$.

When $\mathcal{H}$ is non-separable there is an uncountable number of mutually orthogonal normal pure states. Why then is the representation theorem able to get away with summing over a countable subset? The short answer is that a non-zero probability can be assigned to only a countable number of the members of the uncountable set $\{S_{\psi_a}\}$ of support projections.

If the Pr of Theorem 1 interpreted as the credence function of a Bayesian agent who assigns degrees of belief to the elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$—and nothing in the formal apparatus prevents such an interpretation—then apart from the restriction $\dim(\mathcal{H}) \geq 3$ this theorem fulfills the wish list of expectations for a representation theorem that vindicates the present construal of the PP for QM. It remains to discuss how an agent can learn what the quantum chances are. The short answer is that, in principle, she can do a Yes-No experiment for the support projection $S_\psi$ for a normal pure state state $\psi$. If she receives a Yes answer she can be (subjectively) certain that the chances are those induced by $\psi$. The need for the restriction to $\dim(\mathcal{H}) \geq 3$ and the difficulty this poses for the quantum Principal Principle for $\dim(\mathcal{H}) = 2$ will be discussed in Section 5 below.

To explore the differences in the relation between credence and chance in the classical vs. quantum cases, call a family $\{\psi_a\}$ of mutually orthogonal normal pure states amenable to a quantum probability Pr if it instantiates (QGPP) for Pr. In general, different quantum probability functions, encoding the credences of different rational agents, have different amenable families. Similarly, the family $\{\psi_a\}$ of mutually orthogonal normal pure states amenable to Pr encoding an agent's initial credences in quantum events may not be amenable to that agent's Lüders updated $\Pr(\bullet//F)$ credences for an $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\Pr(F) \neq 0$. The updated $\Pr(\bullet//F)$ has an amenable family $\{\overline{\psi}_b\}$, but this may not be the same as the family $\{\psi_a\}$ that is amenable to the initial Pr.

Further differences with the classical case emerge from asking how the credences of an agent are related to chances when those chances are not amenable to the agent's credence function. The answer is given by the following version of the quantum general PP:

> *Theorem 2.* Let $Pr$ be a completely additive quantum probability measure on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\dim(\mathcal{H}) \geq 3$. If $\{|\varphi_a\rangle\}$ is an arbitrary ON basis for $\mathcal{H}$ with corresponding normal states $\{\varphi_a\}$ and support projections $\{S_{\varphi_a}\}$ then
>
> $$\begin{aligned} \Pr(E) &= \sum_a \varphi_a(E) \Pr(S_{\varphi_a}) \qquad\qquad (\text{QGPP}') \\ &\quad + \sum_{b \neq c} \omega(S_{\varphi_b} E S_{\varphi_c}) \end{aligned}$$
>
> for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, where $\omega$ is the unique normal state that extends Pr to $\mathfrak{B}(\mathcal{H})$.

Proof: By Gleason's theorem Pr extends uniquely to a normal state $\omega$ on $\mathfrak{B}(\mathcal{H})$. Use the fact that $\sum\limits_a S_{\varphi_a} = I$ and, thus, $\omega(E) = \omega((\sum\limits_a S_{\varphi_a})E(\sum\limits_b S_{\varphi_b}))$. From the normality and linearity of $\omega$ it follows that $\omega((\sum\limits_a S_{\varphi_a})E(\sum\limits_b S_{\varphi_b})) = \sum\limits_a \omega(S_{\varphi_a} E S_{\varphi_a}) + \sum\limits_{b \neq c} \omega(S_{\varphi_b} E S_{\varphi_c})$. Since $\omega(S_{\varphi_a} E S_{\varphi_a}) = 0$ when $\omega(S_{\varphi_a}) = 0$ any such terms can be left out of the first sum, and for each of the remaining terms in this sum $\omega(S_{\varphi_a} E S_{\varphi_a}) = \dfrac{\omega(S_{\varphi_a} E S_{\varphi_a})}{\omega(S_{\varphi_a})}\omega(S_{\varphi_a})$. Next use the filter property of the support projection $S_\psi$ for a normal pure state $\psi$, viz. if $\omega$ is any normal state such that $\omega(S_\psi) \neq 0$ then $\dfrac{\omega(S_\psi A S_\psi)}{\omega(S_\psi)} = \psi(A)$ for all $A \in \mathfrak{B}(\mathcal{H})$ (see Earman and Ruetsche 2020). By the filter property of the $S_{\varphi_a}$, $\dfrac{\omega(S_{\varphi_a} E S_{\varphi_a})}{\omega(S_{\varphi_a})} = \varphi_a(E)$, and together with $\omega(S_{\varphi_a}) = \Pr(S_{\varphi_a})$ this gives $\sum\limits_a \omega(S_{\varphi_a} E S_{\varphi_a}) = \sum\limits_a \varphi_a(E) \Pr(S_{\varphi_a})$. Collecting these results yields (QGPP$'$).

Thus, for an arbitrary way of partitioning quantum chance-making normal pure states into a mutually orthogonal and exhaustive family $\{\varphi_a\}$, the credence $\Pr(E)$ in $E$ is represented as a sum of two terms: a sum over the possible objective chances $\varphi_a(E)$ of $E$ weighted by their respective subjective probabilities $\Pr(S_{\varphi_a})$, *plus* a term that can be interpreted as a kind of interference between the chances. Note that the $\omega(S_{\varphi_b}ES_{\varphi_c})$ terms on the rhs of (QGPP′) cannot in general be written as $\Pr(S_{\varphi_b}ES_{\varphi_c})$ since $(S_{\varphi_b}ES_{\varphi_c}) \notin \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ when the projections do not commute. (QGPP′) reduces to (QGPP) when the basis $\{|\varphi_a\rangle\}$ is amenable to Pr. That there always is an amenable basis follows from Theorem 1. The basis is amenable when the density operator corresponding to the normal state $\omega$ extending Pr diagonalizes in the basis $\{|\varphi_a\rangle\}$ and, consequently, the $\omega(S_{\varphi_b}ES_{\varphi_c})$ terms vanish. And, of course, (QGPP′), reduces to (QGPP) when the algebra is classical (= abelian) since then the $\sum\limits_{b\neq c}\omega(S_{\varphi_b}ES_{\varphi_c})$ term vanishes for *any* mutually orthogonal family $\{\varphi_a\}$.

The more complicated form of the quantum general PP given in Theorem 2 may be viewed as a drawback, but on the other hand there would be something very suspicious if the non-abelian nature of quantum events did not make itself felt in the relation between credence and quantum chance. To appreciate this point the reader is invited to do the following exercise. Analyze the two slit experiment using classical probability and Bayes conditioning to get an expression for the probability of a hit in some region of the screen when both slits are open. The classical prediction is just the sum of what one would expect when the left slit only is open, plus what would expect when right slit only is open. This, of course, is not what the experiment yields. Now repeat the probability calculation using Lüders conditioning in place of Bayes. One gets an analog of the classical prediction plus an extra $\omega$-term of a similar form as in representation Theorem 2.

If (QGPP′) deserves to be called a quantum general PP then it should have the quantum special PP as a corollary. This is indeed the case.

> *Cor.* Let $Pr$ be a completely additive quantum probability measure on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\dim(\mathcal{H}) \geq 3$. And let $\psi$ be any normal pure state such that $\Pr(S_\psi) \neq 0$. Then
>
> $$\Pr(E//S_\psi) = \psi(E) \text{ for all } E \in \mathcal{P}(\mathfrak{B}(\mathcal{H})). \qquad \text{(QSPP)}$$

Proof: If Pr satisfies the hypotheses of Theorem 2 then so does $\Pr(\bullet//F)$, $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, provided that $\Pr(F) \neq 0$. So if $\Pr(S_\psi) \neq 0$ Theorem 2 implies that for any ON basis $\{|\varphi_a\rangle\}$

$$\begin{aligned}\Pr(E//S_\psi) &= \sum_a \varphi_a(E) \Pr(S_{\varphi_a}//S_\psi) \\ &\quad + \sum_{b \neq c} \omega'(S_{\varphi_b} E S_{\varphi_c})\end{aligned}$$

where $\omega'$ is the normal state that extends $\Pr(E//S_\psi)$. Choose the basis $\{|\varphi_a\rangle\}$ so that one of the basis vectors defines the normal pure state $\psi$. With this choice the $\omega'$ term vanishes and $\sum_a \varphi_a(E) \Pr(S_{\varphi_a}//S_\psi) = \psi(E)$.[10]

# 5 Reflections

1. *The account of quantum chance.* The account outlined above can be attacked in multiple ways. The most head-on attack comes from the QBians who want to do a de Finetti number on QM. Their basic tactic is a judo move: the Gleason theorem which was used to prove the representation theorems (QGPP) and (QGPP′) can be used to support the view that quantum states are simply bookkeeping devices used to represent and track the credence functions of Bayesian agents. For a skeptical assessment of QBians see Earman (2019). Other philosophers of physics who are open to the idea of objective chance will find different faults with my account. Let the battles begin. When the smoke clears one can hope that one or another result will stand out: if the QBians are victorious then as far as QM goes the relation between credence and chance is a non-topic; if the QBians are vanquished and there emerges an account of quantum chance different and better than the one on offer here, then see whether it can serve as a basis for proving the desired representation theorems and draw conclusions accordingly for the status of the quantum PP.

2. *Why are chances induced by pure states and not by mixed states?* The answer has several interrelated components. The main response is that if, as is being assumed here, Lewis' PP serves as a functional characterization of chance then the probabilities induced on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ by mixed states on $\mathfrak{B}(\mathcal{H})$

[10] A proof of the (QSPP) can also be obtained by combining Gleason's theorem with the filter property of the support projection of a normal pure state.

do not qualify as chances, for they do not lend themselves to theorems such as (QSPP), (QGPP), and (QGPP$'$).

If this response is regarded as too self-serving, additional considerations can be brought to bear. (i) Whatever else chances are they are objective, observer-independent probabilities. All observers attending the Yes-No measurement of the support projection for a normal pure state $\psi$ can agree when there is a Yes answer, and in these circumstances they can agree that the state $\psi$ has been prepared. The probability values induced by $\psi$ can be compared to the frequencies of outcomes in repeated measurements when the system is repeatedly re-prepared in the same initial state. The close match that is obtained in actual experiments is evidence of the objective nature of the probabilities induced by pure states. If mixed states did induce chances then we could not learn what the chances are in the way we learn what the pure-state chances are by preparing the normal pure state by doing a Yes-No experiment and, if receiving a Yes answer, by calculating the probabilities induced by said state. This is because mixed states do not have filters (see Ruetsche and Earman 2020), so there is no Yes-No measurement we can do for some element of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ whose Yes outcome establishes that the mixed state $\xi$ has been prepared. The support projection $S_\xi$ of an impure normal state does not distinguish between $\xi$ and a family of different mixed states (see Reflection 3 below). (ii) Whatever else chance is, it is a non-epistemic probability. Consider the mixed state $\xi := \dfrac{1}{2}\psi_1 + \dfrac{1}{2}\psi_2$ where $\psi_1$ and $\psi_2$ are normal pure states corresponding to the orthogonal unit vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively. This state can be created by programming a robot to flip a classical fair coin and then prepare state $\psi_1$ (respectively, state $\psi_2$) if the coin lands Heads (respectively, if the coin lands Tails). An agent who is told the robot's procedure but not the outcome of the coin flip will give an epistemic reading to the $1/2$ mixture weights in the state $\xi$. If, on the other hand, the agent is simply presented with the mixed state $\xi$ and not told how it has been prepared she will not know how to identify the epistemic component of the probabilities induced by $\xi$, for $\xi$ can be expressed in many ways as a mixture over different pure states. But the stance adopted here is that the fact that there *can* be an epistemic component is enough to disqualify $\xi$ as inducing true chances. The reader is reminded that the physics literature on quantum entanglement takes a similar stance: a quantum state for a system is not counted as inducing true quantum entanglement between the observables associated with two subsystems if said state *can* be written as a mixture of unentangled (= product) states. The motivation is similar: the mixture weights *can* be given an epistemic interpretation, in which case the correlation between the subsystems is the result of the ignorance about which product state is the actual one. Nor is this stance gainsaid by the fact that the mixture can be written in different ways, as different mixtures over different product states. (iii) A property often attributed to chance is that chances are irreducible probabilities. One way to cash in this notion is that information specifying the chances is maximally specific in the sense that there is no further information compatible with said specification that changes the probabilities. In the present context this feature is captured by the fact that the support projection $S_\psi$ for a normal pure state $\psi$ is minimal in the projection lattice, i.e. for any $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ if $E \leq S_\psi$ ($E$ implies $S_\psi$) then $E = S_\psi$. (iv) Finally, the next section indicates why normal pure states are the receivers as well as the givers of chances.

3. *Transition probabilities and other chances.* The quantum representation theorems of Section 4 use a family of mutually orthogonal normal pure states. But to contrast the difference between quantum and classical chance it is useful to consider the chances that arise from non-orthogonal pure states. The support projection $S_\psi$ for normal pure state $\psi$ on $\mathfrak{B}(\mathcal{H})$ is the projection $E_{|\psi\rangle}$ onto the ray spanned by a vector $|\psi\rangle \in \mathcal{H}$ corresponding to $\psi$. For normal pure states $\psi$ and $\psi'$ the expression $\psi'(S_\psi)$, giving the chances of a Yes answer to a Yes-No measurement of $S_\psi$ when performed on a system in state $\psi'$ agrees with the standard expression for transition probability from $\psi'$ to $\psi$: $\psi'(S_\psi) = \psi'(E_{|\psi\rangle}) = \langle\psi'|E_{|\psi\rangle}|\psi'\rangle = \langle\psi'|E^2_{|\psi\rangle}|\psi'\rangle = ||E_{|\psi\rangle}|\psi'\rangle||^2 = |\langle\psi'|\psi\rangle|^2$. Of course, when $|\psi\rangle$ and $|\psi'\rangle$ are orthogonal the transition probability is flatly zero.

Here it is worth noting that two different senses of quantum incompatibility for elements of the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ in general, and for the support projections $S_\psi$ and $S_{\psi'}$ for normal pure states in particular, can be distinguished: the first is that $S_\psi$ and $S_{\psi'}$ are orthogonal, which holds iff $S_\psi\ S_{\psi'} = S_{\psi'}S_\psi = 0$, implying transition probabilities $\psi'(S_\psi)$ and $\psi(S_{\psi'})$ are both 0. The second is that $S_\psi$ and $S_{\psi'}$ are non-commuting, $S_\psi S_{\psi'} \neq S_{\psi'}S_\psi$, in which case quantum doctrine declares that co-determination (or co-measurability) is impossible; in this case the transition probabilities $\psi'(S_\psi)$ and $\psi(S_{\psi'})$ are non-zero. Transition probabilities for non-orthogonal normal pure states deserve to be numbered as a kind of quantum chance.

In the classical case the abelian event structure means that there is no analog of the second sense of quantum compatibility and, hence, no corresponding notion of transition probability for classical chances. This a conclusion that can also be reached by conjuring with the notation philosophers use and applying the classical (SPP). For any credence function $Cr$ obeying the axioms of classical probability we have that for any chance function $ch'$ such that $Cr(C_{ch'}) \neq 0$, $Cr(C_{ch}/C_{ch'}) = \dfrac{Cr(C_{ch}C_{ch'})}{Cr(C_{ch'})} = 0$ when $ch' \neq ch$ since then $C_{ch}$ and $C_{ch'}$ are logically incompatible. But the (SPP) says that Bayes conditionalizing on $C_{ch'}$ is supposed to bring rational credence in line with $ch'$-chances so that $Cr(C_{ch}/C_{ch'}) = ch'(C_{ch})$, with the upshot that $ch'(C_{ch}) = 0$ whenever $ch \neq ch'$ (as long as there is a credence function such that $Cr(C_{ch'}) \neq 0$). This blocks the obvious route to non-trivial transition probabilities in the classical setting.[11] The lack of a notion of transition probability in the classical setting should not be surprising if the only classical chances are the $0-1$, a conclusion that can be derived from the abelian structure of classical events (see Reflection 7 below).

Finally, although the expression $\psi(S_\varphi)$ is mathematically well-defined for any normal states $\psi$ and $\varphi$ it would be untoward to try to interpret it as a transition probability from $\psi$ to $\varphi$ when $\varphi$ is impure. This is a consequence of the fact noted above that there are no filters for impure states. To illustrate the problem consider the impure state $\varphi = \lambda_1\xi_1 + \lambda_2\xi_2$ where $\xi_1$ and $\xi_2$ are orthogonal vector states and $\lambda_1 + \lambda_2 = 1$, $0 < \lambda_1, \lambda_2 < 1$. The support projection $S_\varphi$ for this state is the projection onto the subspace spanned by the vectors corresponding to $\xi_1$ and $\xi_2$. Thus, $\psi(S_\varphi)$ does not distinguish between the transition from $\psi$ to $\varphi$ vs. the transition to any other impure state $\overline{\varphi} = \overline{\lambda}_1\xi_1 + \overline{\lambda}_2\xi_2$ with $\overline{\lambda}_1 + \overline{\lambda}_2 = 1$, $0 < \overline{\lambda}_1, \overline{\lambda}_2 < 1$, and $\overline{\lambda}_1 \neq \lambda_1$ and $\overline{\lambda}_2 \neq \lambda_2$. It is a normal pure state that giveth chance, and it is a normal pure state that receiveth transition chances.

There are undoubtedly other senses of quantum chance worth studying. My program is does not exclude them but is committed to the stance that they are best studied within the framework outlined above.

4. *What goes wrong when* $\dim(\mathcal{H}) = 2$*?* When $\dim(\mathcal{H}) = 2$ Gleason's theorem fails and there are quantum probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that do not extend to any state on $\mathfrak{B}(\mathcal{H})$ and, as a result, they cannot be rep-

[11] Some philosophers who write about classical chance want to allow for the possibility of "self-undermining" chance functions where $ch(C_{ch}) < 1$ and, thus, for the possibility that $ch(C_{ch'}) \neq 0$ when $ch \neq ch'$. To allow for such possibilities (SPP) would need to be modified; see Section 6.2 below.

resented as weighted averages of objective chances in the way required by the desired representation theorem. So do we conclude that here the PP fails? This would be too hasty. Recall that what we want to show if that *if* a Pr satisfies the norms of normative Bayesianism then a representation of Pr as epistemic uncertainty about chance values is a theorem of the theory of quantum chance. The synchronic norm of normative Bayesianism is satisfied since Pr is finitely additive for $\dim(\mathcal{H}) = 2$. But what about the diachronic norm? Recall: When Pr extends to the state $\omega$ and $\Pr(F) \neq 0$ Lüders conditionalization is given by $\Pr(E//F) := \dfrac{\omega(FEF)}{\omega(F)} = \dfrac{\omega(FEF)}{\Pr(F)}$. The numerator cannot be written as $\Pr(EFE)$ since when $E$ and $F$ do not commute, $EFE \notin \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and no Pr value is assigned to $EFE$. So when Pr does not extend to a state on $\mathfrak{B}(\mathcal{H})$ Lüders conditionaliztion is undefined. Should this be described by saying that the diachronic norm is moot for such a Pr, or should we say that such a Pr fails the norm?

The reason that in the $\dim(\mathcal{H}) = 2$ case there are probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that fail to extend to a state on $\mathfrak{B}(\mathcal{H})$ is that such measures are not continuous in the strong or weak operator topology, which they would have to be if they were induced by a normal state, all states being normal for $\dim(\mathcal{H}) = 2$. Perhaps such continuity of credence functions on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ can be justified as a norm of rationality. But that remains to be seen, for none of the familiar Bayesian rationales seem to do the job.

*5. The additivity requirement.* When $\mathcal{H}$ is an infinite dimensional separable space the representation theorems require that Pr is countably additive. Some, but not all, of the justifications for finite additivity as a rationality constraint can be extended to cover countable additivity. There are, however probabilists—including de Finetti and his heirs—who insist that anything beyond finite additivity is not justified as a rationality constraint, and for them Theorems 1 and 2 do not vindicate a quantum PP. When $\mathcal{H}$ is non-separable the theorems require that Pr is completely additive, and some of the justifications for this requirement are fraught (see Skyrms 1992 for a hitch in using a Dutch book argument to justify complete additivity). But for probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ countable additivity suffices for all practical purposes since a countably additive Pr is completely additive unless $\dim(\mathcal{H})$ is as great as the least measurable cardinal (see Eilers and Horst 1975). There are no known applications of QM that require such high dimensional Hilbert spaces.

6. *Beyond ordinary QM.* What are the prospects for the desired represen-

tation theorem when the algebra of observables is more exotic than $\mathfrak{B}(\mathcal{H})$? Gleason's theorem can be extended to cover much more general von Neumann algebras than the plain vanilla $\mathfrak{B}(\mathcal{H})$ of ordinary QM. But these more exotic algebras have features that raise tricky issues about the nature of quantum chance, e.g. the Type III von Neumann algebras encountered in relativistic QFT admit no normal pure states. Such issues will have to be discussed elsewhere (see Earman and Ruetsche 2020 for some pertinent remarks).

7. *Classical chance.* What are the prospects of carrying out for classical probability the program illustrated above for quantum probability? They are either excellent or dismal depending on your account of classical chance. They will seem excellent if you belong to the venerable, but much criticized, school that says the only genuine classical chances are the degenerate ones, i.e. those that assign $0-1$ probabilities to all events, all other probability measures expressing epistemic uncertainty about the exact state of the system which confers $0-1$ probabilities. This result can be obtained from the above apparatus by viewing classical probability as concerned with the special case where the von Neumann algebra is abelian and, thus, the projection lattice is a Boolean lattice. Any pure state on such an abelian von Neumann algebra induces $0-1$ measures on its Boolean projection lattice. The above representation theorems apply, the only differences being that in the Boolean case the chances $\psi_a(E)$ in both Theorem 1 and 2 are always 0 or 1 and the $\omega$-term in Theorem 2 is always 0.

For those who find repugnant the use of Hilbert space apparatus to treat classical probability there is a way to shed some of the trappings of Hilbert space and make connections with the more familiar nomenclature of classical probability. It begins with the Gelfand-Naimark theorem which shows that an abelian von Neumann algebra is $*$-isomorphic to $C(X)$, the algebra of continuous functions on a compact Hausdorff space $X$, the sum and product of functions being defined pointwise.[12] The projections $\mathcal{P}(C(X))$ are characteristic functions $\varkappa_Y$ of subsets $Y \subseteq X$,[13] and they form a Boolean algebra. For any pure state $\phi$ on $C(X)$ and any $f \in \mathcal{P}(C(X))$, $\phi(f) \in \{0, 1\}$. Further if $\phi$ is a pure state then there is a unique $x_\phi \in X$ such that $\phi(f) = f(x_\phi)$ so that pure states can be identified with points of $X$ or the extreme point-measures on $C(X)$. For a pure state $\phi$ and $\varkappa_Y \in \mathcal{P}(C(X))$, $\phi(\varkappa_Y) = \varkappa_Y(x_\phi) = 1$ if $x_\phi \in Y$ and 0 if not. Borel probability measures on $X$ are mixtures of these

[12] I.e. $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for $f, g \in C(X)$ and $x \in X$.

[13] $(\varkappa_Y \varkappa_Y)(x) = \varkappa_Y(x)\varkappa_Y(x) = \varkappa_Y(x)$ for $x \in X$ and $Y \subseteq X$.

extreme point-measures, each of which induces classical chances of $0-1$.

Needless to say, some philosophers will reject this account of classical chance. I urge them to develop their alternative account of classical chance and to use it to prove representation theorems of the kind studied above.

# 6 Twisted knickers

The analytical metaphysicians who write about the PP get their knickers twisted about a number of issues. I will comment on five of them.

1. *Admissible evidence.* Recall the opening example of coin flipping and the "...regardless of other information you may have" clause. The idea is that information of the objective chance of Heads trumps other information you may have about the behavior of the coin, including information about the frequency of Heads in past flips. But exactly which information is trumped and why? There is a wrangle about this in the philosophical literature—not surprising since the discussion takes place in the absence of a substantive theory of chance.

Given the account of quantum chance on offer the answer is straightforward answer to what counts as admissible evidence. If the "other information" is already incorporated in the agent's credence function Pr then unless $\Pr(S_\psi) = 0$ for the support projection for a normal pure state $\psi$, the information that the proposition $S_\psi$ is true and, hence, that the chances are those induced by $\psi$ trumps the "other information" since then $\Pr(E//S_\psi) = \psi(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. If on the other hand the "other information" that $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is true is acquired at the same time as the information that $S_\psi$ is true then standard quantum doctrine takes over, at least if the information is acquired by measurement. For standard doctrine says that $F$ and $S_\psi$ are simultaneously measurable iff they commute. And if $\Pr(S_\psi F) = \Pr(FS_\psi) \neq 0$ then $FS_\psi = S_\psi F \neq 0$. But since $S_\psi$ is a minimal projection $FS_\psi = S_\psi F = S_\psi$, so again $S_\psi$ trumps and $\Pr(E//S_\psi F) = \Pr(E//FS_\psi) = \Pr(E//S_\psi) = \psi(E)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $\Pr(S_\psi F) = \Pr(FS_\psi) \neq 0$.

2. *Self-undermining chances and the New Principal Principle.* In the philosophical literature on the PP a chance function $ch$ is said to be self-undermining if $ch(C_{ch}) < 1$. Self-undermining chances are odd. Just as it would be odd to say in the same breath that "It is raining, but there is a chance that it is not raining," so it is odd to say "The chances are given by $ch$, but there is a $ch$-chance that the chances are not given by $ch$."

(SPP) does not permit such oddities because rules against self-undermining chances. Presumably, for any chance function $ch$ there is a rational credence function $Cr$ such that $Cr(C_{ch}) > 0$. It then follows from (SPP) that $Cr(C_{ch}/C_{ch}) = 1 = ch(C_{ch})$. But for any oddity the chances are good that there are analytical metaphysicians who will take it seriously. Thus, it is not surprising to find proposals for a New Principal Principle that modifies the PP so as to allow for self-undermining chances (see Hall 1994 and Thau 1994). Fortunately, there is no need to open this can of worms here since quantum chances are never self undermining: for any pure normal state $\psi$ (and indeed for any normal state, pure or impure) $\psi(S_\psi) = 1$.

3. *Humean grounding of quantum chances*? In discussing the relation of credence and chance David Lewis climbed aboard another of his hobby horses, Humeanism (see Lewis 1994). A Humean world is one devoid of all of the hidden springs, powers, and potentialities which Hume found so distasteful; at base it is world consisting of prosaic local particular facts—this electron now has spin up, that electron had spin down 10 secs. ago, etc.—and any other fact about such a world must be grounded in the base of Humean facts. (In some formulations the grounding is explained in terms of reducibility, in others supervenience takes the place of reduction. I will not open this can of worms here.) But how can such a world accommodate objective chance which smacks of non-Humean powers?

The standard answer in the literature is that in order to satisfy Humean strictures chance must be grounded in facts about patterns of Humean events, such as relative frequencies (see, for example, Schwarz 2016 and Hoefer 2019). This answer is given in apparent innocence of how chance works in quantum theory where the truth makers for quantum chances are not facts about frequencies or other patterns of events but facts about the support projections for normal pure states. If it is a fact that a Yes-No measurement of the support projection $S_\psi$ for the normal pure $\psi$ has just yielded a Yes answer then the chance that a Yes-No measurement $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ would yield a Yes is $\psi(E)$, and this is so even if the world is such that $E$ is never measured and there are no frequencies of Yes outcomes for $E$ to ground its chances.

Of course, there is a connection here with frequencies; but it involves potential frequencies, and the connection is epistemic rather than ontic. Suppose that the system were to be prepared over and over again in the state $\psi$ and that after each such preparation a Yes-No measurement is made of $E$. Since the trials are independent and identically distributed the classical law of large numbers can be invoked to conclude that in almost all outcome sequences the relative frequency of Yes outcomes will converge to $\psi(E)$ as the number of trials goes to infinity. But strict Humeans will find distasteful the resort to hypothetical or potential frequencies. And even in the measure-zero cases where the frequency of Yes outcomes either does not converge or converges to a value different from $\psi(E)$ it is nevertheless true that the chance of $E$ is $\psi(E)$.

Will the Humean guard admit the proposed truth makers for quantum chances into the Humean base? Or will the guards turn them away, thereby turning their backs on quantum chances? If the latter I will tear up my Humean membership card but nevertheless maintain that I can believe in quantum chances while continuing to be a good (liberal) empiricist since the truth makers for quantum chances can in principle be verified by experiment, although not usually by direct observation.

4. *The usefulness of the quantum principal principle.* The above account of the grounding of quantum chances provides an escape from one of the gripes against Lewis' PP. Assuming, as most writers do, that Humean chances are grounded in either limiting relative frequencies or finite frequencies involving future events, it is hard to see how bounded agents who do not have a God's-eye-view of the universe can ever be in a position to put the PP into action. In the case of quantum chances the assumption is wrong; finite observers do have access to the facts that ground quantum chances, and these agents can put the PP into action by conditionalizing their degrees of belief on the grounding facts.

5. *Are quantum chances lawlike*? Humeans are modally challenged. The modality 'It is a law of nature that ...' poses a particular challenge: to dismiss it runs afoul of modern science which seems to accept a distinction between mere regularities and regularities that are backed by natural laws[14]; to accommodate it requires the Humeans to produce an account of how laws are grounded on Humean facts and to show that the account accords with scientists' usage of the modality. Taking his cue from unpublished work of Frank Ramsey, David Lewis proposed to answer the challenge with the "best system" analysis of laws. With non-probabilistic laws in mind, the initial idea was that the laws of nature are the truths expressed by the axioms or theorems of the best deductive system, where 'best' means the best compromise between the simplicity of the axioms and their strength

[14] For dissenting views see van Fraassen (1989) and Giere (1999).

(or information content). In order to accommodate probabilistic laws the criteria for 'best system' have to be broadened to include some condition on the fit between probabilities and frequencies, the details of which need not concern us here. If we take scientific theorizing as striving towards the best system for the actual world then an explanation is at hand for why scientists tend to cite the axioms and theorems of their most fundamental theories as examples laws of nature.

I want to think that, if the quantum theory is true, quantum chances have a lawlike character; and I think that there is compelling evidence that the theory is true. I am also a would-be Humean who does not see a better Humean alternative to the best system analysis of laws. So I should believe—as indeed I do—that, if true, the quantum theory is the best system for describing phenomena in its domain of application. Consequently, I have to accept that although the truth makers of chance assertions (e.g. 'The chance is 1/2 that this photon will be reflected by a half-silvered mirror') are not facts about frequencies of events, the lawlike character of these assertions is, at least in part, grounded in facts about frequencies.[15]

Non-Humeans will be unimpressed by the thin nature of the lawlike character that Humeans attribute to quantum chances: the lawlikeness does not travel with the truth of the quantum theory since the theory can be true in but not the best system of other possible worlds.[16] Humeans take this as a virtue of the best systems analysis since modalities must be thin to pass their muster. Although as a would-be Humean I wish Humeanism could accommodate a thicker sense of lawlikeness, I will settle for the thin spread if I cannot have the thicker one.

# 7 Conclusion

The program I have recommended for understanding the relation between credence and quantum chance can be described as an exercise in naturalized metaphysics—metaphysics transmuted into the meta-physics, the enterprise of interpreting theories of physics. The conclusions it draws about the nature of quantum chance and its relation to rational credence are every bit as fragile and problematic as the interpretational principles it employs—very

[15]'In part' because because the quantum theory has many non-probabilistic consequences, e.g. the energy levels of atoms.

[16]I owe this remark to Laura Ruetsche.

fragile and problematic, especially so since quantum theory has so many hotly debated interpretations. But even if my conclusions are rejected I believe that an engagement with the issues I have outlined here are more likely to produce progress in understanding the relation between credence and chance than trying to adjudicate the clash of competing intuitions that characterizes much of the analytical metaphysics literature; and in any case, refusing to come to grips with the differences between classical and quantum probability and the ramifications of these differences for credence and chance is not a prescription for progress.

In closing I want to underscore an obvious but overlooked point about Lewis' PP that applies both the classical and quantum probability but is especially potent in the quantum case. Bayesians who worry about the objectivity of inductive inference are apt to cite merger of opinion results: yes, different Bayesian agents can have wildly different prior, but as more and more evidence is accumulated and the agents update their priors there is a tendency of their posterior probabilities to merge. A close look at the merger of opinion results reveal strong limitations; for example, the merger theorems are often in the form of long run or limit results but, as Keynes famously remarked, in the long run we are all dead. The PP offers a remedy, at least if agents can be in a position to conditionalize on propositions about chances. The result is especially striking in the quantum case when the agents have the ability to do Yes-No experiments for the elements of the projection lattice.

Suppose that a Yes-No experiment is performed on the support projection $S_\psi$ for a normal pure state $\psi$, and suppose that all the agents who witness the experiment agree that the outcome of the experiment is a Yes answer. The (QSPP) guarantees that the subset of those agents whose prior probability for $S_\psi$ is non-zero will find that when they update their credence function by Lüders conditionization on $S_\psi$ they agree on their posterior credences for every element of the projection lattice, no matter how otherwise divergent their prior credences were. And this is quite independent of whether learning $S_\psi$ is parsed as learning that the objective chances are those induced by $\psi$ or whether the learning is given some less metaphysically loaded reading. The remaining subset of agents who assigned flatly zero prior credence to $S_\psi$ will be in a quandary about what their new credences should be since Lüders conditionalization, like its classical counterpart of Bayes conditionalization, is undefined for zero probability events. But if they resolve their quandary by agreeing that their new credence functions should assign credence 1 to $S_\psi$ then they too will find that their new credences align perfectly with all of their fellow Bayesian agents, for $\Pr_{new}(S_\psi) = 1$ implies that $\Pr_{new}(E) = \psi(E)$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Call it chance or call it by another name, the thing that lies behind this coup de foudre merger of opinion deserves to be studied more carefully.

*Acknowledgement*: I am grateful to Laura Ruetsche for helpful suggestions on an earlier draft of this paper. Needless to say, this does not imply that she agrees with any of the opinions expressed herein (although she should).

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