Geometric cognition: a hub-and-spoke model of geometric concepts[1]

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Abstract: Geometric cognition is still poorly understood. Concerning pure geometry, it is basically *terra incognita*. In the present paper, we develop a tentative model of the neural representation of geometric concepts; for that we adopt an interdisciplinary approach bringing together elements of the history of geometry and the cognitive neuroscience of semantic cognition. Our interest is to develop a model of the neural representation of abstract geometric objects in the Euclidean practice. To arrive at a coherent model, it is necessary to consider earlier forms of geometry. For that, we address the change from practical to pure geometry, proposing models of the neural representations of geometric concepts for each of these practices. This will enable us to have an understanding of geometric objects in terms of a neural model and in relation to neural models of geometric figures. Our models are based on the hub-and-spoke theory. We will present a tentative model of the neural representation of geometric figures in ancient Greek practical geometry. We then propose a related model for the earliest form of pure geometry – that of Hippocrates of Chios. Finally, we develop the model of the neural representation of geometric objects in Euclidean geometry.

1. Introduction

The scientific study of geometric cognition is still in its infancy. There are, however, important results. Spelke and co-workers have determined what might be the cognitive core at the foundations of geometry. This would be constituted by a system for navigation in the environment and a system for recognizing objects. These systems would become interconnected in this way enabling the arising of some form of geometrical cognition (see, e.g., Spelke 2011).[2] While Spelke and co-works propose to identify this form of geometrical cognition directly with that of Euclidean geometry, this cannot the case (Ferreirós and García-Pérez 2020); the most we might arrive at is some form of proto-geometry, which we might try to relate to early stages of practical geometry.[3]

Presently, if we endeavor to address pure geometry in terms of geometric cognition, we can at best present tentative accounts of the cognitive underpinning of the practice of pure geometry. A philosophically oriented inquire can help in this endeavor. For example, regarding the reasoning during Euclidean proofs, Hohol and Miłkowski, propose "a list of desiderata that the future theory of geometric cognition should satisfy" (Hohol and Miłkowski 2019, 675).

In the present work, our interest rests in the semantic cognition of the abstract geometric objects of pure geometry. More specifically, in the neural representation of geometric concepts that underline our cognitive processes related to the practice of pure geometry. With this objective, we want to propose tentative models of the neural representation of geometric concepts in different geometrical practices that are compatible with the views on these practices that we will set forward in the present work. By addressing the models of the distinct but related practices simultaneously we expect to

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[1] This paper is part of my work made in the context of the research project "the genesis of geometrical knowledge" whose PI is José Ferreirós from the University of Seville.

[2] Exactly how this interconnection is established is rather hypothetical (see, e.g., Spelke, Lee, and Izard 2010; Spelke, and Lee 2012).

[3] It is not clear from Ferreirós and García-Pérez's interpretation of Spelke's 'natural geometry' in terms of a proto-geometry (Ferreirós and García-Pérez 2020), how 'much' can it be identified with early stages of practical geometry as addressed in the present work. However, this is not an issue for the present work. Our approach is unrelated to Spelke's work on core geometrical systems.

achieve a more consistent formulation of the model for pure geometry. In our view the model is only intelligible when taking into account the models for the previous practices.

We will develop our working models using the hub-and-spoke theory (Ralph, Jefferies, Patterson, and Rogers 2017).[4] According to this theory, the neural representation of concepts is made in terms of spokes, which are modality-specific brain regions, involving sensory and motor processing, that codify modal features of concepts. For example, there are the spokes that encode visual, verbal and praxis representations.[5] There are also integrative regions – the hub – which blends, in an amodal format, the different aspects codified in the spokes and gives rise to coherent concepts. The function of the hub goes beyond bringing together the modality-specific aspects of concepts; in the process, it also enables a modality-free codification of further aspects of concepts:

[The hub] allows the formation of modality-invariant multi-dimensional representations that, through the cross-translation of information between modalities, code the higher-order statistical structure that is present in our transmodal experience of each entity. (Ralph 2014, 7)

We can think of a particular concept directly in terms of 'spokes' and a 'hub' not has regions in the brain but as 'parts' of the concept. In this way, "each concept also has a 'hub' – a modality-independent unified representation efficiently integrating our conceptual knowledge" (Eysenck and Keane 2020, 319).

This work has three parts. In section 2 – the first part – we will characterize practical geometry. The features we will mention are common to several historical examples of practices of practical geometry. We have taken into account aspects of the practical geometry of ancient Egypt, the ancient Near East, and ancient Greece. We endeavor to present the main features that Greek practical geometry had before the 'dawn' of pure geometry. In section 3 – the second part – we will consider the earliest (known) form of pure geometry, that of Hippocrates of Chios. We will tentatively propose what kind of change there is in the concept of figure when going from ancient Greek practical geometry to Hippocrates' pure geometry. In our view, in this early form of pure geometry there was not yet a notion of abstract geometric object. The notion was that of what we call a perfect figure. In section 4 – the third part –, we will address the change from Hippocrates' pure geometry to Euclidian pure geometry. We will present a view of how geometrical concepts change from that of a perfect figure to that of an abstract object. This change can be adequately taken into account in the hub-and-spoke models proposed here. In each of these sections we will propose a tentative hub-and-spoke model. These models are developed taking into account the historical material of each section but also of different sections when relevant.

2. Ancient practical geometry

From the perspective of a philosophy of mathematical practices (see, e.g., Ferreirós 2016), we should take into account actual practices in their historical context. This is particularly the case if we need to address the arising of pure geometry. Not only should we consider the practice of pure geometry by the ancient Greeks but also their previous practice of practical geometry. When this methodological ideal is not possible to fulfill, we might try to approach it as much as possible. Not much is known about Greek practical geometry previous or contemporary to pure geometry (see, e.g., Asper 2003, 109-114). So, in this work, we will consider basic aspects of the Greek practical geometric practice that are common to other practical geometries. We will include examples of these practices when helpful for us to grasp what practical geometry is.

[4] For the purpose of this work, we will only consider the original hub-and-spoke model and not the graded hub-and-spoke model. On the differences see Ralph et al. (2017).

[5] The spoke related to vision encodes representations in the visual modality related to visual features of concepts. The spoke related to the praxis can encode, e.g., representations related to object use. The spoke related to verbal (speech) descriptors encodes the 'labels' we use to name concepts.

The basic characteristics that Greek practical geometry shares with others are as follows:

1) There are metrological systems, in particular for length and area measurements.
2) There are recurrent figures whose geometric properties are known (e.g., we have 'formulas' to calculate the area from length measures), like the square, the rectangle, the circle, or the triangle.
3) There are no explicit definitions of geometrical figures, just names.
4) There are instruments to draw and/or measure geometric figures; in particular, the rod or straightedge and the compass.[6]
5) There is a didactic component to practical geometry. In particular, there are written problems that are useful in learning the basics of practical geometry.
6) There is no pure geometry or at least no influence from it.
7) The practice relies on oral and written language.

A key aspect of land measurements is to have a common unit of length so that we have a common standard. In this way, different surveyors will arrive at the same measure when using different measuring instruments (e.g., rods or cords). These measuring instruments are 'calibrated' to the adopted standard. For example, in ancient Egypt, the unit of length measure was the *cubit*. This unit corresponds to the common measure of the forearm and it is divided into 6 *palms* or 24 *fingers*. It is represented by the drawing of a forearm (Imhausen 2016, 47; Rossi 2007, 59). For land-surveying, it seems that Egyptian used ropes having a standard length of 100 *cubits*, which were divided using knots placed at 1-*cubit* intervals (Imhausen 2016, 18; Rossi 2007, 154).[7]

Regarding the Greek length units there was no common unit adopted, and, like with other civilizations, the metrological systems changed with time. A widely used unit during Hellenistic times was the *foot* which was about 30-32 cm (Lewis 2004, xix).

Besides the measurement of the boundaries of fields, it was essential to calculate the areas of these fields. Agricultural plots along the Euphrates relied on a sophisticated irrigation system based on channels but also basins to store water, and structures to control as much as possible recurrent floods (Mori 2007, 42-4). The system of irrigation favored a rectangular shape for the fields (Mori 2007, 47-8). The names adopted for the sides of the rectangle (for the long sides and the short sides) refer to the rectangular shape of agricultural plots. These might be translated as the 'long side' and the 'front'. The term 'front' refers to the original agricultural plots where the rectangular shape was adopted. The 'front' is the narrow side parallel to the irrigation channel (Høyrup 2002, 34). In the case of ancient Greece, there is evidence of the division of land in rectangular plots (Lewis 2004, 3; Cuomo 2001, 7-8).

In the ancient Near East, surveyors adopted a formula to calculate the area of fields that, from our perspective, gives a good approximation to the area of rectangular-like shapes (being exact in the case of rectangles). Let l_1, l_2, l_3, and $_4$ be the sides of a quadrilateral field plot that are measured by a surveyor (e.g., l_1 and l_3 are the 'long sides', and l_2 and l_4 are the 'short sides'). The surveyors' formula gives for the area of the field plot the value $(l_1 + l_3)/2 \times (l_2 + l_4)/2$.

Here, we find an important and common feature of practical geometries. The lengths are measured (or taken to be measured), and as such are given in terms of a unit of measure. The area is calculated from these length measures and given in terms of a unit of area. For example, in the Old-Babylonian period, the main unit of length was the *rod* (approximately 6 m), and the unit of area was the *sar*, corresponding to one square *rod* (36 m^2) (Robson 2008, 294).

For different reasons, in practical geometries, some figures are widely used and crucial aspects of them are well established: how to draw the figure precisely; the measured lengths that are taken into account; the 'formula' to calculate the area. These figures are clearly distinguished from all the other

[6] The difference between a rod and a straightedge is that the second is only for drawing line segments. The first enables also the measurement of length since it has a standard length and may even have markings for its subdivisions. One example is the Egyptian cubit-rod (Imhausen 2016, 168-9).
[7] There are also records of the existence of high-quality ropes made of plant fibers being 1000 or more *cubit* long (Rossi 2007, 155-6).

possible figures by naming them, even if definitions do not exist (contrary to the case of the pure geometry in Euclid's *Elements*). A good example is the circle. In ancient Mesopotamia, a circle, like other geometrical figures, was conceptualized in terms of its boundary. The circle was the shape enclosed in a circumference. In this case, both had the same name. A translation of the name might be "thing that curves" (Robson 2004, 20). The area of the circle was determined from the measure of the length of the circumference. It was given by the square of the length of the circumference divided by 12 (Robson 2004, 18). This is not to say that the only length measure at play was that of the length of the circumference. While the length of the circumference was the main measure taken into account, there is evidence that the length of the diameter could also be adopted (Friberg 2007, 210 and 296). The circle was drawn using a specific instrument – the compass (Høyrup 2002, 105; Friberg 2007, 207). The compass made it possible to have a precisely drawn figure.

In terms of a model based on the hub-and-spoke theory, we can conceive of the concept of circle as relying heavily on a 'visual spoke' that represents aspects related to the visual shape of a circle, a 'verbal spoke' that codifies the name of the circle, and a 'praxis spoke' related to the drawing and measurements on the circle. Here, going beyond the spokes taken into account by Ralph and co-workers (see, e.g., Ralph et al. 2017), we propose another spoke related to measure-numbers, i.e., numbers that result directly from measurements in the case of length, or indirectly in the case of areas, and are addressed in terms of abstract symbols in the context of metrological systems (see figure 1).[8, 9, 10]

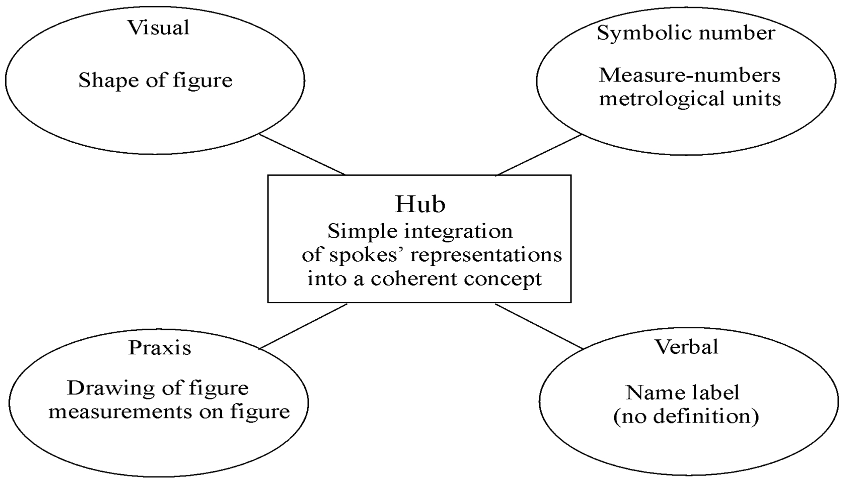

Figure 1. Hub-and-spoke model of the neural representation of geometric figure in practical geometry.

[8] There is evidence for a 'symbolic number spoke'. Accordingly, "adult humans possess two distinct systems to support magnitudes: 1) a symbolic system used specifically to represent symbolic numerical magnitudes, and 2) a general magnitude system used to represent both discrete and continuous magnitudes" (Sokolowski, Hawes, Peters, and Ansari, 2019, 20). In this work we departure from the idea that the spokes consist in modality-specific representations. In our case the symbolic number spoke is amodal. Also, it might be best to address the verbal spoke in terms of an amodal representation if we are to include more than simple words in it. On the possible kind of amodal representation underlying language see Frankland and Greene (2020). See also footnote 24.

[9] We address the neural representation of the concept of geometric figure without trying to include in our account Spelke's views. However, one might try to include one of the two core representational systems of geometry identified by Spelke and co-workers in the hub-and-spoke model. We hypothesize that the spatial layout system might not be too relevant for the neural representation of the concept of geometric figure, while the visual form system might be 'connected' to the visual and praxis spokes, since it is taken to "represent the shapes of 2D visual forms and movable objects" (Spelke 2011, 303). Since this is too speculative, it will not be taken into account in the present work.

[10] In his study of the conceptual representation of triangles (the concept TRIANGLE), in relation to the linguistic label "triangle", Lupyan concluded that "(even formal) concepts have a graded and flexible structure, which takes on a more prototypical and stable form when activated by category labels" (Lupyan 2017, 1). More specifically, he arrived at the following results: "(a) the representations of even a formally defined category like triangle reflect formally irrelevant, but perceptually relevant, properties; and (b), category names help to form a kind of idealized perceptual state—a prototype of sorts" (Lupyan 2017, 9). These results fit well with the model presented here: (a) The praxis spoke might provide for the typicality effects related to, e.g., prototypical drawings being favored; and (b) The verbal spoke – the label "triangle" – is part of the neural representation of triangle.

Another aspect shared by different practical geometries is the existence of written geometrical problems. These are couched in the terminology of practical geometry. But they are a somewhat different way of doing practical geometry. It is not so much that there are suprautilitarian problems since it is still the case that many problems clearly originate from the practitioner's work practice, even if they might be simplified for didactic reasons (see, e.g., Imhausen 2016, 193). The point is that by being written problems that are disconnected from an immediate surveying activity, there are no actual measures taken into account in the problems. The measures are putative measures that could have been made according to the practice of practical geometry. This led, for example, to the adoption of conventional lengths in the problems. In Old Babylonian mathematical texts, all circumferences are taken to have the same standard length. This is so independently of the actual length of the circumference drawn with a compass (Friberg 2007, 207). So, while there are references to length measures in problems, these have not been actually measured, and neither corresponds to the actual measures of the drawn figure.

Whatever cognitive processes are at play during actual measurements these are not active during problem solving with conventionalized measures. One still uses whatever conceptual representation is at play when also making measurements, but only part of it. For example, the concept of line segment must include aspects related to the act of measuring. These are the procedures by which one attributes to the line segment a number (its length). In problems, we address these segments taking into account that there is a number associated with them – it is part of the concept –, but we disregard the use of a measuring instrument and the procedure by which the number is obtained. In our view, it might be the case that some aspects of the concept are only loosely taken into account.

This is less speculative than it might seem. There is a cognitive basis for this loosening of the connection of lengths to metrological units in problem solving in the context of practical geometry.[11] We do not make use of concepts in a rigid way in which the 'full' concept is always taken into account. Conceptual processing is flexible, in the sense that "one aspect of a 'concept' may be used in one context or task, but another aspect of the concept may be used in another" (Mahon and Hickok 2016, 949). A very simple way to take into account conceptual flexibility within a hub-and-spoke model of a concept is by taking a particular spoke not to be fully active during particular tasks. This can be accounted for by reduced neural connectivity between the hub and the spoke depending on the context (Chiou and Ralph 2019). In this way, during problem solving, the praxis spoke of the concept of geometric figure would not be fully taken into account. The only thing that is included is the association of measure-numbers to drawn lines that in a full practical geometric practice are measured.

One example of a geometrical problem is a Hellenistic geometrical problem from a papyrus written in demotic Egyptian in the third-century BCE (here, we take this problem not to have been influenced by the existing pure geometry). The statement of the problem is as follows: "A plot of land that <amounts to> 60 square cubits, [that is rec]tangular, the diagonal (being) 13 cubits. Now how many cubits does it make [to a side]?" (Cuomo 2001, 71). We have a rectangle and are asked to determine the length of its sides. These are calculated to have 12 and 5 *cubits* (Cuomo 2001, 71; Friberg 2005, 125). This is a problem of practical geometry; however, we do not have an actual shape whose relevant lengths were measured. In fact, in a strictly practical practice, this problem is unfeasible. We can only calculate the area after measuring the lengths of the long side and the short side. In any case, as a didactic problem of practical geometry, the rectangle is conceived in terms of practical measures (in this case, the length of the diagonal) or values that are determined from practical measures (in this case, the area that is calculated from the lengths of the sides). In the problem, we consider a rectangle that is not measured nor even drawn, which has an area of 60 square *cubits*, a diagonal of 13 *cubits*, and sides that have (as calculated) 12 and 5 *cubits*.

The aspects we have seen of practical geometry have all traces in written accounts; like 'reports' on land surveying (see, e.g., Robson 2008, 61-6), markings of the use of compasses, or geometric

[11] This loosening is even more drastic in pure geometry. As we will see, one aspect of pure geometry is having concepts of figures in which their lengths are not conceived in relation to measurement practices. The loosening of the connection of lengths to metrological units reaches the point in which one disregards measuring units altogether.

problems. There are also aspects related to oral transmission of which we only have indirect evidence through inscriptions, images, or preserved artifacts, for example. This is the case in particular in what regards the fabrication and use of measuring tools (see, e.g., Rossi 2007, 153-6). Even the geometric knowledge is only attested somewhat indirectly mainly through the existence of written geometrical problems. If we consider Greek practical geometry, we have very little testimony (Asper 2003). We can, however, 'reconstruct' aspects of oral communication from 'traces' in written accounts.

It has been defended that a well-known passage in Plato's *Meno* about finding a square that doubles the area of another square is an example of Greek practical geometry (Valente 2020). Regarding this passage, it had already been argued by Saito (2018) that it is an example of a written rendering of the oral teaching and discussion of geometry in ancient Greece. It might be the closest we get to the ancient oral communication of geometry.[12] Here, we will address this passage taking into account this two-fold aspect of it. Socrates helps a boy having recollections of his knowledge about a geometric figure – the square. We can re-frame it as an episode of geometrical teaching – a lecture on a particular geometrical subject. We may assume that the teacher begins the lecture by drawing a square on a wax tablet (Saito 2018, 928; Netz 1999, 15): "Tell me now, boy, you know that a square figure is like this? —I do. A square then is a figure in which all these four sides are equal? —Yes indeed." (Plato 1997, 881). The teacher draws two lines perpendiculars to the sides of the square and passing by the center (see figure 2): "And it also has these lines through the middle equal? —Yes." (Plato 1997, 881). The teacher then starts to address issues related to the area of the square:

And such a figure could be larger or smaller? —Certainly. If then this side were two feet, and this other side two feet, how many feet would the whole be? Consider it this way: if it were two feet this way, and only one foot that way, the figure would be once two feet? —Yes. But if it is two feet also that way, it would surely be twice two feet? —Yes. How many feet is twice two feet? Work it out and tell me. — Four, Socrates. (Plato 1997, 882)

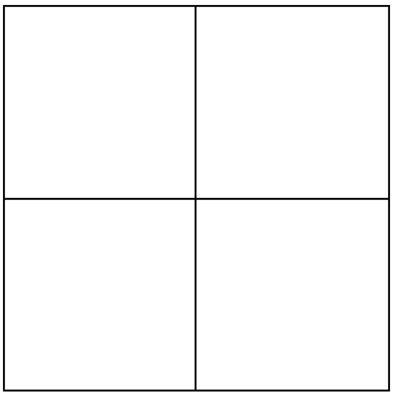

Figure 2. The drawing of the figure at this point of the lecture.

For our purpose, we do not need to go further with the lecture. In this part, we see a feature of practical geometry that we have called attention to. We have a drawing of a figure, but we do not measure its sides. We take it to have measures that are useful in a didactic context. The teacher (Socrates) asks the student (the boy) to consider that the sides of the square measure two *feet*, and asks him to calculate the area, which the student does, arriving at the result of four (square *feet*). We have here a

[12]According to Asper, "the mathematical passages in Aristotle or Plato (minus the general differences between written prose and oral discourse) might be a model of how mathematicians *talked* about their objects" (Asper 2003, 119). It has been suggested that some well-known written works might have been first a series of lectures presented orally. This is the case with Proclus' *Commentary* and Cleomedes' *The Heavens* (Taub 2013, 358). In fact, taking into account Saito's view about traces of oral teaching in Euclid's *Elements* (Saito 2018), these might also have been oral lectures previous to being organized as a written text. These lectures might have consisted of presentations of propositions in which the students had to learn by memory the protasis (which is the general enunciation of the proposition) and the teacher showed with the help of an unlettered diagram, e.g., how to do a particular construction showing its correctness. There is written evidence compatible with these 'lectures' (see footnote 20). We will see more on this in the next sections.

clear example of a written rendering of oral teaching of practical geometry. This is an example of the cognitive loosening we mentioned before. We perceive the square to have a particular dimension – it might look as having approximately 0.5 *feet*, for example. But we conceive it as having two *feet*. All this is made without an actual measurement procedure. The kind of conceptual representation at play is in some way 'vaguer' than when we pick up a rod and measure the sides of the square and determine that they have 0.5 *feet*.

In the next section, we will consider the early Greek pure geometry, having as a background the basic characteristics of practical geometry and the hub-and-spoke model of the neural representation of geometric figure that we presented in the present section.

3. Hippocrates' pure geometry

Hippocrates' quadrature of lunules is taken to be the earliest evidence of Greek pure geometry (see, e.g., Netz 2004; Høyrup 2019). We know of Hippocrates' work by a text of Simplicius from the sixth-century CE. This text is based on two previous accounts, one by Alexander of Aphrodisias and the other by Eudemos. Hippocrates' work is believed to be from the early second half of the fifth century BCE. Written prose was rare; because of this Netz considers that "Hippocrates' treatise on the lunules could well be among the first treatises written in Greek mathematics" (Netz 2004, 247). Regarding Eudemos account, Netz realizes an exercise of reconstruction, trying to determine what in the text is closer to Hippocrates' original.[13] Netz assumes that the text "should have two layers, one closer to Hippocrates' original, and another closer to late fourth century mathematics" (Netz 2004, 259). The main difference between these layers is the adoption or not of lettered diagrams, and the use of letters in the text to refers to parts of these diagrams. Netz's assessment is as follows:

While Eudemus has written his own text, he had before him Hippocrates' text and, even against his will, he would be likely to be influenced by this text. I suggest that he had in front of him an unlettered text, and that he had modernized it in the two more complicated quadratures. Even there, however, he let himself here and there reproduce the original structure of the argument, correlating it however with his own lettered diagram. Thus, some letters are redundant. (Netz 2004, 265)

We can contrast this view with what Høyrup says about the two accounts taken into account by Simplicius. According to Høyrup:

Alexander draws on Hippocrates's teaching, being based either on lecture notes of his or on students' notes; the Eudemos version may instead go back to what Hippocrates published more officially. (Høyrup 2019, 158)

Simplicius' account of Alexander's text includes lettered diagrams. Taking into account Netz's view that the Eudemos text on the first two lunules – relying on unlettered diagrams – might be closer to the original, it might be the case that what is closer to Hippocrates' teaching are these passages.

Here, we want to suggest that the text might refer to an early written rendering of oral teaching by Hippocrates. This is not that a bold suggestion. We know that Hippocrates taught about astronomy and geometry (Høyrup 2019, 160). Even if Netz takes Hippocrates to have written a treatise he also mentions the following:

If you wish to convey an argument which relies, among other things, on a diagram, then you must have at least the 'written', i.e., drawn diagram to accompany it […] Briefly, then, some use of writing, in the sense of a physical drawn object, is a necessary aspect of Greek mathematics. (Netz 2004, 246)

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[13] The fragment that Netz considers is already a reconstruction by Becker in which parts added by Simplicius to Eudemus' text are taken out (Netz 2004, 252-5).

This is perfectly compatible with oral teaching of geometry like we have seen in the case of Plato's *Meno*. Hippocrates might have presented his arguments orally to his students accompanying them with the corresponding drawing. Independently that Hippocrates might have made a written rendering of his lectures on geometry we take these to be the main vehicle of his approach to geometry. We suggest that like in the case of practical geometry, in pure geometry there is also an oral practice which might well be the earliest.

In what follows we will consider the passage about the first quadrature as a rewriting of an initially written rendering of oral teaching. It is as follows:

(1) ... Therefore we shall discuss and quote them <=the quadratures> at length. (2) So he made his starting point by assuming, as the first among the things useful to the quadratures, that both the similar segments of the circles, and their bases in square, have the same ratio to each other. ((3) And this he proved by proving that the diameters have the same ratio, in square, as the circles). (4) This being shown to his satisfaction, he first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. (5) He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the base, <he drew> a segment of a circle, similar to those taken away by the joined <lines>. (6) And, the segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (7) So the lunule, having been proved equal to the triangle, could be squared. (8) In this way, taking the outer circumference of a semicircle as the <outer circumference> of the lunule, he readily squared the lunule. (Netz 2004, 248-9)

Our purpose here is to determine what has changed in relation to practical geometry that leads us to say that here we are in the context of a pure geometric practice.

We can see that the text begins by calling the attention that the starting point of Hippocrates' argumentation is the assumption that the similar arcs of circumference of the circular figures and their bases in square have the same ratio to each other.[14, 15] Here, we do not have postulates like those of the *Elements*. In fact, this presupposition enters the argumentation in the same way that in problems of practical geometry. It is taken to be known by the interlocutor and without any need of justification. According to Høyrup, this presupposition was "known by Near Eastern practical geometers at least since the beginning of the second millennium BCE" (Høyrup 2019, 165). Possibly, the only difference with the use of background knowledge in practical geometrical problems is that the assumption that will be used during the argumentation is stated explicitly at the beginning. Both Høyrup and Netz consider that when in the text is written that this assumption was proved by Hippocrates this was included there by Eudemus to make the text closer to the geometrical practice of his times (Høyrup 2019, 170-1; Netz 2004, 258). So, we take this aspect of the argument to be similar to how background knowledge was used in practical geometry. As it is, by now we could still be considering the written rendering of an oral exposition related to practical geometry.

The first quadrature is that of a lunule whose outer circumference can be seen as a semicircle. Thinking in terms of an oral presentation, Hippocrates after mentioning the assumption to his audience might have drawn an isosceles triangle and using a compass drawn a semicircle circumscribing it (see figure 3).

[14] Whatever notion of ratio Hippocrates is using, we take for granted that it is consistent with his geometry. According to Heath, Hippocrates might be using in a somewhat intuitive way the idea that segments of circles are in the same ratio as the circles (in this case, squares) if they are 'the same part' of the circles respectively (Heath 1981, 182). Knorr agrees with this view. Accordingly, "Hippocrates seems to appeal to a concept of proper 'parts' as the basis of his notion of ratio" (Knorr 1986, 45).

[15] Similar arc segments can be defined from the following: "similar sectors are those which make up the same part of the circle, for example half-circle to half-circle and third-circle to third-circle" (Høyrup 2019, 167).

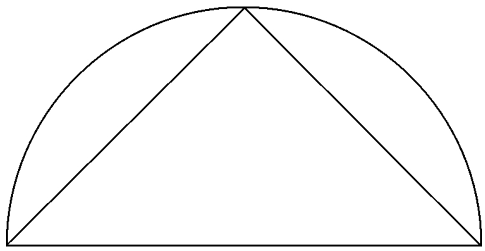

Figure 3. Initial drawing with a semicircle circumscribing an isosceles triangle.

Afterward, he might have completed a square based on the triangle and using a corner of the square as the center of a circle drawn an arc segment that is similar to the two formed previously. According to Netz's rendering of the text in English, "<he drew> a segment of the circle, similar to those taken away by the joined <lines>" (Netz 2004, 249) (see figure 4; this approach is taken from Harper and Driskell 2010).

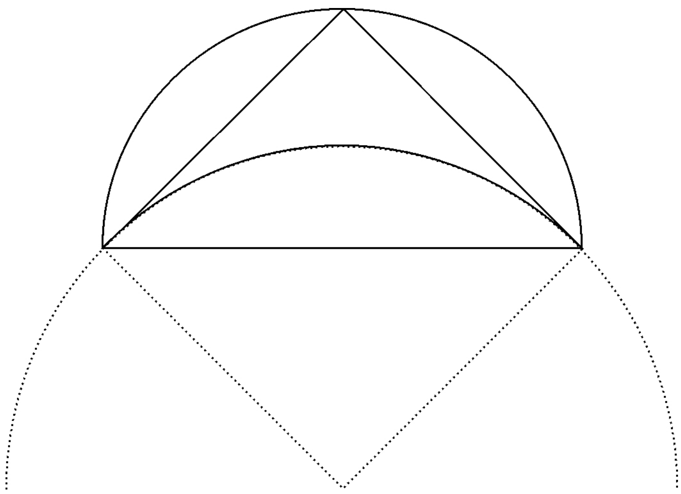

Figure 4. Completion of the drawing of the lunule.

As Høyrup mentions, Hippocrates' arguments have a 'single-level', directly based on the assumptions taken into account (Høyrup 2019, 179). Based on the presupposition mentioned at the beginning, Hippocrates simply mentions that the area of the circular figure about the base is equal to that of both circular figures about the other sides of the triangle. He then proceeds to add the area not included in either of these to each one of them. We have what we might call a visual operation in which we alternatively imagine each of the area addition operations (see figure 5, left and right).[16]

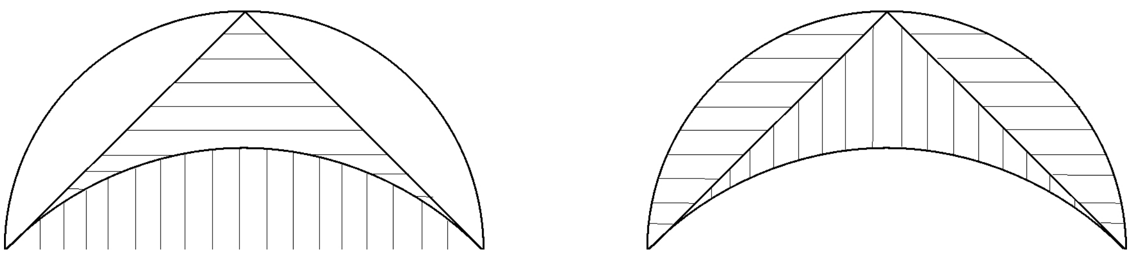

Figure 5. Two ways of doing the 'visual operation' of adding areas of figures.

[16] This aspect of Hippocrates' practice is found also in the *Elements*. For example, in the proof of proposition 1 of book 1, we can see three segments as just that or we can make the visual operation of seeing them as the sides of a triangle (Euclid 1956, 241-242). This corresponds to a practice of seeing a figure in different ways (Macbeth 2010).

Evidently, the area of the two figures is the same and so "the lunule, having been proved equal to the triangle, could be squared" (Netz 2004, 249). This addition of areas is something that we find in practical geometry. In fact, it is one of the basic elements of ancient Near Eastern geometry. It was clear to practitioners that "the size of a figure which consists of partial areas equal the sum of these partial areas" (Damerow 2016, 115). We have what we might call the principle of conservation of area. For example, we can 'cut' a part of a figure and move it around and add it – 'past' it – to the figure in a different part of it (see, e.g., Høyrup 2017). Or, like in the present case, we might conceive of different figures touching each other and unifying them in different ways, like in figure 5, left and right.

What is it then that makes this example a case of early Greek pure geometry and not of Greek practical geometry? The key evidence that we are not engaged in a practical geometrical practice is the lack of reference to metrological units. They are completely absent. This corresponds to a crucial conceptual change that is at the crux of the reformulation of practical geometry as pure geometry. We assist to a perfectioning of the geometrical figures that leads to what we might call the exactification of lengths.

We have seen that within practical geometry we assist to a loosening of the conception of geometric figure by not taking into account directly the measurement practices. As we have seen, we can have geometric figures that we do not measure but conceive as having lengths that cannot be even approximately like those of the figure. We mentioned the case of circles in Old Babylonian problems that adopt a conventional length for the circumference or the case of the passage in Plato's *Meno*, in which the sides of the square are taken to have a particular measure not related to the actual measure of the drawing. In Hippocrates' pure geometry we have what we might call perfect figures. These are figures that look as having no irregularities and that if we were to measure them, we would find measure-numbers that are the same. That is, whatever small differences there are, they are invisible to the eye even when using the available measuring tools. This perfection of the geometric figure does not correspond to doing a more precise practical geometry. It is the opposite; not measuring the figures with better measuring techniques we take them to be perfect. In this way, e.g., the sides of a square are taken to be exactly equal. This is what we mean by the exactification of lengths. In this context, the lengths are exact and 'belong' to the figures. A length as a measure-number is the result of a measurement procedure in which, e.g., we put a measuring rod side by side with the side of a square and check that they are congruent. The length as a measure-number arises from this measurement procedure. In Hippocrates' case, we do not have this anymore. The sides of the square have lengths 'of their own', independently of whatever measurement we might make, and it is senseless to mention a metrological unit in this context.

As it is, early Greek pure geometry is the geometry of perfect figures (not yet of geometric objects). Like in the case of practical geometry, these are not explicitly defined. As mentioned by Høyrup, "there is not the slightest reference to a definition in the Eudemos text" (Høyrup 2019, 179). This has important consequences that we will address in the next section when comparing Hippocrates' pure geometry to that of the *Elements*. The main difference with the previous practical geometry is a further loosening of the concepts in relation to its more practical aspects related to measurements.

In terms of the very simple hub-and-spoke model of geometrical concept that we are using in this work, the main difference in relation to the concept of geometric figure of practical geometry is in the praxis and symbolic number spokes. In the praxis spoke the main features represented are related to the drawing of figures; however, there is still a representation not so much of particular measurement procedures as of the possibility of making measurements on the figure – there are 'traces' of the praxis of measuring.[17] Regarding the symbolic number-magnitude spoke there is no encoding of measure-numbers or metrological units. Instead, we have a representation of length, which, taking into account the praxis spoke, is dissociated from any particular measurement procedure.[18] The visual

[17] As we will see in the next section, this is necessary for a coherent transition between the models corresponding to the different but related geometrical practices.

[18] Here, we hypothesize that the symbolic number spoke is also the spoke that represents symbolic continuous magnitudes, like in the case of the non-symbolic representation of both discrete and continuous magnitudes, which is made by a general

spoke is basically the same; it encodes the visual shape of a geometric figure. The verbal spoke is also the same. It encodes the 'label' for the figure. A 'higher order' change can be taken to occur in the hub that would enable to encode a conceptualization of figure as a perfect figure (see figure 6).

What are then, according to our view, the basic characteristics of early Greek pure geometry?

1) The propositions rely on unproved presuppositions that are part of a shared knowledge whose origins can be found in practical geometry.
2) There is a standard order in presenting propositions, in which the presuppositions are presented previous to the argumentation.
3) The original mode of argumentation is oral, being the extant text a rewriting of the originally written rendering of lectures.
4) Unlettered diagrams are drawn during the lecture (and reproduced in the treatises). These are conceived by the audience as perfect figures even if they are not.
5) The conceptualization of figures is made without resort to notions related to length measurements. This leads to the exactification of lengths.
6) Geometrical figures are drawn using drawing instruments like those used in practical geometry but are (conceived as) visually 'perfect'.
7) There is no linguistic definition of geometrical figure, just a name like in the case of practical geometry.

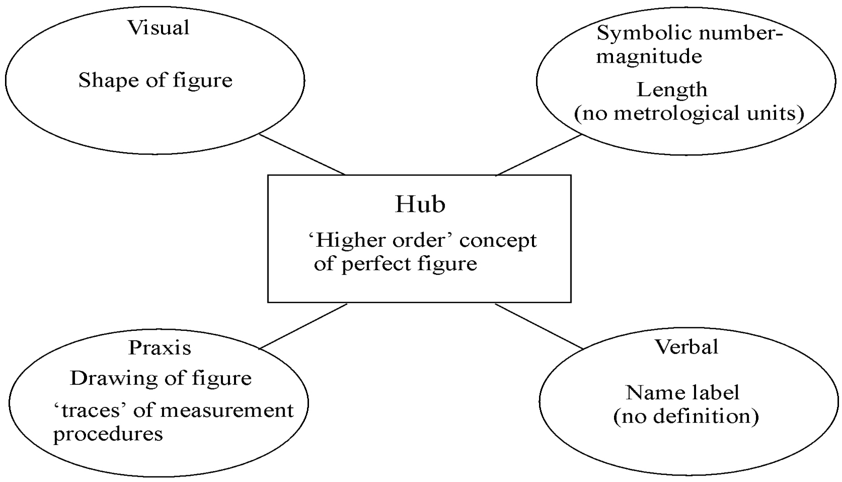

Figure 6. Hub-and-spoke model of the neural representation of geometric figure in Hippocrates' pure geometry.

So far, we have addressed Hippocrates' pure geometry mainly in relation to the previous practical geometry. We developed hub-and-spoke models of the neural representation of geometric figures in both geometrical practices that are compatible with the main changes we have found to have occurred when going from one practice to the other.

We are now at a position where we can address the neural underpinning of the Euclidean geometrical practice. We will consider the changes that occur when going from Hippocrates' pure geometry to Euclid's. By taking into account the model of geometric concept in the earlier form of pure geometry and how it changed from the previous model from practical geometry, we will set forward a hub-and-spoke model of the neural representation of an abstract geometric object that makes more understandable how we go from perfect figures to abstract objects and how this can be encoded neurologically.

magnitude system (see footnote 7). Accordingly, we change the naming of the spoke from 'symbolic number spoke' to 'symbolic number-magnitude spoke'.

4. Euclidean pure geometry

That Hippocrates' pure geometry is not that of Euclid's *Elements* should not be surprising. From the perspective of a philosophy of mathematical practices, this is quite natural. A particular practice is historically situated; we need to analyze a mathematical practice in its historical context, and we need history to address the historical development of practices (Ferreirós 2016).

Nobody would say, e.g., that the geometrical practice of Hilbert's pure geometry is like that of Euclid's. However, some repeat the mistake of Simplicious or Eudemos in relation to the Hippocratic practice by analyzing the Euclidean practice from the perspective of a more recent one, in terms of modern reformulations of Euclid's geometry. We have to address Euclid's geometry from within the Euclidean geometrical practice (Dal Magro and García-Pérez 2019). This is also the case with Hippocrates' pure geometry.

Instead of considering modern reconstructions of ancient practices, it is more interesting and meaningful for our purpose to address the evolution of Hippocrates' pure geometry (PG_1) to that of Euclid's (PG_2) from within these practices. Taking into account the interest of this work on the cognitive underpinning of geometrical practices, we are interested in unveiling cognitive changes that occur when going from PG_1 to PG_2. In this way, like in previous sections, looking back into history is most of all a way to uncover possible aspects of the semantic cognition underpinning these practices.[19]

We think that we can conceive the early stages of the geometry of the *Elements* in terms of lectures in which the teacher helps the student to memorize the enunciation of the proposition (the protasis). According to Saito the role of the protasis was to enable to memorize and refer to the propositions (Saito 2018, 928-9). It suffices to memorize the protasis and the corresponding diagram. Accordingly, "with the [unlettered] diagram in your memory, you can surely understand the protasis" (Saito 2018, 929). With these two elements, a practitioner can recover the whole of the content related to a proposition as manifested in the *Elements*.[20] In Saito's view, "mathematical teaching was very probably performed directly and orally by drawing a diagram in front of pupils, and explaining it" (Saito 2018, 924). We can conceive of a lecture as starting with the teacher reciting an enunciation (protasis) of a proposition. He would then proceed to the 'doing' or the 'showing' (Rodin 2014, 15-35); i.e., the teacher would construct a particular figure by starting to draw a diagram and pointing to the correctness of the successive steps until the final figure is constructed. Alternatively, he would show a particular result with the help of the diagram.

The diagram was probably drawn "on sand or a wax tablet" (Saito 2018, 928), and, importantly, the teacher would indicate the "points and other geometrical objects by finger". (Saito 2018, 928). In the context of an oral presentation, it is very unlikely that the teacher assigned labels to the diagram. As Netz shows, there is a close relationship between lettered diagrams and the type of written text adopted in the demonstrations of pure geometry (Netz 1999). According to him, "the introduction of letters as tools is a reflective use of literacy". (Netz 1999, 62). In this way, with the written form, "the lettered diagram is the tool which […] was made more central" (Netz 1999, 66). By contrast, the oral teaching as rendered in Plato's *Meno* reveals that "the diagram is not lettered, and the geometrical

[19] The nature of this aspect of our work is so tentative that we do not claim to be doing cognitive history. Cognitive history as proposed, e.g., by Netz, would enable us to study as historical phenomena 'central cognitive processes' like the reasoning underlying geometrical proofs (Netz 1999, 7). Netz focus on two 'cognitive tools', the lettered diagrams and the mathematical lexicon (in particular the definitions and the linguistic formulae). It is beyond the scope of this work to address Netz's views; we only make a few remarks we consider necessary. The historical elements taken into account in this work lead us to consider that lettered diagrams were not essential to the arising of PG_1 or even PG_2 and so also not essential to demonstrations. Formulae are also not relevant in the early development of PG_1 and PG_2. Regarding definitions, as we will see, these seem to be central in the arising of PG_2 from PG_2; this would not agree with Netz's account of definitions in geometrical practice. Netz considers that "the existence of a definition must strengthen, to some extent, the tendency to employ the definiendum instead of other" (Netz 1999, 103). In our view, this is not the case. The definitions 'stabilize' the Euclidean idealizations which are at the crux of passing from a pure geometry of perfect geometric figures (PG_1) to a pure geometry of abstract objects (PG_2).

[20] A proposition in the *Elements* can be conceived a formed by the protasis, ekthesis, diorismos, kataskeuē, apodeixis, and sumperasma (see, e.g., Mueller 1981, 11).

objects in it are referred to by the word 'this'" (Saito 2018, 932). According to Saito, "the written text of the *Elements* with lettered diagrams shows certainly a more developed stage" (Saito 2018, 932).

In a more organized lecture, the teacher would first present the necessary definitions, postulates, and common notions, or he would refer to them as needed.[21] This would be the counterpart of Hippocrates' practice, where we would start by mentioning the presuppositions that are used in the demonstration. We can see the Euclidean approach as resulting from the Hippocratic one when the presuppositions are "assumed as principles for which no justification is given" (Cellucci 2013, 68).[22]

Where do we find then a cognitively relevant difference between PG_1 and PG_2? In our view, the difference is in the way we learn to look at the diagrams and use them. That there is a crucial change from the Hippocratic practice can be seen in the fact that in the Euclidean practice we have explicit linguistic definitions at play. For example, a point is defined as "that which has no part" (Euclid 1956, 153). Regarding lines, according to definitions 2 to 4, "A line is breadthless length. The extremities of a line are points. A straight line is a line which lies evenly with the points on itself" (Euclid 1956, 153).

According to Harari, the definition of point makes reference to the idea of measurement by 'contraposition': "a point is characterized as a non-measurable entity, as it has no parts that can measure it" (Harari 2003, 18). In the models of geometrical concept that we tentatively suggest in this work, when going from practical geometry to PG_1 there is a loosening in the praxis spoke of aspects related to measurements. The change from PG_1 to PG_2 might be seen as the 'overwriting' using the verbal spoke of any encoding related to a measurement praxis still existing in PG_2 (and 'inherited' from PG_1). This is achieved by extending the content of the verbal spoke that would consist now of a definition and not just a label. The changes in the spokes go hand and hand with changes in the hub that gives rise to a coherent concept. That this might be so can be seen, e.g., in the definitions related to lines. In PG_1 we already have a perfect figure but one that has a breadth; in PG_2 we move beyond this and conceive of something that is breadthless. This linguistic term points to something that is not even visualizable – it is an abstract object.

Taking this into account together with the definition of point we see that this definition goes beyond the "non-measurability" mentioned by Harari. The point is also an entity that goes beyond a visualized figure – as perfect as it might be. The verbal spoke helps to recreate the concept of perfect figure of PG_1 as the concept of abstract object of PG_2.

What does this imply regarding the diagrams that are drawn during the lectures? These are not conceived anymore as perfect figures. They are representations of geometrical objects as defined and instantiated following the postulates in the *Elements* (Valente 2020, 27-9). This has consequences concerning how we address the diagrams during the oral lectures (or in the written treatises). According to Ferreirós:

The first definitions indeed suggest a *way of reading diagrams*, a perspective for seeing or conceiving what is implied by a diagram, and what is not. And this way of reading is not at all evident, especially if one previously knows only practical geometry. For the definitions and the reading that comes with them lead the practitioner to certain crucial idealizations. More importantly, the definitions suggest

[21] This is not such a bold suggestion as one might think. Several books of the Latin version of the *Elements* that was most influential in the Latin West during the 12th and 13th centuries have these characteristics. In book 1, in the beginning, we find the definitions, postulates, and common notions; then, we have the enunciation of proposition 1, an unlettered diagram, and a commentary giving indications about how to carry out the proof (Busard and Folkerts 1992, 113-115). If we consider the books to be used by the student with the aid of a teacher this is all that is needed. Other examples are, e.g., a second-century CE fragment that contains proposition 9 of book 1 of the *Elements*. It consists only of the enunciation with an unlettered diagram (Brashear 1994), and a fragment from the third century CE with identical characteristics (Cairncross and Henry 2015, 24).

[22] However, even in the Euclidean practice some assumptions function as background presuppositions and not as 'principles'. The demonstration by Hippocrates takes into account, as a background assumption, the 'conservation of area' we mentioned above. This is also the case with Euclidean demonstrations. According to Høyrup, the 'arithmetic of areas' (additivity and subtractivity) is "something not even Euclid considered worth arguing for specifically but just included in his common notions 2 and 3 ("if equals be added to equals, the wholes are equal", etc.)" (Høyrup 2019, 165).

certain forms of response (and of indifference) to some aspects of the diagram: thus, the crossing of two drawn lines will be a (very small) planar region, but we are taught to disregard this and consider in the argumentation that one and only one point has thus been determined. (Ferreirós 2016, 144).[23]

We suggest that the conceptual change leading from PG_1 to PG_2 might have arisen in a practice based on oral lectures with PG_1 where the above-mentioned way of looking into and reading diagrams arose. That is, PG_2 does not lead to this particular way of attending to the diagrams; it would be the other way around. This way of attending to the diagrams (or a very similar precursor) would give rise to an early oral version of PG_2. This change would be made more explicit and stable by developing explicit linguistic definitions that help to stabilize the concepts, and further 'sedimented' by written treatises and a teaching practice that would rely more and more on these.[24]

In terms of the simple model of geometrical concept that we are working with in the present paper, the verbal definition might be encoded in the verbal spoke that would be much more developed than in the cases of practical geometry and Hippocrates' pure geometry. In these cases, the content of the verbal spoke consisted only of the label used to name a geometric figure. Now we have a definition. The main difference would occur in how the hub re-represents the encoding in the spokes (see figure 7).

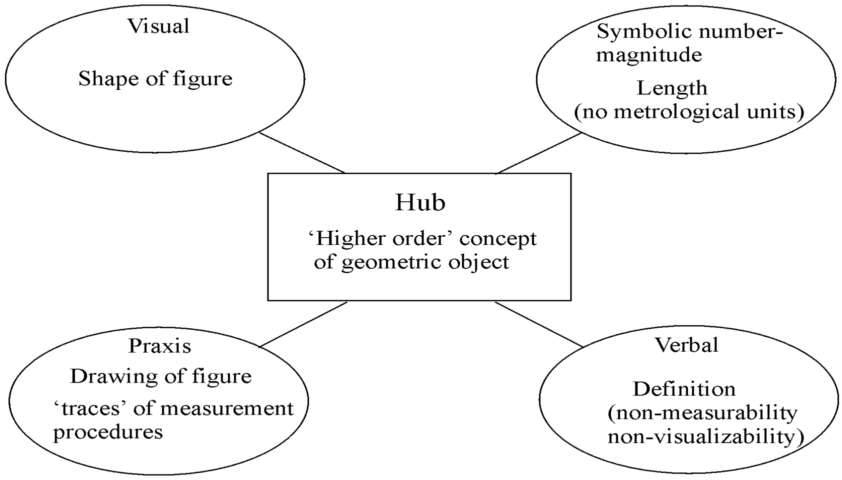

Figure 7. Hub-and-spoke model of the neural representation of geometric object in Euclidean pure geometry.

The representations in the visual, praxis and symbolic number-magnitude spokes are represented in a highly abstract way by taking into account the encoding of the verbal spoke. When we look at a figure there is a particular indifference and responsiveness to its features related to the verbal definition such that we re-conceive the figure in terms of an abstract geometric object and not as a perfect figure anymore (as mentioned, the figure becomes for us a representation of the geometric object).[25] In this way, the verbal spoke becomes decisive in how the representations in the visual,

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[23] In a footnote, Ferreirós mentions that "indifference and responsiveness to features of representations is a topic that I have seen Manders elaborate on in an unpublished conference" (Ferreirós 2016, 144).

[24] Hippocrates himself is said to have written a treatise (but see Netz 2004, 275) and there are references to treatises written previous to the *Elements* during the fourth-century BCE (Knorr 1986, 102).

[25] It must be noticed that this is a very tentative suggestion. In fact, the hub-and-spoke theory has not been developed by taking into account anything like the definitions in the *Elements*, only individual words (for concrete objects like 'apple' and 'rope', or abstract concepts like 'rule' and 'hope'. See, e.g., Hoffman, Binney, and Ralph 2015; Kemmerer 2015, 343-4). A model in terms of cross-modal convergence zones might be more appropriate. In this kind of model instead of one main brain hub one considers a hierarchy of cross-modal convergence zones (Kuhnke, Kiefer, and Hartwigsen 2020). This is so because in a definition we engage with complex features of linguistic semantics that seem to be addressed in different brain regions than the anterior temporal lobes identified in the hub-and-spoke theory as the main hub. For example, there is evidence that what we call thematic role (e.g., agent and patient in an event verbally described as "the dog chased the cat") might be represented in two subregions of the left-mid superior temporal cortex (Frankland and Greene 2020, 289-90). The situation with Euclidean definitions is possibly much more complex at a neural level. Even if

praxis and symbolic number-magnitude spokes are interpreted and recombined in the hub giving rise to a 'higher order' representation of geometric abstract objects.

In our view, the hub-and-spoke models of the neural representation of geometric figure/object make more intelligible what a geometric object might be since we can relate its neural representation to that of a geometric figure in Hippocrates' pure geometry and in practical geometry.[26]

5. Conclusions

The purpose of this work is to set forward a tentative model of the neural representation of abstract geometric objects. This model might be useful in relation to the development of a future theory of the semantic cognition underpinning pure geometry.

To develop the model consistent with the previous geometrical practices, we have considered a historically informed account of practical geometry. The objective was to provide a basic characterization of practical geometry. Taking into account these basic 'characteristics' we build a model of the neural concept representation of geometric figure in practical geometry using in a very simple way the hub-and-spoke theory of neural concept representation.

We then address pure geometry. Previous to the geometrical practice related to Euclid's *Elements*, there was a development of a pure geometry that was an intermediary stage between practical geometry and the pure geometry of the *Elements*.

In this work, we present a basic characterization of this geometrical practice as revealed in Hippocrates' work on the quadrature of lunules. In our view, this pure geometry deals not with abstract objects, but still with geometric figures – perfect figures. We provide a model of the neural concept representation of perfect figures again relying on the hub-and-spoke theory.

We then address what kind of neural representation of geometric concept we have in Euclidian pure geometry. For that, we reconstruct some aspects of the Euclidean practice taking into account how these are different from the corresponding aspects in the Hippocratic practice. Taking these differences into account together with the models of neural concept representation in practical geometry and Hippocrates' pure geometry we proposed a simple model of abstract geometric object. Comparing this model with the ones related to geometric figures makes more intelligible what it is to have a concept of abstract object and how this concept may underline the Euclidean practice.

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this is so, the lack of knowledge on the neural representation of geometric figures/objects and the lack of development of alternative models is such that a hub-and-spoke model of geometrical concepts (with modal and amodal spokes) seems to us as a useful way of framing the discussion at this point. Independently of the issue of 'verbal spoke' being an 'acceptable' working model or not, there is the even more embarrassing issue of what evidence there is of the role of language in geometrical concepts. There is evidence in relation to geometric problem solving. It has been found that the semantic system in the brain supports geometric problem solving, for example in problems where the practitioner must take into account rules that can be expressed linguistically, like "each interior angle for an equilateral triangle is 60º" (Zhou, Li, Li, Zhang, Cui, Liu, and Chen 2018, 366). We take this to be indirect evidence for something like the 'amodal verbal spoke' we adopt here.

[26] This is not to say that the ontogeny of the semantic geometrical cognition must correspond to the stages and kind of models presented here. We would, however, expect that there are some points in common in what regards an earlier neural representation of geometric figure and a later neural representation of geometric object.

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