Reopening the Hole Argument*

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Abstract

The aim of this expository paper is to argue that although Weatherall (2018) and Halvorson & Manchak (2022) claim to 'close the Hole Argument', its philosophical thrust may be resurrected by rephrasing the argument in terms of the theorem of Choquet-Bruhat & Geroch (1969) on the existence and uniqueness of maximal globally hyperbolic solutions to the Einstein field equations for suitably posed initial data. This not only avoids the controversial pointwise identification of manifolds underlying non-isometric spacetimes, but also suggests a precise notion of determinism intrinsic to GR (as an initial value problem), which may be compared e.g. with the one proposed by Butterfield. We also discuss generally covariant special relativity in this context, and finally muse about possible implications for the general philosophy of science.

1 Introduction

Initially, the Hole Argument (*Lochbetrachtung*) was an episode in Einstein's struggle between 1913–1915 to find the gravitational field equations of general relativity. At a time when he was already unable to find generally covariant equations for the gravitational field (i.e. the metric) that had the correct Newtonian limit and satisfied energy-momentum conservation, the Hole Argument confirmed him in at least temporarily giving up the idea of general covariance (which he later recovered without ever mentioning the Hole Argument again). More generally, Einstein's invention of the argument formed part of his analysis of the interplay between general relativity (of motion), general covariance (of equations under coordinate transformations), and determinism. The more recent emphasis on substantivalism versus relationalism (Earman & Norton, 1987) is not Einstein's, but since for him this opposition was closely related to the problem of absolute versus relative motion and hence to general relativity (Earman, 1989), he would certainly have been interested in it.[1]

*If anything, this paper is a tribute to the weekly Cambridge–LSE *Philosophy of Physics Bootcamp*, in which the hole argument has often been discussed. I am especially indebted to the organizers of this seminar, Jeremy Butterfield and Bryan Roberts, as well as to Henrique Gomes, Hans Halvorson, Joanna Luc, and JB Manchak, for comments on earlier drafts. I also wish to thank Michel Janssen for historical comments, some of which have found a way to the Introduction.

[1]See Janssen & Renn (2022) for the final reconstruction of Einstein's struggle, with §4.1 devoted to the Hole Argument. The earliest known reference to the Hole Argument is in a memo by Einstein's friend and colleague Besso dated August 1913, provided this dating is correct (Janssen, 2007). Einstein subsequently presented his argument four times in print; I just cite Einstein (1914)

In modernized form (using a global perspective and replacing Einstein's coordinate transformations by diffeomorphisms), his reasoning was essentially as follows:[2]

- Let (M, g) be a spacetime.[3] The transformation behaviour of the Einstein tensor $\text{Ein}(g)$ under diffeomorphisms ψ of the underlying manifold M is

$$\psi^*(\text{Ein}(g)) = \text{Ein}(\psi^* g). \tag{1}$$

  Similarly, for any healthy energy-momentum tensor $T(g, F)$ constructed from the metric g and the matter fields F that matter we should have

$$\psi^*(T(g, F)) = T(\psi^* g, \psi^* F). \tag{2}$$

  Consequently, if g satisfies the Einstein equations $\text{Ein}(g) = 8\pi\, T(g, \varphi)$, then $\psi^* g$ satisfies these equations for the transformed matter fields $\psi^* F$.

- Now consider an open connected vacuum region H in spacetime possibly surrounded by matter (i.e. $F = 0$ in H); H is referred to as a "hole", whence the name of the argument.[4] Furthermore, find a diffeomorphism ψ that is nontrivial inside H and equals the identity outside H, so that in particular,

$$T(\psi^* g, \psi^* F) = T(\psi^* g, F) = T(g, F), \tag{3}$$

  both outside H (where ψ is the identity) and inside H (where $T(g, F) = 0$).

- It follows from the previous points that if g satisfies the Einstein equations for some energy-momentum tensor T, then so does $\psi^* g$. Hence the spacetimes (M, g) and $(M, \psi^* g)$ satisfy the Einstein equations for the same matter distribution and are identical outside H. But they differ inside the hole.

Einstein saw this as a proof that the matter distribution fails to determine the metric uniquely, and regarded this as such a severe challenge to determinism that, supported by the other problems he had at the time, he retracted general covariance.

as the paper containing his final version. See Stachel (2014), Norton (2019), and Pooley (2022), and references therein for reviews of the Hole Argument in both a historical and a modern context.

[2] We write the Einstein tensor as $\text{Ein}(g)$, where its dependence on the metric g is explicitly denoted; in coordinates we have $\text{Ein}(g)_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

[3] A spacetime is a smooth four-dimensional connected Lorentzian manifold with time orientation (this nomenclature of course hides philosophical issues to be discussed later in this paper). More generally, my notations and conventions follow Landsman (2021) and are standard, e.g. spacetime indices are Greek whereas spatial ones are Latin, the metric has signature $- + ++$, etc.

[4] Einstein's arrangement looks unnatural compared to Hilbert's (1917) reformulation as an initial-value problem in the PDE sense, see below, but Einstein was probably inspired by Mach's principle, where "the fixed stars at infinity" determine the local inertia of matter; see Maudlin (1990), Hofer (1994), and Stachel (2014). There is another argument that actually favours Einstein's curious setting for the Hole Argument: the smaller the hole, i.e. the larger the complement of the hole, the greater the challenge to determinism, for if even things almost everywhere except in a tiny hole fail to determine things inside that hole, then we should really worry (Butterfield, 1989). This pull admittedly gets lost in the initial-value formulation of the argument below. See Muller (1995) for the explicit construction of a hole diffeomorphism (the only one I am aware of).

From a modern point of view the energy-momentum tensor is a red herring in the argument,[5] which may just as well be carried out *in vacuo*, as will be done from now on; this also strengthens my subsequent reformulation of the argument, since the theorem on which this is based is less well developed in the presence of matter.

Earman & Norton (1987) famously revived the Hole Argument. Streamlining:

1. Although (M, g) and $(M, \psi^* g)$ are different spacetimes (unless of course ψ is an isometry of (M, g), i.e. $\psi^* g = g$), physicists—usually tacitly—circumvent this alleged lack of determinism of GR by simply "identifying" the two, i.e. by claiming that (M, g) and $(M, \psi^* g)$ represent "the same physical situation".

2. In this practice they are encouraged by the trivial observation that (M, g) and $(M, \psi^* g)$ are *isometric*; indeed, the pertinent isometry is none other than ψ.[6]

3. *However*—and this is their key point—this spells doom for spacetime substantivalists (like Newton), who (allegedly) should be worried that if in order to save determinism, $x \in M$, carrying the metric $\psi^* g(x)$, must be identified with $\psi(x) \in M$, carrying the same metric, then points have lost their "this-ness": they cannot be identified *as such*, but only as carriers of metric information.

Apart from its primary bearing on spacetime substantivalism and determinism, the Hole Argument also has implications for the general philosophy of science (see §4). But all of this is pointless if the argument is void, as claimed by Weatherall (2018) and his followers (Fletcher, 2020; Bradley & Weatherall, 2021; Halvorson & Manchak, 2022), against critics including e.g. Arledge & Rynasiewicz (2019), Roberts (2020), Pooley & Read (2021), and Gomes (2021). My goal is not to decide who is right or wrong, but, in the light of the indisputable fact that all arguments so far have turned out to be controversial, to present a version of the Hole Argument that should be *un*controversial, while leading to exactly the same philosophical issues.

In order to see some of the advantages of the version of the Hole Argument I prefer (i.e. based on Theorem 2 below), it may be useful to briefly discuss the main point raised by Weatherall (2018) and subsequently by Halvorson & Manchak (2022). The main issue seems to be whether it is a valid move in GR to compare—or even put—two different metrics g and g' on the same manifold M (for example in the Hole Argument above one takes $g' = \psi^* g$). Since this involves comparing $g(x)$ with $g'(x)$, the objection would really be against identifying $x \in M$ *seen as a point in the spacetime* (M, g) with $x \in M$ *seen as a point in a different spacetime* (M, g'):

[5] Continuing footnote 4: Janssen (2007), footnote 98, notes that Einstein formulated his requirement that the matter distribution fully determines the metric only in 1917; in 1913 Einstein still thought of Mach's principle in the light of the relativity of inertia. Furthermore, Einstein (1914) explicitly introduced the final version of the hole argument in terms of a conflict between general covariance and the "law of causality" ("*Kausalgesetz*"), which was contemporary parlance for determinism. In sum, it seems safe to say, with Janssen (2007), that the 'worries about determinism and causality that are behind Einstein's hole argument have strong Machian overtones.' See Norton (1993) for Einstein's general struggle with general covariance, and its aftermath.

[6] We say that $(M', g') \stackrel{\psi}{\to} (M, g)$, where $\psi : M' \to M$ is a diffeomorphism, is an *isometry* iff $g' = \psi^* g$ (in particular, following e.g. Hawking & Ellis, 1973, we always take an isometry to be a diffeomorphism). Now simply take $M' = M$ and $g' = \psi^* g$. See Weatherall (2018) and Halvorson & Manchak (2022) for the meaning of this for GR and for what the alternatives could (not) be.

> The basic principles of general relativity—as encompassed in the term 'the principle of general covariance' (and also 'principle of equivalence')—tell us that there is no natural way to identify the points of one space-time with corresponding spacetime points of another. (Penrose, 1996, p. 591)[7]

There are also philosophical arguments against such trans-world identifications, see e.g. Lewis (1986) and, specifically in connection with the Hole Argument, Butterfield (1989). Such arguments may be convincing, but remarkably, Weatherall (2018) mainly appeals to *mathematical practice*, according to which two Lorentzian manifolds (M, g) and (M', g') may only be "compared" using isometries (or so he claims).

In particular, if $M' = M$ and g are given, then putting a metric ψ^*g on M next to g amounts to the simultaneous use of a (hole) diffeomorphism $\psi : M \to M$, which is used to pull back the metric, and the identity map $\mathrm{id}_X : M \to M$, which is used to compare ψ^*g and g and in particular express the key to the Hole Argument, namely

$$g'(x) \equiv \psi^*g(x) \neq g(x),$$

unless, of course, ψ is an isometry, in which case the Hole Argument would be void.[8] Such double talk, then, is supposed to be at best confused and at worst illegal.

Wheatherall's argument is categorical: he decides to only reason in the category **Lor** that has Lorentzian manifolds as objects and isometries as maps (or at least as isomorphisms).[9] Or, using a logical language, one uses model theory (and only that), where Lorentzian manifolds are models and isometries are isomorphisms thereof. The specific nature of the individual objects or models cannot be used.

But this is not the only valid way of doing mathematics. Much of differential geometry could be thrown away if one is not allowed to compute the pullback of a metric under a diffeomorphism; even the very definition of an isometry rests on the ability to say whether *or not* $\psi^*g(x)$ equals $g(x)$, at each $x \in M$. The usual definition of the action of a diffeomorphism on a tensor (field) would be a "category mistake", and with it, the Lie derivative (see also Gomes, 2021, §2.4). All of the coordinate-based *definitions* of tensors used in the past by Einstein (and even by mathematicians like Ricci and Levi-Civita) would have to go, and so it seems no accident that not only Earman & Norton's but also Einstein's original version of the Hole Argument is deemed suspicious. Even existence proofs of a Riemannnian metric on a (paracompact) manifold (typically through an explicit construction using partitions of unity) would be suspicious in a purely categorical framework. *Et cetera.*

[7]Taken from the penultimate version of Gomes (2021); omitted, alas, from the final one.

[8]The emphasis Halvorson & Manchak (2022) put in this context on their otherwise highly valuable Theorem 1 (see footnote 19) seems like flogging a dead horse. This theorem implies that a hole diffeomorphism of the kind envisaged by Einstein (1914) and Earman& Norton (1987), and explicitly constructed by Muller (1995), cannot be an isometry (which, or so it is suggested, would be the only remaining hope for the Hole Argument to work, accepting Wheatherall's critique). But if it were, then $\psi^*g = g$ all across M and the dilemma of having both (M, g) and (M, ψ^*g) as models with the same matter distribution or other initial data simply would not arise: both (naive) determinism and substantivalism would be safe in GR: the Hole Argument would be a dud.

[9]In view of Theorem 2 below, within such reasoning one should optimally work in the category **ST** of spacetimes (see footnote 3), whose isomorphism are isometries *preserving time orientation.*

However, even accepting the claim that the only valid comparison maps in Lorentzian geometry are isometries, the Hole Argument may survive, for it actually uses neither id_M seen as a map from $(M, \psi^* g)$ to (M, g), nor ψ seen as a map from (M, g) to (M, g), both of which indeed fail to be isometries (unless of course ψ from (M, g) to (M, g) is an isometry). Instead, as is clear from the main text, the Hole Argument relies on *ψ seen as a map from $(M, \psi^* g)$ to (M, g)*, which is surely an isometry, cf. footnote 6. The complaint that $(M, \psi^* g)$ should never have been introduced in the first place seems feeble to me, since any spacetime (M', g') is an object in the category **Lor**, whatever its construction; given (M, g), in making the identifications $M' = M$ and $g' = \psi^* g$ one might see the (time orientation preserving) diffeomorphism ψ of M as the proverbial Wittgensteinian ladder, to be thrown away after it has served its purpose. It should not matter where $(M', g') = (M, \psi^* g)$ comes from: one cannot eat one's cake and have it by on the one hand insisting on the use of categories like **Lor** but on the other hand banning all kinds of objects from them—in fact any object is of the said kind and nothing would be left.

In any case, this discussion, even if it turns out to be misguided, is only included here to show that the nullification arguments in Weatherall (2018) and Halvorson & Manchak (2022) are *controversial*, which seems hardly the case for the Choquet-Bruhat–Geroch theorem reviewed in §2, and which, I claim, leads to similar conclusions as the original Hole Argument(s). If Weatherall *c.s.* turn out to be correct after all, then I would see their principles as a *consequence* of the Hole Argument.

In some sense, discussed e.g. by Stachel (2014), my preferred version of the Hole Argument in §2 goes back to Hilbert (1917), who gave the first analysis of GR from a PDE point of view.[10] The initial-value problem of Einstein's equations is very involved. But due to the efforts of especially the "French school", consisting (in direct lineage of doctoral descent) of Darmois, Lichnerowicz, and Choquet-Bruhat (whose early papers carry the name Fourès-Bruhat), the abstract situation is well understood now, at least *in vacuo* and for initial data given on a spacelike hypersurface.[11]

The culmination of the PDE theory is a theorem due to Choquet-Bruhat & Geroch (1969), which I recall in §2. I see this theorem as the least vulnerable version of the Hole Argument, in that it has exactly the same philosophical implications and seems uncontroversial. It also yields the appropriate notion of determinism; namely existence and uniqueness up to isometry of the very specific geometric intial-value problem posed by the Einstein equations (which on the one hand is unique to GR but on the other hand is close to what one expects in classical mathematical physics). This will be compared with the influential definition of determinism proposed by Butterfield (1987, 1989). In §3 I review the Hole Argument for generally covariant special relativity, which leads to very similar questions as the one for GR. These questions will briefly be discussed in the (provisional and speculative) final section.

[10] Hilbert addresses the indeterminism of Einstein's equations, and also refers to Einstein (1914) in connection with this problem, but does not explicitly relate his analysis to the *Lochbetrachtung*.

[11] See Stachel (1992) and Choquet-Bruhat (2014) for some history, summarized in Landsman (2021), §1.9. It is in fact more popular nowadays to give initial data for the Einstein equations on a *null* hypersurface (Penrose, 1963). See Landsman (2021), §7.6, for a summary of the ideas, and e.g. Christodoulou & Klainerman (1993) and Klainerman & Nicolò (2003) for full treatments.

2 The Choquet-Bruhat–Geroch theorem

The initial-value approach to GR is based on PDE-theory and the following ideology:

- *All valid **assumptions** in* GR *are assumptions about initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$.

Such an initial data triple, assumed smooth, is obtained by equipping some $3d$ Riemannian manifold $(\tilde{\Sigma}, \tilde{g})$ with a second symmetric tensor $\tilde{k} \in \mathfrak{X}^{(2,0)}(\tilde{\Sigma})$, i.e. of the same "kind" as the 3-metric $\tilde{g}$, such that $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ satisfies the vacuum constraints

$$\tilde{R} - \mathrm{Tr}\,(\tilde{k}^2) + \mathrm{Tr}\,(\tilde{k})^2 = 0; \qquad \tilde{\nabla}_j \tilde{k}^j_i - \tilde{\nabla}_i \mathrm{Tr}\,(\tilde{k}) = 0. \tag{4}$$

Here $\tilde{R}$ is the Ricci scalar on $\tilde{\Sigma}$ for the Riemannian metric $\tilde{g}$ and likewise $\tilde{\nabla}$ is the unique Levi-Civita (i.e. metric) connection on $\tilde{\Sigma}$ determined by $\tilde{g}$ (so that $\tilde{\nabla}\tilde{g} = 0$).

- *All valid **questions** in* GR *are questions about "the"* MGHD (M, g, ι) *thereof.*

Among these questions, the one relevant to the Hole Argument concerns the uniqueness of (M, g, ι), whence the scare quotes around 'the'. Roughly speaking, a MGHD (for *maximal globally hyperbolic development*) of $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ is a maximal spacetime (M, g) "generated" by these initial data via the Einstein equations, in that

$$\iota : \tilde{\Sigma} \hookrightarrow M$$

injects $\tilde{\Sigma}$ into M as a "time slice" on which the 4-metric g induces the given 3-metric $\tilde{g}$ and extrinsic curvature $\tilde{k}$. In more detail,[12] A *Cauchy development* or *globally hyperbolic development* of given initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ satisfying the constraints (4) is a triple (M, g, ι), where (M, g) is a spacetime that solves the vacuum Einstein equations $R_{\mu\nu} = 0$ and ι is an injection making $\iota(\tilde{\Sigma})$ a spacelike Cauchy (hyper)surface in M such that g induces these initial data on $\iota(\tilde{\Sigma}) \cong \tilde{\Sigma}$, i.e. $\tilde{g} = \iota^* g$ is the metric and $\tilde{k}$ is the extrinsic curvature of $\tilde{\Sigma}$, induced by the embedding ι and the 4-metric g.[13] It follows that (M, g) is globally hyperbolic, since it has a Cauchy surface.[14]

This formulation of the (spatial) initial-value problem for the (vacuum) Einstein equations was an achievement by itself. In particular, it cleverly circumvents the vicious circle one ends in by trying to find initial data for an already given spacetime (solving the Einstein equations); for it is part of the problem to find the latter from the given initial data, and hence one cannot give say dg/dt at $t = 0$ as initial data.

However, the main achievement concerns the existence and uniqueness of (M, g, ι), which depends on a suitable notion of *maximality* (as in the far simpler case of ODEs, where in order to guarantee uniqueness the time interval on which the solution is defined should be maximal). This notion is also non-trivial, and tied to GR. Namely:

[12] See also the references in footnote 19, or Landsman (2021), §7.6. Tildes adorn $3d$ objects.

[13] Let N be the unique (necessarily timelike) future-directed normal vector field on $\iota(\tilde{\Sigma})$ such that $g_x(N_x, N_x) = -1$. Then $\tilde{k}(X, Y) = -g(\nabla_X N, Y)$ defines the extrinsic curvature of $\iota(\tilde{\Sigma})$.

[14] This procedure by no means excludes the study of non-globally hyperbolic spacetimes in GR, which in this approach emerge as possible extensions of globally hyperbolic spacetimes (which is possible even if they are maximal in the above, i.e. globally hyperbolic sense). This is closely connected to strong cosmic censorship (Penrose, 1979), which in turn is related to a kind of indeterminism in GR that is outside the scope of the Hole Argument and may occur even if we all agree that the MGHD of given initial data is essentially unique. See e.g. Earman (1995), Doboszewski (2017, 2020), Smeenk & Wüthrich (2021), or Landsman (2021), Chapter 10, and references therein.

- A *maximal Cauchy development* or *maximal globally hyperbolic development*,[15] acronym MGHD, of given smooth initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$, satisfying the constraints (4), is a Cauchy development (M, g, ι) with the property that for any other Cauchy development = globally hyperbolic development (M', g', ι') of these same data there exists an embedding $\psi : M' \to M$ that preserves time orientation, metric, and Cauchy surface as defined by ι, i.e., one has

$$\psi^* g = g'; \qquad\qquad \psi \circ \iota' = \iota. \tag{5}$$

The Hole Argument à la Hilbert (1917) then follows from the straightforward observation that if (M, g, ι) is a MGHD of the initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ and $\psi : M' \to M$ is a diffeomorphism (*pace* Weatherall *c.s.*!), then the triple (M', g', ι'), where g' and ι' are defined by (5), i.e. $g' = \psi^* g$ and $\iota' = \psi^{-1} \circ \iota$, with time orientation induced by ψ,[16] is a MGHD of the initial data $(\tilde{g}', \tilde{k}')$ induced on $\tilde{\Sigma}$ via ι' and g'. In particular:[17]

Proposition 1. *Given some* MGHD (M, g, ι) *of the initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$*, let* U *be a neighbourhood of* $\iota(\tilde{\Sigma})$ *in* M*. Take a (time orientation preserving) diffeomorphism* ψ *of* M *that is the identity on* U*, so that in particular* $\iota' = \iota$ *and* $(\tilde{g}', \tilde{k}') = (\tilde{g}, \tilde{k})$*.*

Then the "Hilbert-triple" $(M, \psi^* g, \iota)$ *is a* MGHD *of the same initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$*.*

This is a decent version of the Hole Argument.[18] But since it starts from a diffeomorphism ψ of M that only becomes an isometry from $(M, \psi^* g)$ to (M, g) "with hindsight", it may be vulnerable to the reasoning in Weatherall (2018), Fletcher (2020), Halvorson & Manchak (2022), etc. My claim is that this is not the case for the highly nontrivial *converse* of the reasoning preceding Proposition 1, which nonetheless poses the same philosophical problems as the original Hole Argument(s). This converse is the celebrated theorem of Choquet-Bruhat & Geroch (1969):[19]

[15] It might be thought that isometries enter surreptitiously via this definition of maximality, but this is not the case. The appearance of isometries is a consequence of a local version of Theorem 2: *Any two Cauchy developments* (M, g, ι) *and* (M', g', ι') *of the same (smooth) initial data are locally isometric, in that* $\iota(\tilde{\Sigma})$ *and* $\iota'(\tilde{\Sigma})$ *have open neighbourhoods* U *and* U' *in* M *and* M'*, respectively, such that* (U, g) *and* (U', g') *are isometric through a diffeomorphism* $\psi : U' \to U$ *satisfying* (5). See Choquet-Bruhat (2009), Theorem VI.8.4, or Ringström (2009), Theorem 14.3.

[16] Defining time orientation by (the equivalence class of) a global timelike vector field T on M, so that some causal vector X is future-directed iff $g(X, T) < 0$, this means that $T' = \psi_*^{-1} T$.

[17] This construction also works if $U = J^-(\iota(\tilde{\Sigma}))$, cf. Curiel (2018) and Pooley (2022).

[18] Continuing footnote 4, it is superior to Einstein's and Earman & Norton's formulation in that it has shaken off any implicit reference to Mach's principle and is closer to the usual initial value problem for hyperbolic PDEs (with a special GR twist though). But it may be weaker as a challenge to determinism in that the open set on which initial data are given can be made arbitrarily thin.

[19] The original source is Choquet-Bruhat & Geroch (1969), who merely sketched a proof (based on Zorn's lemma, which they even had to use twice). Even the 800-page textbook by Choquet-Bruhat (2009) does not contain a proof of the theorem (which is Theorem XII.12.2); the treatment in Hawking & Ellis (1973), §7.6, is slightly more detailed but far from complete, too. Ringström (2009) is a book-length exposition, but ironically the proof of Theorem 16.6 is wrong; it is corrected in Ringström (2013), §23. A constructive proof was given by Sbierski (2016), which is streamlined and summarized in Landsman (2021), §7.6. Though never mentioned in statements of the theorem, the isometry ψ is unique. This can be shown by Proposition 3.62 in O'Neill (1983) or the equivalent

Theorem 2. *For each initial data triple* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ *satisfying the constraints* (4) *there exists a* MGHD (M, g, ι). *This is unique up to (unique) time-orientation-preserving isometries fixing the Cauchy surface, i.e. for any other* MGHD (M', g', ι') *there exists an isometry* $\psi : M' \to M$ *that preserves time orientation and satisfies* $\psi \circ \iota' = \iota$.

All reference to diffeomorphisms that are not (yet) isometries has gone! And yet in a sense, this theorem *is* the Hole Argument, for it forces us to choose between:

1. *Determinism*, in the precise version that the Einstein equations for given initial data have a unique solution in the sense that we agree that triples (M, g, ι) and (M', g', ι') as in the statement of the theorem are seen as different representatives of the same physical situation (i.e., are "physically identified").

2. *Space-time substantivalism* in the sense of *denial of Leibniz equivalence*. This denial says that triples (M, g, ι) and (M', g', ι') represent "distinct states of affairs" (although they are observationally indistinguishable). This choice saves the "this-ness" of points at the cost of accepting some invisible indeterminism.

Or, at least, this is the dilemma Earman & Norton (1987), or even Einstein (1914), left us with on the basis of their own versions of the argument. Most philosophical discussions of this dilemma, including more precise formulations thereof (e.g. Butterfield, 1989; Pooley, 2022) or dismissals (e.g. Curiel, 2018), remain relevant if we replace the controversial earlier versions of the Hole Argument by Theorem 2. This is the sense in which the Hole Argument remains alive; which is all I wish to argue.

If we opt for determinism, the specific version thereof in GR that seems enforced by Theorem 2 is that we must "physically identify" all maximal globally hyperbolic spacetimes (M, g, ι) with Cauchy surface $\iota(\tilde{\Sigma})$ that carry fixed (and *a priori* "timeless") initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$. Theorem 2 states that all putatively different possibilities are isometric, and hence isometries (preserving ι) play the role of gauge symmetries.[20] This may be unsurprising, since the isometries in Theorem 2 are a shadow of the diffeomorphism invariance of the Einstein equations. But it is also somewhat surprising, since the isometries of a fixed spacetime (M, g) are not given by freely specifiable functions on M, as in the case of gauge theories.[21] See also §3.

It may be interesting to compare the notion of determinism in GR provided for free by Theorem 2 to some others that have been used in the literature on the Hole Argument. To facilitate this, here is a somewhat awkward weakening of Theorem 2:

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argument in footnote 639 of Landsman (2021), to the effect that an isometry ψ is determined at least locally (i.e. in a convex nbhd of x) by its tangent map ψ'_x at some fixed $x \in M'$. Take $x \in \iota'(\tilde{\Sigma})$. Since ψ in Theorem 2 is fixed all along $\iota'(\tilde{\Sigma})$ by the second condition in (5) and since it also fixes the (future-directed) normal N_x to $\iota'(\tilde{\Sigma})$ by the first condition in (5), it is determined locally. Theorem 1 in Halvorson & Manchak (2022) then applies, which is a rigidity theorem for isometries going back at least to Geroch (1969), Appendix A (as Halvorson & Manchak acknowledge).

[20] See Gomes (2021) for a detailed analysis of the relationship between gauge symmetries in gauge theories and diffeomorphisms in GR, including a discussion of the Hole Argument.

[21] If $\dim(M) = n$, then for any semi-Riemannian metric g the isometry group of (M, g) is at most $\frac{1}{2}n(n+1)$-dimensional. See O'Neill (1983), Lemma 9.28; Kobayashi & Nomizu (1963), Theorem VI.3.3 does the Riemannian case. Thus the Poincaré-group in $n = 4$ has maximal dimension 10.

Corollary 3. *If two globally hyperbolic spacetimes* (M, g) *and* (M', g') *contain Cauchy surfaces* $\tilde{\Sigma} \subset M$ *and* $\tilde{\Sigma}' \subset M'$*, respectively, which carry initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ *and* $(\tilde{\Sigma}', \tilde{g}', \tilde{k}')$ *induced by the 4-metrics* g *and* g' *on* M *and* M'*, respectively, where both* (M, g) *and* (M', g') *are maximal for these initial data, and there is a 3-diffeomorphism* $\alpha : \tilde{\Sigma} \to \tilde{\Sigma}'$ *such that* $\tilde{g} = \alpha^* \tilde{g}'$ *and* $\tilde{k} = \alpha^* \tilde{k}'$*, then there exists an isometry* $\psi : M' \to M$ *that preserves time orientation and reduces to* α *on* $\tilde{\Sigma}$*.*

Recall that by convention an isometry is always a diffeomorphism. This corollary is weaker than Theorem 2, for it lacks the existence claim of (M, g) and (M', g'), which are now taken as given. We mention this corollary because it relates to an influential notion **Dm2** of determinism introduced in this context by Butterfield (1987, 1989):

> A theory with models $\langle M, O_i \rangle$ is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, iff: given any two models $\langle M, O_i \rangle$ and $\langle M', O'_i \rangle$ containing regions S and S' of kind **S**, respectively, and any diffeomorphism α from S onto S': if $\alpha^*(O'_i) = \alpha(O_i)$ on $\alpha(S) = S'$, then: there is an isomorphism β from M onto M' that sends S to S', *i.e.* $\beta^* O'_i = O_i$ throughout M' and $\beta(S) = S'$.
>
> (Butterfield, 1987, p. 29; 1989, p. 9).[22]

To cover GR, this definition should be amended by: firstly adding the extrinsic curvature to the initial data induced on S and S'; secondly specializing to globally hyperbolic solutions to the vacuum Einstein equations; and finally, adding a maximality condition on M and M', as in Corollary 3. In that case, GR is deterministic by Theorem 2. But this theorem does more: whereas **Dm2**-like definitions *assume* the existence of the spacetimes in question, Theorem 2 includes an existence proof.[23]

This closes our discussion of the Choquet-Bruhat–Geroch theorem, at least for the moment. We now turn to a version of this theorem for special relativity.[24]

[22]Butterfield (1987, 1989) contrasts **Dm2** with a Laplacian kind of definition of determinism **Dm1** he attributes to Montague and Earman: 'A theory with models $\langle M, O_i \rangle$ is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, iff: given any two models $\langle M, O_i \rangle$ and $\langle M', O'_i \rangle$ and any diffeomorphism β from M onto M', and any region S of M of kind **S**: if $\beta(S)$ is of kind **S** and also $\beta^* O'_i = O_i$ on $\beta(S)$, then: $\beta^* O'_i = O_i$ throughout M.' If we correct this similarly to **Dm2**, Butterfield's point still stands: the Hole Argument (in any version) shows that GR violates **Dm1**. See also Pooley (2022) for a detailed analysis of similar definitions. Pooley's version of **Dm2** is a bit more general and also applies to GR: 'Theory T is deterministic just in case, for any worlds W and W' that are possible according to T, if the past of W up to some timeslice in W is qualitatively identical to the past of W up to some timeslice in W', then W and W' are qualitatively identical.' Apart from my complaint that also this definition assumes the existence of W and W' (instead of proving it), a definition like this requires a sub-definition of what is meant by 'qualitative', which Theorem 2 also takes care of.

[23]Similarly, an amended version of *Property R* introduced by Halvorson & Manchak (2022) would state that if (M, g) and (M', g') are two maximal globally hyperbolic spacetimes solving the vacuum Einstein equations, and if two time-orientation preserving isometries $\psi : M' \to M$ and $\varphi : M' \to M$ coincide on the causal pasts $J^-(\tilde{\Sigma})$ of some Cauchy surface $\tilde{\Sigma} \subset M$, then $\psi = \varphi$ altogether. This definition is *compatible* with GR because of the uniqueness of the isomorphism ψ in Theorem 2 (see footnote 19). But like Butterfield's definition **Dm2**, Halvorson & Manchak's Property R *assumes* the existence of models and then makes some uniqueness claim about it.

[24]The following section was inspired by correspondence with Henrique Gomes and Hans Halvorson, who specifically proposed to look at special relativity in this context.

3 The case of generally covariant special relativity

It may be instructive to compare Theorem 2 with the corresponding situation in special relativity, perhaps unusually seen as a generally covariant field theory à la vacuum GR, but this time with field equation $R_{\rho\sigma\mu\nu} = 0$ instead of $R_{\mu\nu} = 0$. The initial value problem may then be posed in almost the same way as in GR;[25] the only difference is that the initial data triple $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ now satisfes the constraints

$$\tilde{R}_{ijkl} - \tilde{k}_{il}\tilde{k}_{jk} + \tilde{k}_{ik}\tilde{k}_{jl}; \qquad \tilde{\nabla}_i \tilde{k}_{jk} - \tilde{\nabla}_j \tilde{k}_{ik} = 0. \tag{6}$$

The constraints (6) are stronger than (4), which follows from (6) by contracting with $\tilde{g}^{ik}\tilde{g}^{jl}$ and $\tilde{g}^{ik}$, respectively. The reason is that in GR one merely asks for an embedding of the initial data in a Ricci-flat Lorentzian manifold (M, g), whereas in special relativity (M, g) is (locally) flat altogether, i.e. $R_{\rho\sigma\mu\nu} = 0$. To avoid irrelevant global topological issues (interesting as these might be in a different context), *we assume that* $\tilde{\Sigma}$ *is diffeomorphic to* $\mathbb{R}^3$. By the splitting theorem of Geroch (1970) as improved by Bernal and Sánchez (2003), global hyperbolicity of (M, g) then gives

$$M \cong \mathbb{R} \times \tilde{\Sigma} = \mathbb{R}^4 \tag{7}$$

diffeomorphically. Without loss of generality we may actually take $M = \mathbb{R}^4$, so that

$$(M, g) = (\mathbb{R}^4, \eta) \equiv \mathbb{M} \tag{8}$$

is Minkowski spacetime.[26] As a solution to the Einstein equations, $\mathbb{M}$ is not only globally hyperbolic, but also maximal.[27] We may then invoke the Minkowskian version of the so-called fundamental theorem for hypersurfaces (Kobayashi & Nomizu, 1969, Theorem VII.7.2) to obtain the following "special" case of Theorem 2:[28]

Theorem 4. *For each initial data triple* $(\mathbb{R}^3, \tilde{g}, \tilde{k})$ *satisfying the constraints* (6) *there exists an isometric embedding* $\iota : \mathbb{R}^3 \to \mathbb{R}^4$ *carrying the Minkowski metric, whose extrinsic curvature is the given tensor* $\tilde{k}$. *Such an embedding is unique up to (unique) Poincaré transformations that preserve time-orientation.*

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[25] This upsets the idea that special relativity uses only linear subspaces of space-time as hypersurfaces of simultaneity whereas general relativity uses general curved surfaces, but already Schwinger (1948) employed arbitrary initial data surfaces in relativistic quantum field theory.

[26] This follows from the fundamental theorem of (semi) Riemannian geometry, which states that (M, g) is locally flat iff its Riemann tensor vanishes. See e.g. Landsman (2021), Theorem 4.1.

[27] Maximality of Minkowski spacetime follows from its inextendibility; see e.g. Corollary 13.37 in O'Neill (1983) for the smooth case and Sbierski (2018) for inextendibility even in C^0.

[28] See also Landsman (2021), Theorem 4.18, for the fundamental theorem for hypersurfaces. This theorem is concerned with embeddings of curved surfaces with prescribed second fundamental form into Euclidean space and goes back to the nineteenth century. The proof of the Minkowskian case is practically the same, up to some sign changes: in the Euclidean case the first constraint in (6) is $\tilde{R}_{ijkl} + \tilde{k}_{il}\tilde{k}_{jk} - \tilde{k}_{ik}\tilde{k}_{jl}$, the sign changes going back to the different signs in the Gauss–Codazzi equations in Euclidean and Lorentzian signature, see e.g. eqs. (4.147) - (4.148) in §4.7 in Landsman (2021). These sign changes do affect the outcome. For example, Hilbert (1901) proved that it is impossible to isometrically embed two-dimensional hyperbolic space (H^2, g_H) in Euclidean $\mathbb{R}^3$. But hyperbolic space *can* be isometrically embedded in $\mathbb{R}^3$ with Minkowski metric, cf. e.g. Landsman (2021), §4.4. Hence given (H^2, g_H), a symmetric tensor $\tilde{k}$ such that $(g_H, \tilde{k})$ satisfy the Euclidean constraint do not exist, but such a $\tilde{k}$ can be found satisfying the Minkowski constraints.

There is a clear conceptual analogy between Theorems 2 and 4, except that the former refers to the initial-value problem in general relativity, whilst the latter states the situation in special relativity (albeit in a somewhat unusual way). In particular, the role of isometries in the general theory is now played by Poincaré transformations (i.e. isometries of the Minkowski metric), as was to be expected. And yet, whereas most physicists would be happy to regard isometries in general relativity as gauge symmetries akin to coordinate transformations, few if any would regard Poincaré transformations as physically inert. But in Theorem 4, they are. In a more general context, this is explained by Gomes (2021), partly reflecting on Belot (2018):

> But some familiar symmetries of the whole Universe, such as velocity boosts in classical or relativistic mechanics (Galilean or Lorentz transformations), have a direct empirical significance when applied solely to subsystems. Thus Galileo's famous thought-experiment about the ship—that a process involving some set of relevant physical quantities in the cabin below decks proceeds in exactly the same way whether or not the ship is moving uniformly relative to the shore—shows that subsystem boosts have a direct, albeit relational, empirical significance. For though the inertial state of motion of the ship is undetectable to experimenters confined to the cabin, yet the entire system, composed of ship and sea registers the difference between two such motions, namely in the different relative velocities of the ship to the water.
>
> (Gomes 2021, p. 150)

In other words, in thinking about Poincaré transformations as bringing physical change, as for example in boosts of Galieli's ship or Einstein's train, we apply such transformations to *subsystems* of the universe. But Theorem 4 concerns the action of Poincaré transformations on space-time as a whole. See also Wallace (2021).

This brings us back to the substantivalism versus relationalism debate (Earman, 1989; Pooley, 2013); and indeed I see little difference between general and special relativity in this context. The difference between the former and the latter is merely the one between Theorems 2 and 4, respectively, which are good starting points for this debate. Whatever differences there are seem technical rather than conceptual to me, like the underlying difference between the field equations $R_{\mu\nu} = 0$ and $R_{\rho\sigma\mu\nu} = 0$.

Finally, to some extent Theorems 2 and 4 are reminiscent of the spontaneous breakdown of gauge symmetry through the Higgs mechanism.[29] Here, in order to settle into a minimum of the Higgs potential, the Higgs field φ must "choose" a point φ_c on a circle as its "frozen" vacuum value. The global $U(1)$ symmetry involved in this choice is a finite (in fact one) dimensional shadow of the original infinite-dimensional gauge symmetry of the theory. Different choices of φ_c yield phenomenologically indistinguishable worlds and the analogy is between moving the vacuum value φ_c around on a circle and moving a spacetime (M, g) around in its orbit under its isometry group.[30] Note that also here we are talking about symmetries of the universe as a whole, which is what makes them unobservable; the situation changes completely if different domains in the universe have different values of φ_c.

[29] See Struyve 2011, Landsman (2017), §10.10, or any book on the Standard Model.

[30] This analogy is admittedly weak, since Theorem 2 involves both the embedding maps ι and the possibility that isometries move a given spacetime (M, g) to one (M', g') with a different underlying (but diffeomorphic) manifold M, neither of which have a counterpart in the Higgs mechanism.

4 Confronting the Hole Argument

Despite their denial of the Hole Argument, Weatherall (2018) and Halvorson & Manchak (2022) make some of the most useful comments towards its resolution:

> Mathematical models of a physical theory are only defined up to isomorphism, where the standard of isomorphism is given by the mathematical theory of whatever mathematical objects the theory takes as its models. One consequence of this view is that isomorphic mathematical models in physics should be taken to have the same representational capacities. By this I mean that if a particular mathematical model may be used to represent a given physical situation, then any isomorphic model may be used to represent that situation equally well. Note that this does not commit me to the view that equivalence classes of isomorphic models are somehow in one-to-one correspondence with distinct physical situations. But it does imply that if two isomorphic models may be used to represent two distinct physical situations, then each of those models individually may be used to represent both situations.
>
> (Weatherall, 2018, pp. 331–332)

> Why is it, then, that there has been, and will surely continue to be, a feeling that there is some remaining open question about whether general relativity is fully deterministic? Our conjecture is that the worry here arises from the fact that general relativity, just like any other theory of contemporary mathematical physics, allows its user a degree of representational freedom, and consequently displays a kind of *trivial semantic indeterminism*: how things are represented at one time does not constrain how things must be represented at later times. (Halvorson & Manchak, 2022, p. 19)

These comments could just as well have been made about Theorem 2, which by itself already makes it worth delving into the idea of "representational freedom".[31] In particular, the Hole Argument (if it is correct) and Theorem 2 give us a choice between two positions in the philosophy of mathematics that are traditionally seen as opposites, namely a Hilbert-style structuralism *and* a Frege-style abstractionism.[32]

[31]See also Belot (2018), Fletcher (2020), Gomes (2021), Luc (2022), and Pooley (2022).

[32]See e.g. Hallett (2010), Ebert & Rossberg (2016), Mancosu (2016), Blanchette (2018), Hellman & Shapiro (2019), and Reck & Schiemer (2020). Historically, Frege's abstractionism served his higher goal of logicism, but the former stands on its own and can be separated from the latter. It may be objected that the heart of the Frege–Hilbert opposition does not lie in abstractionism versus structuralism but in differences about the nature of mathematical axioms, definitions, elucidations, and existence, and in particular about Frege's insistence that every mathematical concept (such as "point" or "line") be defined on its own through reference, against Hilbert's revolutionary idea of implicit and "holistic" definition of concepts through an entire axiom system in which they occur. But these issues are closely related. For example, Hilbert's contextual and relational way of defining concepts naturally implies that whatever makes them concrete is given only up to isomorphism. Abstractionism of the kind considered here arguably goes back to Aristotle, since the kind of equivalence relation lying at the basis of Frege's abstraction principle is typically obtained by Aristotle's procedure of *abstraction by deletion* (Mendell, 2019). For example, a mathematician sees a bronze sphere as a sphere, deleting its bronzeness. Also in so far as Hilbert famously claimed that mathematical objects exist as soon as the axioms through which they are implicitly defined are consistent (leaving their precise manner of existence in the dark, like Plato), the Frege–Hilbert opposition has its roots in the Aristotle–Plato one (Bostock, 2009).

- *Structuralism:* spacetimes (with fixed initial data) are mathematical structures which by their very nature can only be studied up to isomorphism. Since isometry is the pertinent notion of isomorphism, the identitifcation of isometric spacetimes called for by the Hole Argument or Theorem 2 was to be expected.
- *Abstractionism:* the relevant mathematical object is the equivalence class of all spacetimes (with fixed initial data) up to isometry. Quoting Wilson (2010):

  > Appeals to equivalence classes will seem quite natural if one regards the novel elements as formed by *conceptual abstraction* in a traditional philosophical mode: one first surveys a range of concrete objects and then *abstracts* their salient commonalities. (Wilson, 2010, p. 395)

  In the case at hand, the 'salient commonalities' seem to be the property that all members of a given equivalence class satisfy the vacuum Einstein equations with identical initial data. In the spirit of the abstractionist programme, this commonality may be expressed by the function f from the class of all triples (M, g, ι) to the class of all triples $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ that maps (M, g, ι) to the initial data it induces on $\iota(\Sigma) \subset M$, where it is assumed that each (M, g, ι) is a maximal globally hyperbolic space-times with given Cauchy surface $\iota(\tilde{\Sigma})$.

These two options are put in perspective by the opening quote of Benaceraff (1965):

> The attention of the mathematician focuses primarily upon mathematical structure, and his intellectual delight arises (in part) from seeing that a given theory exhibits such and such a structure, from seeing how one structure is "modelled" in another, or in exhibiting some new structure and showing how it relates to previously studied ones ... But ... the mathematician is satisfied so long as he has some "entities" or "objects" (or "sets" or "numbers" or "functions" or "spaces" or "points") to work with, and he does not inquire into their inner character or ontological status.
>
> The philosophical logician, on the other hand, is more sensitive to matters of ontology and will be especially interested in the kind or kinds of entities there are actually ... He will not be satisfied with being told merely that such and such entities exhibit such and such a mathematical structure. He will wish to inquire more deeply into what these entities are, how they relate to other entities ... Also he will wish to ask whether the entity dealt with is sui generis or whether it is in some sense *reducible* to (or *constructible* in terms of) other, perhaps more fundamental entities.
>
> —R.M. Martin, *Intension and Decision*

Against abstractionism (both in the context of the Hole Argument and in Frege's original application to the definition of Number), one may claim extravagance by noting that an equivalence class $[x]$ with respect to any equivalence relation $\sim$ on some given set X is typically huge;[33] no theoretical or mathematical physicist ever works with such equivalence classes of spacetimes, or even a tiny fraction of it.[34]

[33]Recall that an equivalence class $[x] \subset X$ consists of all $y \in X$ such that $y \sim x$

[34]See Gomes (2021) for an analysis of physical practice, which in the context of gauge theories and GR amounts to the smart choice of cross-sections of the canonical projection from X to $X/\sim$.

In practice, one picks some representative (M, g, ι), from which one may switch to an equivalent triple (M', g', ι') now and then, but one never uses the entire equivalence class. And yet it is, strictly speaking, the entire equivalence class that Frege would invoke in order to obtain a proper definition or reference of the word "spacetime" (provided the analogy with his definition of natural numbers is valid). See also Benaceraff (1965). To resolve this, one might try to work with the single object $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$, i.e. the initial data that give rise to all of these isometric spacetimes, but no one does this either; all actual work in GR is done in terms of just a few of the triples (M, g, ι), whose choice (within its isometry class) is made for convenience.

If instead we go for a structuralist resolution, seemingly incompatible philosophical points of view remain possible. For example, fixing a triple (M, g, ι) within its equivalence class, Newton may be perfectly right in thinking of elements $x \in M$ as points in spacetime, which carry a metric $g(x)$ as a secondary quality. But Gelfand may be equally justified in regarding points of M as nonzero multiplicative linear functionals $C^\infty(M) \to \mathbb{R}$, which by definition carry fields as a primary quality.[35]

Moreover, within mathematical structuralism, the Hole Argument seems compatible with both structural realism (Ladyman, 2020) and empiricist structuralism (van Fraassen, 2008); in the former, the structures in question are so to speak parts of reality whereas in the latter they model empirical phenomena. Let me quote:

> 1. Science represents the empirical phenomena as embeddable in certain *abstract structures* (theoretical models).
> 2. Those abstract structures are describable only up to structural isomorphism.
>
> (...) How can we answer the question of how a theory or model relates to the phenomena by pointing to a relation between theoretical and data models, both of them abstract entities? The answer has to be that the data model represent the phenomena; but why does that not just push the problem [namely: *what is the relation between the data and the phenomena it models*] one step back? The short answer is this:
>
> > construction of a data model is precisely the selective relevant depiction of the phenomena *by the user of the theory* required for the possibility of representation of the phenomenon.
> >
> > (van Fraassen, 2008, pp. 238, 253)

This last comment seems to describe the practice of physicists and mathematicians working in GR: some *user of the theory* chooses a member (M, g, ι) of its equivalence class, whilst some other *user* (or even the same one) may pick another member.[36]

In conclusion, empiricist structuralism seems to have strong cards in confronting the Hole Argument (in both its original versions or rephrased as Theorem 2): it does not suffer from the abstractionist extravaganza and warrants scientific practice.

[35]Gelfand then uses an isometric triple (M', g', ι'), where M' consists of the said functionals.

[36]Van Fraassen's emphasis on the user also explains why say Kerr spacetime, even with fixed parameters m and a, can be used to describe different black holes, despite the mathematical identity of the two models. Indeed, one user models the phenomena produced by one black hole, whilst another user uses (!) the "spacetime" in question to model the phenomena produced by another.

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