On the reality of the global phase

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Abstract

I respond to a recent argument by David Schroeren (*Phil.Sci.*, forthcoming; `philsci-archive.pitt.edu/20419/`) that — contrary to a very wide consensus — global phase in quantum mechanics is real. I argue that Schroeren's argument rests on a mistaken assumption about projective representation theory and that, when this is corrected, the argument no longer follows. In doing so I give a brief introduction to projective representation theory.

1 Introduction

In a recent paper, David Schroeren (2022) (henceforth DS) has argued against a standard assumption in quantum mechanics which he calls RAYS: that Hilbert-space vectors that differ by a phase factor represent the same physical state. ("I will argue that the global phase is not a representational redundancy or a mere mathematical degree of freedom without a physical counterpart, but that it corresponds to a real physical parameter" — DS p.2[1])

This would have radical consequences if true: Schroeren rightly describes RAYS as an orthodoxy common to physics and philosophy of physics; it has been central in mathematically-precise statements of quantum mechanics since at least von Neumann; it underpins the general theory of mixed states (where the projector $|\psi\rangle\langle\psi|$ is taken to be equivalent to the state $|\psi\rangle$), the algebraic approach to quantum mechanics, the Bloch sphere approach to qubits, and more besides.

However, I don't think it is true: Schroeren's argument relies on certain technical claims about projective representation theory (the theory of group actions on projective Hilbert space) that I believe are incorrect. In this short note I first briefly review projective representation theory, then present Schroeren's argument and point out what I take to be its flaw.

(The mathematics here is standard and I do not give original references; for a more detailed account see, e. g. , Weinberg (1995, ch.2.7), whose terminology I largely follow. I take $\hbar = 1$.)

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[1]Page numbers refer to the philsci-archive preprint.

2 A brief review of projective representation theory

2.1 Definitions

Given a Lie group $\mathcal{G}$, a (unitary) *vector representation* of that group on a Hilbert space $\mathcal{H}$ is a homomorphism of $\mathcal{G}$ into the unitary transformations of $\mathcal{H}$: that is, a representation V assigns to any group element $g \in \mathcal{G}$ a unitary operator $V(g)$ on $\mathcal{H}$, such that $V(g)V(g') = V(gg')$.

Motivated perhaps by the view that quantum states are defined only up to phase, or perhaps by simple mathematical curiosity, we could weaken the homomorphism requirement so that it applies only up to phase: that is, so that it becomes

$$V(g)V(g') = \mathrm{e}^{i\phi_V(g,g')}V(gg') \tag{1}$$

where $\phi_V(g,g') : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is some smooth function, the *phase function* for the representation. A map V satisfying this weaker condition defines a *projective* representation of $\mathcal{G}$.

Given one projective representation, we can obtain many: if we let $f : \mathcal{G} \to \mathbb{R}$ be any smooth function and define $V_f(g) = \mathrm{e}^{if(g)}V(g)$, then V_f is a projective representation with $\phi_{V_f}(g,g') = \phi_V(g,g') + f(g) + f(g') - f(gg')$.

The relation

$$f_2 \sim f_1 \text{ iff } f_2(g,g') = f_1(g,g') + f(g) + f(g') - f(gg') \tag{2}$$

defines an equivalence relation on phase functions; equivalence classes under this relation are called *cocycles*, and the *trivial cocycle* is the cocycle containing $f(g,g') = 0$. Any two projective representations whose phase functions lie in the same cocycle are called *phase-equivalent*. A representation of $\mathcal{G}$ is *intrinsically projective* if it is not phase-equivalent to any vector (i. e. non-projective) representation of $\mathcal{G}$; equivalently, a representation is intrinsically projective iff its phase function is not in the trivial cocycle.

A projective representation of $\mathcal{G}$ on $\mathcal{H}$ determines a well-defined representation of $\mathcal{G}$ by isometries on the projective Hilbert space $\mathbb{P}\mathcal{H}$ (the space of rays in $\mathcal{H}$), since $V(g)V(g')\,|\psi\rangle$ equals $V(gg')\,|\psi\rangle$ up to phase; phase-equivalent projective representations define the same action. By Wigner's theorem, any isometric representation of a connected[2] Lie group on $\mathbb{P}\mathcal{H}$ may be so represented; hence, there is a one-to-one correspondence between isometric representations of $\mathcal{G}$ on $\mathbb{P}\mathcal{H}$ and phase-equivalence classes of projective representations of $\mathcal{G}$ on $\mathcal{H}$.

Projective representations can be classified as reducible or irreducible just as for vector representations: a projective representation on $\mathcal{H}$ is irreducible *iff* $\mathcal{H}$ cannot be written as the direct sum of Hilbert spaces each separately invariant under the group action.

[2]This restriction serves to rule out antiunitary representations.

2.2 Physical examples

Non-relativistic quantum physics gives us two important classes of inherently projective representations. The subgroup of the Galilei group defined by (mutually-commuting) spatial translation operators $T_{\mathbf{x}}$ and velocity boosts $B_{\mathbf{v}}$ has a family of phase-inequivalent projective representations labelled by a real parameter m:

$$V(T_{\mathbf{x}})V(B_{\mathbf{v}}) = \mathrm{e}^{im\mathbf{x}\cdot\mathbf{v}/2}V(T_{\mathbf{x}}B_{\mathbf{v}}) \tag{3}$$

which leads (once momentum is identified as the generator of spatial translation, and position as m times the generator of boosts) to the familiar commutation relation $[X_i, P_j] = i\delta_{ij}$.

More important for our present purposes is that the group $SO(3)$ of rotations in three dimensions has two classes of projective representation, one of which is inherently projective. The different irreducible representations can be identified by an integer or half-integer J; following DS I write S_J for the irreducible representation associated to J (while noting that strictly S_J is only defined up to projective equivalence). The *integer-spin* representations are defined by ordinary vector representations of $SO(3)$; the *half-integer-spin* (strictly: half-odd-integer-spin) representations are irreducibly projective, with each being phase-equivalent to a common phase factor $\phi_J(g, g')$, the explicit form of which we will not need.[3] There are thus two cocycles for $SO(3)$: the trivial cocycle for integer-spin representations, and the 'spinor' cocycle for half-integer spin.

The projective representations can be (and, in physics, usually are) specified in terms of the vector representations of $SU(2)$. As is well-known, there is a 2:1 homomorphism π of $SU(2)$ onto $SO(3)$ which associates each rotation with a pair of elements of $SU(2)$: if $\pi(g) = \pi(g')$ then $g = \pm g'$. It can be shown that (i) if S' is an irreducible vector representation of $SU(2)$, then we can find an irreducible representation S of $SO(3)$ satisfying

$$S(\pi(g)) = \pm S'(g). \tag{4}$$

and (ii) all the irreducible projective representations of $SO(3)$ can be obtained this way, with each projective representation S_J of $SO(3)$ associated with a unique irreducible vector representation S'_J of $SU(2)$. (This is actually a special case of a fairly general recipe relating the projective representations of a group to the vector representations of its topological covering group.)

2.3 Combining projective representations

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are distinct Hilbert spaces with respective vector representations V_1, V_2 of $\mathcal{G}$. V_1 and V_2 can always be combined to give a reducible

[3] We can state it as follows: for each $g \in SO(3)$, fix some path $l(g)$ in $SO(3)$ that connects the identity id to g, and define $l(g') * l(g)$ as the path obtained by first going along $l(g)$ from id to g, and then going from g to gg' along the path obtained by applying g to $l(g')$. $\phi_J(g, g')$ equals π if $l(g') * l(g)$ is homotopic to $l(g'g)$, and equals zero otherwise. (This is a slightly cleaned-up version of Schroeren's own definition, which omits a rule to associate paths to group elements)

representation $V_1 \oplus V_2$ of $\mathcal{G}$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$: if $|\psi_1\rangle, |\psi_2\rangle$ are vectors in $\mathcal{H}_1$, $\mathcal{H}_2$ respectively then

$$(V_1 \oplus V_2)(g)(|\psi_1\rangle + |\psi_2\rangle) = V_1(g)\,|\psi_1\rangle + V_2(g)\,|\psi_2\rangle\,. \tag{5}$$

If instead V_1, V_2 are projective representations, this becomes

$$(V_1 \oplus V_2)(g)(|\psi_1\rangle + |\psi_2\rangle) = \mathrm{e}^{i\phi_{V_1}(g,g')}V_1(g)\,|\psi_1\rangle + \mathrm{e}^{i\phi_{V_2}(g,g')}V_2(g)\,|\psi_2\rangle\,. \tag{6}$$

Unless $\phi_{V_1} = \phi_{V_2}$, this does not satisfy the requirement (1) for a projective representation.

If ϕ_{V_1} and ϕ_{V_2} lie in the same cocycle, then we could replace V_1 with a phase-equivalent projective representation with the same phase factor as V_2, so that their sum would be well defined. But if there is no such function, there is no way to define a direct sum of the two projective representations. So the rules for combining projective representations are more restrictive than those for combining vector representations: only representations with the same cocycle can be combined.

This is the origin of the well-known superselection rules in non-relativistic quantum mechanics. Representations of the Galilei group with different masses have different cocycles; hence, no quantum state can be in a superposition of different masses (*mass superselection*). Representations of $SO(3)$ have the same cocyle only if they all have integer or half-odd-integer spin; hence, no quantum state can be in a superposition of spins differing by a half-odd-integer value (*univalence superselection*). It is the latter principle that is central to Schroeren's argument, to which I now turn.

3 Schroeren's argument

In outline, Schroeren's argument against RAYS (the assumption that phase-related Hilbert-space vectors are physically equivalent) is as follows (he works in a system consisting of some number of subsystems each transforming irreducibly under rotations, though this detail will not be significant):

1. There are two distinct candidates for representing rotations on a quantum system (DS p.6): via projective representations of $SO(3)$ ('PROJECTIVE-SO(3)' in Schroeren's terminology) and via vector representations of $SU(2)$ ('LINEAR-SU(2)').

2. If we accept RAYS, we are obliged to regard PROJECTIVE-SO(3) and LINEAR-SU(2) as equivalent (DS pp.6-7).

3. But they're not equivalent: PROJECTIVE-SO(3) implies univalence superselection ('UNIVALENCE' – DS p.7) and LINEAR-SU(2) does not.

4. So we have to reject RAYS and recognize that phase-related quantum states are physically distinct.

It is worth pausing to note that if RAYS is abandoned, it's not at all clear how we should understand *either* PROJECTIVE-SO(3), *or* LINEAR-SU(2), as representations of spatial rotation. The rationale for representing rotation via a *projective* representation of $SO(3)$ is precisely that phase-related states are equivalent: if they are not, shouldn't we be insisting on a *vector* representation of $SO(3)$? If our system has half-odd-order spin, then successively applying twice the operator corresponding to rotation by 180° about some axis will change the phase of the quantum state by a factor of -1. But group-theoretically, two such rotations compose to the identity; how is this consistent? Similarly, we use $SU(2)$ to represent rotations via the double-cover map — but to every element of $SO(3)$ there corresponds two elements of $SU(2)$, and on half-odd-integer spin systems their actions differ by a factor of -1. If phase is physical, there ought to be a fact of the matter as to which of these transformations actually represents a given rotation — how is it to be decided.

However, this is moot, because Schroeren's argument for (2) above — that RAYS obliges us to treat PROJECTIVE-SO(3) and LINEAR-SU(2) as equivalent — is flawed.

The argument is presented on p.6 of DS. First, Schroeren correctly notes (his equation (7)) that any irreducible vector representation of $SU(2)$ induces a projective representation of $SO(3)$. He writes $\hat{\gamma}$ for the map from Hilbert-space vectors into Hilbert-space rays, $\mathcal{H}_s$ for the Hilbert space of his system, T for the representation of $SO(3)$ on $\mathbb{P}\mathcal{H}_s$, and S_J for the 'family of linear [vector] and projective representations of $SO(3)$', i. e. the possibly-reducible projective representation of $SO(3)$ on $\mathcal{H}_s$ that is determined by T. He then writes:

> Since $\hat{\gamma}$ maps any two unitary operators on $\mathcal{H}_s$ *that differ merely by an overall phase* to the same projective automorphism on $\mathbb{P}\mathcal{H}_s$, the linear [vector] representation S' of $SU(2)$ and the family S_J of linear and projective representations of $SO(3)$ agree about which unitary projective automorphisms count as rotations: exactly those in the codomain of T. [Emphasis mine.]

It is true that $\hat{\gamma}$ maps any two unitary operators on $\mathcal{H}_s$ that differ merely by an overall phase to the same projective automorphism on $\mathbb{P}\mathcal{H}_s$. And so if $\pi(g) = \pi(g')$ and $S'(g)$ and $S'(g')$ differ merely by an overall phase, Schroeren's argument would go through. But this need not be the case. Writing elements of $SU(2)$ as 2x2 matrices, define

$$e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathrm{id} \tag{7}$$

and note that $\pi(e) = \pi(\mathrm{id})$. On an irreducible spin-J representation, we have

$$S'(e) = (-1)^{2J} \times S'(id). \tag{8}$$

So $S'(e)$ and $S'(\mathrm{id})$ differ merely by an overall phase if S' is irreducible; indeed, they do so if S' is a reducible sum of representations all of which have integer or half-odd-integer J. But they *do not* differ merely by an overall phase in general.

If S' is a sum of representations differing in spin by $n + 1/2$, then $S'(e)$ acts trivially on the subspaces of integer spin and acts by multiplication by -1 on the subspaces of half-odd-integer spin. So $S'(e)$ determines a different projective automorphism from $S'(\mathrm{id})$, *even though* $\pi(e) = \pi(\mathrm{id})$.

So it is not true that a generic vector representation of $SU(2)$ determines a projective representation of $SO(3)$. The condition for a reducible representation of $SU(2)$ to determine a projective representation of $SO(3)$ is precisely univalence superselection.

This should not be surprising given our earlier discussion: irreducible vector representations of $SU(2)$ determine irreducible projective representations of $SO(3)$, but the latter cannot be combined to form reducible projective representations unless they have the same cocycle, i. e. unless univalence superselection holds. So we should expect sums of vector representations of $SU(2)$ to determine sums of projective representations of $SO(3)$ exactly when the latter are defined.

Schroeren goes on to re-present his argument in a different way (DS p.6):

> Suppose someone were to insist that, instead of T, rotational symmetry should be implemented as the homomorphism $\hat{\gamma} \cdot S'$... Whereas T is faithful, $\hat{\gamma} \cdot S'$ is not: if for any $\varphi \in \mathbb{R}$ we denote by g_φ the element of $SU(2)$ that corresponds to rotation about some axis,[4] then although $g_{2\pi} \neq g_{4\pi}$, $\hat{\gamma} \cdot S'(g_{2\pi}) = \hat{\gamma} \cdot S'(g_{4\pi})$. Implementing rotations in terms of $\hat{\gamma} \cdot S'$ therefore results in exactly the same projective automorphisms as does implementing rotations in terms of T.

Again, this holds only for representations of S' satisfying univalence superselection. As a concrete illustration, suppose that $\mathcal{H}_s$ is the direct sum of $\mathcal{H}_0$ and $\mathcal{H}_{1/2}$, the spin-zero and spin-half representations of $SU(2)$; let $|0\rangle$ and $|1/2\rangle$ denote, respectively, arbitrary states in each subspace. Then

$$S'(g_{2\pi})(\alpha\,|0\rangle + \beta\,|1/2\rangle) = \alpha\,|0\rangle - \beta\,|1/2\rangle \tag{9}$$

but

$$S'(g_{4\pi})(\alpha\,|0\rangle + \beta\,|1/2\rangle) = \alpha\,|0\rangle + \beta\,|1/2\rangle \tag{10}$$

and these states are not phase-related: $\hat{\gamma} \cdot S'(g_{2\pi}) \neq \hat{\gamma} \cdot S'(g_{4\pi})$.

Allowing for the subtleties of combining irreducible projective representations requires us to replace LINEAR-SU(2) in Schroeren's argument with 'UNIVALENT-LINEAR-SU(2)', the assumption that rotations are represented by representations of $SU(2)$ satisfying univalence superselection. But thus modified, the argument no longer implies a physical difference between the two candidates for representing rotations: both candidates equally require univalence superselection. I conclude that Schroeren's argument fails to establish the reality of spin.

[4] Taken literally this doesn't quite make sense: SU(2) is not a rotation group and none of its elements can be identified uniquely as 'the element that corresponds to rotation about some axis'. But the intended meaning is clear enough: picking the z axis for definiteness, $g_\varphi = \mathrm{e}^{i\varphi\sigma_z/2}$.

References

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Weinberg, S. (1995). *The Quantum Theory of Fields, Volume I: Foundations.* Cambridge: Cambridge University Press.