A generic approach to the quantum mechanical transition probability

Gerd Niestegge
gerd.niestegge@web.de

Abstract

In quantum theory, the modulus-square of the inner product of two normalized Hilbert space elements is to be interpreted as the transition probability between the pure states represented by these elements. A probabilistically motivated and more general definition of this transition probability was introduced in a preceding paper and is extended here to a general type of quantum logics: the orthomodular partially ordered sets. A very general version of the quantum no-cloning theorem, creating promising new opportunities for quantum cryptography, is presented and an interesting relationship between the transition probability and Jordan algebras is highlighted.

Keywords: quantum transition probability; no-cloning theorem; quantum logics; Jordan algebras; quantum cryptography; quantum information

1 Introduction

Max Borns statistical interpretation [11] made probability play a major role in quantum theory. He postulated that the modulus-square of the inner product of two normalized Hilbert space elements should be interpreted as a transition probability between the pure states represented by the two Hilbert space elements. The mathematical formalism does not provide any reason for this interpretation, but the experimental evidence forces us to accept it.

Since then various approaches have emerged to find a better motivated axiomatic access to Born's postulate, using convex sets [3, 36, 37], transition probability spaces [33, 47], continuous geometries [51] or quantum logics [25, 34]. Among these, only the convex set approach does not postulate the existence of the transition probability by an extra axiom. Here we use the quantum logics, which were originally pioneered by Birkhoff and von Neumann [10].

Earlier quantum logical approaches by the author were based on projective quantum measurement (Lüders - von Neumann quantum measurement process) or on an extension of the classical conditional probability [39, 40]. A different approach was undertaken in a previous paper [43]. Its intention was to point to the algebraic origin of the quantum probabilities and their inherent difference from the classical probabilities. Therefore, only the Hilbert space quantum logics were considered, and the transition probability was defined in a new way. The new definition includes physically meaningful and experimentally verifiable novel cases that are covered neither by the usual quantum mechanical transition probability nor by the approaches of other authors [3, 25, 33, 34, 36, 37, 47, 51].

The extension of this definition to general quantum logics (*orthomodular partially ordered sets* [24, 45, 50]) is quite straight forward and is studied in the present paper. This transition probability does neither require any extra axiom nor the extended conditional probability, but is a characteristic of the algebraic structure of the quantum logic; in some cases it exists and in others it does not. A very general version of the quantum *no-cloning theorem* [20, 54] is presented and proven in this setting, which creates promising new opportunities for the quantum key distribution protocols [9, 22] - even in the common quantum mechanical Hilbert space setting. An interesting class of quantum logics are the projection lattices of the *JBW-algebras* (the Jordan analogue of the von Neumann algebras or W*-algebras [3, 29, 49]); it is shown how the transition probability is linked to the algebraic structure in this case and that the Jordan algebras provide the appropriate framework for a structural analysis of the transition probability.

This paper requires some knowledge of two mathematical subjects: the orthomodular partially ordered sets and the JBW-algebras. Some basics needed in the paper are briefly sketched in sections 2 and 6, respectively. Beyond that, it is referred to the monographs [45, 50] for the first subject and to the monographs [3, 29] for the second one.

The paper is organized as follows. In section 2, some basics of the orthomodular partially ordered sets (the general mathematical model for the quantum logic), the states on them and their morphisms are recapitulated for later use. In section 3, the transition probability is introduced and its basic properties are presented. The relation between this transition probability and the quantum logical notion of *compatibility* is studied in section 4. Its main result is a certain product rule for the transition probability and becomes the major tool for the derivation of the no-cloning theorem in section 5. In section 6, some basics of the JBW-algebras and their associated quantum logics are recapitulated for later use. How the transition probability in these quantum logics is linked to the algebraic structure is studied in section 7. The relation between the transition probability defined here and the usual transition probability in the common quantum mechanical Hilbert space formalism can then easily be disclosed. Moreover, it becomes apparent that a certain property known for the von Neumann algebras as *isoclinicity* [18, 28, 35] or *equiangularity* [23] is very closely related to the transition probability. This property is then studied in the Jordan algebraic framework in section 8, the results of which are used in section 9 for the structural analysis of the transition probability in the Jordan algebraic quantum logics. Theorem 9.3 and, prior to this, the product rule (section 4), the no-cloning theorem (section 5) and the link between the transition probability and the Jordan algebraic structure (section 7) represent the main results of the paper.

2 Quantum logics and states

The Boolean algebra is a mathematical structure playing an important role in many scientific and technical fields such as formal logic, classical probability theory, circuitry, computer science. Only quantum theory challenges the general applicability of this structure, since the dichotomic observables (those with spectrum $\{0, 1\}$) do not form a Boolean algebra, but an orthomodular lattice where the distributivity law fails [8, 10, 31, 50]. The dichotomic observables are identical with the self-adjoint projections and can also be identified with the closed linear subspaces of the Hilbert space. The new mathematical structure which they form is generally called a *quantum logic*.

The lattice structure used to play an important role in the early quantum logical approaches [10, 44, 50]. However, there is no physical motivation for the existence of the lattice operations for elements of the quantum logic that are not compatible, and later a quantum logic was often assumed to be an orthomodular partially ordered set only [24, 45, 46]. This more general structure will also be sufficient here.

A *quantum logic* shall be an *orthomodular partially ordered set* L with order relation $\leq$, smallest element 0, largest element $\mathbb{I}$ and an orthocomplementation $'$. This means that the following conditions are satisfied by the $p, q \in L$:

(a) $q \leq p$ *implies* $p' \leq q'$.

(b) $(p')' = p$.

(c) $p \leq q'$ *implies that* $p \vee q$, *the supremum of* p *and* q, *exists.*

(d) Orthomodular law: $q \leq p$ *implies* $p = q \vee (p \wedge q')$.

Here, $p \wedge q$ denotes the infimum of p and q, which exists iff $p' \vee q'$ exists. Note that $p \vee p' = \mathbb{I}$ holds for $p \in L$; this follows from (d).

In a lattice, $p \wedge q$ and $p \vee q$ would exist for any elements p and q. An element $e \in L$ with $e \neq 0$ is called *minimal* if there is no $q \in L$ with $q \leq e$ and $0 \neq q \neq e$. The minimal elements are also called *atoms* in the common literature. Two elements p and q in L are *orthogonal*, if $p \leq q'$ or, equivalently, $q \leq p'$; in this case, $p \vee q$ exists and shall be noted by $p + q$ in the following. Moreover, if p and q are orthogonal, $p \wedge q$ exists and $p \wedge q = 0$. In a Boolean algebra, the identity $p \wedge q = 0$ is the same as the orthogonality of p and q. However, this does not hold any more in a quantum logic.

The interpretation of this mathematical terminology is as follows: orthogonal elements represent mutually exclusive potential measurement outcomes, p' represents the negation of p, $p + q := p \vee q$ is the disjunction (logical *or*-operation) of p and q which generally exists only if these two elements are orthogonal, and $\leq$ is the logical implication relation ($q \leq p$ with $q, p \in L$ means that the measurement outcome q implies the outcome p).

A *state* μ shall allocate probabilities to the elements of the quantum logic. Therefore it becomes a map from L to the unit interval $[0, 1] \subseteq \mathbb{R}$ with $\mu(\mathbb{I}) = 1$

and $\mu(p+q) = \mu(p) + \mu(q)$ for any two orthogonal elements p and q in L. A set S of states on L is called *strong* if, for any $p, q \in L$,

$$\{\mu \in S \mid \mu(q) = 1\} \subseteq \{\mu \in S \mid \mu(p) = 1\} \quad \Rightarrow \quad q \leq p.$$

Note that a strong set S contains a state $\mu \in S$ with $\mu(q) = 1$ for each $q \in L$ with $q \neq 0$; if $\{\mu \in S \mid \mu(q) = 1\} = \emptyset$, we would get $q \leq p$ for all $p \in L$ and thus $q = 0$.

A *morphism* from a quantum logic L to another quantum logic K is a map $\pi : L \to K$ satisfying the following two conditions:

(a) $\pi(\mathbb{I}) = \mathbb{I}$

(b) If $p \in L$ and $q \in L$ are orthogonal, then $\pi(p)$ and $\pi(q)$ are orthogonal in K and $\pi(p+q) = \pi(p) + \pi(q)$.

With a state μ on K, a state $\mu\pi : L \ni p \to \mu(\pi p)$ can then be defined on L. However, if S is a certain set of states on L, S' a certain set of states on K and $\mu \in S'$, the state $\mu\pi$ need not lie in S. Therefore, an *S-S'-morphism* becomes a morphism satisfying the following additional condition:

(c) If $\mu \in S'$, then $\mu\pi \in S$.

In most interesting cases, S will be the set of *all* states on L and the additional condition (c) will be needless, but a few cases will require the use of a smaller set of states, as we shall see later.

3 Transition probability in quantum logics

In a preceding paper [43], the *transition probability* was defined in a new way for the Hilbert space quantum logics, but its following extension to the general situation is quite straightforward.

Definition 3.1 *Let L be a quantum logic and S a strong set of states on L. If a pair $p, q \in L$ with $p \neq 0$ and some $s \in [0,1]$ satisfy the identity*

$$\mu(q) = s \text{ for all } \mu \in S \text{ with } \mu(p) = 1,$$

then s is called the transition probability from p to q *and is denoted by $\mathbb{P}(q|p)$.*

The identity $\mathbb{P}(q|p) = s$ then becomes equivalent to the set inclusion

$$\{\mu \in S \mid \mu(p) = 1\} \subseteq \{\mu \in S \mid \mu(q) = s\}$$

and means that, whenever the probability of p is 1, the probability of q is fixed and its numerical value is s; particularly in the situation after a quantum measurement that has provided the outcome p, the probability of q becomes s, independently of any initial state before the measurement.

Two elements p and q in L are orthogonal iff $\mathbb{P}(q|p) = 0$, and $p \le q$ holds iff $\mathbb{P}(q|p) = 1$. The second part here holds since S is a strong set of states, and the first part follows by considering q'.

The following lemma provides a collection of some further basic properties of the transition probability.

Lemma 3.2 *Let L be a quantum logic and S a strong set of states on L.*

(i) *If $\mathbb{P}(q|p)$ exists and $0 \neq p_o \le p$ with $p_o, p, q \in L$, then $\mathbb{P}(q|p_o)$ exists and $\mathbb{P}(q|p_o) = \mathbb{P}(q|p)$.*

(ii) *If $\mathbb{P}(q|p)$ exists for $p, q \in L$ and $\mathbb{P}(q|p) \neq 1$, then $p \wedge q = 0$.*

(iii) *If $0 \neq q < p$ with $p, q \in L$, then $\mathbb{P}(q|p)$ does not exist.*

(iv) *If $\mathbb{P}(q|p)$ exists for all $q \in L$ with $0 \neq p \in L$ (in this case, p defines a state $L \ni q \to \mathbb{P}(q|p)$), then p is an atom (a minimal element in L).*

(v) *Let K be a second quantum logic, let S' be a strong set of states on K, and let $\pi : L \to K$ be a S-S'-morphism. If $\mathbb{P}(q|p)$ exists for $0 \neq p$ with $p, q \in L$, then $\mathbb{P}(\pi q|\pi p)$ exists and $\mathbb{P}(\pi q|\pi p) = \mathbb{P}(q|p)$.*

Proof. (i) Suppose $\mathbb{P}(q|p)$ exists and $0 \neq p_o \le p$ with $p_o, p, q \in L$. Then

$$\{\mu \in S|\mu(p_o) = 1\} \subseteq \{\mu \in S|\mu(p) = 1\} \subseteq \{\mu \in S|\mu(q) = \mathbb{P}(q|p)\}.$$

This means $\mathbb{P}(q|p_o) = \mathbb{P}(q|p)$.

(ii) Suppose $p_o \le p$, $p_o \le q$ and $p_o \neq 0$ with $p_o, p, q \in L$. If $\mathbb{P}(q|p)$ exists, we get by (i): $\mathbb{P}(q|p) = \mathbb{P}(q|p_o) = 1$.

(iii) Suppose $0 \neq q < p$ with $p, q \in L$. The orthomodularity implies that $p \wedge q' \neq 0$ and, since S is strong, there are states $\mu_1, \mu_2 \in S$ with $\mu_1(q) = 1$ and $\mu_2(p \wedge q') = 1$. Then $\mu_1(p) = \mu_2(p) = 1$, but $\mu_2(q) = 0 \neq 1 = \mu_1(q)$.

(iv) If there were an element $q_o \in L$ with $0 < q_o < p \in L$, then $\mathbb{P}(q_o|p)$ does not exist by (iii).

(v) Suppose that $\mathbb{P}(q|p)$ exists for $p, q \in L$ with $p \neq 0$ and that $\mu\pi p = 1$ holds for $\mu \in S'$. Then $\mu\pi \in S$ and thus $\mu\pi q = \mathbb{P}(q|p)$. Therefore $\mathbb{P}(\pi q|\pi p)$ exists and $\mathbb{P}(\pi q|\pi p) = \mathbb{P}(q|p)$. □

When the transition probability $\mathbb{P}(q|p)$ exists for two elements $p \neq 0$ and q in the quantum logic L, part (i) of Lemma 3.2 means that, after a quantum measurement with the outcome p, a further measurement of any $p_o \in L$ with $0 \neq p_o \le p$ cannot alter the probability of q.

We shall see later in Corollary 7.2 that, in many important cases, the reverse implication of (iv) is also true, and a unique state can then be allocated to each atom such that the atom carries the probability 1.

Part (v) of Lemma 3.2 shows that the transition probability is invariant under morphisms.

The relation between the transition probability defined here and the quantum mechanical transition probability $|\langle\xi|\psi\rangle|^2$ for two normalized Hilbert space elements (or wave functions or pure states) ξ and ψ is not obvious and has been elaborated in Ref. [43]. We shall come back to this in section 7.

4 Compatibility

Two elements p and q in a quantum logic L are said to be *compatible*, if there exist three pairwise orthogonal elements $a_1, a_2, a_3 \in L$ such that $p = a_1 + a_2$ and $q = a_2 + a_3$; it is well-known that the elements a_1, a_2, a_3 are unique if they exist, namely $a_1 = p \wedge q'$, $a_2 = p \wedge q$ and $a_3 = q \wedge p'$ [12, 45]. Here, $\wedge$ can be interpreted as the logical *and*-operation. However, this interpretation is not reasonable for the infimum of two non-compatible elements when it exists. Note that two elements p and q are compatible in the cases when they are orthogonal or when $p \leq q$ or $q \leq p$.

It is well-known and easy to see that two elements p and q in the usual Hilbert space quantum logic of common quantum mechanics are compatible iff they commute; in this case $p \wedge q = pq = qp$.

Lemma 4.1 *Let L be a quantum logic and S a strong set of states on L. If $0 \neq p \in L$ and $q \in L$ are compatible and if $\mathbb{P}(q|p)$ exists, then either p and q are orthogonal and $\mathbb{P}(q|p) = 0$, or $p \leq q$ and $\mathbb{P}(q|p) = 1$.*

Proof. Suppose that $\mathbb{P}(q|p)$ exists for $p, q \in L$, $p \neq 0$, and let a_1, a_2, a_3 be three pairwise orthogonal elements in L with $p = a_1 + a_2$ and $q = a_2 + a_3$. If $a_2 = 0$, p and q are orthogonal and $\mathbb{P}(q|p) = 0$. If $a_2 \neq 0$, there is a state $\mu \in S$ with $\mu(a_2) = 1$ (since S is a strong set of states); then $\mu(p) = 1$ and $\mathbb{P}(q|p) = \mu(q) = 1$. □

Lemma 4.1 means that a non-trivial transition probability requires incompatibility and does not exist in the classical logics which are the Boolean algebras and where all elements are compatible with each other.

Two elements p and q in a quantum logic L are compatible iff there exists a Boolean subalgebra of L containing p and q. However, if three or more elements are pairwise compatible, they need not lie in a joint Boolean subalgebra of L in general. They do so if L is a lattice [12, 50]. In the general case, we have only the following lemma stating that three elements lie in a joint Boolean subalgebra of L, if they are pairwise compatible and two among them are orthogonal.

Lemma 4.2 *If p, q_1 and q_2 are three elements in a quantum logic L such that q_1 and q_2 are orthogonal, p and q_1 are compatible and p and q_2 are compatible, then p and $q_1 + q_2$ are compatible and we have*

$$p \wedge (q_1 + q_2) = p \wedge q_1 + p \wedge q_2.$$

Proof. Suppose $p, q_1, q_2 \in L$ such that q_1 and q_2 are orthogonal, p and q_1 are compatible and p and q_2 are compatible. The five elements $p \wedge q_1$, $p \wedge q_2$, $p' \wedge q_1$, $p' \wedge q_2$ and $p \wedge (q_1 + q_2)'$ are then pairwise orthogonal and therefore they lie in a Boolean subalgebra of L [12, 13, 21, 45]. It includes $p = p \wedge q_1 + p \wedge q_1' = p \wedge q_2 + p \wedge q_2'$, $q_1 = p \wedge q_1 + p' \wedge q_1$ and $q_2 = p \wedge q_2 + p' \wedge q_2$. This implies the the compatibility of p and $q_1 + q_2$ and the above identity. □

Now we shall see that the compatibility implies a certain product rule for the transition probability.

Proposition 4.3 *Let K_1, K_2, L be three quantum logics and let S_1, S_2 and S be strong sets of states for them. Moreover, suppose that $K_1 \ni p_1 \to \bar{p_1} \in L$ is an S_1-S-morphism and that $K_2 \ni p_2 \to \tilde{p_2} \in L$ is an S_2-S-morphism such that $\bar{p_1}$ and $\tilde{p_2}$ are a compatible pair in L for each $p_1 \in K_1$ and each $p_2 \in K_2$.*

If $\mathbb{P}(q_1|p_1)$ and $\mathbb{P}(q_2|p_2)$ exist for some $q_1, p_1 \in K_1$, $q_2, p_2 \in K_2$ and if $\bar{p_1} \wedge \tilde{p_2} \neq 0$, then $\mathbb{P}(\bar{q_1} \wedge \tilde{q_2}|\bar{p_1} \wedge \tilde{p_2})$ exists and

$$\mathbb{P}(\bar{q_1} \wedge \tilde{q_2}|\bar{p_1} \wedge \tilde{p_2}) = \mathbb{P}(q_1|p_1)\mathbb{P}(q_2|p_2).$$

Proof. Suppose that $\mathbb{P}(q_1|p_1)$ and $\mathbb{P}(q_2|p_2)$ exist with $q_1, p_1 \in K_1$, $q_2, p_2 \in K_2$ and $\bar{p_1} \wedge \tilde{p_2} \neq 0$. Let μ be a state on L with $\mu(\bar{p_1} \wedge \tilde{p_2}) = 1$. We shall show that $\mu(\bar{q_1} \wedge \tilde{q_2}) = \mathbb{P}(q_1|p_1)\mathbb{P}(q_2|p_2)$.

Note that $\mu(\bar{p_1}) = 1 = \mu(\tilde{p_2})$ and define states μ_0 and μ_1 on K_1 by $\mu_0(q) := \mu(\bar{q})$ and $\mu_1(q) := \mu(\bar{q} \wedge \tilde{p_2})$ for $q \in K_1$; Lemma 4.2 ensures that μ_1 is a state. Then $\mu_0(p_1) = 1 = \mu_1(p_1)$ and thus $\mu(\bar{q_1}) = \mu_0(q_1) = \mathbb{P}(q_1|p_1) = \mu_1(q_1)$.

If $\mathbb{P}(q_1|p_1) \neq 0$, define a state μ_2 on K_2 by

$$\mu_2(q) := \frac{1}{\mathbb{P}(q_1|p_1)}\mu(\bar{q_1} \wedge \tilde{q})$$

for $q \in K_2$; again Lemma 4.2 ensures that μ_2 is a state. Then

$$\mu_2(p_2) = \frac{1}{\mathbb{P}(q_1|p_1)}\mu(\bar{q_1} \wedge \tilde{p_2}) = \frac{1}{\mathbb{P}(q_1|p_1)}\mu_1(q_1) = 1$$

and thus $\mu_2(q_2) = \mathbb{P}(q_2|p_2)$. Therefore

$$\mathbb{P}(q_2|p_2) = \frac{1}{\mathbb{P}(q_1|p_1)}\mu(\bar{q_1} \wedge \tilde{q_2}),$$

which is the desired result.

In the case $\mathbb{P}(q_1|p_1) = 0$, q_1 and p_1 are orthogonal. Therefore $\bar{q_1}$ and $\bar{p_1}$ are orthogonal, and then $\bar{q_1} \wedge \tilde{q_2}$ and $\bar{p_1} \wedge \tilde{p_2}$ are orthogonal. Thus $\mathbb{P}(\bar{q_1} \wedge \tilde{q_2}|\bar{p_1} \wedge \tilde{p_2}) = 0 = \mathbb{P}(q_1|p_1)\mathbb{P}(q_2|p_2)$. □

5 The no-cloning theorem

Using Lemma 3.2 (v) and Proposition 4.3, we shall now prove a very general version of the well-known quantum no-cloning theorem. This version does neither require any type of state nor any tensor product. Instead of the tensor product, just compatibility is sufficient. Instead of states, elements of a quantum logic are considered; they represent properties of a quantum system and potential measurement outcomes.

The following situation is assumed: A measurement on a system was performed. However, it is unknown which observable was tested. Available is only the information that the measurement outcome was one among the system properties p_1,...,p_n, but it is not known which one. It is assumed that the transition probability exists for each pair chosen from p_1,...,p_n. The following theorem then states that the unknown property cannot be cloned if the properties p_1,...,p_n are not pairwise orthogonal.

In the usual quantum mechanical setting, the cloning is performed by a unitary transformation on a Hilbert space tensor product or by the corresponding inner automorphism on the tensor product of the operator algebras; in this paper, it shall be performed by a morphism from the quantum logic L to itself, where L represents a large system containing the subsystem where the copy is taken from and the subsystem where the copy is to be transferred to.

Theorem 5.1 *Let K and L be quantum logics and let S_K and S_L be strong sets of states for them. Moreover, suppose that there are two S_K-S_L-morphisms from K to L: $K \ni p \to \bar{p} \in L$ and $K \ni p \to \tilde{p} \in L$ such that $\bar{p}$ and $\tilde{q}$ are a compatible pair in L for each $p \in K$ and each $q \in K$ and $\bar{p} \wedge \tilde{q} \neq 0$ for for each $p \in K$ and each $q \in K$ with $p \neq 0 \neq q$.*[1]

A cloning transformation for $0 \neq p_1, ..., p_n \in K$ is an S_L-S_L-morphism $T : L \to L$ with

$$T\left(\bar{p_k} \wedge \tilde{q_o}\right) = \bar{p_k} \wedge \tilde{p_k}$$

for $k = 1, ..., n$, where $0 \neq q_o \in K$ represents the known initial property of the second system which shall be replaced by the copy of the unknown property (one among $p_1, ..., p_n$) of the first system.

Suppose that the transition probabilities $\mathbb{P}\left(p_j|p_k\right)$ exist on K for $j, k = 1, ..., n$. If a cloning transformation T exists for these $p_1, ..., p_n \in K$, any two elements chosen from $p_1, ..., p_n$ must be either orthogonal or identical.

Proof. Note that both Lemma 3.2 (v) and Proposition 4.3 are used repeatedly in the following equation; Lemma 3.2 (v) is applied with the morphisms $p \to \bar{p}$, $p \to \tilde{p}$ and the cloning transformation T.

$$\begin{aligned}
\left(\mathbb{P}\left(p_j|p_k\right)\right)^2 &= \mathbb{P}\left(\bar{p_j}|\bar{p_k}\right)\mathbb{P}\left(\tilde{p_j}|\tilde{p_k}\right) = \mathbb{P}\left(\bar{p_j} \wedge \tilde{p_j}|\bar{p_k} \wedge \tilde{p_k}\right) \\
&= \mathbb{P}\left(T\left(\bar{p_j} \wedge \tilde{q_o}\right)|T\left(\bar{p_k} \wedge \tilde{q_o}\right)\right) = \mathbb{P}\left(\bar{p_j} \wedge \tilde{q_o}|\bar{p_k} \wedge \tilde{q_o}\right) \\
&= \mathbb{P}\left(\bar{p_j}|\bar{p_k}\right)\mathbb{P}\left(\tilde{q_o}|\tilde{q_o}\right) = \mathbb{P}\left(\bar{p_j}|\bar{p_k}\right) \\
&= \mathbb{P}\left(p_j|p_k\right)
\end{aligned}$$

and thus $= \mathbb{P}\left(p_j|p_k\right) \in \{0, 1\}$ for $j, k = 1, ..., n$. If $= \mathbb{P}\left(p_j|p_k\right) = 1$, then $p_k \leq p_j$ and, since $\mathbb{P}\left(p_k|p_j\right)$ exists, we get $p_j = p_k$ by Lemma 3.2 (iii). If $= \mathbb{P}\left(p_j|p_k\right) = 0$, p_j and p_k are orthogonal. □

[1] The last assumption ($\bar{p} \wedge \tilde{q} \neq 0$ for $p, q \in K$ with $p \neq 0 \neq q$) means that the two copies of K are *logically independent* in L. Logical independence is usually defined for von Neumann subalgebras [27, 48] and becomes a necessary and sufficient condition for the *C*-independence* of two commuting von Neumann subalgebras [48]; C*-independence was introduced by Haag and Kastler in the framework of algebraic quantum field theory [26].

The rather general and abstract version of the no-cloning theorem presented here helps to identify its deeper origin which is hidden in the common quantum mechanical Hilbert space formalism like the needle in the haystack. We see that only the existence of the transition probabilities and two properties of them are needed in the proof: their invariance under morphisms (Lemma 3.2 (v)) and the product rule (Proposition 4.3). The requirement that the transition probabilities exist does not occur in the original quantum no-cloning theorem [20, 54], but it is automatically fulfilled since only pure states or atoms are considered and since the transition probabilities always exist for the pure states or atoms in the usual Hilbert space setting of quantum mechanics (see Ref. [43] or Corollary 7.2 and the subsequent remarks).

Theorem 5.1 is substantially more general than the original no-cloning theorem. Even in the usual Hilbert space setting, Theorem 5.1 includes physically meaningful interesting new cases where $p_1, ..., p_n$ are not atomic and states cannot be allocated. This becomes possible by considering the cloning of system properties instead of states and by using the transition probability $\mathbb{P}(\ |\ )$ defined in section 3.

The original no-cloning theorem has been extended into different other directions: to mixed states [7], to C*-algebras [19], to finite-dimensional generic probabilistic models [5, 6], and to universal cloning [38]. Possible is only the approximate or imperfect cloning [14, 17, 32]. However, none of these extensions includes the above result Theorem 5.1.

The original no-cloning theorem is essential for the quantum key distribution protocols [9, 22]. How these protocols can be extended to a much more general setting, using the above no-cloning Theorem 5.1, is shown in Ref. [41]. The existence of the extended conditional probabilities is assumed in Ref. [41], but is not relevant for the key distribution protocols. It is needed there to derive the transition probabilities, which have been derived here in a different way that does not require the conditional probabilities. Particularly the non-atomic cases of the no-cloning Theorem 5.1 go beyond the usually considered situation and create promising new opportunities for the quantum key distribution protocols - even in common Hilbert space quantum mechanics.

6 The quantum logic of a Jordan algebra

The *formally real Jordan algebras* were introduced and classified by Jordan, von Neumann and Wigner [30]. Much later, this theory was extended to include infinite dimensional algebras; these are the so-called *JB-algebras* and *JBW-algebras* [3, 29], which represent the Jordan analogue of the C*-algbras and the W*-algebras (von Neuman algebras [49]). In this section, some basics of the theory of the JB-/JBW-algebras shall be recapitulated for later use.

A *real Jordan algebra* is a $\mathbb{R}$-linear space A equipped with an abelian (but not associative) product $\circ$ satisfying

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$$

for any $x, y \in A$. For any three elements $x, y, z \in A$, their *triple product* is defined as follows:

$$\{x, y, z\} = x \circ (y \circ z) - y \circ (z \circ x) + z \circ (x \circ y).$$

A *JB-algebra* is a real Jordan algebra A that is a Banach space with a norm satisfying $\|x \circ y\| \leq \|x\| \|y\|$, $\left\|x^2\right\| = \|x\|^2$ and $\left\|x^2\right\| \leq \left\|x^2 + y^2\right\|$ for any $x, y \in A$. The subset $A_+ := \left\{x^2 \mid x \in A\right\}$ of a JB algebra A is a closed convex cone, and a partial ordering is defined via $x \leq y :\Leftrightarrow y - x \in A_+$. Then $\{y, x, y\} \geq 0$ for any $x \in A_+$ and any $y \in A$.

A *JBW-algebra* is a JB-algebra that is the dual of a Banach space. Any JBW-algebra has a unit denoted by $\mathbb{I}$. In the finite-dimensional case, the JBW-algebras are identical with the JB-algebras and with the formally real Jordan algebras.

Finite-dimensional formally real Jordan algebras are the matrix algebras $H_n(\mathbb{R})$, $H_n(\mathbb{C})$, $H_n(\mathbb{H})$ $(n = 2, 3, 4, ...)$ and $H_3(\mathbb{O})$. They consist of the self-adjoint $n \times n$-matrices over the real numbers($\mathbb{R}$), the complex numbers ($\mathbb{C}$), the *quaternions* ($\mathbb{H}$) and the *octonions* ($\mathbb{O}$) with the usual Jordan product $x \circ y := (xy + yx)/2$. The quaternions and octonions are also called *Hamilton numbers* and *Cayley numbers*, respectively. For x, y in $H_n(\mathbb{R})$, $H_n(\mathbb{C})$ or $H_n(\mathbb{H})$, the Jordan triple product $\{x, y, x\}$ coincides with the simple matrix product xyx, since $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are associative. However, this does not hold for x, y in $H_3(\mathbb{O})$, because the octonions are not associative. Furthermore, there are the *spin factors* or *type I_2-factors*; examples for them are $H_2(\mathbb{R})$, $H_2(\mathbb{C})$, $H_2(\mathbb{H})$ and $H_2(\mathbb{O})$, but there are many more (including infinite-dimensional ones). Every finite-dimensional formally real Jordan algebra can be decomposed into a direct sum of spin factors and matrix algebras of the above types [29, 30].

With the order relation $\leq$ defined by the cone A_+, the idempotent elements (projections) in a JBW-algebra A form an orthomodular lattice L_A (projection lattice) and thus

$$L_A := \left\{p \in A \mid p^2 = p\right\}$$

becomes a quantum logic; its orthocomplementation is $p' := \mathbb{I} - p$ for $p \in L_A$. For any $p, q \in L_A$, we have: $\{p, q, p\} = 2p \circ (p \circ q) - p \circ q$; $p \leq q$ iff $p \circ q = p$ iff $\{p, q, p\} = p$ iff $\{q, p, q\} = p$; p and q are orthogonal iff $p \circ q = 0$ iff $\{p, q, p\} = 0$ iff $\{q, p, q\} = 0$ [3, 29]. Moreover, two elements $p, q \in L_A$ are compatible iff they operator-commute (this means: $p \circ (q \circ x) = q \circ (p \circ x)$ for all $x \in A$ [3]).

A linear functional $\mu : A \to \mathbb{R}$ is called positive, if $\mu(A_+) \subseteq [0, \infty[$. For each $0 \neq x \in A$ there is such a positive linear functional μ with $\mu(x) \neq 0$. The restrictions of the positive linear functionals to L_A provide a strong (see footnote 2) state space S_A for the quantum logic L_A. It is this natural set of states that will always be used in the remaining part of this paper.

If none of the direct summands in the decomposition of the JBW-algebra A is a spin factor, S_A includes *all* states on L_A; this is the Gleason theorem for JBW-algebras [16]. There is a spin factor for each cardinality except 0, 1 and 2. The most simple one is the $H_2(\mathbb{R})$; it has the real dimension three. The next one is $H_2(\mathbb{C})$ with the real dimension four; it is the self-adoint part of $M_2(\mathbb{C})$

(the 2×2-matrices over the complex numbers $\mathbb{C}$). The matrix algebra $M_2(\mathbb{C})$ represents the two-dimensional version of usual quantum mechanics with the complex numbers and is the model for the spin 1/2 or for the single qubit in quantum information theory. Only to include these cases, the effort with the distinction between the positive linear functionals and the set of *all* states on the quantum logic L_A is made here. The other spin factors are more exotic and less interesting. The distinction becomes needless when the algebras of $n\times n$-matrices with $n \neq 2$ (or Hilbert spaces with dimension $n \neq 2$) are considered.

In the following sections, a little knowledge of the *Shirshov-Cohn theorem* and some further results from the theory of the JB- and JBW-algebras will be required. We shall go into this at those places where it will be needed. For more information, it is referred to Refs. [3, 29].

7 Transition probability in Jordan algebras I

The following proposition shows how the transition probability in the quantum logic L_A of a JBW-algebra A is linked to the algebraic structure of A.

Proposition 7.1 *Suppose that $p \neq 0$ and q are elements in the quantum logic L_A of any JBW-algebra A. The following statements are equivalent:*

(a) *The transition probability from p to q exists and $\mathbb{P}(q|p) = s$.*

(b) *p and q satisfy the algebraic identity $\{p, q, p\} = sp$.*

Proof. For any state $\mu \in S_A$ and $x, y \in A$, the following Cauchy-Schwarz inequality holds (see e.g. 3.6.2 in [29]):

$$|\mu(x \circ y)| \leq \left(\mu(x^2)\right)^{1/2} \left(\mu(y^2)\right)^{1/2}.$$

This implies that, for $y \in L_A$ with $\mu(y) = 0$, $\mu(x \circ y) = \mu(y \circ x) = 0$ for all $x \in A$. Note that $y^2 = y$ holds for $y \in L_A$.

(b) $\Rightarrow$ (a): Suppose $\{p, q, p\} = sp$. Note that $q = \{p, q, p\} + 2p' \circ (p \circ q) + p' \circ q$. If $\mu \in S_A$ and $\mu(p) = 1$, then $\mu(p') = 0$ and $\mu(q) = \mu(\{p, q, p\}) = s\mu(p) = s$. Therefore $\mathbb{P}(q|p) = s$.

(a) $\Rightarrow$ (b): Now suppose $\mathbb{P}(q|p) = s$ and let μ be any state in S_A. If $\mu(p) = 0$, then $\mu(\{p, q, p\}) = 2\mu(p \circ (p \circ q)) - \mu(p \circ q) = 0 = \mu(sp)$. If $\mu(p) > 0$, define a state $\mu_p \in S_A$ by

$$\mu_p(x) = \frac{1}{\mu(p)} \mu(\{p, x, p\})$$

for $x \in A$. Then $\mu_p(p) = 1$ and therefore

$$s = \mu_p(q) = \frac{1}{\mu(p)} \mu(\{p, q, p\}).$$

We have $\mu(\{p, q, p\}) = s\mu(p)$ for all $\mu \in S_A$ and thus $\{p, q, p\} = sp$.[2] □

[2]Note that, with $s = 1$, this also shows that S_A is strong, since $\{p, q, p\} = p$ iff $p \leq q$.

We now come back to Lemma 3.2 (iv) and show that its reverse implication is true in the JBW-algebras.

Corollary 7.2 *Suppose that p is an atom (minimal element) in the quantum logic L_A of any JBW-algebra A. Then $\mathbb{P}(q|p)$ exists for all $q \in L_A$.*

Proof. This follows from Proposition 7.1. Note that $\{\{p,x,p\}\,|x \in A\} = \mathbb{R}p$ holds for the atoms p in the projection lattice of a JBW-algebra (see e.g. 3.29 in [3]). □

The self-adjoint part of a von Neumann algebra on a Hilbert space H with inner product $\langle\ |\ \rangle$ becomes a JBW-algebra A with $x \circ y := (xy + yx)/2$ for $x, y \in A$. Here the Jordan triple product $\{y, x, y\}$ coincides with the operator product yxy. If ξ and ψ are two normalized elements in H and if the projections p and q on the one-dimensional linear subspaces that ξ and ψ each generate belong to A, we get

$$pqp = |\langle\xi|\psi\rangle|^2\, p \text{ and } qpq = |\langle\xi|\psi\rangle|^2\, q$$

and therefore

$$\mathbb{P}(q|p) = \mathbb{P}(p|q) = |\langle\xi|\psi\rangle|^2 .$$

This discloses the relation between Definition 3.1 and the usual quantum mechanical transition probability. If p remains as above, but q is any projection in A, we get

$$pqp = \langle\xi|q\xi\rangle\, p$$

and

$$\mathbb{P}(q|p) = \langle\xi|q\xi\rangle\, .$$

In this way, p then defines the pure state $q \to \mathbb{P}(q|p) = \langle\xi|q\xi\rangle$. However, the existence of $\mathbb{P}(q|p)$ for some, but not for all projections q in A does not require that p is a projection on a one-dimensional subspace or an atom. An explicit example with real 4×4-matrices was already presented in Ref. [43]. Some more general examples shall now be introduced. All these examples demonstrate how the transition probability of Definition 3.1 differs from the usual quantum mechanical version and the approaches in Refs. [3, 25, 33, 34, 36, 37, 47, 51].

Consider a $m \times n$-matrix u ($m, n \in \mathbb{N}$) with entries from $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $uu^* = \mathbb{I}_m$. Here u^* denotes the transpose of u in the real case and the conjugate transpose of u in the other cases; $\mathbb{I}_m$ is the identity matrix of size $m \times m$. Then u^*u is an $n \times n$-matrix with $(u^*u)^2 = u^*(uu^*)u = u^*u$; this requires $m \leq n$. Now choose any real number $0 \leq s \leq 1$ and define the following two matrices

$$p := \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q := \begin{pmatrix} s\mathbb{I}_m & s^{1/2}(1-s)^{1/2}u \\ s^{1/2}(1-s)^{1/2}u^* & (1-s)u^*u \end{pmatrix}$$

in $H_{m+n}(K)$, $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Some simple matrix calculations yield $p^2 = p$, $q^2 = q$, $pqp = sp$ and $qpq = sq$. By Proposition 7.1 $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist and we have $\mathbb{P}(q|p) = \mathbb{P}(p|q) = s$. Selecting $0 \neq s \neq 1$ and $m \geq 2$ provides many examples of non-trivial transition probabilities $\mathbb{P}(q|p)$ where p is not an atom. Using any unitary element $w \in H_{m+n}(K)$ (this means $w^*w = ww^* = \mathbb{I}_{m+n}$), further examples where p gets a different form can be constructed by replacing p and q with w^*pw and w^*qw.

8 Isoclinic pairs in Jordan algebras

In the above example, the projections p and q satisfy the two identities $pqp = sp$ and $qpq = sq$ with $0 \leq s \leq 1$. In a von Neumann algebra M, such a pair is called *isoclinic* [18, 28, 35]. This algebraic property is equivalent to a geometric property called *equiangularity* [23]. The subalgebra of M that an isoclinic pair generates is an isomorphic copy of the matrix algebra $M_2(\mathbb{C})$ (the 2×2-matrices over the complex numbers $\mathbb{C}$). By Proposition 7.1, the pair p and q is isoclinic iff the transition probabilities $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist and coincide. We are now going to study this situation in the Jordan algebraic framework.

With any elements $a_1, ..., a_n$ in a Jordan algebra A, $A_{\{a_1,...,a_n\}}$ shall denote the Jordan subalgebra generated by $a_1, ..., a_n$. Note that this subalgebra need not include the unit element $\mathbb{I}$ of A.

By the Shirshov-Cohn theorem [29], any Jordan algebra generated by the unit element $\mathbb{I}$ and two further elements a_1 and a_2 is *special*. This means that we can assume that $A_{\{\mathbb{I},a_1,a_2\}}$ is a part of an associative algebra and that the product in $A_{\{\mathbb{I},a_1,a_2\}}$ can be derived from this associative algebra in the following way: $x \circ y = \frac{1}{2}(xy + yx)$ for any $x, y \in A_{\{\mathbb{I},a_1,a_2\}}$. Then $\{x, y, x\} = xyx$. This holds as well for $A_{\{a_1,a_2\}} \subseteq A_{\{\mathbb{I},a_1,a_2\}}$ and will be very useful for the study of the Jordan algebra generated by two projections with existing transition probability.

Note that an element v in a JBW-algebra A is called a *symmetry* if $v^2 = \mathbb{I}$ [29] and that $H_2(\mathbb{R})$ denotes the Jordan algebra that consists of the self-adjoint 2×2-matrices over the real numbers.

Lemma 8.1 *Suppose that the transition probabilities $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist for two non-zero elements p and q in the quantum logic L_A of any JBW-algebra A.*

(i) *Then $\mathbb{P}(q|p) = \mathbb{P}(p|q)$.*

(ii) *If $\mathbb{P}(q|p) \neq 0 \neq \mathbb{P}(p|q)$, there is a symmetry $v \in A_{\{\mathbb{I},a_1,a_2\}} \subseteq A$ such that $q = \{v, p, v\}$ and $p = \{v, q, v\}$.*

(iii) *$\mathbb{P}(q|p) = \mathbb{P}(p|q) = 1$ iff $p = q$. In this case, $A_{\{p,q\}} = \mathbb{R}p$.*

(iv) *$\mathbb{P}(q|p) = \mathbb{P}(p|q) = 0$ iff p and q are orthogonal. In this case, $A_{\{p,q\}} = \mathbb{R}p \oplus \mathbb{R}q$.*

(v) *$0 < \mathbb{P}(q|p) = \mathbb{P}(p|q) < 1$ iff p, q and $p \circ q$ are linearly independent. The three elements p, q and $p \circ q$ generate $A_{\{p,q\}}$ and, in this case, $A_{\{p,q\}}$ is isomorphic to $H_2(\mathbb{R})$.*

Proof. Consider the subalgebras $A_{\{p,q\}} \subseteq A_{\{\mathbb{I},p,q\}}$ of A. By the Shirshov-Cohn theorem, we can assume that the Jordan product on them stems from an associative product of some larger algebra.

(i) Define $r := \mathbb{P}(q|p)$ and $s = \mathbb{P}(p|q)$. Then $pqp = rp$ and $qpq = sq$ by Proposition 7.1 and we get $rpq = pqpq = spq$. If $pq \neq 0$, then $r = s$. If $pq = 0$, then $pqp = 0$ and $qpq = 0$; therefore $r = s = 0$. Thus we have (i).

(ii) will be proved later, since (v) will be needed. Items (iii) and (iv) follow immediately from the general properties of the transition probabilities (see section 3).

(v) If p, q and $p \circ q$ are linearly independent, the cases $p \circ q = 0$ and $p = q$ are ruled out. The first one is identical with the case that p and q are orthogonal and $\mathbb{P}(q|p) = 0$. The second one is identical with the case that $p \leq q$ and $q \leq p$, which is the same as $\mathbb{P}(q|p) = 1$ and $\mathbb{P}(p|q) = 1$.

Now suppose $\mathbb{P}(q|p) = \mathbb{P}(p|q) = s$ and $0 \neq s \neq 1$; by Proposition 7.1 this means $pqp = sp$ and $qpq = sq$.

Using $A_{\{\mathbb{I},p,q\}}$ and the Shirshov-Cohn theorem, we first show the linear independence of p, q and $p \circ q$. Suppose $0 = r_1 p + r_2 q + r_3 p \circ q$ with $r_1, r_2, r_3 \in \mathbb{R}$. Then $0 = q'(r_1 p + r_2 q + r_3 p \circ q)q' = r_1 q' p q'$. We have $q'pq' \neq 0$, since otherwise p and q' are orthogonal and thus $p \leq q$ and $s = 1$. Therefore $r_1 = 0$. Using p' instead of q', it follows in the same way that $r_2 = 0$. Finally $r_3 = 0$, since $p \circ q = 0$ would mean that p and q become orthogonal and then $s = 0$.

Furthermore, from $sp = pqp = 2p \circ (p \circ q) - p \circ q$ we get

$$p \circ (p \circ q) = (sp + p \circ q)/2,$$

and from $sq = qpq = 2q \circ (p \circ q) - p \circ q$ we get

$$q \circ (p \circ q) = (sq + p \circ q)/2.$$

Moreover

$$\begin{aligned}(p \circ q)^2 &= (pq + qp)^2/4 = (pqpq + pqp + qpq + qpqp)/4 \\ &= (spq + sp + sq + sqp)/4 \\ &= (2sp \circ q + sp + sq)/4.\end{aligned}$$

Therefore $p \circ (p \circ q)$, $q \circ (p \circ q)$ and $(p \circ q)^2$ lie in the linear hull of p, q and $p \circ q$ which thus becomes a three-dimensional Jordan algebra and identical with $A_{\{p,q\}}$. Now consider the following matrices in $H_2(\mathbb{R})$:

$$a := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } b := \begin{pmatrix} s & s^{1/2}(1-s)^{1/2} \\ s^{1/2}(1-s)^{1/2} & 1-s \end{pmatrix}.$$

Then $a^2 = a$, $b^2 = b$, $aba = sa$ and $bab = sb$. Note that these matrices are rather simple versions of those used in the example at the end of section 7 with $K = \mathbb{R}$, $m = n = 1$ and $u = 1$. The same line of reasoning as above with p

and q or some simple matrix calculations then show that a, b and $a \circ b$ become linearly independent with $0 \neq s \neq 1$ and that the following identities hold:

$$\begin{aligned} a \circ (a \circ b) &= (sa + a \circ b)/2, \\ b \circ (a \circ b) &= (sb + a \circ b)/2 \text{ and} \\ (a \circ b)^2 &= (2sa \circ b + sa + sb)/4. \end{aligned}$$

Now define a linear map $\pi : A_{\{p,q\}} \to H_2(\mathbb{R})$ by $\pi(r_1 p + r_2 q + r_3 p \circ q) := r_1 a + r_2 b + r_3 a \circ b$ for $r_1, r_2, r_3 \in \mathbb{R}$. Since p, q and $p \circ q$ are linearly independent, π is well-defined. The above identities for p and q and for a and b imply that π is multiplicative. Since the real dimension of $H_2(\mathbb{R})$ is three, the Jordan algebras $A_{\{p,q\}}$ and $H_2(\mathbb{R})$ are isomorphic.

(ii) By Proposition 7.1 and part (i) of this lemma, we can assume that $pqp = sp$ and $qpq = sq$ with $s = \mathbb{P}(q|p) = \mathbb{P}(p|q) \neq 0$. With part (v) of this lemma, we can conclude that $A_{\{\mathbb{I},p,q\}}$ has a finite dimension; it thus becomes a JBW-algebra. By Lemma 5.2.1 in Ref. [29], there is a symmetry $v \in A_{\{\mathbb{I},p,q\}}$ such that $vpqpv = qpq$. Then $sq = qpq = vpqpv = svpv$ and thus $q = vpv$. Furthermore, $vqv = v^2pv^2 = p$. $\square$

When we apply Lemma 8.1 (ii) to the usual quantum mechanical setting or to the self-adjoint part of a von Neumann algebra, it tells us that the two projections p and q must be unitarily equivalent, if $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist and $\mathbb{P}(q|p) \neq 0 \neq \mathbb{P}(p|q)$ holds.

The complex $*$-algebra that such an isoclinic projection pair p and q in a von Neumann algebra generates is a copy of $M_2(\mathbb{C})$, but the Jordan algebra $A_{\{p,q\}} \cong H_2(\mathbb{R})$ that it generates, by Lemma 8.1 (v), is smaller than the self-adjoint part of $M_2(\mathbb{C})$. The self-adjoint part of $M_2(\mathbb{C})$ has the real dimension 4, while $H_2(\mathbb{R})$ has the real dimension 3. Therefore, the Jordan algebra that an isoclinic pair in a von Neumann algebra generates provides more structural information about the pair than than complex $*$-algebra that it generates.

9 Transition probability in Jordan algebras II

We now return to the case when only $\mathbb{P}(q|p)$ exists for the elements $p \neq 0$ and q in the quantum logic of a JBW-algebra. The following proposition shows how this is still related to the isoclinicity.

Proposition 9.1 *Suppose that the transition probability $\mathbb{P}(q|p)$ exists for two elements $p \neq 0$ and q in the quantum logic L_A of any JBW-algebra A and that $\mathbb{P}(q|p) \neq 0$. Then there are two orthogonal elements q_o and q_1 in L_A such that*

(a) $q = q_o + q_1$,

(b) *q_1 and p are orthogonal, and*

(c) *$\mathbb{P}(q_o|p)$ as well as $\mathbb{P}(p|q_o)$ exist and both coincide with $\mathbb{P}(q|p)$. This means that p and q_o form an isoclinic pair.*

Proof. Use the Shirshov-Cohn theorem again and consider the subalgebra $A_{\{p,q\}}$ embedded in a larger associative algebra defining the special Jordan product on $A_{\{p,q\}}$. Define $s := \mathbb{P}(q|p)$ and $q_o := \frac{1}{s}qpq$. By Proposition 7.1 we have the identity $pqp = sp$, which will be used repeatedly in the following equations:

$$q_o{}^2 = \frac{1}{s^2}qpqpq = \frac{1}{s}qpq = q_o$$

and therefore $q_o \in L_A$.

$$pq_op = \frac{1}{s}pqpqp = \frac{1}{s}(pqp)^2 = sp$$

and therefore $\mathbb{P}(q_o|p) = s$ by Proposition 7.1.

$$q_opq_o = \frac{1}{s^2}qpqpqpq = \frac{1}{s^2}q(pqp)(pqp)q = qpq = sq_o$$

and therefore $\mathbb{P}(p|q_o) = s$ by Proposition 7.1. Moreover

$$q_o \circ q = \frac{1}{2s}(qpq^2 + q^2pq) = \frac{1}{2s}(qpq + qpq) = q_o$$

and therefore $q_o \leq q$. Now define $q_1 := q - q_o \in L_A$. Then

$$q_1 \circ p = q \circ p - q_o \circ p = q \circ p - \frac{1}{2s}(qpqp + pqpq) = q \circ p - \frac{1}{2}(qp + pq) = 0$$

and therefore q_1 and p are orthogonal. □

A *trace* on a JBW-algebra A is a map $\tau : A_+ \to [0, \infty]$ with $\tau(rx + sy) = r\tau(x) + s\tau(y)$ for $x, y \in A_+$, $r, s \in \mathbb{R}$, $0 \leq r, s$, and $\tau(\{x, y^2, x\}) = \tau(\{y, x^2, y\})$ for all $x, y \in A$ [15]. For the projections $p, q \in L_A$ we then have $\tau(\{p, q, p\}) = \tau(\{q, p, q\})$.

Corollary 9.2 *Let A be a JBW-algebra, τ a trace on A and $p, q \in L_A$, $0 \neq p, q$.*

(i) *If $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist and $\mathbb{P}(q|p) = \mathbb{P}(p|q) \neq 0$, then $\tau(p) = \tau(q)$.*

(ii) *If $\mathbb{P}(q|p)$ exists and $\mathbb{P}(q|p) \neq 0$, then $\tau(p) \leq \tau(q)$.*

Proof. (i) $\mathbb{P}(q|p) = \mathbb{P}(p|q) = s \neq 0$, i.e. $\{p, q, p\} = sp$ and $\{q, p, q\} = sq$ by Proposition 7.1. Then $s\,\tau(p) = \tau(\{p, q, p\}) = \tau(\{q, p, q\}) = s\,\tau(q)$ and thus $\tau(p) = \tau(q)$. Another way to prove this is to use Lemma 8.1 (ii).

(ii) With q_o and q_1 as in Proposition 9.1, wet get from (i): $\tau(p) = \tau(q_o) \leq \tau(q_o + q_1) = \tau(q)$. □

As an example, consider the JBW-algebra, consisting of the self-adjoint linear operators on a Hilbert space H, with the usual trace. In this case, the trace of a projection p is identical with the dimension of the associated closed linear subspace pH. Corollary 9.2 then tells us that $dim(pH) = dim(qH)$

must hold if $\mathbb{P}(q|p)$ and $\mathbb{P}(p|q)$ both exist and $\mathbb{P}(q|p) = \mathbb{P}(p|q) \neq 0$, and that $dim(pH) \leq dim(qH)$ must hold if $\mathbb{P}(q|p)$ exists and $\mathbb{P}(q|p) \neq 0$. This can also be derived directly from Proposition 7.1 in a simpler way: $pqp = sp$ with $s \neq 0$ implies $pH = pqpH \subseteq pqH$ and therefore $dim(pH) \leq dim(pqH) \leq dim(qH)$. In the general Jordan algebraic setting, however, this line of reasoning will not work.

The following theorem finally provides a complete characterization of the Jordan algebra $A_{\{p,q\}}$ generated by a pair of projections $p \neq 0$ and q for that $\mathbb{P}(q|p)$ exists.

Theorem 9.3 *Suppose that the transition probability $\mathbb{P}(q|p)$ exists for two elements $p \neq 0$ and q in the quantum logic L_A of any JBW-algebra A.*

(i) *If $\mathbb{P}(q|p) = 0$, then $A_{\{p,q\}} = \mathbb{R}p \oplus \mathbb{R}q$.*

(ii) *If $\mathbb{P}(q|p) = 1$, then $A_{\{p,q\}} = \mathbb{R}p \oplus \mathbb{R}(q-p)$.*

(iii) *If $0 < \mathbb{P}(q|p) < 1$, then $A_{\{p,q\}} = A_{\{p,q_o\}} \oplus \mathbb{R}(q-q_o)$, where q_o is an element in L_A with $0 \neq q_o \leq q$ such that $\mathbb{P}(q_o|p)$ as well as $\mathbb{P}(p|q_o)$ exist and both coincide with $\mathbb{P}(q|p)$; p and $q-q_o$ are orthogonal, and $A_{\{p,q_o\}}$ is isomorphic to $H_2(\mathbb{R})$.*

Proof. Part (i) follows from the orthogonality of p and q in the case when $\mathbb{P}(q|p) = 0$. Part (ii) follows from the inequality $p \leq q$ in the case when $\mathbb{P}(q|p) = 1$. Part (iii) follows by combining Lemma 8.1 (v) and Proposition 9.1. □

10 Conclusions

An extension of the usual quantum mechanical transition probability to a very general setting has been presented. These are the quantum logics the mathematical structure of which is an orthomodular partially ordered set.

An interesting aspect of the transition probability considered here is that it does not require any state, but it has a purely algebraic origin. The transition probability $\mathbb{P}(q|p)$, if it exists, becomes a characteristic of the algebraic relation between two elements $p \neq 0$ and q in the quantum logic; p needs not be an atom and no state can then be allocated to p. This is a major difference from other approaches [3, 25, 33, 34, 36, 37, 47, 51].

If p and q are compatible, only three cases are possible for the transition probability: either it does not exist or $\mathbb{P}(q|p) = 1$, which is equivalent to $p \leq q$, or $\mathbb{P}(q|p) = 0$, which is equivalent to the orthogonality p and q. Only the same three cases would be possible, if p and q were elements in a Boolean algebra. The inequality $p \leq q$ is a logical relation between the propositions p and q; it means that p implies q. The orthogonality of of p and q is another logical relation between the propositions p and q; it means that p rules out q. Therefore, $\mathbb{P}(q|p)$ can be considered an extension of these two logical relations to certain pairs p and q that are not compatible. This extended relation, however, is associated with a probability and introduces a continuum of new cases ($0 < \mathbb{P}(q|p) < 1$)

between the two classical cases 'p implies q' ($\mathbb{P}(q|p) = 1$) and 'p rules out q' ($\mathbb{P}(q|p) = 0$).

The no-cloning theorem [20, 54] plays an important role in quantum information theory and particularly in quantum cryptography. Theorem 5.1 becomes a very general version of this theorem in the quantum logical setting and still covers new cases in common Hilbert space quantum mechanics, neither requiring the tensor product nor the atomic elements in the quantum logic. Particularly the non-atomic case goes beyond the usually considered situation and creates promising new opportunities for the quantum key distribution protocols [9, 22].

The approach presented here has enabled us to see that the transition probabilities, their invariance under morphisms and the product rule are sufficient to derive the no-cloning theorem in the quantum logical framework without falling back to Hilbert spaces. This approach might be considered rather general and abstract, but such approaches may be needed to identify and understand the deeper origins of the quantum mysteries. Searching for their origins in the common quantum mechanical Hilbert space formalism can be as difficult as finding a needle in the haystack.

Another rewarding approach with a different focus are the generalized probabilistic theories used in Refs. [5, 6]. Instead of the quantum logic and its algebraic structure, their starting point are the state space and its convex structure, but it is not evident how the transition probability can be defined in that framework. One possibility is to construct a quantum logic from the *projective units* of Alfsen and Shultz's theory [2, 3] first and to use the same definition as here then.

An interesting class of quantum logics are the projection lattices of the JBW-algebras. We have seen that they provide the appropriate framework for a structural analysis (sections 8 and 9) of the transition probability and how it is linked to the Jordan algebraic structure (Proposition 7.1). Beyond the usual quantum mechanical model based on the complex Hilbert space or von Neumann algebras, the JBW-algebras include versions based on the real numbers or the quaternions [3, 29], and the no-cloning theorem remains valid in these cases, although a reasonable tensor product is not available [42, 52, 53].

Furthermore, there is the exceptional Jordan algebra $H_3(\mathbb{O})$ [1, 3, 4, 29, 30], which is not special (see section 8), since the product of the octonions is not associative, and which cannot be represented as linear operators on any kind of Hilbert space. Nevertheless, its quantum logic possesses the transition probabilities. It contains many atoms and many non-orthogonal pairs of atoms; by Proposition 7.2, the transition probability then exists in many cases and non-orthogonal pairs of non-identical atoms result in non-trivial transition probabilities. Explicit examples can be constructed using the same matrices p and q as at the end of section 7, but now with entries from $\mathbb{O}$, $m = 1$ and $n = 2$. Note that any two octonions (here the components of u) and their conjugates generate an associative subalgebra of $\mathbb{O}$ [4]. The automorphisms of $H_3(\mathbb{O})$ map the matrices p and q to further examples.

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