Trans-statistical behavior of a multiparticle system in an ontology of properties

Matías Pasqualini[1] & Sebastian Fortin[2]

[1] CONICET – Instituto de Investigaciones “Dr. Adolfo Prieto”, Universidad Nacional de Rosario

[2] CONICET – Universidad de Buenos Aires

sfortin@conicet.gov.ar

Abstract

In the last years, the surprising bosonic behavior that a many-fermion system may acquire has raised interest because of theoretical and practical reasons. This trans-statistical behavior is usually considered to be the result of approximation modeling methods generally employed by physicists when faced with complexity. In this paper, we take a tensor product structure and an ontology of properties approach and provide two versions (standard and algebraic) of a toy model in order to argue that trans-statistical behavior allows for a realistic interpretation.

Keywords composite bosons – non-individual bundle – ontology of properties – tensor product structure

Section 1 Introduction

1.1. Indistinguishability and statistics

In classical mechanics, a composite system of two or more identical particles rearranged because of a permutation between them is statistically considered a different microstate. This fact leads to Maxwell-Boltzmann statistics. In quantum mechanics (QM), an analogous permutation does not yield a statistically different possibility. For this reason, it is said that quantum identical particles are indistinguishable. That means that any permutation between them cannot yield any observable consequence. The indistinguishability postulate (IP) of QM may be formulated as follows (see Butterfield 1993):

> IP: If the vector $|\psi\rangle$ represents the state of a composite system whose components are indistinguishable particles, then the expectation value of any observable represented by an operator O must be the same for $|\psi\rangle$ and for any permutation $|\psi'\rangle$

$$|\psi'\rangle = P|\psi\rangle : \langle\psi'|O|\psi'\rangle = \langle\psi|O|\psi\rangle \tag{1}$$

In order to satisfy IP, a restriction to states is usually introduced in QM: the symmetrization postulate (SP). IP is satisfied by symmetric $|\psi_S\rangle$ or antisymmetric $|\psi_A\rangle$ states with respect to permutation operator P. Both of them are eigenvectors of P with eigenvalues (1) and (-1)

$$\begin{aligned} P|\psi_S\rangle &= |\psi_S\rangle \\ P|\psi_A\rangle &= -|\psi_A\rangle \end{aligned} \tag{2}$$

So, a formulation for SP may be (see Fortin and Lombardi 2021)

> SP: Any system of many identical particles is represented by either a totally symmetric quantum state (bosons) or a totally antisymmetric quantum state (fermions), where symmetry and antisymmetry are defined in terms of permutations P.

$$|\psi'\rangle = P|\psi\rangle = \pm|\psi\rangle \tag{3}$$

In order to obtain a symmetric $|\psi_S\rangle$ or antisymmetric state $|\psi_A\rangle$ from a generic state $|\psi\rangle$, symmetrizer S and antisymmetrizer A operators should be applied to it

$$\begin{aligned} S|\psi\rangle &= |\psi_S\rangle \\ A|\psi\rangle &= |\psi_A\rangle \end{aligned} \tag{4}$$

It must be noted that fermions (half-integer spin) and bosons (integer spin) may have very different behaviors. A many-fermion system is represented by an antisymmetric state, and, therefore, as the Pauli Exclusion Principle states, it is not possible to find two fermions in the same state. This gives rise to the Fermi-Dirac statistics for fermions. On the contrary, in accordance with Bose-Einstein statistics, two different bosons can be in the same state.

1.2. Trans-statistical behavior

Taking into account SP and Pauli Principle, it is quite clear that fermions must behave according to Fermi-Dirac statistics. However, under certain circumstances, physicists found surprising bosonic behavior in many-fermion systems. That is the well-documented phenomenon of composite bosons or simply co-bosons. In this paper, we refer to it more generally as trans-statistical behavior of identical quantum particles. On the one hand, the issue has raised theoretical interest among many researchers. Law (2005) proposed a model based on creation and annihilation operators and found that the degree of entanglement between constituent fermions in a multiparticle system determines how close it behaves as a system of composite bosons. As a result, interactions are not strictly needed for this particular phenomenon to arise. If there were interactions, they apparently only reinforce correlations which are the determinant factor for bosonic behavior. Chudzicki *et al*. (2010) and Tichy *et al*. (2014) obtained a generalization of Law's approach. On the other hand, the issue is also relevant for practical reasons since it has connections with several applications such as quantum information processing (Gigena and Rossignoli 2015), Bose-Einstein condensates (Avancini *et al*. 2003, Rombouts *et al*. 2003), excitons (Combescot *et al*. 2001) and Cooper pairs in superconductors (Belkhir *et al*. 1992). Recently, some of these studies have been applied to describe both fermionic and bosonic behavior of confined Wigner molecules (Cuestas *et al*. 2020).

1.3. A non-realistic approximation

It is a usual assumption that trans-statistical behavior is a phenomenon that should not be interpreted realistically, but simply as a result of approximation methods frequently employed in experimental physics. Most phenomena are so complex that they just cannot be modeled in a realistic manner. In turn, it is necessary to work with models that only provide an approximate description of the object under scrutiny. Physicists are well aware that, in these circumstances, approximate models may predict behavior that cannot be expected from a physical real object. For the sake of clarity, it is not really expected that a real pendulum will exhibit perpetual motion. That is only predicted for an approximate model. Analogously, it is not believed that a many-fermion system really behaves as a system of bosons. Trans-statistical behavior of identical particles –it is believed– is only a suitable description for the observed phenomenon that arises from approximate models of many-fermion systems under specific conditions, in which entanglement is apparently a key factor. This usual assumption is generally reinforced by the fact that creation and annihilation operators-based models do not allow exact but only approximated bosonic behavior. According to this picture, composite bosons could not be quantum systems in an ordinary sense.

1.4. A TPS approach

In this work, we will leave aside, for a moment, the models and interpretations that appear in the works cited. We will take the ideas that appear in these works as inspiration to ask ourselves whether or not it is possible to build a realistic interpretation in which a strong ontological status is assigned to the bosonic behavior of fermions. So, we set aside temporarily the issue of entanglement and tackle trans-statistical behavior from a different and complementary perspective focused on the tensor product structure (TPS) approach (see Harshman and Wickramasekara 2007). As it is well-known, a TPS is a particular way (among many) to factorize the Hilbert space into subspaces or, from an algebraic approach, decompose the algebra of observables into subalgebras in order to split a system into subsystems. We benefit from studies that defend the idea that the notion of separability between subsystems is not absolute but relative to a particular partition (Zanardi 2001). In this work, we explore the possibility that the relativity of separability extends to the notions of fermionic and bosonic when applied to composite systems of identical particles. Such relativity of separability leads also to the question of which of the many mathematically possible TPSs should be endowed with physical and ontological significance. The very idea of what a system is has also been put into discussion (Dugić and

Jeknić 2008). If it were indeed the case that the fermionic and bosonic character of a composite system of identical particles is TPS-relative, then the question of what is the corresponding ontological picture is at issue.

1.5. A realistic interpretation

We are proposing a toy model in which different TPSs correspond respectively to a fermionic or bosonic composite system. We aim to show that composite bosons are on an equal ontological footing with elementary fermions or elementary bosons, in a way that favors a realistic interpretation of trans-statistical behavior. It is important to emphasize that we are not intending to create an approximate model to capture such systems empirically, as usually performed by experimental physicists. We just play mathematically with QM formalism to create a toy model. Before proceeding, it is also necessary to make clear in what sense we are talking about reality. We are not referring to it as a *noumenon* in a naïve manner. Our concept of reality is a relative one. It is reality as it is constituted by the theory, in our case QM. It is a categorical-conceptual framework endowed with ontological significance (see Lombardi 2021). In simple terms, we talk of reality as if QM were true.

1.6. Towards an ontological lesson

If trans-statistical behavior could be realistically interpreted, we would learn a lesson about QM ontology from this phenomenon. A topic of debate in QM ontology is what ontological concept is adequate to refer to a quantum system. There are traditional ontologies, which are favored by the familiar particle-picture in physics, in that properties are attributes of individuals. The ontology of individuals and properties suggests that fermion-pairs should retain their identity when merged into a composite and only in a merely descriptive manner could be or behave as bosons. There are also ontologies based exclusively on properties (see da Costa and Lombardi 2014). From this perspective, a system is just a non-individual bundle of properties. If a system is a bundle, there is no need that it preserves its identity when it enters in a composite. This ontology of non-individual bundles would allow us to claim that the fermionic or bosonic character of a multiparticle system does not depend upon identity conditions previously possessed by elementary quantum systems. This move would also make it possible to construe trans-statistical behavior in a realistic manner.

1.7. Content of the next sections

In Section 2, the first version of our toy model will be proposed. In this first version, we work with a standard Hilbert space formalism. States will have logical priority over observables. Consequently, systems will be identified from their vector state and standard indistinguishability (IP), and the symmetrization (SP) postulates will be employed. In this version, we settle two specific TPSs that allow us to apply the notions of fermionic or bosonic relatively with respect to each of them. On the one hand, alpha-partition corresponds to the fermionic character of the compound system. On the other, beta-partition corresponds with its bosonic character.

In Section 3, the basic lines of an ontology of properties for QM will be exposed. This ontology was originally suggested by the algebraic formalism of QM, which grants priority to observables over states. So, the second version of our toy model will be proposed, in which the two partitions settled in Section 2 are reconsidered from the algebraic approach. The main idea is to show that the set of observables allows us to define the composite system as fermionic or bosonic, without varying its state.

Also, it will be concluded that if we make complementary use of a model based on TPS (such as the toy example that we propose) and a model based on creation and annihilation operators (such as Law's), the trans-statistical behavior may allow a realistic interpretation that assumes and at the same time strengthens a non-individual bundle ontological picture.

Section 2 The toy model in Hilbert space

In this section, we present a toy model in which it is possible to treat fermion-pairs as composite bosons. Although we deal with only four fermions in this model, it could be easily generalized to any even number of fermions. Our interest is to argue in favor of the relativity of the fermionic or bosonic nature of a system composed of elementary fermions with respect to a previously chosen partition. The striking feature of this model is that an alternative symmetrization or antisymmetrization of the system state is not required. The different decompositions are performed in this section in terms of different tensor product structures of the Hilbert space of the system.

2.1. Fermion-like decomposition (alpha-TPS)

The toy model is a system composed of 4 elementary systems of the following type.

The component systems: Let us consider a spin ½ quantum system S represented in its own Hilbert space $\mathcal{H}$. Its Hamiltonian $\hat{H}$ has eigenstates $|n\rangle$ with energy E_n, that is $\hat{H}|n\rangle = E_n|n\rangle$. Then, each state $|\varphi\rangle \in \mathcal{H}$ can be written as $|\varphi\rangle = \sum_n c_n |n\rangle$.

The composite system: Now we will consider a quantum system $U = S_1 \cup S_2 \cup S_3 \cup S_4$ with an associated Hamiltonian $\hat{H}_U = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$ whose eigenstates $|N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle$ generate the Hilbert space $\mathcal{H}_U = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$. Then, $\hat{H}|N\rangle = E_N|N\rangle$ where $E_N = E_{n_1} + E_{n_2} + E_{n_3} + E_{n_4}$, and each state $|\psi\rangle \in \mathcal{H}_U$ can be written as

$$\left|\psi\right\rangle = \sum_N c_N \left|N\right\rangle = \sum_{n_1,n_2,n_3,n_4} c_{n_1,n_2,n_3,n_4} \left|n_1\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_4\right\rangle \tag{5}$$

Since they are fermions (spin ½), the wave function is antisymmetric under the exchange of the labels of any pair of particles. So, permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ and $P_{3\leftrightarrow 4}$ are defined as

$$\begin{aligned}
&P_{1\leftrightarrow 2}\left|N\right\rangle = \left|n_2\right\rangle \otimes \left|n_1\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_4\right\rangle,\ P_{1\leftrightarrow 3}\left|N\right\rangle = \left|n_3\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_1\right\rangle \otimes \left|n_4\right\rangle \\
&P_{1\leftrightarrow 4}\left|N\right\rangle = \left|n_4\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_1\right\rangle,\ P_{2\leftrightarrow 3}\left|N\right\rangle = \left|n_1\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_4\right\rangle \\
&P_{2\leftrightarrow 4}\left|N\right\rangle = \left|n_1\right\rangle \otimes \left|n_4\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_2\right\rangle,\ P_{3\leftrightarrow 4}\left|N\right\rangle = \left|n_1\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_4\right\rangle \otimes \left|n_3\right\rangle
\end{aligned} \tag{6}$$

This condition imposes a restriction on the possible states. Then, the only possible coefficients c_N (or c_{n_1,n_2,n_3,n_4}) are those such that

$$P_{1\leftrightarrow 2}\left|\psi\right\rangle = P_{1\leftrightarrow 3}\left|\psi\right\rangle = P_{1\leftrightarrow 4}\left|\psi\right\rangle = P_{2\leftrightarrow 3}\left|\psi\right\rangle = P_{2\leftrightarrow 4}\left|\psi\right\rangle = P_{3\leftrightarrow 4}\left|\psi\right\rangle = -\left|\psi\right\rangle \tag{7}$$

Because of the very way it is constructed, the Hilbert space of the composite system can be trivially factorized into four equivalent subspaces. This is the alpha tensor product structure (TPS_A) that had to be considered. In summary, U is a composite system of fermions whose wave function is antisymmetric with respect to TPS_A.

2.2. Boson-like decomposition (beta-TPS)

The decomposition of the state $\left|N\right\rangle = \left|n_1\right\rangle \otimes \left|n_2\right\rangle \otimes \left|n_3\right\rangle \otimes \left|n_4\right\rangle$ is not the only one that can be done on the complete system U . For example, it can be described as a system composed of components systems of the following type.

The component systems: Let us consider the system $S_i = S_1 \cup S_2$ represented in its own Hilbert space $\mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2$. Its Hamiltonian $\hat{H}_i = \hat{H}_1 + \hat{H}_2$ has eigenstates $\left|m_i\right\rangle = \left|n_1\right\rangle \otimes \left|n_2\right\rangle$ with energy $E_{m_i} = E_{n_1} + E_{n_2}$, that is $\hat{H}\left|m_i\right\rangle = E_{m_i}\left|m_i\right\rangle$. Then, each state $\left|\varphi^i\right\rangle \in \mathcal{H}_i$ can be written as $\left|\varphi^i\right\rangle = \sum_{m_i} c_{m_i}\left|m_i\right\rangle$. Let us also consider another system $S_{ii} = S_3 \cup S_4$ represented in its own Hilbert space $\mathcal{H}_{ii} = \mathcal{H}_3 \otimes \mathcal{H}_4$. Its Hamiltonian $\hat{H}_{ii} = \hat{H}_3 + \hat{H}_4$ has eigenstates $\left|m_{ii}\right\rangle = \left|n_3\right\rangle \otimes \left|n_4\right\rangle$ with energy $E_{m_{ii}} = E_{n_3} + E_{n_4}$, that is $\hat{H}\left|m_{ii}\right\rangle = E_{m_{ii}}\left|m_{ii}\right\rangle$. Then, each state $\left|\varphi^{ii}\right\rangle \in \mathcal{H}_{ii}$ can be written as $\left|\varphi^{ii}\right\rangle = \sum_{m_{ii}} c_{m_{ii}}\left|m_{ii}\right\rangle$.

If we consider these components, the toy model is a composed system of two subsystems, since the Hilbert space that defines system *U* can be factorized into two subspaces. This is beta tensor product structure (TPS_B).

The composite system: Now the same system *U* can be described as $U = S_i \cup S_{ii}$ with an associated Hamiltonian $\hat{H}_U = \hat{H}_i + \hat{H}_{ii}$, whose eigenstates $|N\rangle = |m_i\rangle \otimes |m_{ii}\rangle$ generate the Hilbert space $\mathcal{H}_U = \mathcal{H}_i \otimes \mathcal{H}_{ii}$. Then, $\hat{H}|N\rangle = E_N |N\rangle$ where $E_N = E_{m_i} + E_{m_{ii}}$, and each state $|\psi\rangle \in \mathcal{H}_U$ can be written as

$$|\psi\rangle = \sum_N c_N |N\rangle = \sum_{m_i, m_{ii}} c_{m_i, m_{ii}} |m_i\rangle \otimes |m_{ii}\rangle \tag{8}$$

To study its fermionic or bosonic nature, it is necessary to define new permutation operators. This is because, for example, the labels 1 and 2 from the old operator $P_{1\leftrightarrow 2}$, no longer refer to subsystems that are present in this partition. To be able to permute the new particles it is necessary to define the operator

$$P_{i\leftrightarrow ii}|N\rangle = P_{i\leftrightarrow ii}|m_i\rangle \otimes |m_{ii}\rangle = |m_{ii}\rangle \otimes |m_i\rangle \tag{9}$$

This is the only permutation operator that exists in this partition. Since the particles S_i and S_{ii} are linked with the particles S_1, S_2, S_3, and S_4 in a direct way, it is easy to see that there is a relation between the permutation operators

$$P_{i\leftrightarrow ii}|N\rangle = |m_{ii}\rangle \otimes |m_i\rangle = |n_3\rangle \otimes |n_4\rangle \otimes |n_1\rangle \otimes |n_2\rangle = P_{1\leftrightarrow 3} P_{2\leftrightarrow 4} |N\rangle \tag{10}$$

So, the relation between permutation operators from both TPS is

$$P_{i\leftrightarrow ii} = P_{1\leftrightarrow 3} P_{2\leftrightarrow 4} \tag{11}$$

It should be noted that so far we have not changed the state, we have only written it in a new way. Therefore, the coefficients c_N have the same restrictions as before. Then, it is possible to compute how $P_{i\leftrightarrow ii}$ operates on the state $|\psi\rangle$

$$P_{i\leftrightarrow ii}|\psi\rangle = P_{1\leftrightarrow 3} P_{2\leftrightarrow 4}|\psi\rangle = -P_{1\leftrightarrow 3}|\psi\rangle = |\psi\rangle \tag{12}$$

In summary, under this decomposition (TPS_B) *U* is a composite system of bosons whose wave function is symmetric.

2.3. The relativity of fermionic and bosonic character

Having arrived at this result, it is important to note that the fact that a set of fermions happens to form a new non-fundamental particle with bosonic behavior is not new. Indeed, it has long been known that a group of protons and neutrons, all spin 1/2, can join together to form an atomic nucleus. For example, two protons together with two neutrons join together through nuclear forces to form a nucleus of Helium 4. These atomic nuclei are bosons that exhibit empirically testable bosonic behavior such as superfluidity (Brooks and Donnelly 1977). In this case, the strong nuclear force holds the particles of the nucleus together so tightly that it is possible to think that the nucleus is a new entity. However, in the toy model presented in this work, the particles do not interact with each other and this argument is not valid. There are also other more recent examples such as atomic Bose-Einstein condensates (Avancini *et al*. 2003, Rombouts *et al*. 2002), excitons (Combescot *et al*. 2008, Rombouts *et al*. 2002), and Cooper pairs in superconductors (Belkhir and Randeria 1992). Nevertheless, the mathematical treatment of all these models includes important approximations that obscure the ontological question about this type of physical systems. Then, in the case of bosons composed of non-interacting fermions, the question arises that a group of bosons should be able to share the same quantum state (*i. e.* should be able to have empirically testable bosonic behavior) but the fermions that compose them cannot, due to the Pauli exclusion principle.

That question finds an answer in the models of trans-statistical behavior based on the definition of creation and annihilation operators mentioned above (*e. g.* Law 2005, Chudzicki *et al.* 2010, Tichy *et. al* 2014). In those models, quantum entanglement allows fermion-pairs to approximately overcome Pauli Principle, at least to a certain degree. Exact bosonic behavior is not possible for fermion-pairs, just an approximate one. That issue led many to think that trans-statistical behavior could not be realistically construed and should be reduced to underlying fermionic behavior.

In our toy model the situation is different, the same state which is antisymmetric under TPS_A is exactly symmetric under TPS_B. That is, the state $|\psi\rangle$ is totally antisymmetric with respect to the permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ y $P_{2\leftrightarrow 3}$ in the TPS_A perspective but also totally symmetric with respect to the permutation operator $P_{i\leftrightarrow ii}$ in the TPS_B perspective. This mathematical fact together with SP allows us to claim that, according to our model, fermion-pairs are, although composite, exact bosons. The relativity of fermionic and bosonic nature with respect to partition is suggested by this model. In the next section, we deal with the ontological consequences of this claim. However, our model does not guarantee that the fermion-pairs or composite bosons actually exhibit a full bosonic behavior when we try to make this system interact with others and change its state. This is because our model has the characteristic that the system is in a very particular state, which is the eigenstate of the Hamiltonian. This is understood as a restriction because if the state or its Hamiltonian were different, it would not be possible to draw the same conclusions. Since this is a constrained system, it is not possible for the TPS_B

bosons to form a Bose-Einstein condensate, because their constraints and dynamics do not allow it.

So, we have the following scenario. On the one hand, if we account for the phenomenon that we are studying by means of creation and annihilation operators-based models, we obtain only approximated bosonic behavior. On the other hand, if we employ a model built in a TPS framework like the one we have previously proposed, we obtain fermion-pairs that have a perfect bosonic nature but do not necessarily exhibit a full bosonic behavior. One way out of this dilemma is to make use of both models in a complementary manner. Models such as those of Law 2005, Chudzicki *et al.* 2010 and Tichy *et. al* 2014 on their own would not allow a full-realistic interpretation of trans-statistical phenomena, provided that it is assumed as usual that fermionic or bosonic notions are not relative but that they correspond univocally to a single partition that entails elementary systems. Since for these models bosonic behavior is only approximated, composite bosons and their behavior could not be as real as the underlying elementary fermions with their Fermi-Dirac statistics, which are not approximated. Contrarily, from a TPS perspective we are able to consider that a system of fermion-pairs is truly bosonic since from this perspective fermionic and bosonic notions are relative. If we assume a TPS perspective together with an appropriate ontology, it would be possible to argue that bosonic behavior as accounted for by creation and annihilation operators-based models (albeit approximated) is the real behavior of particles (albeit composite) that really have bosonic nature.

But not so fast. An ontology of individuals allows at best an emergentist conception of inter-theory relation (Nagel 2008) in which composite bosons and, correspondingly, their bosonic behavior, although real, still depend on underlying elementary fermions with their corresponding statistical behavior (to illustrate how emergence can be applied to a similar concrete case, it is possible to see the emergence of phonons in the paper by Franklin & Knox 2018). According to this ontological approach, fermion-like TPS is fundamental with respect to boson-like TPS. That is, TPSs are not on an equal ontological footing. The toy model built in a TPS framework that we proposed is perhaps a suitable tool to argue in favor of the relativity of the fermionic and bosonic notions. But surely that tool and its corresponding argument are much more empowered if we assume an ontology of properties. The reason is quite simple. If we assume that physical systems are individuals, we have a TPS that splits the total system into individuals while others do not. In our model, TPS_A splits the total system into individuals, but not so TPS_B. In fact, the states of the subsystems in TPS_B can be written as tensor products of subsystems states of TPS_A, but not otherwise. As it will be suggested in the next section, if we employ algebraic formalism together with an ontology of properties, there are no clear means to draw a distinction between a fundamental TPS and emergent TPSs.

Section 3 Trans-statistical behavior in an ontology of properties

3.1. Classical and quantum particles as individual objects

From a philosophical perspective, an individual is an object that possesses an identity that makes it distinguishable from other objects and that is able to retain its identity over time. It is also believed to be the bearer of a set of properties, such as location in space and time. The individuality of such an object may be regarded to be granted by something over and above the properties that it possesses, such as a substance. Alternatively, Leibniz Identity Principle (PII) establishes that individual identity depends only on the properties possessed by the object. Identification of the individual object over time is made possible by its spatiotemporal trajectory. The ontological category of individual fits properly when referring to classical particles. But it runs into trouble when applied to quantum particles. Quantum indistinguishability is known to prevent particles of the same kind to be re-identified once a permutation is performed between them. Moreover, contextuality prevents quantum particles to possess well-defined properties. As a consequence, the omnimode determination principle that is expected to be satisfied by any individual object is violated by particles in the quantum domain. They do not even have well-defined spatiotemporal trajectories, which would have allowed identifying them over time and keeping track the of particles being permuted.

These features led some of the founding fathers of QM (Born and Heisenberg) to radically discard the category of individual to refer to quantum particles. They are simply not individuals. This idea was reflected in early discussions (see Weyl 1931). This constitutes the so-called Received View concerning this matter, which eventually entailed the development of non-standard formal systems to represent non-individual objects (Krause 1992). Recently, a variety of authors criticized the Received View claiming that the category of individual may hold if we drop PII or at least some of its strongest forms. In order to make this view consistent with quantum statistics, van Fraassen (1985) argued that it is not necessary to admit equiprobability for each possible configuration as usually assumed in statistical mechanics. Alternatively, French (1989) proposed that states that are neither symmetric nor antisymmetric are ontologically possible but physically inaccessible. From this perspective, quantum particles are considered individual objects that are contingently in states that make them indistinguishable. Muller and Saunders (2008) explored the possibility of weakly discerning between quantum identical particles in relational terms.

3.2. An ontology of properties for quantum systems

In the context of modal interpretations of QM, some authors proposed a new quantum ontology of properties without individuals (see da Costa, Lombardi and Lastiri 2013; da Costa and Lombardi 2014; Lombardi and Dieks 2016). The choice for this ontology is strongly suggested by the aforementioned quantum features (contextuality and indistinguishability). Our guiding hypothesis is that also trans-statistical behavior best matches with a non-individual ontology.

3.2.1. Ontology of properties and algebraic formalism

Usual presentations of QM employ Hilbert space formalism and the Schrodinger picture. It is mathematically built from a set of vectors, which in turn represent possible physical states of the system. System observables are represented by operators that act on already defined state vectors. The logical priority of system states over observables that characterizes Hilbert space formalism favors an ontology of individuals and properties (unless the Heisenberg picture is applied to it). Systems are individuals identified by their state space and observables are properties that inhere in them (see Ballentine 1998, 234-235).

As it is known, it is also possible to employ an algebraic formalism in QM where the set of physical observables is represented by an algebra of operators. The system's state is represented by a functional that acts upon those already defined operators, in order to compute expected values. In this case, the logical priority of observables over states suggests an ontology of properties, where there may be no individuals. Systems are defined exclusively by their algebra of observables. The state functional is simply a device that codifies quantum probabilities (see Ballentine 1998, 48).

3.2.2. Ontology of properties. Semantic correspondences

To put it more formally, an ontology of properties without individuals is defined by the following semantic correspondences (see Fortin and Lombardi 2021):

- The algebra of *self-adjoint operators* represents the set of *physical observables* that define a quantum system, which in turn correspond to the set of *instances of universal type-properties* in the ontological domain.

- *Eigenvalues* of self-adjoint operators represent possible *physical values*, which in turn corresponds to possible *case-properties* belonging to each type-property.

- *Probability functions* represent physical *probability distributions* for each physical observable, which in turn corresponds to *ontological propensities* of each possible case-property

- *Functionals* over the algebra of observables represent *physical states*. This last item has no ontological counterpart since physical states are just devices that assign a probability distribution for each observable.

It is important to notice that we do not talk about physical outcomes but about physical values because the ontology of properties was first developed in the context of modal interpretations of QM. In this family of interpretations, the observables may have determined values regardless of a measurement context. A preferred context is defined *a priori* and each modal interpretation postulates a particular actualization rule. Nonetheless, the ontology of properties is equally suitable for the standard interpretation or for others not belonging to the modal family.

3.2.3. Quantum systems as non-individual bundles of possible properties

The ontology of properties yields a picture of quantum systems in which they are just bundles of possible case-properties without any individual identity. The familiar particle-picture assumed in physical practice is generally discarded and could be retained only under peculiar circumstances. It is important to stress that it is not the traditional bundle of properties where all the properties can adopt actual values of actual properties, designed in metaphysics to account for classical individual objects without the notion of substance. In the quantum case, the Kochen-Specker theorem (1967) proves the impossibility of ascribing precise values to all observables of a quantum system simultaneously, while preserving the functional relations between commuting observables. If it is assumed, as it is usual both in the classical and in the quantum domain, that the observables are categorical determinable properties and the values of the observables are categorical (non-dispositional and non-probabilistic) determinate properties, then the Kochen-Specker theorem is an obstacle to a traditional bundle theory. In fact, the theorem states that not all the determinable categorical properties are determinate and, as a consequence, the individual cannot be conceived as the bundle of the categorical determinate properties corresponding to all its categorical determinable properties. Even more important is to emphasize that bundles of possible case-properties are no longer object of PII. It is not a matter that PII is false. It simply does not applies to them. Bundles of possible case-properties do not retain any identity each time they merge into a composite bundle or split into them. These features of the ontology of properties make it adequate to overcome the difficulties that quantum contextuality and quantum indistinguishability impose upon the design of a QM ontology. As it will be soon formally stated, this ontology also fits properly with the relativity of the notions of fermionic and bosonic when applied to systems composed of an even number of fermions and thus allows a realistic interpretation of trans-statistical behavior, in which it is observed that a set of fermions loses its identity and becomes a set of bosons that acquire testable bosonic behavior under certain circumstances. It is quite obvious that an ontology based on individuals would have serious difficulties construing this phenomenon in a realistic manner. The ontology of properties certainly does. Of course, a basic assumption that is previously needed to choose for this ontology is to endow modality with an ontological meaning.

3.2.4. Ontology of properties and indistinguishability

An additional result of the ontology of properties for QM is a re-statement of the traditional indistinguishability postulate (IP, see eq. (1) in Section 1) that makes the symmetrization postulate (SP see eq. 3 in section 1) a natural consequence of the ontology. When two or more indistinguishable bundles are combined, it is natural to expect that the instances of universal type-properties belonging to the composite bundles do not distinguish between those component bundles. More simply, when two indistinguishable bundles merge into a single whole, which component bundle is taken first and which second does not matter at all. Mathematically, the restriction that yields the observed statistics is no longer imposed over states (as in SP) but directly over observables. IP_{obs} is formulated as (see Lombardi and Castagnino 2008, and Fortin and Lombardi 2021 for a complete justification)

$$O' = P^{\dagger}OP : \langle\psi|O'|\psi\rangle = \langle\psi|O|\psi\rangle \tag{13}$$

Then, the observables that obey this condition will be symmetric, that is $O_{sym} = P^{\dagger}O_{sym}P$ and form the space $\mathcal{O}_{sym}$ (see Fortin and Lombardi 2021 for details). In contrast with standard IP, IP_{obs} is ontologically motivated, since a bundle is symmetric if its constituents are identical. Let us consider two bundles h^1 and h^2 defined by different instances of the same algebra of observables $\mathcal{O}_1 = \mathcal{O}_2$ such that $h^1 \triangleq h^2$. That means that these bundles are represented in the physical domain by systems or "particles" of the same kind and must be considered indistinguishable. Of course, different indices in this case do not mean physical distinguishability. These two bundles merge into a composite bundle h^U such that $h^U = h^1 * h^2$. Consequently, the algebra $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_2 \vee \mathcal{O}_1$ defines bundle h^U. Now the restriction over observables $O_U \in \mathcal{O}_U$ established in IP_{obs} (eq. (13)) must be carried out. This requires that observables $O_U = \sum_{ij} k_{ij}\left(O_{1i} \otimes O_{2j}\right)$ are such that $O_{1i} \otimes O_{2j} = O_{2i} \otimes O_{1j}$. This means that observables O_U belonging to bundle h^U are symmetric with respect to permutation of bundles h^1 and h^2 (see Fortin and Lombardi 2021).

The restriction imposed by (eq. (13)) includes both the case of fermions and bosons. This is because the permutation operator appears twice, then both in the case that the state ($|\psi_S\rangle$) is eigenstate of P with eigenvalue 1

$$\langle\psi_S|O_{sym}|\psi_S\rangle = \langle\psi_S|P^{\dagger}O_{sym}P|\psi_S\rangle = (1)^2\langle\psi_S|O_{sym}|\psi_S\rangle = \langle\psi_S|O_{sym}|\psi_S\rangle \tag{14}$$

and in the case that it is -1 ($|\psi_A\rangle$)

$$\langle\psi_A|O_{sym}|\psi_A\rangle = \langle\psi_A|P^{\dagger}O_{sym}P|\psi_A\rangle = (-1)^2\langle\psi_A|O_{sym}|\psi_A\rangle = \langle\psi_A|O_{sym}|\psi_A\rangle \tag{15}$$

the eigenvalue appears squared. To account for bosons or fermions separately, it is necessary to further restrict the space of observables. Usually, to obtain the symmetric/antisymmetric state $|\psi_S\rangle/|\psi_A\rangle$ from a generic state $|\psi\rangle$, it is necessary to apply the operator *S*/*A* respectively $|\psi_S\rangle = S|\psi\rangle/|\psi_A\rangle = A|\psi\rangle$, then the expectation value of an observable O is

$$\langle O\rangle_{|\psi_S\rangle} = \langle\psi_S|O|\psi_S\rangle = \langle\psi|S^{\dagger}OS|\psi\rangle = \langle\psi|O_{\mathcal{S}}|\psi\rangle = \langle O_{\mathcal{S}}\rangle_{|\psi\rangle} \tag{16}$$

$$\langle O\rangle_{|\psi_A\rangle} = \langle\psi_A|O|\psi_A\rangle = \langle\psi|A^{\dagger}OA|\psi\rangle = \langle\psi|O_A|\psi\rangle = \langle O_A\rangle_{|\psi\rangle} \tag{17}$$

It is easy to see that observables of the type $O_S = S^\dagger O S$ form the subspace $\mathcal{O}_S \subset \mathcal{O}_{sym}$ and observables of the type $O_A = A^\dagger O A$ form the subspace $\mathcal{O}_A \subset \mathcal{O}_{sym}$. Therefore, the same empirical reason that imposes the restriction to symmetric-bosonic states $|\psi_S\rangle$ or antisymmetric-fermionic states $|\psi_A\rangle$ in the usual presentations, from the present perspective imposes the restriction to bosonic observables $O_S \in \mathcal{O}_S$ or fermionic observables $O_A \in \mathcal{O}_A$.

3.3. Algebraic version of the toy model

3.3.1. Definition of the total system based on its observable space

Let us consider an aggregate h^U of indistinguishable bundles $h^1 \triangleq h^2 \triangleq h^3 \triangleq h^4$ such that $h^U = h^1 * h^2 * h^3 * h^4$. This aggregate of bundles h^U, which is itself a new bundle, is in the physical domain a composite system U of indistinguishable subsystems $S_1 = S_2 = S_3 = S_4$ each of them with spin ½ such that $U = S_1 \cup S_2 \cup S_3 \cup S_4$. We are adopting here an ontology of properties suggested by the algebraic approach of QM, so subsystems $S_1 = S_2 = S_3 = S_4$ are not defined in Hilbert space but by the algebras of observables $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4$, where each algebra represents each subsystem's type-properties. System U is defined in terms of an algebra $\mathcal{O}_U$ such that $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 \vee \mathcal{O}_3 \vee \mathcal{O}_4$, which is the minimal algebra generated by the subsystems algebras.

Since these subsystems are indistinguishable and consequently the bundle h^U is symmetrical with respect to any permutation of component bundles, the operators representing observables $O_U \in \mathcal{O}_U$ are symmetric in accordance with $\mathrm{IP}_{\mathrm{obs}}$ (eq. (13))

$$O'_U = P_\alpha^\dagger O_U P_\alpha = O_U \tag{18}$$

where P_α represents each element of the set $\{P_\alpha\}$ of all possible permutation operators relative to alpha-partition ($\mathrm{TPS_A}$) of system U. In addition, the observables O_U are symmetric with respect to the only admissible permutation relative to beta-partition ($\mathrm{TPS_B}$), since $P_\beta = P_{i\leftrightarrow ii}$ is equivalent to one of the elements of the set $\{P_\alpha\}$, *i. e.* the product of the permutation operators $P_{1\leftrightarrow 3} P_{2\leftrightarrow 4}$ (see eq. 11). Consequently, if observables O_U satisfy condition $O'_U = P_\alpha^\dagger O_U P_\alpha = O_U$ (eq. (18)), they also satisfy

$$O'_U = P_\beta^\dagger O_U P_\beta = O_U \tag{19}$$

This means that every observable O_U belonging to $\mathcal{O}_U$ is permutation invariant with respect to both partitions

$$\forall O_U \in \mathcal{O}_U, O_U = P^{\dagger} O_U P \tag{20}$$

But this is not the whole story, since $\mathcal{O}_U$ includes both fermionic and bosonic observables (see eq. (14) and (15)). It is necessary to introduce specifically the fermionic character of the TPS$_A$ subsystems.

3.3.2. Fermionic subalgebra of observables

In Section 2, because of the value of spin ½ of the component systems in TPS$_A$, we demanded that the state of the system U were antisymmetric with respect to permutation operators $P_{1\leftrightarrow 2}, P_{1\leftrightarrow 3}, P_{1\leftrightarrow 4}, P_{2\leftrightarrow 3}, P_{2\leftrightarrow 4}$ and $P_{3\leftrightarrow 4}$ (eq. 7). However, in this section, we are adopting an ontology of properties. The system state will be considered just a device that assigns a probability at each possible event. It plays no role in identifying the system. The fermionic character that our bundle may assume ought to be defined exclusively in terms of its properties. So, the fermionic character of our bundle will be obtained by imposing a further restriction on its observables. Consider the antisymmetrizer projector corresponding to the TPS$_A$

$$A_{\mathrm{A}} = \frac{1}{\sqrt{N!}} \sum_{i=1}^{\alpha} \pm P_{\alpha} \tag{21}$$

Notice that the projector A is alpha-indexed in correspondence with the permutations that define it. The operator P_{α} (also alpha-indexed) represents each possible permutation (including the identity I) belonging to TPS$_A$, $N!$=24 is the quantity of those permutations and $(\pm)$ depends on the parity of P_{α}: $(+)$ if it is even or $(-)$ if it is odd. Usually in QM, the antisymmetrizer projector is applied to a generic state $A_A |\psi\rangle = |\psi_{A_{\mathrm{A}}}\rangle$. Instead, we are applying it to our observables

$$A_{\mathrm{A}}^{\dagger} O_U A_{\mathrm{A}} = O'_U \tag{22}$$

That operation allows us to define a fermionic subalgebra $\mathcal{O}^F \subset \mathcal{O}_U$ such that

$$\forall O_F \in \mathcal{O}^F, O_F = A_{\mathrm{A}}^{\dagger} O_U A_{\mathrm{A}} \tag{23}$$

which is the algebra with respect to which any generic state $|\psi\rangle$ will behave as antisymmetric

$$\langle \psi | O_F | \psi \rangle = \langle \psi | A_{\mathrm{A}}^{\dagger} O_U A_{\mathrm{A}} | \psi \rangle = \langle \psi_{A_{\mathrm{A}}} | O_U | \psi_{A_{\mathrm{A}}} \rangle \tag{24}$$

3.3.3. Bosonic subalgebra of observables

In Section 2, we found that the same coefficients c_N that make the system state antisymmetric with respect to the set $\{P_\alpha\}$ of permutations, turn it symmetric with respect to operator $P_{i\leftrightarrow ii}$. That is

$$P_\alpha \left|\psi_{A_\mathrm{A}}\right\rangle = -\left|\psi_{A_\mathrm{A}}\right\rangle \Rightarrow P_{i\leftrightarrow ii}\left|\psi_{A_\mathrm{A}}\right\rangle = \left|\psi_{A_\mathrm{A}}\right\rangle \tag{25}$$

Then, every antisymmetric state in the $\mathrm{TPS_A}$ $\left|\psi_{A_\mathrm{A}}\right\rangle$ is a symmetric state $\left|\psi_{S_\mathrm{B}}\right\rangle$ in the $\mathrm{TPS_B}$

$$\forall \left|\psi_{A_\mathrm{A}}\right\rangle / \left|\psi_{A_\mathrm{A}}\right\rangle = A_\mathrm{A}\left|\psi\right\rangle \rightarrow \left|\psi_{A_\mathrm{A}}\right\rangle = \left|\psi_{S_\mathrm{B}}\right\rangle \tag{26}$$

However, the inverse relationship is not valid

$$P_{i\leftrightarrow ii}\left|\psi_{A_\mathrm{A}}\right\rangle = \left|\psi_{A_\mathrm{A}}\right\rangle \not\Rightarrow P_\alpha \left|\psi_{A_\mathrm{A}}\right\rangle = -\left|\psi_{A_\mathrm{A}}\right\rangle \tag{27}$$

This is easy to see in a trivial example. Let us consider the state

$$\begin{aligned}\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|n_1\right\rangle\otimes\left|n_2\right\rangle\otimes\left|n_3\right\rangle\otimes\left|n_4\right\rangle + \left|n_3\right\rangle\otimes\left|n_4\right\rangle\otimes\left|n_1\right\rangle\otimes\left|n_2\right\rangle\right)\\ \left|\psi\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|m_i\right\rangle\otimes\left|m_{ii}\right\rangle + \left|m_{ii}\right\rangle\otimes\left|m_i\right\rangle\right)\end{aligned} \tag{28}$$

If we apply the operator $P_{i\leftrightarrow ii}$ we obtain the same state

$$\begin{aligned}P_{i\leftrightarrow ii}\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|m_{ii}\right\rangle\otimes\left|m_i\right\rangle + \left|m_i\right\rangle\otimes\left|m_{ii}\right\rangle\right) = \left|\psi\right\rangle\\ P_{i\leftrightarrow ii}\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|n_3\right\rangle\otimes\left|n_4\right\rangle\otimes\left|n_1\right\rangle\otimes\left|n_2\right\rangle + \left|n_1\right\rangle\otimes\left|n_2\right\rangle\otimes\left|n_3\right\rangle\otimes\left|n_4\right\rangle\right) = \left|\psi\right\rangle\end{aligned} \tag{29}$$

But if we apply, for example $P_{1\leftrightarrow 2}\left|\psi\right\rangle$, we do not obtain the same state with changed sign

$$P_{1\leftrightarrow 2}\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|n_2\right\rangle\otimes\left|n_1\right\rangle\otimes\left|n_3\right\rangle\otimes\left|n_4\right\rangle + \left|n_4\right\rangle\otimes\left|n_3\right\rangle\otimes\left|n_1\right\rangle\otimes\left|n_2\right\rangle\right) \neq -\left|\psi\right\rangle \tag{30}$$

This means that to build the bosonic subalgebra of this model we cannot simply apply the canonical symmetrizer operator to a generic state in $\mathrm{TPS_B}$. If we define this operator in the standard way, $\tilde{S}_\mathrm{B} = \frac{1}{2}(I + P_{i\leftrightarrow ii})$, the space of observables of the form $O_U = \tilde{S}_\mathrm{B}^\dagger O_U \tilde{S}_\mathrm{B}$ is larger than $\mathcal{O}^F$. Then, we will build the bosonic subspace in this model by means of a symmetrizer operator defined as

$$S_\mathrm{B} = A_\mathrm{A} \tag{31}$$

Oddly enough, this operator is a legitimate symmetrizer for the TPS_B because all antisymmetric states in the TPS_A are symmetric in the TPS_B. The fact of adopting this symmetrizer and not the canonical operator means that we are restricting the bosonic space. We will not take into account all possible bosonic states, but some of them. This restriction is what prevents these bosons from forming a Bose-Einstein condensate. Then, we can apply it to our observables

$$S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}} = O'_U \tag{32}$$

That operation allows us to define a bosonic subalgebra $\mathcal{O}^B \subset \mathcal{O}_U$ such that

$$\forall O_B \in \mathcal{O}^B, O_B = S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}} \tag{33}$$

The observables generated with the operator S_{B} are “less” than those generated by $\tilde{S}_{\mathrm{B}}$; however it generates all that is necessary to describe this model. Then, $\mathcal{O}^B$ is the algebra with respect to which any generic state $|\psi\rangle$ will behave as symmetric

$$\langle\psi|O_B|\psi\rangle = \langle\psi|S_{\mathrm{B}}^{\dagger} O_U S_{\mathrm{B}}|\psi\rangle = \langle\psi_S|O_U|\psi_S\rangle \tag{34}$$

Since there is a direct relation between S_{B} and A_{A}, we have

$$\langle\psi|O_F|\psi\rangle = \langle\psi|A_A^{\dagger} O_U A_A|\psi\rangle = \langle\psi|S_B^{\dagger} O_U S_B|\psi\rangle = \langle\psi|O_B|\psi\rangle \tag{35}$$

Then, the same observables can be interpreted as fermionic observables from the TPS_A and as bosonic observables from the TPS_B. This fact invites us to change the notation with which we call the algebra of observables, instead of $\mathcal{O}^F$ we will use $\mathcal{O}_{\mathrm{A}}^F$ and instead of $\mathcal{O}^B$ we will use $\mathcal{O}_{\mathrm{B}}^B$. In this way, we can say that both algebras are the same, that is

$$\mathcal{O}_{\mathrm{A}}^F = \mathcal{O}_{\mathrm{B}}^B = \mathcal{O} \tag{36}$$

The difference in the notation is that, if $\mathcal{O}$ is considered from different partitions the system has fermionic or bosonic behavior.

Section 4 Conclusions

In the Hilbert space version of our toy model, we found that the system state was totally antisymmetric with respect to TPS_A and totally symmetric with respect to TPS_B. That result made it possible to consider that the fermionic or bosonic nature of a composite system of fermion-pairs is TPS-relative. That is

Fermionic character of U relative to partition $S_1 \cup S_2 \cup S_3 \cup S_4$ (see Subsection 2.1.1)

Bosonic character of U relative to partition $S_i \cup S_{ii} = U$ (see Subsection 2.1.2)

The algebraic version of the model makes that point much clearer since such relativity allows to build both a relative fermionic subalgebra and a relative bosonic subalgebra that can be equated to define the same composite system. Namely

System U defined by the algebra $\mathcal{O}_{\mathrm{A}}^{F} = \mathcal{O}_{\mathrm{B}}^{B} = \mathcal{O}$ (see eq. 36)

Nevertheless, this relativity of fermionic and bosonic conditions cries out for a proper ontology. One option is the one that arises from the traditional ontology of individuals that allows an emergent status of composite bosons and their behavior. In this scheme, a hierarchy of levels is maintained in which the fundamental particles are found at the basal level and the emerging particles are found at a higher level. However, there is another choice, to adopt an ontology of properties. The choice for an ontology of properties is motivated by a number of ontological challenges posed by quantum mechanics which are not studied in this work: quantum indistinguishability, quantum contextuality, and quantum non-locality obtain a simpler ontological interpretation by means of such ontology. The aim of this paper was to show that by adopting this ontology, a full-realistic interpretation of trans-statistical phenomenon is achievable, and, in turn, that such interpretation strengthens the choice for an ontology of properties already motivated to account for the quantum features mentioned above.

As a first point, we proposed that the ontology of properties fits properly with the aforementioned relativity of fermionic and bosonic notions. If an even number of identical bundles of properties with half-integer spin merge into a composite system, the fermionic or bosonic nature of the whole system can only be defined with respect to a previously specified partition. From an ontology of properties perspective, since quantum systems are not individuals, they generally do not retain identity conditions or keep singular reference when entering in a composite. As a consequence, the fermionic or bosonic character of the composite system is not attached anymore to identity conditions belonging to subsystems considered in their singularity. This is equivalent to saying that there is no a fundamental partition or TPS that gives rise to subsystems with identity conditions with respect to which the properties of the whole system can be defined in an absolute sense. From our toy model, we learned that composite bosons are exact bosons. If additionally, we interpret this model from an ontology of properties perspective, composite bosons are true bosons just as much as elementary bosons.

As a second point, we suggested that it is possible to complement our toy model built in a TPS-framework with the models based on creation and annihilation operators in order to realistically account for trans-statistical behavior. From the former, we learned that composite bosons are exact bosons and may be interpreted as real bosons when assuming the ontology of properties. Thanks to the latter we capture the empirical difference between statistical behavior of composite bosons and that of elementary bosons. The former alone would not let us account for full bosonic behavior. The latter alone would not allow us to consider composite bosons as real bosons, since bosonic behavior is approximated and we do not have, from within these models, other means to judge about the bosonic nature of a system than its statistical behavior. If they complement each other, it may be possible to interpret that fermion-pairs are exact composite bosons with real bosonic behavior. But a full account of this suggestion depends on a further assessment of the interplay between our model and those of Law 2005, Chudzicki *et al.* 2010 and Tichy *et. al* 2014, which is left for future work.

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