The Status of the Born Rule and the Role of Gleason's Theorem and Its Generalizations: How the Leopard Got Its Spots and Other Just-So Stories

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Approaching a hundred years since the publication of Born's epochal 1926 papers, the status of the Born rule is still the subject of lively discussion in the physics and philosophy literatures. Here I examine some approaches to justifying the Born rule within the mathematical framework that construes quantum probability theory as the study of probability measures on the projection lattices of von Neumann algebras. Of particular concern is the role of Gleason's theorem and its generalizations. A common line is to credit the Gleason theorems with providing a derivation of the Born rule, but then to complain that the theorems offer little physical insight into the emergence of quantum probabilities and the Born rule and/or that they commit the sin of "non-contextuality." It is argued that both the credit and the complaints are off the mark.

1 Introduction

What is the Born rule? What is its status in quantum theory? Is it an independent axiom or can it be justified by appeal to more basic principles? These questions will receive different answers if posed in different contexts.

One important context is historical: the struggle of the new quantum theory to be born. The story of the birth pangs takes a crucial turn with the advent of Schrödinger's wave mechanics and Schrödinger's electromagnetic interpretation of the ψ-function (Schrödinger 1926a-d). Schrödinger proposed that $|\psi|^2$ determines the charge distribution $\rho := e|\psi|^2$ of the electron, an interpretation he took to to be confirmed by the fact that his time-dependent wave equation implies that ρ obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} S = 0$$

where S is the current density

$$S := \frac{e\hbar}{2\pi i m}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

and m is the mass of the electron.

Born rejected Schrödinger's proposal because the spreading of the wave function seemed incompatible with the corpuscular nature of the electron, of which Born was convinced by the results of scattering experiments. Born proposed instead a statistical interpretation of ψ-function (Born 1926a-c), a proposal whose success was crowned by the award of the Nobel Prize in 1954. In his acceptance speech Born credited Einstein for inspiration:

> He [Einstein] had tried to make the duality of particles - light quanta or photons - and waves comprehensible by interpreting the square of the optical wave amplitudes as probability density for the occurrence of photons. This concept could at once be carried over to the ψ-function: $|\psi|^2$ ought to represent the probability density for electrons (or other particles) (1964, p. 262).

It was in fact Wolfgang Pauli (1927) who first proposed interpreting $|\psi|^2$ as the particle probability density.[1]

Despite—or perhaps, because of—the success of Born's proposal, others were not content to treat it as a bare postulate but were led to ask whether it can be derived from other postulates of the theory or, if not, whether it can be justified by appeal to other physically motivated principles. This discontent was expressed already in 1927 by von Neumann:

> The method hitherto used in statistical quantum mechanics was essentially deductive: the square of the norm of certain expansion coefficients of the wave function or of the wave function itself was fairly dogmatically set equal to a probability, and agreement with experience was verified afterwards. A systematic derivation of quantum mechanics from empirical facts or fundamental

[1] For discussions of how Born's statistical interpretation evolved see Wessels (1981), Pais (1982), and Jammer (1989, Ch. 5).

> probability-theoretic assumptions, i.e., an inductive justification, was not given. Moreover, the relation to the ordinary probability calculus was not sufficiently clarified: the validity of its basic rules (addition and multiplication law of the probability calculus) was not sufficiently stressed. (von Neumann, 1927, p, 246)[2]

The quest to establish a foundation for the Born rule continues today, over nine decades after Born made his inspired proposal.[3]

I am interested here in posing and answering the opening questions in the ahistorical setting of the now standard mathematical framework of quantum theory, both in its concrete Hilbert space incarnation and in the more abstract algebraic formulation; and more particularly, I am concerned with the implications of Gleason's theorem for the Born rule. The implications are often misconstrued and mischaracterized: the theorem is seen as providing a derivation of the Born rule, but the derivation is faulted for committing the sin of assuming "non-contextuality" and/or for not offering physical insight into the basis of the Born rule. I will argue that both parts of this received wisdom are flawed, and I aim to provide a more accurate picture of the relevance of Gleason's theorem and its generalizations for the Born rule and, more generally, the role of these theorems in the quantum theory construed as a theory of probability.

No violation of the Born rule has been detected in laboratory experiments. Nevertheless, reasons have been advanced for thinking that the rule is inadequate in the cosmological setting (see Page 2009). This fascinating matter will not be addressed here, and I proceed on the assumption that the Born rule is empirically adequate.

[2]Translation from Duncan and Janssen (2013, p. 246). This reference contains a comprehensive discussion of von Neumann (1927). von Neumann's own justification for the Born rule will be taken up below in Section 8.1.

[3]For a sampling of recent attempts to derive the Born rule see Auffèves and Grangier (2015), Barnett et al. (2014), Brezhnev (2019, 2020), Carroll and Sebens (2015), Drezet (2020), Frachinger and Renner (2017), Han and Choi (2016), Hosenfelder (2020, 2021), Illyin (2016), Lesovik (2014), Logiurato and Smerzi (2012), Masenes et al. (2019), Modelesi (2020), Saunders (2003), Schlosshauer and Fine (2005), Shrapnel et al. (2018), Vaidman (2020), Zurek (2003, 2005, 2018). Vaidman (2020) can be recommended for an overview of the topic.

2 The framework of quantum theory

2.1 Quantum theory in the algebraic setting: algebras and states

Here I will work with an algebraic framework that is broad enough to encompass not only ordinary non-relativistic QM but relativistic QFT and quantum statistical mechanics as well. A quantum system is characterized by two elements. First, there is a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$, which may be separable or non-separable. To characterize a quantum system by a von Neumann algebra $\mathfrak{N}$ is to posit that the self-adjoint elements of $\mathfrak{N}$ correspond to the observables of the system. A way to generate $\mathfrak{N}$ is to start with the subset $\mathcal{O}$ of self-adjoint operators (bounded or unbounded) that act on $\mathcal{H}$ and that are regarded as corresponding to observables of the system, and then define $\mathfrak{N}$ as the double commutant $\mathcal{O}'' := (\mathcal{O}')'$ of $\mathcal{O}$.[4] Every von Neumann algebra is generated in this manner by its self-adjoint elements.

Second, there is the set of algebra states $\mathcal{S}(\mathfrak{N})$ which consist of the normed positive linear functionals $\omega : \mathfrak{N} \to \mathbb{C}$. $\mathcal{S}(\mathfrak{N})$ is closed under convex linear combinations, $\lambda\omega_1 + (1-\lambda)\omega_2$ for $\omega_1, \omega_2 \in \mathcal{S}(\mathfrak{N})$ and $\lambda \in [0,1]$. (If you prefer a more abstract approach that does not use Hilbert spaces ab initio you can formulate much of the discussion to follow in terms of W^*-algebras which are C^*-algebras that are $*$-isomorphic to a concrete von Neumann algebra.) In physical applications a subset $\mathcal{S}_A(\mathfrak{N}) \subset \mathcal{S}(\mathfrak{N})$ of the set of admissible states may be singled out. The most familiar states are vector states: where ω is a vector state just in case there is a $|\omega\rangle \in \mathcal{H}$ such that $\omega(A) = \langle\omega|A|\omega\rangle$ for all $A \in \mathfrak{N}$. The state ω is said to be pure if it cannot be written as a non-trivial convex combination of two distinct states, otherwise ω is said to be mixed. Ordinary QM (sans superselection rules) concerns the special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on $\mathcal{H}$, typically assumed to be separable. But separable or not, for $\mathfrak{B}(\mathcal{H})$ the pure states coincide with the vector states. In the simple case of a spinless particle moving in three-space $\mathcal{H}$ is usually chosen to be $L^2_{\mathbb{C}}(\mathbb{R}^3)$, the elements of which are Schrödinger wave functions. Outside of ordinary QM—or even inside ordinary quantum mechanics with superselection rules—the von Neuman algebra can be more exotic than $\mathfrak{B}(\mathcal{H})$, and the vector states

[4] $\mathcal{O}'$ consists of all bounded operators A acting on $\mathcal{H}$ such that $[O, A] = 0$ for all $O \in \mathcal{O}$. If O is unbounded $[O, A] = 0$ means that A commutes will all of the spectral projections of O.

may not coincide with the pure states.

2.2 Hilbert space states vs. algebra states

Most philosophers as well as physicists who do not work in the algebraic framework tend to use the notion of quantum state in a somewhat different sense. For a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ "states" are often identified with unit vectors of $\mathcal{H}$ (more properly with equivalence classes of unit vectors where $|\omega\rangle \sim |\omega'\rangle$ iff $|\omega\rangle = e^{i\phi}|\omega'\rangle$) or more generally with the density operators $\mathcal{D}(\mathcal{H})$ (positive self-adjoint operators of trace one) acting on $\mathcal{H}$. Hilbert space states are associated with algebra states through the trace prescription i.e. the algebra state ω^{ϱ} associated with the density operator ϱ is $\omega^{\varrho}(A) = Tr(\varrho A)$, $A \in \mathfrak{N}$. The association is not onto since a non-completely additive algebra state does not correspond to any Hilbert space state. There is the potential here for confusion of terminology since in the Hilbert space state vernacular the pure vs. mixed state distinction is thought to correspond to the vector vs. non-vector state distinction (or in terms of density operators to $\varrho^2 = \varrho$ vs. $\varrho^2 \neq \varrho$). But when $\mathfrak{N}$ is not $\mathfrak{B}(\mathcal{H})$ the algebra state associated with a vector state can be mixed. In the algebraic language, the identification of quantum states with Hilbert space states amounts to the implicit stipulation that the physically admissible algebra states $\mathcal{S}_A(\mathfrak{N})$ are the normal states.

Normal algebra states have a number of equivalent characterizations, among which the following three are the most useful for present purposes (see Kadison and Ringrose 1997, Vol 2, Theorem 7.1.12):

> (i) $\omega \in \mathcal{S}(\mathfrak{N})$ is completely additive on any family of mutually orthogonal projections in $\mathfrak{N}$
>
> (ii) ω is represented by a density operator ϱ_ω, i.e. $\omega(A) = Tr(\varrho_\omega A)$ for all $A \in \mathfrak{N}$
>
> (iii) ω is weak-operator (or strong-operator) continuous on the unit ball of $\mathfrak{N}$.

Some reasons to identify the physically admissible algebra states with the normal states are discussed in Ruetsche (2011).

When discussing the Born rule the use of algebra states has the undesirable feature of appearing to beg the question because as expectation value functionals algebra states have a natural probabilistic interpretation. Nevertheless, the presentation of Gleason's theorem and its generalizations and other aspects of quantum theory is best framed in terms of algebra states. But the discussion below of the justification of the Born will be framed in terms of Hilbert space states.

2.3 Probability states

In the present approach quantum probabilities are assigned to projections $E \in \mathfrak{N}$, self-adjoint operators such that $E^2 = E$. The collection of projections is a lattice $\mathcal{P}(\mathfrak{N})$ when equipped with the natural partial order $\leq$ where, for $E, F \in \mathcal{P}(\mathfrak{N})$, $E \leq F$ iff $range(E) \subseteq range(F)$. Meet '$\wedge$' and join '$\vee$' are defined respectively as the least upper bound and greatest lower bound. Lattice complementation E^c is taken to be orthocomplementation, i.e. $E^c := E^{\perp} = I - E$. The elements of $\mathcal{P}(\mathfrak{N})$ are variously called events, propositions, or Yes-No questions, and they serve as the bearers of quantum probabilities.

Quantum probability theory can be construed as the study of quantum probability measures on the projection lattice $\mathcal{P}(\mathfrak{N})$ (see Hamhalter 1993a). A quantum probability measure is a map $P : \mathcal{P}(\mathfrak{N}) \to [0, 1]$ such that

A1. $P(I) = 1$ (I the identity operator)

A2. $P(E_1 \vee E_2) = P(E_1 + E_2) = P(E_1) + P(E_2)$ for all $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$ such that $E_1 \perp E_2$.

The axiom A2 of finite additivity is to be viewed as a minimal requirement on quantum probabilities. The strongest additivity axiom, called complete additivity, takes the form

A2′. $P(\sum_{a \in \mathcal{I}} E_a) = \sum_{a \in \mathcal{I}} P(E_a)$ for any family $\{E_a\} \subset \mathcal{P}(\mathfrak{N})$ such that $E_c \perp E_d$ when $c \neq d$

where the summation on the lhs of the equality is understood in the sense of the strong operator topology.[5] Finite (respectively, countable) additivity concerns the case where the index set $\mathcal{I}$ is finite (respectively, countable). When the Hilbert space $\mathcal{H}$ on which $\mathfrak{N}$ acts is separable complete additivity reduces to countable additivity, and when $\dim(\mathcal{H}) < \infty$ complete and countable reduce to finite additivity.[6] In the special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ countable additivity implies complete additivity unless $\dim(\mathcal{H})$ is as great as the least measurable cardinal (see Drish 1979 and Eilers and Horst 1975); the implication does not hold for more general von Neumann algebras (see Earman 2020c).

The set of probability measures on $\mathcal{P}(\mathfrak{N})$ is denoted by $\mathfrak{P}(\mathcal{P}(\mathfrak{N}))$. Each measure $P \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ constitutes a probability state of the system. Just as one might want to narrow the mathematically possible algebra states $\mathcal{S}(\mathfrak{N})$ to a proper subset of admissible states, so there may be motivation to narrow the mathematically possible probability states $\mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ to a proper subset of admissible probability states $\mathfrak{P}_A(\mathcal{P}(\mathfrak{N}))$, e.g. the completely additive states and/or the continuous states.

2.4 Probability states and algebra/Hilbert space states

Any algebra state $\omega \in \mathcal{S}(\mathfrak{N})$ determines a probability state $P_\omega \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$; namely, $P_\omega(E) := \omega(E)$ for all $E \in \mathcal{P}(\mathfrak{N})$ since, as the reader can easily verify, $P_\omega(\bullet)$ satisfies the basic axioms A1 and A2 for a quantum probability. When ω is normal and, thus, admits a density operator representation ϱ_ω, $P_\omega(E) := \omega(E) = Tr(\varrho_\omega E)$ for $E \in \mathcal{P}(\mathfrak{N})$. When the Hilbert space $\mathcal{H}$ on which $\mathfrak{N}$ acts is finite dimensional all algebra states are normal, and by the characteristic (iii) of normal states any probability measure on $\mathcal{P}(\mathfrak{N})$ induced by an algebra state is continuous. Thus with $\mathfrak{N}$ acting on a finite dimensional $\mathcal{H}$, if $\mathcal{P}(\mathfrak{N})$ admits discontinuous measures (in particular, dispersion free or $0-1$ measures) they do not extend to an algebra state. The relevance is this remark will become apparent shortly.

[5] When the index set $\mathcal{I}$ is uncountable the sum $\sum\limits_{a\in\mathcal{I}} P(E_a)$ is understood as $\lim_F \sum_{a\in F} P(E_a)$ where the F are finite subsets of $\mathcal{I}$, and $\lim_F \sum_{a\in F} P(E_a) = L$ means that for any $\epsilon > 0$ there is a finite $F_0 \subset \mathcal{I}$ such that for any finite F with $\mathcal{I} \supset F \supset F_0$, $|\sum_{a\in F} P(E_a) - L| < \epsilon$.

[6] Complete additivity implies that a fair infinite quantum lottery is impossible, e.g. a lottery with an infinite number of mutually exclusive and exhaustive outcomes, each of which has the same probability, namely, zero.

Do algebra-state induced probability measures exhaust the possible quantum probability measures, i.e. is it the case that $\{P_\omega(\bullet) : \omega \in \mathcal{S}(\mathfrak{N})\} = \mathfrak{P}(\mathcal{P}(\mathfrak{N}))\}$? And, if so, under what conditions do the admissible probability states $\mathfrak{P}_A(\mathcal{P}(\mathfrak{N}))$ coincide with the probability states $\{P_\omega(\bullet) : \omega \in \mathcal{S}_A(\mathfrak{N})\}$ induced by admissible algebra states? These questions arise in another form in evaluating the possibility of an alternative way of characterizing a quantum system, the success of which could be used to support the slogan that quantum theory is the theory of quantum probabilities (see Pitowsky 2006). Since a von Neumann algebra $\mathfrak{N}$ is determined by its projections in the sense that $\mathfrak{N}$ is the weak closure of $\mathcal{P}(\mathfrak{N})$ or, by von Neumann's double commutant theorem $\mathfrak{N} = \mathcal{P}(\mathfrak{N})''$, there is the possibility of inverting the above way of presenting a quantum system. Instead of starting with a von Neumann algebra and a set of algebra states one might start with a lattice of projections $\mathcal{P}$ on a Hilbert space $\mathcal{H}$ and the set of probability states $\mathfrak{P}(\mathcal{P})$ on $\mathcal{P}$, and from this basis construct the von Neumann algebra $\mathfrak{N}^{\mathcal{P}} =: \mathcal{P}''$ and the algebra states $\mathcal{S}(\mathfrak{N}^{\mathcal{P}})$. Is it the case that for any such $\mathcal{P}$, $\mathfrak{P}(\mathcal{P}) = \{P_\omega(\bullet) : \omega \in \mathcal{S}(\mathfrak{N}^{\mathcal{P}})\}$?

Gleason's theorem helps to answer such questions. But before turning to these matters let's add some physical content to the bare mathematical skeleton sketched above.

2.5 Minimalist interpretation rules

The commitment behind using a von Neumann algebra to characterize a quantum system and using the projection lattice of the algebra as the bearer of probabilities is contained in an observability rule which repeats and expands on what was said above:

> (O). To characterize a quantum system by a von Neumann algebra $\mathfrak{N}$ is to posit that the self-adjoint elements $\mathfrak{N}_{sa}$ of $\mathfrak{N}$ correspond to the observables of the system. In particular, the elements of the projection lattice $\mathcal{P}(\mathfrak{N})$ are observables in the sense that for any $E \in \mathcal{P}(\mathfrak{N})$ there is in principle a measurement procedure that will give a Yes (1)-No (0) answer to the the question "Is E true?"

Can a Yes answer be interpreted to mean that the measurement revealed the pre-existing fact that E true, or is the measurement implicated in making E true? Section 4 will provide an answer.

Additional empirical content can be added to the theory by providing a rule for updating algebra/Hilbert space states on measurement outcomes and an account of state preparation. The generally accepted updating rule is known as Lüders rule:

> (LR) If a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{N})$ yields a Yes answer and the pre-measurement state $\omega \in \mathcal{S}(\mathfrak{N})$ (pure or mixed) is normal and $\omega(F) > 0$ then the post-measurement state is $\omega_F(A) := \dfrac{\omega(FAF)}{\omega(F)}$, $A \in \mathfrak{N}$.[7]

This rule enables an account of state preparation for a class of algebra/Hilbert space states.

2.6 Algebra/Hilbert space state preparation

Recall that the support projection S_ω for a normal $\omega \in \mathcal{S}(\mathfrak{N})$ is the smallest projection in $\mathcal{P}(\mathfrak{N})$ such that $\omega(S_\omega) = 1$. If ω is a vector state then S_ω is the projection onto the ray spanned by a vector representative of ω. A projection $F_\omega \in \mathcal{P}(\mathfrak{N})$ is said to be filter for $\omega \in \mathcal{S}(\mathfrak{N})$ iff $\dfrac{\overline{\omega}(F_\omega A F_\omega)}{\overline{\omega}(F_\omega)} = \omega(A)$ for all $A \in \mathfrak{N}$ and any normal state $\overline{\omega} \in \mathcal{S}(\mathfrak{N})$ such that $\overline{\omega}(F_\omega) \neq 0$. Two basic facts about filters (see Earman and Ruetsche 2020):

> *Fact 1.* The support projection S_ω for a normal pure $\omega \in \mathcal{S}(\mathfrak{N})$ is a filter for ω.
>
> *Fact 2.* Mixed states do not have filters.

Now suppose that a Yes-No measurement of the support projection S_ω for a normal pure state $\omega \in \mathcal{S}(\mathfrak{N})$ gives a Yes answer. Then by (LR) no matter what the pre-measurement state $\overline{\omega} \in \mathcal{S}(\mathfrak{N})$ is, as long as $\overline{\omega}$ is normal and $\overline{\omega}(S_\omega) \neq 0$, the post-measurement state is $\overline{\omega}_{S_\omega}(A) := \dfrac{\overline{\omega}(S_\omega A S_\omega)}{\overline{\omega}(S_\omega)}$ for all $A \in \mathfrak{N}$. Since S_ω is a filter for ω we have $\overline{\omega}_{S_\omega} = \omega$, and the state ω may be

[7]Exercise: Show that if ω is normal state then so is ω_F.

deemed to have been prepared. Fact 2 explains why this procedure doesn't work for mixed states.

Several comments are in order. First, if the Born rule is added as an additional interpretational rule then to prepare the system in an algebra state/Hilbert space state ω is ipso facto to prepare the system in the probability state $P_\omega(E) := \omega(E)$, $E \in \mathcal{P}(\mathfrak{N})$. (This and other forms of the Born rule will be discussed below in Section 7). But since the Born rule is what is at issue here it would be desirable to have an independent account of probability state preparation, a matter that will be taken up below in Sections 8.2-8.3.

Second, the above account is purely formal.[8] The actual physics of algebra state/Hilbert space state preparation can be highly non-trivial, as seen from examples discussed in Lamb (1969).

Third, the Lüders rule (LR) embodies a generalized form of "collapse of the wave function" which leads directly to the measurement problem, the mother of all foundations problems in quantum theory. There are no-collapse interpretations of quantum theory, but they all give rise to their own problems. And what is important for present purposes is that any adequate interpretation of the theory must explain why actual experimental results make it seem that (LR) is true. With the help of (LR) the apparatus of the theory can be used to give a formal account of state preparation, which is essential to testing the operational content of the theory. Pending the demonstration of a viable alternative account of state preparation, I make no apology here for the use of (LR) in discussing the status of the Born rule.

Fourth, apologies are due for the limited scope and faux generality of the proffered account of state preparation: it applies only to normal pure states, and only Type I algebras admit such states. For the Type III algebras encountered in relativistic quantum field theory there is a work-around using the physical postulate that the Type III local von Neumann algebras $\mathfrak{N}(\mathcal{R})$ associated with open bounded regions $\mathcal{R}$ of Minkowski spacetime satisfy the split property. For then a normal state on $\mathfrak{N}(\mathcal{R})$—which is necessarily mixed—has a filter in the algebra a $\mathfrak{N}(\widetilde{\mathcal{R}})$ associated with a slightly larger local region $\widetilde{\mathcal{R}} \supset \mathcal{R}$ and, thus, states on $\mathfrak{N}(\mathcal{R})$ can be prepared by operations performed in $\widetilde{\mathcal{R}}$ (see Earman and Ruetsche 2020). Further wiggle room de-

[8] And it involves the conceit that we can choose not only *what* is measured but *when* the measurement of performed, a conceit that is highly questionable for reasons discussed in Pashby (2017).

rives from the observation that a mixed state ω on a system with algebra $\mathfrak{N}$ may result from the restriction to $\mathfrak{N}$ of a normal pure state $\widetilde{\omega}$ on the algebra $\widetilde{\mathfrak{N}} \supset \mathfrak{N}$ for a larger system of which the said system is a subsystem. But this wiggle room provides no real comfort if the larger system is so large, e.g. the entire cosmos, that we finite beings have no hope of preparing a pure state on such a system. Limited though it is, the account of algebra state/Hilbert space state preparation gives us some grip on the operational content of the Born rule; this content will be further explored below.

3 Gleason's theorem and its generalizations

3.1 The theorems

Adapted to the present setting the original version of Gleason's theorem concerns the special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ with $\mathcal{H}$ separable, the case typically assumed in ordinary non-relativistic QM.

> *Theorem 1* (Gleason 1957). Let P be a countably additive quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\mathcal{H}$ separable and $\dim(\mathcal{H}) > 2$. Then P extends uniquely to a normal state ω^P on $\mathfrak{B}(\mathcal{H})$.

The heart of Gleason's proof is to show that with $\mathcal{H}$ separable and $\dim(\mathcal{H}) > 2$ a countably additive probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is continuous in the weak operator topology. From there it is a small step to the theorem's conclusion since the projections of $\mathfrak{B}(\mathcal{H})$ are weakly dense in $\mathfrak{B}(\mathcal{H})$. The restriction to $\dim(\mathcal{H}) > 2$ is necessitated by the fact that when $\dim(\mathcal{H}) = 2$, where countable additivity collapses to finite additivity and all algebra states are normal, there are discontinuous probability measures (among which are dispersion free measures) that perforce are not induced by any normal state on $\mathfrak{B}(\mathcal{H})$.

Using a different proof technique Theorem 1 was generalized to cover merely finitely additive probability states on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$:

> *Theorem 2.* Let P be a quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\mathcal{H}$ separable and $\dim(\mathcal{H}) > 2$. Then P extends uniquely to a state ω^P on $\mathfrak{B}(\mathcal{H})$. ω^P is normal if P is countably additive and non-normal if P is merely finitely additive.

The restriction to separable $\mathcal{H}$ can also be removed by strengthening the additivity requirement:

> *Theorem 3.* Let P be a quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\mathcal{H}$ may be non-separable and $\dim(\mathcal{H}) > 2$. Then P extends uniquely to a state ω^P on $\mathfrak{B}(\mathcal{H})$. ω^P is normal if P is completely additive and non-normal otherwise.

Separable Hilbert spaces suffice for most applications of quantum theory, but non-separable spaces may have a role to play in some approaches to quantum gravity, and they are needed to describe some idealized systems such as infinite spin chains (see Earman 2020a for more details).

The Born rule is typically stated for $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, the case for ordinary non-relativistic quantum mechanics, but it has a straightforward generalization to arbitrary von Neumann algebras (see Section 7); and the validity of the generalized rule should extend to all the algebras encountered in applications of quantum theory. Thus, if Gleason's theorem is to be relevant to the Born rule it too should have a generalization to arbitrary von Neumann algebras. The extension required the labor of several mathematicians and mathematical physicists over the span of two decades. The labor resulted in

> *Theorem 4.* Let P be a quantum probability measure on $\mathcal{P}(\mathfrak{N})$ where $\mathfrak{N}$ is a von Neumann algebra with no Type I_2 summands. Then P extends uniquely to a state ω^P on $\mathfrak{N}$. ω^P is normal if P is completely additive and non-normal otherwise.[9]

3.2 Upshot

Anticipating the discussion of the Born rule below it will be helpful to state an implication of the Gleason theorems for the relationship between Hilbert space states and probability states. To save ink I will use "P is completely additive/continuous" to mean that P is a completely additive measure on $\mathcal{P}(\mathfrak{N})$ and, when complete additivity does not imply continuity, as can happen when $\mathfrak{N}$ contains Type I_2 summands, P is also continuous.

[9] For proofs and discussion of the generalizations Gleason theorems see Christensen (1982), Yeardon (1984), and Hamhalter (1993a).

(U) For any von Neumann algebra $\mathfrak{N}$ which acts on a Hilbert space $\mathcal{H}$ there is a one-one correspondence between the completely additive/continuous probability states on $\mathcal{P}(\mathfrak{N})$ and Hilbert space states $\mathcal{D}(\mathcal{H})$ on $\mathcal{H}$, wherein the corresponding probability state $P \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ and Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ satisfy $P(E) = Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$.

4 Can one do better than Gleason?

Why isn't Gleason's theorem completely general? For the case $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ Gleason's theorem does not apply when $\dim(\mathcal{H}) = 2$. In this case there are probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that are not weak-operator (or strong-operator) continuous, among which are the dispersion free probability measures (see the following Section). Such discontinuous measures do not extend to a state on $\mathfrak{B}(\mathcal{H})$ since (to repeat) for $\dim(\mathcal{H}) < \infty$ all algebra states are normal and normal states are weak-operator (and strong-operator) continuous on the unit ball of $\mathfrak{B}(\mathcal{H})$.

There is a way to do better than Gleason by changing the subject; namely, by changing the space of events to which probabilities are assigned. Such is the way of the approach to quantum theory championed by Busch and collaborators (see Busch et al. 1995) which challenges the view that observables in quantum theory correspond to self-adjoint operators and that outcomes of measurements correspond to projection operators. The proposed alternative is to treat the outcomes of measurements in terms of "effects." Define the effect algebra $\mathcal{A}(\mathfrak{N})$ associated with a von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$ by $\mathcal{A}(\mathfrak{N}) := \{A \in \mathfrak{N}_{sa} : 0 \preceq A \preceq I\}$ where $A \preceq B$ iff $B - A$ is a positive operator, i.e. $\langle (B - A)\psi, \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$.[10] A countably additive "generalized probability measure" on $\mathcal{A}(\mathfrak{N})$ is a map $p : \mathcal{A}(\mathfrak{N}) \to [0, 1]$ such that

A1*. $p(I) = 1$ (I the identity operator)

A2*. $p(\sum_{a \in I} A_a) = \sum_{a \in I} p(A_a)$ where I is a countable set and the $A_a \in \mathcal{A}(\mathfrak{N})$ are any effects such that $\sum_{a \in I} A_a$ is an effect (i.e.

[10] Note that $\mathcal{A}(\mathfrak{N})$ is not a lattice under $\precsim$.

$\sum_{a\in I} A_a \preceq I$).[11]

For the case of ordinary QM with $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and separable $\mathcal{H}$ Busch (2003) shows that a countably additive generalized probability measure on the effect algebra $\mathcal{A}(\mathfrak{B}(\mathcal{H}))$ has, regardless of $\dim(\mathcal{H})$, a unique extension to a normal quantum state. I am unaware of a proof that this result extends to the case of a non-separable $\mathcal{H}$ and to more general von Neumann algebras. There is also the problem of crafting an attractive rule for updating probabilities of effects. But this is not the place to debate the merits of the effects approach.

The other way to overcoming the exceptions to Gleason's theorem is to keep $\mathcal{P}(\mathfrak{N})$ as the event space and to supply reasons for counting as inadmissible the discontinuous probability measures on $\mathcal{P}(\mathfrak{N})$ that arise when $\mathfrak{N}$ has a Type I_2 summand. Prospects for success of this tack will be receive attention in Sections 9.1-9.2.

5 The nature of quantum probabilities, dispersion free measures, hidden variables, and all that

The ramifications of the Gleason theorems are felt throughout the structure of quantum theory, so it would be surprising if these theorems did not have important implications for the status of the Born rule. I begin with an explanation of how the Gleason theorems imply that quantum probabilities have to be understood as propensity probabilities and not as ignorance probabilities; that is, why the probability $P(E)$, $E \in \mathcal{P}(\mathfrak{N})$, cannot be uniformly interpreted as the probability that a Yes–No measurement of E will reveal an unknown but preexisting Yes-No value.

As is well known, $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ admits dispersion free $(0-1)$ probability states when $\dim(\mathcal{H}) = 2$. Gleason's theorem implies that there are no dispersion free and completely additive probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ when $\dim(\mathcal{H}) \geq 3$. When the probability measure is completely additive Gleason's theorem tells us that the measure extends uniquely to a normal state ω, which can be written in the form

[11] Note that when applied to projections the condition $\sum_{a\in I} A_a \preceq I$ implies that the A_a are mutually orthogonal.

$$\omega = \sum_{n=1}^{\infty} c_n \omega_{|\psi_n\rangle}$$

where the $\omega_{|\psi_n\rangle}$ are vector states corresponding to the members of an orthonormal family of vectors $|\psi_n\rangle \in \mathcal{H}$. If ω were dispersion free (i.e. $\omega(F) \in \{0, 1\}$ for all $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$) then since $\omega(E_{|\psi_n\rangle}) = c_n$, where $E_{|\psi_n\rangle}$ is the projection onto the ray spanned by $|\psi_n\rangle$, it follows that $\omega = \omega_{|\psi_{n*}\rangle}$ for some n^*. But this contradicts the fact that a vector state takes values other than 0 and 1 on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ (Hamhalter 1993a, 89-91). This no-go result can be extended to cover merely finitely additive probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\dim(\mathcal{H}) \geq 3$ (see Hamhalter 2003a, Theorem 3.4.1). No-go results for a finite family of projections on $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 3$ were first obtained by Kochen and Specker (1967) and for even smaller families by subsequent researchers (see Cabello 1996).

These no-go results for the special case of $\mathfrak{B}(\mathcal{H})$ takes advantage of the fact that every projection on $\mathcal{H}$ belongs to the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. This luxury is unavailable for more general von Neumann algebras; for example if $\mathfrak{N}$ is Type III then $\mathcal{P}(\mathfrak{N})$ contains no finite dimensional projections. Nevertheless, go and no-go results are available for general von Neumann algebras. An arbitrary von Neumann algebra $\mathfrak{N}$ with Type I_2 summands $\mathcal{P}(\mathfrak{N})$ admits a dispersion free measure (see Hamhalter 1993b, 185-186). On the other hand, if $\mathfrak{N}$ has no Type I_2 summands the generalized Gleason theorems can be used to prove

> *Theorem 5* (Hamhalter 1993b). Let $\mathfrak{N}$ be a von Neumann algebra without Type I_2 summands, and let $\mathcal{Z}(\mathfrak{N}) := \mathfrak{N} \cap \mathfrak{N}'$ denote the center of $\mathcal{Z}$. If P is a finitely additive dispersion free measure on $\mathcal{P}(\mathfrak{N})$ then there is an abelian projection $E \in \mathcal{Z}(\mathfrak{N})$ such that $P(E) = 1$.

So if $\mathfrak{N}$ has no Type I_2 summands the dispersion free measures on $\mathcal{P}(\mathfrak{N})$ have to be concentrated on an abelian (= classical) part $E\mathcal{P}(\mathfrak{N})$ of the projection algebra. Put the other way round, if $\mathfrak{N}$ has no Type I_1 or I_2 summands then there are no dispersion free measures on $\mathcal{P}(\mathfrak{N})$ (see Döring 2004). The condition that $\mathfrak{N}$ has no Type I_1 summands is necessary for rule out dispersion free measures.[12] Putting these pieces together, we arrive at

[12] Suppose that there is a non-zero central abelian projection $E \in \mathcal{P}(\mathfrak{N})$. Since $\mathcal{Z}(\mathfrak{N})$ is

Upshot: $\mathcal{P}(\mathfrak{N})$ admits no dispersion free states iff $\mathfrak{N}$ is without either Type I_1 or Type I_2 summands.

I am unaware of results showing that some finite substructure of $\mathcal{P}(\mathfrak{N})$ admits no dispersion free states iff $\mathfrak{N}$ is without either Type I_1 or Type I_2 summands.

A no-go result on dispersion free probability measures implies in turn a no-go result on truth valuations for the propositions $\mathcal{P}(\mathfrak{N})$. A truth value assignment is a map $V : \mathcal{P}(\mathfrak{N}) \to \{1, 0\}$, with 1 standing for $true$ and 0 for $false$. Natural constraints on such an assignment follow the constraints for classical propositional logic with orthogonality of projections taking the place of logical inconsistency in classical logic. For present purposes all that is needed are the following constraints on V:

(α) $V(I) = 1$

(β) For any mutually orthogonal $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$, if $V(E_1) = 1$ then $V(E_2) = 0$.

(γ) For any mutually orthogonal $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$, $V(E_1 \vee E_2) = V(E_1 + E_2) = 1$ if either $V(E_1) = 1$ or $V(E_2) = 1$, and $V(E_1 \vee E_2) = 0$ if both $V(E_1) = 0$ and $V(E_2) = 0$.

If such an assignment existed then $\Pr : \mathcal{P}(\mathfrak{N}) \to \{1, 0\}$, where $\Pr(E) = 1$ if $V(E) = 1$ and $\Pr(E) = 0$ if $V(E) = 0$, would define a dispersion free probability measure. Thus, when the no-go results for dispersion free measures apply to $\mathcal{P}(\mathfrak{N})$ a probability measure on $\mathcal{P}(\mathfrak{N})$ cannot be consistently interpreted as assigning probabilities that Yes-No experiments reveal pre-existing truth values of the propositions in $\mathcal{P}(\mathfrak{N})$ if those values satisfy the minimal requirements (α)-(γ).

The connection to Kochen-Specker type no-go results for von Neumann algebras is easily made. With $\mathfrak{N}_{sa}$ denoting the self-adjoint elements of $\mathfrak{N}$, a Kochen-Specker valuation function for a von Neumann algebra $\mathfrak{N}$ is a mapping $V_{ks} : \mathfrak{N}_{sa} \to \mathbb{R}$ which satisfies

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abelian there are dispersion free measures on $\mathcal{P}(\mathcal{Z}(\mathfrak{N}))$—any pure state on $\mathcal{Z}(\mathfrak{N})$ induces such a measure. Pick such a P with $P(E) \neq 0$. Define $\overline{P}(F) := \dfrac{P(EF)}{P(E)}$ for $F \in \mathcal{P}(\mathfrak{N})$. Obviously, $\overline{P}(I) = 1$. To show that $\overline{P}(F_1 \vee F_2) = \overline{P}(F_1 + F_2) = \overline{P}(F_1) + \overline{P}(F_2)$ for all $F_1, F_2 \in \mathcal{P}(\mathfrak{N})$ such that $F_1F_2 = F_2F_1 = 0$ use the fact that $EF = FE = EFE$ for any $F \in \mathcal{P}(\mathfrak{N})$ and, thus if $F_1F_2 = F_2F_1 = 0$ then $(EF_1E)(EF_2E) = (EF_2E)(EF_1E) = 0$. So $\overline{P}$ is a dispersion free measure on $\mathcal{P}(\mathfrak{N})$.

(i) (Spectrum rule) $V_{ks}(A) \in spec(A)$ for all $A \in \mathfrak{N}_{sa}$.

(ii) (FUNC rule) For all Borel functions $f : \mathbb{R} \to \mathbb{R}$, $V_{ks}(f(A)) = f(V_{ks}(A))$ for $A \in \mathfrak{N}_{sa}$.

These conditions imply that $V_{ks}(E) \in \{0, 1\}$ for $E \in \mathcal{P}(\mathfrak{N})$, $V_{ks}(I) = 1$, and $V_{ks}(E_1 + E_2) = V_{ks}(E_1) + V_{ks}(E_2)$ for orthogonal $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$. Hence, the existence of a K-S valuation function entails the existence of a dispersion free probability measure on $\mathcal{P}(\mathfrak{N})$, and a no-go result on dispersion free measures on the projection lattice of a von Neumann algebra entails a no-go K-S result for $\mathfrak{N}$ (see Döring 2004).

Note that these no-go results preclude the class of hidden variable theories that seek to "complete" quantum theory by adjoining additional variables in way that expands but does not distort the structure of $\mathcal{P}(\mathfrak{N})$; in particular, if the no-go results apply to $\mathcal{P}(\mathfrak{N})$ then $\mathcal{P}(\mathfrak{N})$ cannot be embedded in an abelian algebra.

6 Quantum symmetries and quantum dynamics

Not only do the generalized Gleason theorems help to codify the probabilistic structure of quantum theory and preclude some types of hidden variable interpretations, they also show that the structure of the event space $\mathcal{P}(\mathfrak{N})$ underwrites important aspects of the symmetries and dynamics of quantum theory.

The concept of a Jordan $*$-automorphism[13] captures the idea of a struc-

[13] A Jordan $*$-automorphism of $\mathfrak{N}_{sa}$ is defined as a bijection $\theta : \mathfrak{N}_{sa} \to \mathfrak{N}_{sa}$ such that for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}_{sa}$

$$\begin{aligned} \theta(\lambda A + \mu B) &= \lambda\theta(A) + \mu\theta(B) \\ \theta(A^*) &= \theta(A)^* \\ \theta(A \circ B) &= \theta(A) \circ \theta(B) \end{aligned}$$

where Jordan multiplication $\circ$ is defined by $A \circ B := \frac{1}{2}(AB + BA)$. A Jordan $*$-automorphism θ can be extended to a bijection $\Theta : \mathfrak{N} \to \mathfrak{N}$ using the fact that any $A \in \mathfrak{N}$ can be uniquely decomposed as $A = R + iS$ with $R, S \in \mathfrak{N}_{SA}$, and by setting $\Theta(A) := \theta(R) + i\theta(S)$. So a Jordan $*$-automorphism of $\mathfrak{N}$ is defined as a bijection Θ of $\mathfrak{N}$

ture preserving transformation of the self-adjoint elements of the von Neumann algebra $\mathfrak{N}$ of a quantum system. And this concept links together other notions of quantum symmetry, such as preservation of the structure of the state space $\mathcal{S}(\mathfrak{N})$, preservation of transition probabilities, and preservation of expectation values (see Earman 2020b). This link underscores the importance of a perhaps less familiar notion of quantum symmetry. An event space symmetry (aka quantum logic symmetry) is a bijection $\Pi : \mathcal{P}(\mathfrak{N}) \rightarrow \mathcal{P}(\mathfrak{N})$ preserving the lattice structure; in particular, Π preserves orthogonality relations between projections. Call Π an orthoautomorphism if it preserves orthogonality in both directions.

> *Theorem 6* (Dye 1955). If $\mathfrak{N}$ is a von Neumann algebra containing no Type I_2 summands then any orthoautomorphism of $\mathcal{P}(\mathfrak{N})$ extends to a unique Jordan $*$-automorphism of $\mathfrak{N}$.

Dye's theorem can be obtained as consequence of the generalized Gleason theorems (see Bunce and Wright 1993).

When restricted to $\mathcal{P}(\mathfrak{N})$ any Jordan $*$-automorphism of $\mathfrak{N}$ induces a symmetry of $\mathcal{P}(\mathfrak{N})$ and, conversely, every Jordan $*$-automorphism of $\mathfrak{N}$ is obtained as a unique extension of a symmetry of $\mathcal{P}(\mathfrak{N})$. Thus, thanks to the Dye and Gleason theorems, all of the familiar symmetries of quantum theory can be seen to derive from event space symmetries. This opens the road to fulfilling von Neumann's quest for a derivation of quantum theory from "fundamental probability-theoretic assumptions" by building up the theory from $\mathcal{P}(\mathfrak{N})$, its symmetries, and probability measures on $\mathcal{P}(\mathfrak{N})$.

The road to dynamics for factor algebras is relatively straight.[14] For a factor algebra $\mathfrak{N}$ a Jordan $*$-automorphism is either a $*$-automorphism or

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such that for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}$

$$\begin{aligned} (i)\ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \Theta(A^*) &= \Theta(A)^* \\ (iii)\ \Theta(AB + BA) &= \Theta(A)\Theta(B) + \Theta(B)\Theta(A). \end{aligned}$$

[14] That $\mathfrak{N}$ is a factor means that $\mathfrak{N} \cap \mathfrak{N}' = \emptyset$, where $\mathfrak{N}'$ is the commutant of $\mathfrak{N}$. The story for non-factors is more complicated and will not be treated here.

a $*$-anti-automorphism.[15] Any von Neumann algebra is $*$-isomorphic to a von Neumann algebra in standard form, and any $*$-automorphism Θ (respectively, $*$-anti-automorphism) of an algebra $\mathfrak{N}$ in standard form is unitarily (respectively, anti-untarily implementable); that is there is a unitary map $U : \mathcal{H} \to \mathcal{H}$ (respectively anti-unitary $V : \mathcal{H} \to \mathcal{H}$) such that $\Theta(A) = U^*AU$ (respectively, $\Theta(A) = V^*A^*V$) for all $A \in \mathfrak{N}$. If the dynamics for observables is supplied by a one-parameter group Δ_t, $t \in \mathbb{R}$, of symmetries of $\mathfrak{N}$ it must be in the form of $*$-automorphisms of $\mathfrak{N}$ since $*$–anti-automorphisms do not form a group (the composition of two $*$–anti-automorphisms being a $*$-automorphism). The group Δ_t embodies the Heisenberg form of dynamics for observables $\mathfrak{N} \ni A \mapsto A^t$, $A^t := \Delta_t(A)$, with accompanying probability dynamics $P \mapsto P^t$, $P^t(E) := P(\Delta_t(E))$ for $E \in \mathcal{P}(\mathfrak{N})$. The corresponding Schrödinger form of dynamics for algebra states is $\omega \mapsto \omega^t$ with $\omega^t(A) := \omega(\Delta_t(A))$ for $A \in \mathfrak{N}$. With the algebra in standard form the group Δ_t is implementable by a unitary group U_t, $\Delta_t(A) = U_t^* A U_t$, and if U_t is strongly continuous it has a self-adjoint generator which serves as the Hamiltonian of the system.

7 The Born rule and Gleason's theorems

We are finally in a position to discuss the Born rule and its relationship with the Gleason theorems.

[15] A $*$-automorphism of $\mathfrak{N}$ is a bijection $\Theta : \mathfrak{N} \to \mathfrak{N}$ satisfying

$$\begin{aligned} (i)\ \ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \ \Theta(A^*) &= \Theta(A)^* \\ (iii')\ \Theta(AB) &= \Theta(A)\Theta(B). \end{aligned}$$

A $*$-anti-automorphism is a bijection of $\mathfrak{N}$ satisfying

$$\begin{aligned} (i)\ \ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \ \Theta(A^*) &= \Theta(A)^* \\ (iii'')\ \ \Theta(AB) &= \Theta(B)\Theta(A). \end{aligned}$$

7.1 Formulating the Born rule

Consider a quantum system characterized by a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$. In the terminology used here, the Born rule is a rule that links algebra states/Hilbert space states to probability states on the projection lattice $\mathcal{P}(\mathfrak{N})$. The use of algebra states permits a generalized and elegant formulation of the Born rule that applies to arbitrary von Neumann algebras:

> (BRA) If a system characterized by a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ is in the algebra state $\omega \in \mathcal{S}(\mathfrak{N})$ then it is in the probability state $P_\omega \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ where $P_\omega(E) := \omega(E)$ for all $E \in \mathcal{P}(\mathfrak{N})$.

But because algebra states have a natural interpretation as expectation value functionals such a formulation might seem to beg questions about the nature and status of the Born rule, the formulation to be considered here is couched in terms of Hilbert space states. Additionally, this formulation is congruent with the bulk of the physics and philosophy literature which tends to identify the Born rule with the "trace rule":

> (BR) If a system characterized by a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ is in the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ then it is in the probability state $P_\varrho \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ where $P_\varrho(E) := Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$.

Under the mild Assumption (which seems to be implicit in the literature) that at any given time a system is in one and only one Hilbert space state and one and only one probability state, (BR) has some immediate consequences for what counts as a physically realizable probability state. In particular, a physically realizable probability state must be completely additive and also continuous, regardless of whether $\mathfrak{N}$ contains Type I_2 summands.[16] For by the Assumption a system is always in some Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ and thus, by (BR), is in the probability state P_ϱ which is continuous and completely additive; so by the Assumption the system can never be in a

[16] (BRA) does not imply that physically realizable probability states are completely additive; but this is because it leaves hanging the issue of whether physically realizable algebra states must be normal.

non-continuous or non-completely additive probability state. Furthermore, under the Assumption (BR) presupposes that every physically realizable probability state consistent with (BR)—which as we have just seen must be continuous and completely additive—extends to a Hilbert space state, a condition guaranteed by the generalized Gleason theorems.

A little further work shows that under the Assumption (BR) entails the converse Born rule

> (CBR) If a system characterized by a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ is in probability state $P \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ then it is in the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ such that $P = P_\varrho$ where $P_\varrho(E) = Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$.

Suppose that, contrary to (CBR), the system is in probability state P but not in Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ such that $P = P_\varrho$. By the Assumption the system is in some other Hilbert space state $\overline{\varrho} \in \mathcal{D}(\mathcal{H})$ distinct from ϱ. But then by (BR) the system is in probability state $P_{\overline{\varrho}}$ where $P_{\overline{\varrho}}(E) = Tr(\overline{\varrho}E)$ for all $E \in \mathcal{P}(\mathfrak{N})$. Since $\overline{\varrho} \neq \varrho$, probability states $P_{\overline{\varrho}}$ and P_ϱ are distinct, producing a contradiction with the Assumption.[17]

And a little more work shows that (CBR) entails (BR) so that under the Assumption the Born rule is equivalent to

> $(BR*)$ At any time a system characterized by a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ is in a Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ and a probability state $P \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ where the co-instantiated states satisfy $P(E) = Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$.

7.2 Born and Gleason

Sprinkled throughout the literature are comments that begin by paying lip service to Gleason's theorem, granting that it provides a derivation of the Born rule, but then go on to complain that the theorem offers little

[17] If $\overline{\varrho} \neq \varrho$ then it is not the case that $Tr(\overline{\varrho}E) = Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$. Since $\mathfrak{N}$ is the weak closure of $\mathcal{P}(\mathfrak{N})$ the equality $Tr(\overline{\varrho}E) = Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$ would imply $Tr(\overline{\varrho}A) = Tr(\varrho A)$ for all $A \in \mathfrak{N}$ and, hence, $\overline{\varrho} = \varrho$.

physical insight into the basis of the Born rule[18] and/or to fault the justification for assuming "non-contextuality" of probability assignments. The non-contextuality complaint will be taken up below in Appendix 2, and Sections 8 and 9 will address the first part of the complaint. But here I want to indicate why the complaint does not go far enough: without the help of additional assumptions about the relation between probability states and Hilbert space states the Gleason theorems do not provide a straightforward derivation or justification of the Born rule.

As noted above, the Born rule in the guise (BR) presupposes that for a system with von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$ every continuous and completely additive probability state on $\mathcal{P}(\mathfrak{N})$ extends to a Hilbert space state on $\mathcal{H}$, a presumption that is guaranteed by the generalized Gleason theorems. So far so good. But the Born rule in the guise (BR) also presupposes that physically realizable probability states are continuous and completely additive, a presumption untouched by the Gleason theorems. More importantly, while there is a striking resemblance between the Born rule in guise $(BR*)$ and the upshot (U) of the Gleason theorems (recall Section 3.2) in that both concern the same one-one correspondence between probability states and Hilbert space states, as mathematical theorems the Gleason theorems do not, without the help of further assumptions, guarantee the co-instantiation of the corresponding states (i.e. that a system is in Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ if and only if it is in the Born rule probability state P_ϱ) which is what $(BR*)$ requires. In particular, while the Gleason theorems guarantee that any completely additive/continuous probability measure on $\mathcal{P}(\mathfrak{N})$ can be calculated from the trace rule for some $\varrho \in \mathcal{D}(\mathcal{H})$, the theorems do not guarantee that when a system is in the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ the trace rule probabilities calculated from *that* ϱ, rather than some other $\overline{\varrho} \in \mathcal{D}(\mathcal{H})$, are the probabilities that govern outcomes of Yes-No measurements of elements of $\mathcal{P}(\mathfrak{N})$.

What it means for a system to be "in" a probability state depends, of course, on the interpretation of probability. It seems appropriate to begin with the statistical interpretation of probability intended by Born, von Neumann, Einstein and other founders of the new quantum theory. Their in-

[18]For example: "In spite of its mathematical elegance, Gleason's theorem is usually considered as giving rather little physical insight into the emergence of quantum probabilities and the Born rule" (Schlosshauer and Fine 2004, 198). These authors are not endorsing the view, but they are accurately reporting a prevalent attitude.

terpretation is often referred to as an ensemble interpretation[19], a label that carries with it an anti-Copenhagen tinge wherein the Hilbert space state is not taken to characterize an individual quantum system but rather an ensemble of systems in the same Hilbert space state and, correspondingly, the Born rule probabilities calculated from the Hilbert space state are not taken to refer to single-case outcomes of measurements but rather to the statistics of outcomes of measurements on systems in the ensemble. This raises a tangle of issues in the foundations of physics and the foundations of probability theory that need not sidetrack us. Advocates of a single-case interpretation of probability as well as advocates of a statistical interpretation agree that to claim a probability measure governs the outcomes of measurements implies that the probabilities should be reflected in the statistics of outcomes of measurements on a large ensemble of identically prepared systems. Some early advocates of the statistical interpretation thought that probabilities could be identified with limiting relative frequencies of outcomes, but it is now well understood that this is not a live option for countably additive probability measures (see van Fraassen 1977). On both the statistical interpretation and the single-case interpretation a link between probabilities and limiting relative frequencies is forged by the law of large numbers for identically prepared systems; but since no actual experiment involves an infinite number of trials the practical link is one of confirmation/disconfirmation between probability assertions and frequencies observed in a finite number of trials. The details

[19] Duncan and Janssen (2013) suggest that von Neumann's ensemble interpretation may have derived from an acquaintance with the work of Richard von Mises. Einstein's allegiance to the ensemble interpretation is found in several places, including his essay "Physics and Reality":

> It seems to be clear, therefore, that Born's statistical interpretation of quantum theory is the only possible one. The ψ function does not in any way describe a state which could be that of a single system; it relates rather to many systems, to "an ensemble of systems" in the sense of statistical mechanics. If, except for certain special cases, the ψ function furnishes only *statistical* data concerning measurable magnitudes, the reason lies not only in the fact that the *operation of measuring* introduces unknown elements, which can be grasped only statistically, but because of the very fact that the ψ function does not in any sense, describe the state of *one* single system. (Einstein 1936, p. 375)

Born's own version of the statistical interpretation is difficult to characterize, and it is left to the historians of science to provide it.

of these matters are discussed in Appendix 1 and need not detain us here.

The next Section discusses routes to a justification of the Born rule under the assumption that quantum probabilities, whether given a statistical or single-case interpretation, correspond to objective, observer-independent features of quantum systems.

8 Some strategies for justifying the Born rule

In this section I will examine some strategies for justifying the Born rule that operate under two presumptions. The first (Objective) is that both Hilbert space states and probability states record objective, observer-independent features of quantum systems. The second presumption (None) is really a non-presumption; namely, no a priori assumption is made about dependency relationships between probability states and Hilbert space states—in particular, no assumption to the effect that either type of state is ontologically prior to the other, or that one grounds the other, or that one supervenes on the other, or whatever form of symmetric or asymmetric dependence one cares to use. Dependency relations there may be but, if so, None presumes that they must derive from the laws of the theory rather than from metaphysical doctrines about the nature of states. In Section 9 I will discuss how relaxing one or both of these presumptions changes how the status and the justification of the Born rule are regarded.

8.1 A non-effective strategy

One commonly followed strategy for justifying the Born rule consists of drawing up a list of conditions that are deemed desirable in a rule for using Hilbert space states to assign probabilities to outcomes of measurements, and then proving that the list is uniquely satisfied by the Born rule. The earliest example comes from von Neumann (1927) who gave a justification for the more general trace prescription $\langle 0 \rangle = Tr(\varrho 0)$ for calculating the expectation value $\langle 0 \rangle$ of a self-adjoint operator 0 from a state $\varrho \in \mathcal{D}(\mathcal{H})$. Von Neumann relied on the controversial condition that $\langle 0_1 \rangle + \langle 0_2 \rangle = \langle 0_1 + 0_2 \rangle$ for all self-adjoint 0_1 and 0_2, including those that do not commute. Later writers have appealed to a variety of desiderata including envariance (Zurek 2005, 2018), bit symmetry (Galley and Masanes 2017), and constraints on compositional structure such as the principles of purification and tomography (Galley and

Masenes 2018). Some of uniqueness results are impressive, especially those that operate within a framework much broader than the standard version of quantum theory considered here. But most of the results in this line of research tend to focus on special cases such as $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and $\dim(\mathcal{H}) < \infty$. Even assuming that these results can be generalized to cover non-Type I algebras and infinite dimensional and Hilbert spaces, they would still fall short of what is needed. If, as Born and other founders of quantum mechanics intended, quantum probabilities are given a statistical interpretation then there is a gap between showing that the Born rule is picked out as the unique rule satisfying a list of preferred constraints vs. showing that the Born rule probabilities calculated from a Hilbert space state via the trace prescription should govern the statistics of measurement outcomes. It is, of course, a physical postulate that the statistics of outcomes of measurements performed on actual physical systems reflect Born rule probabilities, and the justification of this postulate ultimately rests on experimental evidence. But a satisfying justification of the Born rule should explain why, prior to testing, one should expect that *if* the quantum theory as characterized here (sans the Born rule) is true then the statistics of outcomes of measurements will reflect Born rule probabilities.

The strategy I will explore resists the temptation to read the standard formulation of the Born rule in the guise (BR) as implying that a quantum system acquires its probability state from its Hilbert space state. The fact that (BR) is equivalent to its converse (CBR) and, thus, to the biconditional form ($BR*$) should serve as a check to this temptation, and it suggests that the task of justifying the Born rule should be construed not as explaining why Hilbert space states induce Born rule probabilities but as solving the concordance problem: Why (if the theory of true) are the co-instantiated probability states and Hilbert space states in accord with ($BR*$)? I will not offer a solution to the concordance problem in general but only to an operational version that focuses on state preparation.

8.2 Probability state updating

Section 2.6 above offered an account of Hilbert space state preparation, albeit an account limited to pure Hilbert space states. The operational content of the notion that Born rule probabilities govern measurement statistics is, roughly, that when a system is prepared over and over again in the same Hilbert space state, the long run frequencies of Yes and No answers to Yes-No measurements of elements of $\mathcal{P}(\mathfrak{N})$ should conform to the Born rule probabilities calculated via the trace prescription from the prepared Hilbert space state. A more precise specification of this idea is given in the Appendix, but the details need not concern us for the moment. What is needed is an account of probability state preparation that can provide a solution to the concordance problem, or at least to an operationalized version of the problem. We can work our way there by fasioning a probability updating rule.

For a probability measure P on $\mathcal{P}(\mathfrak{N})$ and $F \in \mathcal{P}(\mathfrak{N})$ such that $P(F) \neq 0$ we seek a conditional probability $P(\bullet//F)$ with two properties:

(a) $P(\bullet//F)$ is a probability measure on $\mathcal{P}(\mathfrak{N})$

(b) For any $E \in \mathcal{P}(\mathfrak{N})$ such that $EF = FE$ the conditional probability of E on F reduces to the classical conditional probability, i.e.

$$P(E//F) = \frac{P(EF)}{P(F)} = \frac{P(E \wedge F)}{P(F)}.$$

Note that if $E \leq F$ then $EF = FE = E$, and condition (b) reduces to $P(E//F) = \dfrac{P(E)}{P(F)}$.

To get the desired conditional probability measure we can appeal to a result that relies on the generalized Gleason theorems. Stated in terms of algebra states the result asserts:

Theorem 6. (Cassinelli and Zanghi 1983)[20]. Let $\mathfrak{N}$ be a von Neumann algebra without Type I_2 summands and let P be a completely additive measure on $\mathcal{P}(\mathfrak{N})$. Then for $F \in \mathcal{P}(\mathfrak{N})$ such that $P(F) \neq 0$ there is a unique probability measure $P(\bullet//F)$ on $\mathcal{P}(\mathfrak{N})$ such that $P(E//F) = \dfrac{P(E)}{P(F)}$ when $E \leq F$; namely,

$$P(E//F) := \frac{\omega(FEF)}{\omega(F)} = \frac{\omega(FEF)}{P(F)}.$$

where by the generalized Gleason theorems ω is the unique normal state that extends P to $\mathfrak{N}$.

[20] The Cassinelli and Zanghi result is stated in terms of countably additive probability measures and separable Hilbert spaces. But it can be generalized to the form stated here.

Note that Gleason's theorem is essential in obtaining the sought-after conditional probability since when E and F do not commute $\omega(FEF)$ cannot be written as $P(FEF)$ since $FEF \notin \mathcal{P}(\mathfrak{N})$.

Repurposed for the present context this result implies:

> *Theorem 6′*. Let $\mathfrak{N}$ be a von Neumann algebra acting on $\mathcal{H}$ and let P be a completely additive/continuous measure on $\mathcal{P}(\mathfrak{N})$. Then for $F \in \mathcal{P}(\mathfrak{N})$ such that $P(F) \neq 0$ there is a unique probability measure $P(\bullet//F)$ on $\mathcal{P}(\mathfrak{N})$ such that $P(E//F) = \dfrac{P(E)}{P(F)}$ when $E \leq F$; namely,
>
> $$P(E//F) := \frac{Tr(\varrho FEF)}{Tr(\varrho F)} = \frac{Tr(\varrho FEF)}{P(F)}.$$
>
> where by the generalized Gleason theorems $\varrho \in \mathcal{D}(\mathcal{H})$ is the unique Hilbert space state that extends P to $\mathfrak{N}$.

If it is agreed that a Yes answer to a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{N})$ means that the resulting post-measurement probability state $P_F(\bullet)$ is the $P(\bullet//F)$ of Theorem 6′ then we have our rule for updating probabilities on Yes-No measurement results:

> (PUR) If a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{N})$ yields a Yes answer and the pre-measurement probability state P is completely additive/continuous with $P(F) > 0$ then the post-measurement probability state is $P_F(\bullet)$ is $P(\bullet//F) = \dfrac{Tr(\varrho FEF)}{P(F)}$, where $\varrho \in \mathcal{D}(\mathcal{H})$ is the unique Hilbert space state that extends P to $\mathfrak{N}$.

8.3 Concordance, probability state preparation, and the Born rule

The proffered procedure for preparing a pure Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ is to measure its support projection S_ϱ until a Yes answer is obtained. If the pre-measurement Hilbert space state is any $\overline{\varrho} \in \mathcal{D}(\mathcal{H})$ such that $Tr(\overline{\varrho} S_\varrho) > 0$ then the Lüders rule (LR) plus the filter property of S_ϱ imply that the post-measurement state is ϱ. At this juncture the Born rule could be invoked to conclude that, as a consequence of preparing the system in Hilbert space state ϱ, the system has thereby been prepared in the probability state $P_\varrho(E) = Tr(\varrho E)$, $E \in \mathcal{P}(\mathfrak{N})$. But since the status of the Born rule is what is at issue this would be to beg the question.

The desired conclusion can be reached in a non-question-begging way using the above account of probability state updating and by noting that the preparation of a Hilbert space pure state can also be viewed as a probability state preparation. According to (PUR), if the pre-measurement probability state $\overline{P}$ is completely additive/continuous and $\overline{P}(S_\varrho) > 0$ for the support projection S_ϱ of a pure $\varrho \in \mathcal{D}(\mathcal{H})$ then, when a Yes-No measurement of S_ϱ returns a Yes answer, the post-measurement probability state $\overline{P}_{S_\varrho}(E)$ is $\overline{P}(E//S_\varrho) = \dfrac{Tr(\overline{\varrho}S_\varrho E S_\varrho)}{Tr(\overline{\varrho}S_\varrho)}$, $E \in \mathcal{P}(\mathfrak{N})$, where by the Gleason theorems $\overline{\varrho} \in \mathcal{D}(\mathcal{H})$ is the unique extension of $\overline{P}$. And by the filter property of S_ϱ, $\overline{P}_{S_\varrho}(E) = Tr(\varrho E)$, in accordance with the Born rule.

8.4 Assessment

The justification on offer fits neatly with the ensemble interpretation of quantum probabilities intended by Born, von Neumann, Einstein, and the other founders of quantum theory, and it has the virtue of not making any a priori assumption about dependency relations between Hilbert space states and probability states. But its scope is limited by the restriction to pure Hilbert space states and to the context of state preparation. The first limitation is serious but hardly fatal in light of the fact that most attempts to justify the Born rule focus on pure states. The second limitation is justifiable if one is concerned with the operational content of the theory as revealed by the statistics of measurement outcomes when the system is repeatedly prepared in the same pure Hilbert space state. But one should be clear that what has been justified is not even a special case of (BR) but rather

> $(\widetilde{BR})$ If the system is prepared in a pure Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ then it is in the probability state $P_\varrho(E) = Tr(\varrho E)$, $E \in \mathcal{P}(\mathfrak{N})$.

Physicists want to employ the Born rule in contexts, such as early universe cosmology, where state preparation is a viable operation only for the gods, and in such contexts the mere mortals need recourse to (BR) rather than $(\widetilde{BR})$.

It seems then that a wider-scope justification of the Born rule calls for weakening, if not abandoning, one or both of the presuppositions—Objective and None—of this section. The following section examines some of the possibilities opened by such a weakening.

9 Alternative strategies

To give an example of how abandoning None can lead to a justification (of sorts) of (BR), note that (BR) implies that the probability state P "knows" the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ in the sense that $P(S_\varrho) = 1$, where S_ϱ is the support projection for ϱ. For pure states this knowing is sufficient for (BR). Suppose that the system is in the completely additive/continuous probability state P and in the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$. By the knowledge assumption $P(S_\varrho) = 1$, and since (thanks to the Gleason theorems) P extends uniquely to a $\widetilde{\varrho} \in \mathcal{D}(\mathcal{H})$, $P(E) = Tr(\widetilde{\varrho}E) = \dfrac{Tr(\widetilde{\varrho}S_\varrho E S_\varrho)}{Tr(\widetilde{\varrho}S_\varrho)}$ for all $E \in \mathcal{P}(\mathfrak{N})$. When ϱ is pure S_ϱ is a filter for ϱ, and so $P(E) = Tr(\varrho E)$ as required by (BR).

When a pure Hilbert space state is prepared the accounts of state preparation given in Sections 2.6 and 8.2-8.3 imply that the prepared probability state "knows" the Hilbert space state. But postulating that the probability state "knows" the Hilbert space state, mixed as well as pure, outside the context of state preparation seems little better than simply postulating (BR).

A non-question-begging motivation is needed for violating None. Two will now be discussed; first, a radical one that jettisons both Objective and None; and then a less radical one that maintains Objective but abandons None.

9.1 qbism, QBism, and Born-again

Thus far I have been assuming that the Born rule is to be understood on an objectivist interpretation of probability. But quantum probability theory as the study of probability measures on the projection lattice of a von Neumann algebra, the Gleason theorems, and much of the rest of the formal apparatus of quantum theory sketched above are neutral with respect to the interpretation of probability. Bayesians who want to ply their wares to the quantum realm will take advantage of this flexibility and feel free to interpret a probability measures on $\mathcal{P}(\mathfrak{N})$ as codifying the personal degrees of belief that Bayesian agents assign to elements of $\mathcal{P}(\mathfrak{N})$. And presumably they will give a Bayesian twist to the probability updating rule discussed in Section 6.1: if an agent with prior probability measure P learns that the outcome of a Yes-No measurement of an $F \in \mathcal{P}(\mathfrak{N})$ is Yes then she should update her measure per (PUR) to $P_F(\bullet) = P(\bullet//F)$. Building on this basis these qbians hope to construct an account of the inductive reasoning used by physicists to navigate among quantum events. This lower-case, epistemic form of quantum Bayesianism is not, or at least need not be, in competition with the more orthodox interpretation of quantum probabilities as objective, observer-independent probabilities that govern the statistics of measurement outcomes.

Upper-case quantum Bayesians (QBians) have larger ambitions: they not only want to give an account of the inductive reasoning of quantum physicists but also an account of probabilities delivered by the theory. They propose to do this by jettisoning the orthodox statistical interpretation of quantum probabilities which, they believe, is responsible for many of the paradoxes and conundrums that have bedeviled quantum theory from its beginnings, and they propose to make do with personal probabilities alone.[21] Abandoning the Objective presupposition does not logically imply also abandoning None, but one can easily imagine that a Born-again who has converted to QBianism would construe the Gleason theorems as showing that Hilbert space states can be treated as bookkeeping devices used to track and calculate a QBian agent's degrees of belief, at least if the personal probabilities of QBian agents are completely additive/continuous. For then the Gleason theorems show that any agent's degree of belief function is represented by a unique $\varrho \in \mathcal{D}(\mathcal{H})$.

Such a stance seems to threaten the objectivity of quantum theory, for if Hilbert space states are merely devices for representing personal probability measures on $\mathcal{P}(\mathfrak{N})$ then at any time there are as many different Hilbert space states as there are QBian agents at that time with different probability measures on $\mathcal{P}(\mathfrak{N})$. But does not Hilbert space state preparation described above in Section 2.6 guarantee that there is an objective, observer-independent

[21] For a readable introduction to QBianism see von Baeyer (2016); for a skeptical examination see Earman (2019).

quantum state? Born-again has a response. The notion of a filter as applied above to Hilbert space states can be repurposed for probability states. Define a filter for a probability state $P \in \mathfrak{P}(\mathcal{P}(\mathfrak{N}))$ as a projection $F_P \in \mathcal{P}(\mathfrak{N})$ such that for any completely additive/continuous $\overline{P}$ where $\overline{P}(F_P) > 0$, the updating of $\overline{P}$ on a Yes answer to a Yes-No measurement of F_P results in $\overline{P}_{F_P}(\bullet) = \overline{P}(\bullet // F_P) = P(\bullet)$. If S_ϱ is the projection onto a ray corresponding to a pure state $\varrho \in \mathcal{D}(\mathcal{H})$ then S_ϱ is the filter for the probability state given by $P(E) := Tr(\varrho E)$, $E \in \mathcal{P}(\mathfrak{N})$. Born-again reads this as showing that all Bayesian agents whose prior probability states are completely additive/continuous and give S_ϱ a non-zero value will have the same posterior probability state upon updating on a Yes outcome of a Yes-No measurement of S_ϱ, and their common posterior probability state will be represented by the same Hilbert space state ϱ. This merger of the opinions of QBian agents is the meaning that Born-again assigns to the objectivity of state preparation.

What is the status of the Born rule for Born-again, and what justification can Born-again offer for the rule? The short answer is that for Born-again these are non-issues. Born-again rejects the notion that Hilbert space states and probability states are properties of quantum systems, so he does not worry about whether the two are linked per (BR), nor does he fret about whether the link is forged by a primitive physical postulate or other means. For Born-again probability states are epistemic states of Bayesian agents who assign degrees of belief to $\mathcal{P}(\mathfrak{N})$. Hilbert space states do not induce or explain these epistemic states but quite the opposite; the epistemic states come first and the Hilbert space states serve as mathematical representations. All there is to the Born-rule for Born-again is the fact that, thanks to the Gleason theorems, the Hilbert space state representing an agent's personal probabilities for $\mathcal{P}(\mathfrak{N})$ allows her probabilities to be calculated per the trace rule.

Or more precisely, the Gleason theorems enable the trace rule calculation when the agent's probability measure is completely additive/continuous. Here there is a real issue for Born-again and the QBians. For the QBians the admissible quantum probability measures are deemed to be those that can codify rational degrees of belief of a Bayesian agents who assign degrees of belief to $\mathcal{P}(\mathfrak{N})$. De Finetti, the patron saint of the personalist interpretation of probability, thought that the requirements of rationality end with finite additivity. Latter day personalists have noted that de Finetti's Dutch book argument for finite additivity can be generalized to cover countable additivity if it is required that a rational agent stands ready to accept any countable family of bets, each of which she regards as favorable. But to make the argument work for complete additivity it must use the stronger and less plausible requirement that the agent stands ready to accept any family of bets, countable or uncountable, each of which she regards as fair but not necessarily favorable (see Skyrms 1992). Other personalist justifications for additivity, such as scoring rule arguments, do not reach beyond finite additivity. Moreover, Dutch book arguments and scoring rule arguments do not serve to disqualify as non-rational those non-continuous probability measures on a $\mathcal{P}(\mathfrak{N})$ that do not extend to a Hilbert space state, as can arise in the case of $\mathfrak{N}$ with Type I_2 summands (e.g. $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and $\dim(\mathcal{H}) = 2$). If rationality of personal probability measures over $\mathcal{P}(\mathfrak{N})$ requires a conditionalization rule that reduces to classical Bayesian conditionalization for an abelian $\mathfrak{N}$ then the QBians have a motivation for continuity (and for complete additivity as well) since the Gleason theorems need to be invoked to secure the conditionalization rule; but the antecedent needs argument.[22]

9.2 Born-modern

Meet Born-modern. While he may accept qbianism as an account of inductive reasoning of quantum physicists he stoutly rejects the QBian personalist reading of the probability statements delivered by the theory. But Born-modern does sing from a similar hymnal as Born-again as regards the relationship between probability states and Hilbert space states.

Born-modern, unlike the historical Born, initially knows nothing of Schrödinger and his wave mechanics. Born-modern's route to quantum theory is through mathematics, specifically through a study of von Neumann algebras. He invents a new branch of probability theory, the study of probability measures on the projection lattices of von Neumann algebras. Being the brilliant mathematician he is, Born-modern proves what we know as the generalized Gleason theorems, Dye's theorem, etc., With the mathematics in place, Born-modern realizes that he has the makings of a new physical theory that he calls Q-theory (or Queer-theory), for he proposes that the probability measures he has been studying describe the queer statistics that experimental physicists

[22] The representation theorems of Deutsch (1999) and Wallace (2002, 2010), intended to show that rational agents act as if the Born rule is satisfied, are supposed to apply to the $\dim(\mathcal{H}) = 2$ case. Thus, the rationality requirements of their theorems must rule out the measures not continuous on $\mathcal{P}(\mathfrak{N})$. This requires further investigation which will not be attempted here.

have been reporting in their investigation of microscopic phenomena. Using the Gleason theorems he is led to the updating rule (PUR) for probability measures and, thence, to a procedure for preparing a class of probability states that possess what he calls probability filters. And he proceeds to test his proto Q-theory by preparing a Q-system over and over again in the same probability state and comparing the statistics of measurement outcomes of the identically prepared systems to the theoretical values. Encouraged by the success of the tests, he studies the symmetries of the projection lattice (here Born-modern elaborates on the considerations of Section 6) and is led to a dynamics for Q-theory in the form of a one-parameter group of automorphisms of the von Neumann algebra. He calls the unitary Hilbert space implementation of the algebra automorphism group W-dynamics, in honor of his physicist friend, Werner. Not wanting to neglect his other friend Erwin, he calls S-dynamics the Hilbert space state version of W-dynamics.

If Born-modern should learn the real-world story of the development of quantum mechanics he might feel some self-congratulation is in order because his Q-theory fulfills the second alternative in von Neumann's desire for a "derivation of quantum mechanics from empirical facts or fundamental probability-theoretic assumptions." Moreover, Born-modern would think that the concern about the status of the Born rule is part of a tempest in a tea pot that results from accidents of history; in particular, the accident that Schrödinger's wave mechanics came first and Born's probability interpretation came second. That history seems to have generated a felt need for a just-so story of how the leopards (quantum systems) get their (probability) spots. The just-so story behind the concern about the Born rule is that quantum systems get their probability spots from Schrödinger's wave functions, and this story in turn generates a felt need to justify the use of the wave function (and Hilbert space states in general) to assign probability spots in accord with Born's rule. Born-modern counters that if history had been different and the course of discovery had developed along the lines of his Q-theory, where probability states came first and Hilbert space states came second, there would be no felt need for just-so origin stories; for it would have been clear from the start that Q-systems always have probability spots, and there is no need to explain where they come from. From Born-modern's alternative historical perspective it is natural to follow his very distant cousin Born-again in construing Hilbert space states as mathematical representations of probability states, although for Born-modern the probability states being represented by Hilbert space states are objective, observer-independent states of Q-systems rather than the observer dependent epistemic states favored by Born-again. Alternatively, Born-modern (perhaps having read too much analytical metaphysics literature) might construe Hilbert space states as physical states of Q-systems that are grounded in or supervenient on the probability states they represent. Either way for Born-modern the Born-rule (BR) is true as a calculational device, and no further justification is called for.

Born-modern has to face an analog of an issue that arose for Born-again. The Gleason theorems justify Born-again and Born-modern in using the trace rule calculation of probabilities for probability states that are completely additive/continuous. For Born-again the issue was: Must the personal probabilities rational agents assign to $\mathcal{P}(\mathfrak{N})$ be completely additive/continuous? For Born-modern the analog issue is: Must the physically realizable objective probability measures on $\mathcal{P}(\mathfrak{N})$ be completely additive/continuous? Born-modern doesn't seem to have available any natural criteria for deciding which probability states are physically realizable, apart from the question-begging criterion that the realizable probability states are those induced by Hilbert space states.

9.3 Assessment

Although the perspectives on quantum theory offered by Born-again and Born-modern are quite different they have the common feature that the Born rule is justified—if that is the right word—by reducing it to a calculational device whose validity is vouchsafed by the Gleason theorems, at least supposing quantum probability measures must be completely additive/continuous. Given their different takes on the nature of quantum probabilities Born-again and Born-modern need to travel different roads to justify this supposition.

QBism certainly has some attractive features, especially if the QBians can make good on the promises to dissolve the measurement problem and soothe worries about quantum non-locality. But it also has some disturbing features. To name one, if Hilbert space states are just representations of the epistemic states of QBian agents then there is no Schrödinger evolution since these states do not continuously evolve but change sporadically in reaction to the agents' receipt of new information. This is not the place to evaluate the prospects and perils of QBianism. Suffice it to say that an assessment of the status and justification of the Born rule based on a radical and still programmatic interpretation of quantum theory must be regarded as tentative.

Because Born-modern hews to an objectivist interpretation of probability, most physicists would probably find Born-modern's story more congenial than Born-again's. But before buying into the Born-modern's story they would need to be convinced that the theory they use to describe quantum phenomena really can be entirely built up starting with probability measures on the projection lattices of von Neumann algebras. And even if they are convinced of this, there remains the fact that, as a matter of practical procedure, physicists start with an ansatz about the initial Hilbert space state, solve the Schrödinger equation for this initial condition, and then plug the evolved state into the Born-rule (BR) to get a prediction about the probabilities of outcomes of measurements. Born-modern can acknowledge as much but respond that his view is about the foundations of the Born rule, not its practical applications. This response leaves one nervous: when principle and scientific practice do not line up, principle is suspect.

10 Conclusion

It may seem more than a little curious that, approaching a hundred years since the publication of Born's epochal 1926 papers, the status of the Born rule is still the subject of lively discussion in the physics and philosophy literatures. But on second thought the continued scrutiny is not so strange given the central importance of Born rule to the functioning of the quantum theory. There are undoubtedly a number of other components to the explanation, two of which stand out.

The first is that many of the contributors to the literature on the Born rule are vested in a particular interpretation of quantum theory—many worlds, Bohmian mechanics, GRW collapse theory, etc.—and for them the task of deriving or justifying the Born rule amounts to showing how their favored interpretation accommodates the rule. Naturally, their accounts differ; and, not surprisingly, these accounts rely on controversial assumptions and moves, provoking critical notices and rebuttals.

The second part of the explanation derives from the misunderstandings of the role of the Gleason theorems that have become ingrained in the literature. To repeat, a common line is to credit the Gleason theorems with providing (apart from the exceptional cases when $\mathfrak{N}$ admits Type I_2 summands) a derivation of the Born rule, but then to complain (a) that the theorems offer little physical insight into the emergence of quantum probabilities and the Born rule and/or (b) that they commit the sin of non-contextuality. The credit is undue. The theorems do not (and do not pretend to) provide a rule for assigning probabilities to quantum systems, i.e. they do not specify when or under what circumstances a quantum system is in one probability state rather than another. Nor do they explain (or pretend to explain) why quantum probabilities must be completely additive and continuous, as required by (BR). The complaints are also off the mark. As for the non-contexuality complaint, any theory of probability, classical or quantum, must propose an algebra of events to which probabilities are assigned, and in doing so it opens itself to the complaint that its event algebra fails to take into account the context that derives from various exogenous or endogenous variables. There may, of course, be specific empirical or theoretical objections to using the projection lattice $\mathcal{P}(\mathfrak{N})$ as the quantum event algebra, but these would need to be detailed and considered on their merits (see Appendix 2 for more details on contextuality). The complaint that the Gleason theorems offer little physical insight into the emergence of quantum probabilities betrays a prejudice that derives from the notion that the actual history of the way in which the new quantum theory developed tells us something important about the foundations the theory. As illustrated by Born-modern's alternative history, the theory might have developed from the get-go as a new theory of probability—the study of probability measures on the project lattices of von Neumann algebras—and there would then be no felt need to explain how quantum probabilities emerge.

Granting all of this, the question remains: What do the Gleason theorems tell us about the Born rule? In the first instance they tell us that any completely additive/continuous probability measure P on $\mathcal{P}(\mathfrak{N})$ can be represented by a Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$ via the trace prescription $P(E) = Tr(\varrho E)$, $E \in \mathcal{P}(\mathfrak{N})$, and so the Born rule can be used to calculate such a P. To go further requires assumptions about the nature of quantum probabilities and/or the relation between probability states and Hilbert space states.

If Hilbert space states and probability states are viewed as objective observer-independent states that are on equal ontological footing the task of justifying the Born rule becomes the task of explaining why Hilbert space states and probability states are coordinated in the way required by (BR) and the apparently stronger but equivalent $(BR*)$. Only partial progress towards this goal was offered here by showing how the coordination falls out of accounts of the preparation of Hilbert space states and probability states when the former are pure states. The Gleason theorems play a role in this story by underwriting the account of probability state preparation. The founders of the new quantum theory would be pleased with the story because it fits with their notion that quantum probabilities refer to the statistics of measurement outcomes on an ensemble of identically prepared systems. But the story leaves unexplained the non-operational content of the Born rule in general and even the operational content for mixed Hilbert space states in particular.

Turning to approaches that posit dependency relations between probability states and Hilbert space states, Born-modern hews to an objectivist reading of quantum probabilities whereas Born-again defends a personalist reading; but, for different reasons, both construe probability states as basic and Hilbert space as devices for representing probability states. On both accounts the role of the Gleason theorems is to ensure that every completely additive/continuous probability state has a unique Hilbert space representation. For both Born-modern and Born-again the Born rule is not a substantive assertion, and for Born-again (BR) ill-stated since he rejects the presupposition that Hilbert space states and probability states are states of physical systems. But for both Born-modern and Born-again (BR) can be said to be correct as a calculational device for completely additive/continuous probability measures.

For those who are not ready to concede that the Born rule is merely a calculational device and who are not satisfied with explaining why the rule works for pure states in the context of state preparation it may seem time to try again for a justification that is part of the more general project of answering von Neumann's call for a "systematic derivation of quantum mechanics from empirical facts or fundamental probability-theoretic assumptions."[23] Failing a satisfactory completion of such a project the remaining option is simply to take the Born rule as an empirical postulate whose justification is its agreement with experiment. Being forced to this option might be regarded as an admission of failure, but one can take solace in the realization that the many attempts to justify the Born rule have led to deeper appreciation of quantum theory as a theory of probability.

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[23]Saunders (2003) may be regarded as an exemplar of this approach.

Appendix 1: statistical probability

The purpose of this appendix is to indicate what commitment is being made in asserting that the statistics of outcomes of measurements performed on elements of the projection lattice $\mathcal{P}(\mathfrak{N})$ of a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ are "governed" by a probability measure P on $\mathcal{P}(\mathfrak{N})$. The assertion is operationalized in terms of probability state preparation. Prepare the system in probability state P (recall Sections 8.2-8.3) and make a Yes-No measurement of some $E \in \mathcal{P}(\mathfrak{N})$ and record the result. Repeat over and over again ad infinitum. Since the trials are independent and identically distributed the classical law of large numbers can be invoked to conclude that in almost all outcome sequences the relative frequency of Yes outcomes approaches $P(E)$ as the number of trials goes to infinity.

This procedure, which might seem fishy because it is mixing the classical and the quantum mechanical, can also be described in terms of a single quantum mechanical measurement of the idealized observable $\otimes_{i=1}^{\infty} E_i$ made on the idealized infinite tensor product system $\otimes_{i=1}^{\infty} \mathfrak{N}_i$, where $\mathfrak{N}_i = \mathfrak{N}$ and $E_i = E$ for all i, and where each component system is prepared in the same probability state P on $\mathcal{P}(\mathfrak{N}_i)$. One then appeals to a theorem of Landsman showing that the two procedures are equivalent: they both result in the infinite Bernoulli process probability measure $P^{\infty}(E)$ on the space of the countably infinite outcome sequences (see Landsman 2017).

To be committed to the claim that P governs the statistics of outcomes is to be committed to seeing confirmation (respectively, disconfirmation) of the claim according as for any $E \in \mathcal{P}(\mathfrak{N})$ there is sufficiently good apparent convergence (respectively, non-convergence) of the relative frequency of Yes outcomes to $P(E)$ as the number of trials becomes larger and larger. Apply your chosen account of confirmation—be it Bayesianism, a version of significance testing, or other—to decide whether or not "sufficiently good" apparent convergence is obtained. This is rather vague, but fortunately in typical experiments there is such rapid apparent convergence to the values predicted by the quantum theory that there is little room to quibble about confirmation. Except from inductive skeptics. But if scientists listened to them then not just quantum mechanics but all empirical science would stymied.

Needless to say, this account is limited to cases where there is in principle an operational procedure for preparing the probability state. On the account given above the in-principle operational procedure covered only probability states P that admit a filter $F_P \in \mathcal{P}(\mathfrak{N})$, which turns out to be a projection onto a ray of $\mathcal{H}$ and, thus, coincides with the support projection of a normal pure algebra/Hilbert space state, making the procedure simultaneously a preparation of a probability state and an algebra/Hilbert space state.

Appendix 2: the non-contextuality complaint

An often heard complaint about the role the Gleason theorems play in justifying the Born rule is that the theorems assume "non-contextuality." What is the sin of non-contextuality and how do the Gleason theorems commit this sin? One way to think about the contextual/non-contextual distinction is the difference between rules for assigning probability measures to $\mathcal{P}(\mathfrak{N})$: a non-contextual rule, such as the Born rule, assigns probabilities purely on the basis of the Hilbert space state $\varrho \in \mathcal{D}(\mathcal{H})$; a contextual rule assigns probability measures $P_{\varrho,C}$ on the basis of the Hilbert space state *and* the context $C \in \mathcal{C}$ chosen from the set $\mathcal{C}$ of what are regarded as the as the relevant contextual factors $\mathcal{C}$. On this way of reading the distinction it should be clear that the Gleason theorems and whatever legitimate use is made of them do not commit the sin of non-contextuality. For as explained above ad nauseam, the theorems are not about rules for assigning probability measures to $\mathcal{P}(\mathfrak{N})$ on the basis of Hilbert space states simply because they are not about rules for assigning probability measures period; they are about the extension of probability measures on $\mathcal{P}(\mathfrak{N})$—however they are thought to arise—to algebra/Hilbert space states on $\mathfrak{N}$.

Nevertheless, the Gleason theorems do have implications for contexturalized probability assignment rules. In one class of applications the contextual factors are supplied by the values of exogenous or "hidden" variables. The goal is to show that the contextual assignment rule can be so designed and $\mathcal{C}$ can be equipped with a measure such that averaging over $\mathcal{C}$ returns standard quantum mechanical probabilities while the contextualized assignments $P_{\varrho,C}$ exhibit desired features, such as determinism and/or locality. We have noted that the Gleason theorems frustrate the desire for determinism, at least if this determinism requires that the $P_{\varrho,C}$ be dispersion free since the theorems preclude the existence of such measures except in special cases. The Gleason theorems, plus the probability updating rule (Section 8.2) enabled by the theorems, also place constraints on the dynamics of the hidden variables. For example, a non-Born rule assignment rule might assign probability measures $P_{\varrho,C}$ to $\mathcal{P}(\mathfrak{N})$ such that $P_{\varrho,C}(E) \neq Tr(\varrho E)$ for all $E \in \mathcal{P}(\mathfrak{N})$. Of course, by Gleason's theorem, for any completely additive/continuous $P_{\varrho,C}$ there always is a unique $\overline{\varrho}_C \in \mathcal{D}(\mathcal{H})$ such that $P_{\varrho,C}(E) = P_{\overline{\varrho}_C}(E) = Tr(\overline{\varrho}_C E)$ for all $E \in \mathcal{P}(\mathfrak{N})$, so it is as if the system in total state (ϱ, C) obeys (BR) with $\overline{\varrho}_C$ as the Hilbert space state of the system. The 'as-if' can become 'is' when state preparation is taken into account. Choose a pure state $\overline{\overline{\varrho}} \in \mathcal{D}(\mathcal{H})$ such that $P_{\varrho,C}(S_{\overline{\overline{\varrho}}}) = P_{\overline{\varrho}_C}(S_{\overline{\overline{\varrho}}}) > 0$, where $S_{\overline{\overline{\varrho}}}$ is the support projection for $\overline{\overline{\varrho}}$. According to the accounts of updating and state preparation developed above, if a Yes-No measurement of $S_{\overline{\overline{\varrho}}}$ returns a Yes answer then the post measurement Hilbert space state and probability state are respectively $\overline{\overline{\varrho}}$ and $P_{\varrho,C}(E//S_{\overline{\overline{\varrho}}}) = \dfrac{Tr(\overline{\varrho}_C S_{\overline{\overline{\varrho}}} E S_{\overline{\overline{\varrho}}})}{Tr(\overline{\varrho} S_{\overline{\overline{\varrho}}})} = Tr(\overline{\overline{\varrho}} E)$. The non-Born assignment rule says that when the system state is Hilbert space state $\overline{\overline{\varrho}}$ and the context is C',which may have changed with the performance of the Yes-No measurement, the probability state is $P_{\overline{\overline{\varrho}},C'}(\bullet)$. Putting things together we have that $P_{\overline{\overline{\varrho}},C'}(\bullet) = Tr(\overline{\overline{\varrho}}\bullet)$ after the preparation of $\overline{\overline{\varrho}}$, which is to say that as a result of the updating the values of the hidden variables must have adjusted themselves so that the contextual assignment aligns with the Born rule, and the adjustment must anticipate which pure state $\overline{\overline{\varrho}}$ the experimenter will choose to prepare.

The sin of non-contextuality can be seen in the neglect of endogenous factors rather than exogenous or hidden variables. For example, Saunders (2003) perceives the sin of non-contextuality in any probability assignment rule that assumes that "probabilities can be defined for a projector independent of the family of projectors of which it is a member." There are various ways to implement the perceived need for this second kind of contextuality, two of which will be mentioned here. The first, less radical, idea is to continue to assign probabilities to elements of $\mathcal{P}(\mathfrak{N})$ but in a piecemeal fashion, making the assignment to an element $E \in \mathcal{P}(\mathfrak{N})$ conditional on a family $\{F_\alpha\} \in \mathcal{P}(\mathfrak{N})$ of which E is a member. A motivation for this idea is the notion that it makes no sense to speak of Yes-No measurements of elements of $\mathcal{P}(\mathfrak{N})$ per se; rather (the story goes) one must speak of a measurement of $E \in \mathcal{P}(\mathfrak{N})$ in a 'measurement context' as specified by a family $\{F_\alpha\}$. The accompanying notion of contextual probability could then be modeled by a conditional probability $p(E \wr \{F_\alpha\})$ defined on pairs $(E, \{F_\alpha\})$, $E \in \mathcal{P}(\mathfrak{N})$ and $\{F_\alpha\}$ a measurement context for E. Unless one is prepared to define a measure over measurement contexts this conditional probability would have to be taken as a primitive rather than derived concept. Context dependence is explicitly exhibited in cases where $p(E \wr \{F_\alpha\}) \neq p(E \wr \{G_\beta\})$ for $\{F_\alpha\} \neq \{G_\beta\}$. Scare quotes should perhaps be placed around the use of probability here since more details would have to be provided to make the case that something sufficiently analogous to classical or quantum probability is in the offing that it deserves to be called probability. Before conferring this honorific on the numbers assigned one would certainly want the satisfaction of an analog of the basic additivity axiom (A2) (Section 2.3) for orthogonal $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$. Context dependence gives every reason to believe that $p(E_1 \wr \{F_\alpha\}) + p(E_2 \wr \{G_\beta\}) = p(E_1 + E_2 \wr \{H_\gamma\})$ will fail when the measurement contexts $\{F_\alpha\}$, $\{G_\beta\}$, and $\{H_\gamma\}$ are different. And $p(E_1 \wr \{F_\alpha\}) + p(E_2 \wr \{F_\alpha\}) = p(E_1 + E_2 \wr \{F_\alpha\})$ will not be well-defined when $E_1, E_2 \in \{F_\alpha\}$ but $E_1 + E_2 \notin \{F_\alpha\}$, as will always be the case when $\{F_\alpha\}$ is a partition of the identity to which E_1 and E_2 belong.[24] This is unfortunate since a partition of the identity is an attractive candidate for a measurement context.

A more radical implementation of Saunders contextuality would replace $\mathcal{P}(\mathfrak{N})$ as the event space and assign non-conditional probabilities to contextualized events that result from fine-graining the events of $\mathcal{P}(\mathfrak{N})$ by measurement contexts, i.e. an $E \in \mathcal{P}(\mathfrak{N})$ is fine-grained to $E{-}qua\{F_\alpha\}$, $E{-}qua\{G_\beta\}$, etc. It is far from clear how to even get started on building a probability theory for such an event space in a way that maintains contact with standard quantum probability. Do the $E - qua\{...\}$ objects correspond to projection operators on some larger Hilbert space? If so, the contextuality complaint can be raised anew, calling for $qua - quaing$. If not, in what sense are the numbers attached to these events quantum probabilities?

This and other ways to try to implement contextualized probabilities take us far away from standard quantum theory. In this distant land the task of justifying the Born rule would involve precisely the quashing of the present notion of contextuality, taking us back closer to the land of standard quantum theory; and Gleason's theorem is no help in this task since it is inapplicable in the face of the contextuality. A journey into the land of contextuality is justified if there is some discernible payoff in prospect, either for empirical adequacy or conceptual insight. The contextuality implemented by introducing exogenous 'hidden variables' offers promise of the latter. What discernible payoff is offered by the second kind of contextuality?

[24] That $\{F_\alpha\}$ is a partition of the identity means that the F_α are mutually orthogonal projections such that $\sum_\alpha F_\alpha = I$.

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