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A Metaphysical Foundation for Mathematical Philosophy

Abstract. Although mathematical philosophy is flourishing today, it remains subject to criticism, especially from non-analytical philosophers. The main concern is that even if formal tools serve to clarify reasoning, they themselves contribute nothing new or relevant to philosophy. We defend mathematical philosophy against such concerns here by appealing to its metaphysical foundations. Our thesis is that mathematical philosophy can be founded on the phenomenological theory of ideas as developed by Roman Ingarden. From this platonist perspective, the "unreasonable effectiveness of mathematics in philosophy"—to adapt Wigner's phrase—is analogous to that of mathematical explanations in science. As success-criteria for mathematical philosophy, we propose that it should be correct, responsive, illuminating, promising, relevant, and adequate.

Keywords: mathematical philosophy, mathematical explanations in science, phenomenology, platonism, ideas, ideal qualities.

Introduction

Many engaged in mathematical philosophy (MP) today claim to be making important contributions through a subtly nuanced deployment of mathematical tools. By contrast, many of those working in the tradition of non-analytical philosophy (NA) claim that technical investigations add nothing of value, and are instead mere formal artifacts devoid of substantial content. Though taking the form of a convoluted and complicated philosophical investigation, MP is said to be fruitless because, allegedly, neither formalization nor explication really does justice to the depth of the philosophical problems involved. The application of formal methods to such matters thus appears to result from a misunderstanding of what should come first.

In presenting some of the virtues of MP here in ways that should be especially "palatable" for those working in the NA tradition, we are seeking to build a bridge between different methods and "thought styles" (Fleck, 1979, p. 99). Throughout, we deliberately use terms taken from the otherwise unrelated traditions of *phenomenology* and modern *philosophy of mathematics*. Our aim is not to show that MP is doing fine, but that its methods and results should be of interest to NA. Despite apparently deep-seated differences of meta-philosophical attitude, tradition, or thought style, we think there are good arguments to the effect that MP can shed light on important philosophical problems. Inspiring and facilitating a better understanding of crucial disagreements—beyond the usual *cris de guerre* that "NA is nonsense" or "MP is idle symbol-shuffling"—therefore requires us to step a little outside of our own "formal comfort zone". We nevertheless offer, as a common language, that of the philosophy of mathematics, albeit *enriched* with a phenomenological approach.

Although MP is often associated (partly for historical reasons) with positivism and naturalism, our own view of mathematics is strictly "platonist"—a term we prefer to "realist". Rather than treating mathematics as a set of mere conventions or useful fictions, we construe it as a body of formal truths to which our cognitive faculties afford access.[1] Thus, beyond proving theorems in a formal system, the praxis of mathematics rests on *cognitive acts of intuition*, where such intuitions allow mathematicians to see a reality independent of themselves. This reality is itself composed of qualities that constitute the content of the ideas that mathematicians study. It is not just that these quality-complexes *intersect* with what is investigated by MP: philosophers and mathematicians are actually studying *the same* subject matter. This (partially) explains the success of MP. Hence, we hold that MP can also be philosophically fertile for those NA philosophers who have embraced this same Platonist stance in terms compatible with the phenomenological approach. The reader is invited to perform a thought experiment by tentatively accepting this stance while reading the present paper.

The motivation to engage in such an experiment might well have a different character for MP and NA philosophers, respectively. Practitioners of the former are often skeptical as regards the phenomenological attitude. However, such a thought experiment need not involve any metaphysical commitments of a kind that someone might be reluctant to make: we just seek to encourage the reader to look for a common denominator in analyzing mathematical and philosophical investigations in terms of performing certain cognitive acts. It is enough to accept the claim that there is some intuitive element present in the pre-formal phase of mathematical investigations.

For NA philosophers, it might perhaps be problematic to accept the claim that intuition could be adequately captured in formalized systems. But it is not necessary to become a "believer" in formal tools in order to accept the claim that formalization is a way of grasping otherwise vague notions, and that there is a very strong semantic, intuitive element present. In particular, what this shows is that rigor need not be tantamount to formalization: mathematical proofs are rigorous, but not often fully formalized.

We also think that the working attitude of many mathematicians might serve as an inspiration for this thought experiment.[2] They repeatedly report feeling that their subject is somehow directly given—i.e. that they are often making *discoveries* rather than *inventions*. So there is a creative moment of mathematical imagination, but it is nevertheless limited. The eminent mathematician Hardy famously compared himself to an observer who gazes at a distant range of mountains and notes down his observations (Hardy, 1929, p. 18). Cantor claimed that the reality of the whole numbers struck him as being much stronger than the

[1] There are also diverse variants of realism: for instance, realism in respect of truth value, which is different from realism in ontology. A good example of the former, but not the latter, is Hellman's (1989) modal structuralism: while he denies the existence of mathematical entities, he wants to preserve the notion of the truth of mathematical sentences (which must be defined in a different way than in terms of correspondence). What we have in mind in this paper is realism both in respect of truth value and in ontology: the truth of mathematical sentences is defined in terms of correspondence with an objectively existing realm of abstract objects.

[2] We wish to thank an anonymous referee for pressing us to be more explicit on these matters: in particular, as regards furnishing the motivation for engaging in such a thought experiment.

reality of the physical world (letter to Hermite; see Hallett, 1984, p. 149). Gödel was convinced that there is an objective mathematical (set-theoretic) reality we can just perceive and describe (Gödel, 1964, p. 271), and Thom (1971, p. 697) directly asserted that "Platonic ideas give shape to the universe"—to mention just a few examples. The saying that "the typical working mathematician is a platonist during the week and becomes a formalist on Sunday" is becoming increasingly familiar. During working days, they are convinced that they are dealing with an objective mathematical reality that is independent of them, and when on Sunday they meet a philosopher who begins to question this reality, they claim that mathematics is in fact the juggling of formal symbols (see Davis et al., 2012, p. 359). The Platonist attitude of the working (rather than philosophizing) mathematician is so common that Monk (1976, p. 3) was tempted to make a subjective estimate to the effect that sixty-five percent of mathematicians are platonists, thirty percent formalists, and five percent intuitionists.[3]

The phenomenological approach, while highly influential in the "wider world", has also become increasingly popular in discussions pertaining to the philosophy of mathematics. This raises the hope that it might serve as a link between the two parties to the dispute. Husserl was trained in mathematics, and it is no coincidence that so many examples of the application of phenomenological tools pertain to mathematics. It is reported that Gödel considered Husserl the greatest philosopher since Leibniz (Rota, 1992, p. 175). Phenomenology and mathematics were, and still are, connected by many threads—a fact to which many works testify (see Tieszen, 1995, 2005; Fine, 1995; Berghofer, 2020; Hartimo, 2007, 2010; Skowron & Wójtowicz, 2021). In order to achieve our goal, we shall therefore adopt the phenomenological approach as proposed by Husserl and substantially developed by his pupil Ingarden.

What was less foreseeable is that practitioners of MP have also faced criticism from working mathematicians. The latter often assert that the findings of mathematical philosophy are trivial from a purely mathematical point of view, and do not contribute to the development of mathematics itself. Mathematicians (Rota, 1991, pp. 133-134; Mac Lane, 1986, p. 443) are given to wondering why it is that philosophers have become so fond of logic, overlooking other branches of modern mathematics. That criticism is not the main issue to be addressed in this paper, but we shall certainly be proposing a view of mathematical philosophy that seeks to respond to such criticisms—both those from the direction of NA and those of mathematicians themselves.

The present article falls into three parts. In the first, we discuss the metaphysical foundations of MP. Mathematics plays an *explanatory* role in science, and should also play such a role in MP, so we start with an analysis of the process of furnishing mathematical

[3] Maddy, in a series of papers and a monograph (Maddy, 1989), defends her version of mathematical (set-theoretic) realism: in general terms, it is based on Quine's influential indispensability argument, but according to Maddy this has some weaknesses, as it leaves out large fragments of set theory (which do not seem to play a role in applied mathematics). Rejecting Gödel's notion of mathematical intuition, she aims to construct a naturalistic epistemology that will allow one to not only account for our knowledge of the truths of applied mathematics, but also to include set theory.

explanations in science—in particular, those carried out from a platonist position. We briefly outline the Platonist stance in the philosophy of mathematics, and the role of intuition in mathematical cognition, referring to the experience of working mathematicians. Then, following Ingarden, we present the ontological scenario in which the mathematician and mathematical philosopher find themselves. Our claim will be that just like the mathematician, the philosopher seeks to directly confront, and understand, a horizon constituted out of ideal qualities and ideas. The former examines the content of mathematical ideas, the latter that of philosophical ones. We then define MP as an approach to philosophy based on the analysis of the same ideas as are examined in mathematics, in that MP is concerned with the domain of overlap between mathematical and philosophical ideas. We also propose an answer to the question of what the (in)effectiveness of mathematics in philosophy rests on.

In the second part, we give two examples of undertakings in the field of MP that lend support to our thesis that the latter is, in fact, a way of analyzing the intersectional content of philosophical and mathematical ideas. Given that MP has, in the 20th century, mainly employed tools from logic, we think it interesting to focus on other branches of mathematics. The first of these will be the formalization of the Husserlian theory of part-whole relations carried out by Kit Fine, which exhibits an interesting interplay of mereological and topological ideas. The second case will concern conditionals, and the use of probabilistic ideas for their analysis.

Our third part moves in a methodological direction, discussing certain criteria that mathematical philosophy can be expected to meet: namely, that it should aim to be correct, responsive, illuminating, promising, relevant, and adequate. Taking each of these criteria in turn, we consider the risks posed to the work of the mathematical philosopher who fails to satisfy them.[4] We also believe that they can be used for purposes of self-evaluation in the context of MP, and that while we ourselves derive them from our vision of the latter, they can also be entertained and explored without reference to any specific metaphysics.

Part I. The metaphysical foundations of MP

1.1. Mathematics in science (and philosophy)

It is obvious that mathematics plays a crucial role in science, particularly in scientific explanations. However, what the exact role of mathematics is, and what makes mathematics so effective when it comes to scientific theorizing and explanation, are matters of dispute. A large part of this debate is connected with the realism-antirealism controversy, and both camps take strong positions here. Not unsurprisingly, our own sympathies are with those authors who claim that mathematics is not just a tool, or a language performing some auxiliary role, and is in no way merely Hempel's "theoretical juice extractor". Rather, it functions to identify some deeper structure underlying phenomena that make up the explanandum. Thus, the claim is that mathematics identifies certain fundamental modalities stronger than physical ones, such that the explanations of certain phenomena are furnished by

[4] These criteria are gradable—it is not a 0/1 matter whether some conception is promising or illuminating or not.

mathematical theorems themselves.[5] Hence, identification of the relevant causal nexus is not the only significant factor in scientific explanations, and we can say that the explanatory power of mathematics exhibits a *sui generis* character. The source of this is ultimately to be found in mathematical constraints, not causal relationships. Pincock's account of abstract explanations[6] is very similar, the subtle differences notwithstanding: the aim is to identify some abstract properties of the system that, though not physically causally efficacious, are nevertheless responsible for the course of events. Moreover, the notion of "explanation by constraints" operates in a very similar vein (Lange, 2017)—it being aimed at identifying fundamental modal constraints stronger than the natural modalities.[7]

We seek to develop our ideas here in line with the first way of approaching the problem, endorsing the constraint-based account of mathematical explanations in science. We believe that mathematics plays an explanatory role because it identifies some fundamental modal facts—some metaphysical constraints (and does not just serve as a convenient computing device). We therefore try to think about mathematics in terms of the identification or "grasping" of deep truths, rather than in terms of its being a computational tool. These terms are not entirely precise, to be sure, yet we believe they may serve as a general and guiding idea of sorts. From this point of view, practicing mathematics is not only a matter of proving theorems: rather, "according to a now-forgotten etymology, a theorem is above all the object of a vision" (Thom, 1971, p. 697).

Our point is that mathematics is as important as it is in philosophy *not* by virtue of being a convenient language, and so not because of its pragmatic features; instead, it is so because of its ability to recognize certain fundamental relations internal to the metaphysics of qualities and ideas—ones we shall of course be exploring below. These are primitive in the metaphysical sense, and it is *upon them* that such mathematical explanations are actually built. We can say that metaphysical constraints are represented by mathematics, or that metaphysical truths are expressed by mathematics. Mathematics identifies constraints on what is possible. Of course, our line of argumentation will only have traction if one is in principle willing to grant that metaphysical considerations make sense, and that one can ascribe some

[5] Of the numerous examples in the literature we shall select just two, which we find especially persuasive. (1) The particular geometrical structure of honeycombs is explained by an optimization theorem to the effect that hexagonal tiling minimizes total perimeter length (Hales, 2001), where minimizing the amount of wax gives the bees an evolutionary advantage. (2) The Borsuk-Ulam theorem states that for any continuous function f from a sphere into R^2, there will exist two antipodal points x, y, such that f(x) = f(y). This then provides a mathematical explanation of the fact that there are always two antipodal points on the surface of the Earth where temperature and pressure are both equal (Baker, 2005, 2009; Baker & Colyvan, 2011).

[6] The general idea of abstract explanation was introduced by Pettit and Jackson (2009), who discuss the example of a glass cracking after being filled with hot water: the inevitability of this scenario follows from general features of the system rather than from any particular sequence of micro-events. Pincock (2015) also discusses Plateau's laws, stressing that although causally irrelevant to the explanandum, these abstract entities are nevertheless explanatorily relevant (see Pincock, 2015, p. 857). In particular, "the mathematical entities central to the explanation are more abstract than the fact being explained" (Pincock, 2015, p. 864). From our point of view, it is important to stress that mathematics can identify some objective abstract dependence relations. It is not just a computational tool.

[7] "Explanations by constraint work not by describing the world's causal relations, but rather by describing how the explanandum arises from certain facts ("constraints") possessing some variety of necessity stronger than ordinary laws of nature possess" (Lange, 2017, p. 10).

meaning—perhaps vague, but still "graspable"—to metaphysics as such. So it is a premise of our inquiry that metaphysics *per se* does indeed make sense.

1.2 Platonism and intuition in the (philosophy of) mathematics

As we see it, the work of mathematicians is not reducible to proving theorems. Seen from the perspective of the experience of practitioners working in the field, what they are doing is recognizing the mathematical qualities present in acts of ideation, and then analyzing the content of mathematical ideas using the methods furnished by modern mathematics. Mathematicians, naturally (and just like phenomenologists), are primarily interested in necessary, not accidental, connections. Ingarden (1971, p. 324) claimed that we tend to forget that the results of mathematics are grounded in eidetic insights into relations between ideal qualities, and there is much more to mathematical cognition than just examining conventional deductive systems. It is the intuiting of ideal qualities that prompts the insights on which appropriate formal systems can then be built.[8]

Before presenting our approach to MP in detail, we intend to discuss the role of intuition in mathematics—something recognized by many outstanding mathematicians and logicians. Gödel, for example, pointed out (as did the phenomenologists) the similarity between sensory perception and mathematical intuition:

> Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future (Gödel, 1964, pp. 120–121).

In 20th-century philosophy of mathematics,[9] the Platonist stance towards mathematical objects that, to put it simply, treats them as existing independently of both the human mind and any kind of formalism, was by no means prevalent.[10] What was dominant, rather, was the trust placed in formalism and logic—a fact that also influenced the very shape of mathematics itself, and not only its foundations or its philosophy. Although the purpose of this article is not to set out a defence of platonism, we should nevertheless mention the broad outlines of the latter, which have emerged from centuries of experience in mathematics, and not from metaphysics itself. Indeed, metaphysics generally does not receive a good press in logical circles. Thom (1971, p. 697) noted that in the history of mathematics it has not been the case

[8] Gödel stresses the role of intuition in mathematics in many places—for instance, when discussing Carnap's syntactic interpretation: "in whatever manner the syntactic rules are formulated, the power and usefulness of the mathematics resulting is proportional to the power of mathematical intuition necessary for their proof of admissibility... [I]t is clear that mathematical intuition cannot be replaced by conventions, but only by conventions plus mathematical intuition" (Gödel, 1953/9, p. 358).
[9] Though not in mathematics itself—if we agree with Monk (1976, p. 3) that something like sixty-five percent of working mathematicians are platonists.
[10] There are diverse variant of mathematical realism—see footnote 1.

that an error in the reasoning of a single mathematician put the whole field of mathematics on the wrong track, while, on the other hand, exaggerated formalism has often led mathematical cognition to embrace irrelevant or somewhat uninteresting theories. An error in mathematics is most often detected very quickly and, what is more, mathematicians are extremely consistent in their results. So what is this broad consensus in mathematics based on? Thom (1971, p. 697) claims that it is "the result of the mind's struggle with permanent, timeless, and universal constraints".

The existence of these permanent and timeless constraints, which Thom, following Plato, called "ideas", allows the working mathematician to remain in a world of comfort, independent of any formalism. For Thom argued that the world of ideas infinitely exceeds our technical possibilities. Ideas are not given all and at once to the mathematician; he or she always sees only a part of their world, making appropriate reconstructions for the purposes of mathematical cognition. Thus, Thom (1971, p. 697) states that "[i]t is in the intuition that the *ultima ratio* of our faith in the truth of a theorem resides". The truthfulness of a mathematical statement consists for Platonists in stating a relation obtaining between ideas, not in making deductions from axioms according to rules accepted in some formal system, as formalists would like to believe.

It is an empirical fact that real proofs—such as we know of from papers on differential geometry or seminars on complex analysis—are rigorous, yet by no means fully formalized. The "formal layout" of the proof is not what is crucial. A mathematical proof is a sequence of inferential steps recognized by mathematicians as legitimate. There is an ongoing discussion in the philosophy of mathematics concerning the relationship between the real proofs known from mathematical practice and their formal counterparts (see, for instance, Hamami (2018), who provides an extensive bibliography). The key question is: What makes the proof legitimate? Is it the formal or the conceptual structure? And what is the relationship between these? One natural-seeming proposal is to think of it in terms of a sequence of cognitive acts in which the validity of the inferential steps is recognized—and this tradition can be traced back at least to Descartes.[11] Ultimately, doing mathematics is possible, because we have a kind of intuition of the subject matter. The object of mathematical investigations appears to mathematicians' minds as directly given, and this, indeed, is a common experience on their part. This experience of "grasping" ideas can provide a valuable link between NA philosophers and philosophers making use of mathematical methods. It leads us to claim that what phenomenologists describe as qualities—and which they propose some specific ways of "becoming acquainted" with—have their counterparts in mathematics. Thus, the gap between different styles of doing philosophy turns out to be less significant than had seemed to be the case, and MP turns out to offer something that may well prove highly attractive for these philosophers, too—ones who, quite probably, had never looked to mathematics as a potential source of inspiration.[12]

[11] Regarding this, Detlefsen (2005) offers an excellent historical presentation.
[12] We are not asserting any strong dichotomy between mathematical philosophy and non-analytical philosophy (or, in particular, phenomenology). We should rather say that while mathematical philosophy is more similar in its form and methods to analytical philosophy, it might nevertheless be still more similar in substance to some

1.3. What is mathematical philosophy?

Why is it the case that mathematics makes such a valuable contribution to philosophy? This, we believe, is the counterpart of the question that asks in similar fashion about the role of mathematics in physics.[13] The crucial factor is *not just* that mathematical philosophy makes philosophy more precise thanks to what mathematical tools contribute: being precise is not the only goal of philosophers! Strictness is a means for achieving philosophical insight, not the aim or end *per se*. The reason for the success of MP has to be far more deeply rooted.

On our view, there are profound and far-reaching analogies between mathematical and philosophical investigations: ones that reside in the fact that in the two cases researchers are engaged in examining the contents of the very same ideas. Indeed, applying mathematical methods in philosophy is only possible and fruitful when both philosophers and mathematicians are studying the same ideas and qualities, making use of certain cognitive acts (to be addressed in detail later). The work of MP has a metaphysical foundation in the ideal realm of being. As philosophers, we analyze certain philosophical ideas, but see that mathematicians have already grasped certain properties of some of the qualities found in the contents of those ideas, and so use those identifications for our own philosophical purposes.

In brief, we hold that mathematical philosophy is mathematical not only because it employs mathematical tools, but also in virtue of analyzing philosophical ideas that intersect with mathematical ones. In this regard, MP may be said to rest on an adjunction, at the level of what is ideal, between philosophy and mathematics.

1.4. Qualities, ideas, and the phenomenological attitude

In order to further discuss ideas and qualities, let us take as an example the simple experience of viewing an orange ball lying on a table in a brightly lit room. The first thing that is striking when we look at this ball is that it is orange. After walking around the table several times, we can see that the ball is uniformly colored all over. When we continue looking at it, we are able to distinguish its brightness, saturation, and shade of orange, along with other such qualities. Moreover, we can distinguish its shape: the object we are inspecting is a sphere and not, for example, a cube. The orange color, and its spherical shape, are defining properties of this orange sphere. If the sphere is lying on a real table in a real room, we can also assign it a real manner or mode of existence. Our perceptions of the ball would be slightly different if it were only an imaginary ball in an imaginary room, or a virtual ball in a computer game. Let us, then, take a fundamental step forward, following in the footsteps of the phenomenologists: after carefully observing this sphere and becoming aware of some of its qualities, let us try to see the qualities themselves: not this particular sphere anymore, but the *sphericity and*

strands of non-analytical philosophy. Moreover, when discussing these possible positions we have in mind their more radically orthodox variants. This is why we feel able to contrast them so starkly with each other.

[13] Mathematics does indeed identify fundamental modalities underlying physical phenomena, so one might well think of the relationship between mathematics and philosophy along similar lines.

orangeness themselves. Phenomenologists would say that we are performing an act of *ideation*. Many philosophers question the possibility of carrying out such an act; nevertheless, the phenomenological approach—or attitude—lends support to its effective realization. Husserl introduces it in the following terms:

> Let us now shift from our natural-scientific psychological standpoint to an ideal-scientific, phenomenological one. We must exclude all empirical interpretations and existential affirmations, we must take what is inwardly experienced or otherwise inwardly intuited (e.g. in pure fancy) as pure experiences, as our exemplary basis for acts of Ideation. We must ideate universal essences and essential connections in such experiences—ideal Species of experiencing of differing levels of generality, and ideally valid truths of essence which apply a priori, and with unlimited generality, to possible experiences of these species. We thus achieve insights in a pure phenomenology which is here oriented to real (*reellen*) constituents, whose descriptions are in every way 'ideal' and free from 'experience', i.e. from presupposition of real existence (Husserl, 2001, pp. 112-113).

Thus, in the act of ideation, we abstract from the *modus* of existence. It is not important whether the orange ball is imagined, or virtual, or real: in each of these cases we are dealing with its orangeness or its sphericity. The act of ideation does not confirm or deny the existence of the sphere; it only distinguishes certain qualifications of the sphere and treats them as qualifications *per se*, regardless of their mode of existence. The act is not sensitive to the environment of the sphere: it can change, but once sphericity has been isolated and captured in the act of ideation, it will not do so. The phenomenological term for such properties is "qualities". More particularly, Roman Ingarden referred to them as *ideal* or *pure* qualities, while Jean Hering called them "*Wesenheiten*" (Ingarden, 2013, p. 67): "An ideal quality is perfectly its very self and nothing more" (Ingarden, 2013, p. 111). We ourselves shall continue to use the term *quality* here for the sake of simplicity, often omitting the adjective *ideal* or *pure*.

When considering qualities, visual ones, such as the color of an object, have been the examples favored by phenomenologists. But according to the latter, we have access to many types of qualities, not only visual ones. Let us note that kinds of shape, such as sphericity or squareness, are also qualities, just as are straightness (e.g., that of a straight line) or parallelism (e.g., that of parallel lines) or flatness (e.g., planes). According to Ingarden, it is precisely on our intuiting spatial qualities of this type that geometry is built. However, in order to reach this sort of quality, additional cognitive procedures are needed, which Ingarden (1971, pp. 299-311) characterized as functioning to sharpen or clarify the quality and its variation. We shall be focusing below on the former, leaving the latter as a subject to be studied separately.

Ideational acts are a familiar experience: imagine that our orange ball is a virtual ball in a computer game, and let it continuously change its color, but not its shape or size. Let us direct the intention of our perception at this ball only. Our perception of it can be the basis for an ideational act with respect to sphericity: in perceiving the changing color, we are allowing ourselves to be led to a certain "limit" (though not in the strict mathematical sense). We see what was constant (invariant) in it—and this is sphericity. We are making precise the "boundaries" of this quality, and it is only thanks to such a movement towards grasping the limits involved that we have access, according to Ingarden, to the quality *itself*. For the latter, this idealizing and radicalizing attitude towards this or that quality is an instance of *eidetic intuition*. From the phenomenological perspective, the work of a practicing mathematician is based on simple acts of eidetic intuition, in which mathematical qualities are captured. This holds not only for straightness or flatness as mentioned earlier, but also the whole repertoire of mathematical qualities and their configurations. The associativity and commutativity of a group operation, isomorphism of two groups, and commutativity of a diagram in category theory are algebraic qualities. Continuity, compactness, separateness, dimensionality, metrizability and connectedness are topological qualities. The validity of a line of reasoning, truth of a sentence (in a structure), and provability of a claim are logical qualities. Optimality is a quality in game theory, probability a quality examined in probability theory, and so on. Of course, acts of ideation pertaining to qualities are not the only ones that a mathematician is concerned with, but they do form a specific basis for extremely complex forms of mathematical cognition.[14]

A mathematician is not normally only required to deal with a single quality in his or her work, but rather, most often, with rich collections of such qualities. Qualities co-exist, and in their concretized versions constitute separate ideal entities. Ingarden called these entities *ideas*. Ideas are distinguished by their specific bilateral formal structure: (i) an idea has a number of properties as an idea, and (ii) an idea has a content (Ingarden, 2013, p. 68). Among the properties of ideas as ideas, Ingarden lists invariance, atemporality, the property of possessing content, etc. On the other hand, every idea possesses content: for example, the idea 'any human being' traditionally contained rationality in its content. Intuitively speaking, concretized qualities are the building blocks of the content of ideas. The content of an idea contains constant and variable elements. From an ontological point of view, these elements are concretizations of ideal qualities. (We will come back to the subject of this concretization process in due course, below.) The bilateral formal structure distinguishes ideas from individual objects:[15] individual objects do *not* have bilateral formal structure, i.e. they are unidimensional in their structure (Ingarden, 2016, p. 227). Also, having constant and variable elements in the content makes ideas different from individual objects. Let us now discuss the importance of the content of ideas, and the difference between ideas and individual objects, in more detail.

It is the content of an idea that makes it refer to this or that object and not some other: for example, the content of the idea of a square makes it refer to individual squares, both real and ideal. At the same time, the content of an idea has a variable element which renders that idea

[14] Several types of operation responsible for structuring mathematical cognition were distinguished by Mac Lane (1986, pp. 434-438).

[15] The grammar suggests that when we speak of ideas we are treating them as if they were objects. Nevertheless, it is not a grammatical object that is at stake in the present context. The term "object" is understood here in an ontological sense. Ingarden (2016, p. 75), in his ontology, distinguishes an individual object by defining it as follows: "An individual object, being what its nature makes it into (e.g. into a specific table or into a particular human being, say, I. Kant), contains a peculiar form which is best explicated as the *immediately qualified subject of properties* (or, more generally: of characteristics). In these properties it finds the consolidation [*Ausgestaltung*], and precisely therewith also the imprint [*Ausprägung*], of its self. It unfolds in them, as it were, and precisely therewith makes its imprint in them."

more detailed or more general. Ideas have generality, so—even if the generality is gradable—they cannot be individual. Hence, for Ingarden, ideas are not individual objects and, especially, mathematical ideas are not individual mathematical objects. An individual square is not an idea, it is an ideal individual object that falls under the idea of squareness. There is *one* general idea of a square under which *many* individual squares fall. There is the hierarchy obtaining between ideas: some are more general than others, and particular ideas form the bottom of this hierarchy. In contrast to ideas, individual objects are all "equally individual": there is no comparable hierarchy between them, no difference in generality. Furthermore, individual objects are not underdetermined in any respect, precisely because they are individual, whereas ideas have variables in their content. Regardless of these ontological differences, however, individual objects are connected by a strong relationship to the ideas under which they fall. The necessary relationships that obtain within the co-occurrence of the qualities concretized in the contents of the ideas are transferred to the necessary relationships of the qualities concretized in the objects that fall under the relevant idea, and are so regardless of whether a given individual object exists in a real or ideal way, or indeed some other—e.g., intentional—one (cf. Ingarden, 2013, p. 72).[16]

As was said, the content of an idea is built from concretized qualities. To be precise, ideas have variables and constants in their content (when we consider, say, Euclidean geometry): "A constant belonging to an idea's content is the ideal concretization of a quite specific ideal (pure) quality. In the idea 'any square' occur the constants 'squareness', 'quadrilateralness', 'equilateralness', 'orthogonality', and so on" (Ingarden, 2013, p. 70). Meanwhile, what is variable in the content of an idea is not the concretization of a specific quality, but the concretization of the *possibility* of concretizing some specific quality in an individual object. Within the idea of "any square", the variable is "having a certain side-length". In order to get to an individual square, we have to determine the exact length of its sides, and the very possibility of accurately doing so rests on something tantamount to a variable within the content of the idea of "any square". Pursuing this further, it is only thanks to the qualities and their concretization in the content of ideas that we can talk about pure *possibilities* on the one hand, and *necessary* relations on the other. Take an example: within the content of the idea of "any square", there obtains a necessary existential interconnection between the constants "squareness" and "quadrilateralness". This interplay between constant and variable factors within the content of an idea is what both mathematical and philosophical cognition have to contend with. Thanks to it, some ideas will overlap. A given pair will do so when there is an ideal concretization of the same quality as what is constant in the content of them both. The ideas of "any square" and "any parallelogram" overlap, because in their content (as the constant factor) there is a concretization of the same quality: the quadrilateral. This case of

[16] The issue of modes of existence was extensively researched by Ingarden (2013, pp. 95-161), but we lack the space to discuss it in detail here. In this article, we follow him in assuming that there are many ways of existing, where such ways (or modes of being) consist of some combination or other of existential moments. To illustrate this, we give here three pairings of such moments: autonomy—heteronomy; originality—derivativeness; self-sufficiency—non-self-sufficiency (Ingarden, 2013, p. 109ff). Existence, for Ingarden, is gradable: from weak intentional existence (e.g., works of art), through real existence (desks, stones and trees), and then ideal existence (mathematical objects, ideal qualities, ideas), to absolute existence (e.g., the God of philosophers). For a gentle introduction to Ingardenian ontology, we recommend Piwowarczyk (2020).

overlapping ideas, as can be noted, is special, because the first idea falls under the second. However, as the examples below show, this need not always be the case.

Part II. Mathematical and philosophical ideas: two illustrative examples

In this second part, we discuss two examples of significant results emerging from mathematical philosophy, presenting the relations between philosophical and mathematical ideas that are—in our opinion—the source of success in such cases. We have taken our examples from *outside of* logic, hoping that this helps show the diversity of mathematical tools within formal philosophy. The authors of these results may disagree with us about the metaphysical foundations of their results: certainly, anyone who definitively rejects the existence of ideal qualities and ideas will not accept our point of view. However, we think that even in this case there will be some possibility for agreement about the practical basis for the pursuit of philosophy—and this regardless of whether it be in the context of a Platonist or non-Platonist stance. This prompts us to hope that the two camps will be able to agree that these examples bring some important insights to philosophy.[17]

2.1 Ontological momentarity and mathematical closedness

One formalization of Husserlian part-whole theory has been carried out by Fine (1995), whose paper begins with the following remark:

> Husserl's third *Logical Investigation* is perhaps the most significant treatise on the concept of part to be found in the philosophical literature. In it Husserl attempts to analyze the notion of dependent part, to lay down the principles governing its use, and to relate it to more general considerations concerning the nature of necessity and unity (Fine, 1995, p. 463).

In this particular investigation, Husserl carried out detailed analytical studies of many important ontological concepts, including that of a *moment*, which is crucial for the phenomenological tradition. He offered highly precise analyses of the notions of *part* and *whole*, and of *independent* and *non-independent part*, analyzed the notions of *ontological foundation* and *fragmentation*, and defined the notions of *extended whole* and *pregnant whole*, as well as of *unity*, etc. He was aware of the fact that his theory called for a mathematical formulation and, indeed, many philosophers have since undertaken attempts at formalizing his part-whole theory (see Null, 1983; Null & Blecksmith, 1991; Cassari, 2000; Simons, 2003; also, for a general overview of mereology, see Gruszczyński & Varzi, 2015). Each of these

[17] Taking Kelly's ideas from (1996), Schulte and Juhl (1996) point out certain similarities between topological and epistemological qualities. For instance, the Popperian notion of falsifiability is represented as one of the topological notions. (We shall not go into details here). Thanks to the procedure proposed by Kelly (1996), epistemology has been significantly enriched by new concepts, and a number of interesting and surprising results have been established—something that could hardly have been imagined before. There are many more examples of the phenomenon of "overlap" between topological and philosophical ideas in epistemology, philosophy of science and ontology (Gruszczyński & Varzi, 2015; Kaczmarek, 2019; Skowron, Kaczmarek & Wójtowicz, unpublished).

attempts is based on some kind of recognition of the relevant groups of qualities—of mereological ideas and of logical or mathematical ideas.[18]

We shall skip over the technical details here and focus on the metaphysical aspects and foundations of Fine's work. He distinguishes several possible ways of formalizing. One of them is formalization using the *foundational closure* f(x) of an object x. Husserl defined the basic concept of foundation for his theory in the following way:

> If a law of essence means that an *A* cannot as such exist except in a more comprehensive unity which associates it with an M, we say that an *A as such requires foundation by an M* (quoted after: Fine, 1995, p. 474).

Fine seeks to place appropriate conditions on the operation of foundational closure f(x), where these point to the fact that the algebraic structure behind this concept is that of a pre-closure[19] algebra, and he comments on this as follows:

> Given that Husserl accepts the axioms for a pre-closure algebra, it is natural to wonder whether he would have accepted the additional additivity axiom for a closure algebra. His theory could then be seen to be essentially topological in character (Fine, 1995, p. 475).

This type of topological interpretation of Husserl's part-whole theory leads Fine, among other things, to a clear definition of the concept of a *dependent moment*: namely, that an object[20] x is a dependent moment if and only if $f(x) \neq x$. That is, an object is a dependent moment when its foundational closure is distinct from it, while it is independent when its closure is equal to it. But what makes his formalization relevant—or, to put it another way, what does the validity of such a conception rest on? Fine's idea is based on the discovery that a certain group of algebraic and topological qualities belongs to the group of mereological qualities. More particularly, he notes that certain ontological qualities exhibiting what we shall call *momentarity* overlap with a topological and algebraic group of qualities exhibiting what we may refer to as *closedness*. Whether or not a formalization is relevant will depend on a recognition of the appropriate group of qualities and relations between them: i.e. it is based, metaphysically, on a relation of identity obtaining between the contents of the respective ideas. If it does not successfully target this identity between groups of qualities, even the most sophisticated mathematical construction designed to reflect some philosophical problem will be considered irrelevant for philosophical purposes—even if it counts as highly subtle from a technical perspective.

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[18] An interesting (but not very well-known) proposal is Vopěnka's theory of semisets, where the notion of vagueness plays an important role. The conception is inspired by Husserl, and in particular by Husserl's challenge to go "back to the things themselves". Also, the notion of horizon has been an important source of inspiration. Interestingly, Vopěnka's aim was not to put forward another formal version of set theory (and to investigate its metamathematical features within, for example, ZFC), but rather to treat it as a "naïve" theory, based on some new fundamental notions. For a thorough presentation, see Trlifajová (2022).

[19] Here, by "pre-closure algebra" is meant an algebra defined by three axioms: A1. $x \leq f(x)$, A2. $f(f(x)) \leq f(x)$ and A3. if $x \leq y$, then $f(x) \leq f(y)$, where f is the operation of foundational closure which assigns objects their closures and $\leq$ is the relation of part to whole. If we add the axiom of additivity: A4. $f(x \cup y) = f(x) \cup f(y)$, we obtain the so-called "closure algebra" (see Fine, 1995, p. 475-485).
[20] The concept of object is understood here in a broad sense: for example, the brightness of a color is an object.

2.2 The mathematical analysis of probabilities of conditionals

Conditional sentences such as *If Reagan works for the KGB, I'll never believe it* (Lewis 1986, p.156) are puzzling. Numerous problems arise—for instance, whether conditional sentences express propositions. Conditionals are of interest to a wider philosophical audience, as many problems have this form: e.g., ethical ones, such as "what are the chances that if I had acted differently, then…". Consider also such assertions as "I'm sure that I would have acted in the right way if I had had to choose", etc. Do they contain genuine information about me, or only about my degree of self-confidence?

We intuitively ascribe some credence to such sentences, not always being aware of the traps and paradoxes involved in making intuitive judgements. This makes the need for formal analysis all the more pressing. A part of the discussion surrounding the probability of conditionals is conducted in a semi-formal and intuitive way. However, probability theory is permeated with paradoxes and difficulties, so it is particularly important to situate the discussion in a proper mathematical setting: i.e. that of sound mathematical models pertaining to the probability of conditionals.[21]

Many proposals rely on the notion of possible worlds (with Stalnaker (1968, 1970) and Stalnaker & Thomason (1970) providing theoretical contributions now considered classic) and related metaphysical notions. For instance the model of McGee (1989) is based on the idea of the closest (most similar) possible world—an idea originating in Stalnaker (1968). The mathematical counterpart is a selection function, and a probability distribution defined on the class of selection functions. A probability space is defined in this way, and this construction has important consequences. (However, it is formally quite intricate). Philosophical ideas such as possible worlds, the accessibility of worlds, and the likelihood of a conditional are linked to such mathematical ideas as probability distribution, selection function, etc. This leads to a fusion of the world of mathematics and philosophy, expressed in theorems like the following: every probability distribution on atomic propositions has a unique extension to right-nested conditionals (i.e. of the form $A\to(B\to C)$) (cf. McGee, 1989, p. 507). This is reminiscent of the situation with classical propositional calculus: once the valuation on atomic sentences is defined, the logical values of complex formulas are determined uniquely. Here we are dealing with probability assignments—and the crucial fact is that it is sufficient to know the probabilities of the simple sentences A, B, C… in order to compute the probabilities of the right-nested conditionals, as in $A\to(B\to C)$.

The same phenomenon is illustrated by the proposal of van Fraassen (1976), which Kaufmann (2004, 2005, 2009, 2015) has since developed over the course of a series of papers. The idea is to think of conditionals in terms of sequences of possible worlds. This philosophical idea (of a "journey between possible worlds") is formalized within the Stalnaker-Bernoulli probability space, in which elementary events are infinite sequences of possible worlds. Such a model allows one to compute the probabilities not only of simple conditionals, but also of right-nested conditionals, conjoined conditionals $(A\to B)\&(C\to D)$

[21] Here we focus on work dealing with the probability of conditionals, not even mentioning the vast subject that is "the logic of conditionals". What are the appropriate axioms and rules of inference? See, for instance, Leitgeb (2012) for a comprehensive formal account of a probabilistic semantics that also addresses such logical aspects. We, on the other hand, focus on non-logical investigations here.

and conditional conditionals $(A \to B) \to (C \to D)$. This is an impressive result, as the interpretation of such conditionals is quite intricate. However, the Stalnaker-Bernoulli model imposes a particular interpretation of right-nested conditionals that leads to unintuitive consequences in some cases.[22] The approach is not compatible with McGee's model, as the Import-Export Principle generally fails.[23] This shows how presenting a mathematical counterpart of the intuitions involved can help with making the latter explicit and identifying the consequences they entail.

Still another model, based on the theory of Markov chains,[24] has been provided by Wójtowicz & Wójtowicz (2021) (and see, also, Wójtowicz & Wójtowicz, 2022). Broadly speaking, the idea is to associate a Markov chain with the conditional $A \to B$. The Markov chain generates a probability space, in which the conditional receives an interpretation as an event. This leads to a mathematically sound construction, in which philosophical arguments can be reformulated and given mathematical (or, perhaps, mathematized) versions. In particular, it is possible to give a formal counterpart of Lewis' triviality proofs, and to identify some controversial assumptions on which this is based. Without a formally defined probability space, it would not be possible to identify the problematic places in the philosophical argumentation. Informally, we can say that the formal machinery sheds light on the problem, and one of the results is that mathematical ideas generate a novel way of thinking, together with novel philosophical concepts.[25]

In all these models there is an interplay between mathematical (probabilistic and stochastic) and philosophical ideas such as chance, credibility, or likelihood. Probability theory represents the essence of these philosophical notions, and recognition of the appropriate groups of mathematical qualities enables us to discuss and deepen the corresponding philosophical ideas.[26]

2.3. How to choose mathematical tools?

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[22] Consider the classic "wet match" example: *If the match is wet, then it will light if you strike it*. Assume that the probabilities are as follows (Kaufmann, 2005, p. 206): (a) that the match is wet = 0.1; (b) that you strike it = 0.5; (c) that it lights given that you strike it and it is dry = 0.9; (d) that it lights given that you strike it and it is wet = 0.1. Moreover, striking the match is independent of its wetness. Kaufmann's formula gives a result of 0.46, which is counterintuitive, as we would expect the probability to be 0.1.

[23] The probabilistic version of the Import-Export Principle for conditionals states that the probabilities of $\alpha \to (\beta \to \gamma)$ and $(\alpha \wedge \beta) \to \gamma$ are the same.

[24] Informally speaking, Markov chain theory deals with random events, which unfold in time and are memoryless—like tossing a coin many times. The coin does not remember its history, and the $(n+1)^{st}$ result is independent from what was happening before. A classic example is the Gambler's Ruin Problem: two players toss a coin (it might be fair or not), and at every turn the loser gives one penny to the winner. The game lasts until one of the players is ruined, i.e. has no pennies left. Markov chain theory allows one to answer such questions as "What is the chance that the gambler will be ruined?", "What is the average time of the game?", etc. One of the advantages of the theory is that it furnishes computationally simple methods for dealing with such problems.

[25] Once we have set up a formal model for conditionals, we will inevitably find ourselves postulating truth conditions for these that are of a different nature than for factual sentences. The existence of such truth conditions is another matter—but if we do postulate them, they will have to be very different.

[26] *Probability* is a classic example of a mathematical explication of the common-sense notion of chance; see Carnap (1950), or Brun (2016) for a still more contemporary treatment. Of course, the problem of whether some concept is originally philosophical, or has perhaps been "imported" into philosophical discussions from other sciences, is a highly delicate one: just what is the status of a notion such as *propensity*?

One of the questions to be addressed in this context concerns which formal methods should be used in a given branch of philosophy (whether it be ontology, ethics, or the philosophy of science). In our view, this is a question about how the philosophical and mathematical ideas in question overlap. The examples presented in this part of our article will—we hope—show how this sort of correspondence can make possible the justification of philosophical claims. An undertaking in mathematical philosophy will be successful if it alights upon the appropriate overlap of philosophical and mathematical qualities—as constant factors in the corresponding ideas. It cannot be ruled out that, for example, a set of ethical qualities (such as duty) or aesthetic qualities (such as beauty) is connected to some mathematical ones we have not yet identified. The lesson we take from this can be thought of as a revised version of Carnap's Principle of Tolerance: the mathematical philosopher is free to adopt mathematical methods and tools, but this freedom will be limited by the demand for adequate recognition of the overlapping contents of the respective ideas involved.[27] Thus, any mere "technical *tour-de-force*"—in the sense of formalizing a philosophical issue without identifying the relationships between the contents of the respective philosophical and mathematical ideas—will be doomed to failure: it will make no lasting contribution to philosophy.

Part III. How to practice MP?

Taking our proposed founding of mathematical philosophy in a world of ideas and ideal qualities as a basis, we also seek to draw practical conclusions—to the effect that good philosophical work should be correct, responsive, illuminating, promising, relevant and adequate.[28] "Correctness" means, simply, that the reasoning involved is formally flawless: i.e. the conclusions are logical consequences of the premises, according to the rules of inference adopted. "Responsiveness" refers to whether philosophical considerations solve some traditional philosophical problem, or help resolve some issue currently under consideration. Philosophical inquiries are "illuminating" when they shed new light on an issue and/or present it from a new perspective. Whether a given set of philosophical considerations are "promising" depends on whether they can be expected to impact on the direction of future inquiries, rather than just contributing nominally to discussions currently in vogue. "Relevance" means that the investigations in question pertain to a well-chosen subset of the common philosophical and mathematical qualities, so as to ensure a choice of formal tools

[27] The famous quotation from Carnap runs as follows: "*In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments" (Carnap, 1937, §17). So there are no limitations, apart from pragmatic and logical ones. Our proposal is to some extent similar—but we acknowledge that there are some objective limitations apart from logical ones when setting up formal systems (provided we do not want to just play a conceptual game). This situation exactly recalls that of the working mathematician who, on the one hand, is free to adopt any (consistent) set of premises, axioms and conventions, but, on the other, strives for natural concepts, reliable assumptions, etc.

[28] We have borrowed the first five of these terms from Mac Lane (1986, pp. 440-446). Although Mac Lane refers to the development of mathematics and not philosophy, these criteria, with minor changes and additions, can also be applied to philosophy (and not only of a mathematical kind). We shall not indicate here exactly the points where the content of a given criterion has been altered, and where we have literally transferred it, as this would require a separate treatment extraneous to the goals of the present article.

appropriate to the problem in hand. Finally, a piece of MP-work will be deemed adequate if it is based on an accurate intuition of both philosophical and mathematical qualities.

3.1. Formal correctness in MP

Correctness, understood as formal correctness, is a necessary condition for our deliberations to be valid. It also imparts a special cognitive status to our findings: if formally correct, they will be certain and ineluctable.[29] Moreover, formal correctness is also connected with the precision and rigor of one's considerations, and no other science is a match for mathematics in this regard. By "precision" and "rigor" we mean here not only the use of formal language, but also—above all—the number of conceptual distinctions available to be introduced within any given conceptual framework.[30]

The focus on logic exhibited by 20th-century mathematical philosophy can be interpreted as marking a reliance on formal correctness alone, because it is logic that provides dependably sound reasoning schemes. However, mathematics is not reducible to logic, and mathematical disciplines (such as calculus, knot theory, topology, probability theory, differential geometry, etc.) have their own notions, which are interesting *per se*, not just as counterparts of certain logical notions. And it is the "dynamics of mathematical ideas" which is important in this respect, not just the logical, formal reconstruction of these theories. Hence, seen from the perspective of the vision of MP outlined in this paper, the logic-centeredness of the last century represents an excessive investment in one single epistemic value—namely, correctness as such, or accuracy and clarity—at the expense of others. (For an interesting quality we take to have been unjustly neglected, consider *spatiality*; see Thom (1971) and Mormann (2013) for details on how geometry and topology were pushed into the background in this regard).

Formal correctness is related to philosophy's desire for clarity of thought—which, in turn, often does issue from rigorousness. However, as rigor and accuracy grow, so does the overall number of details under consideration, and this can engender an excess of formality or strictness. If it so happens that this transpires when a mathematical philosopher considers a mathematical model addressing some philosophical issue, then by trying to make this model more detailed and precise we can end up with little more than a philosophical caricature.[31]

3.2. The responsiveness of MP

[29] "Once the axioms of a system are set, all the statements of the system are either demonstrable, refutable, or (thanks to Gödel) undecidable. No collusion, no political influence, no second thoughts can alter the fact of the matter that the theorem can be proved. Given the straightforward definition of a finite simple group, the existence of the 'monster' is ineluctable" (Mac Lane, 1986, p. 442).

[30] There is an ongoing discussion concerning the nature of mathematical proofs. The proofs known from mathematical practice are not formalized—however, they are rigorous. This issue was briefly addressed by us in Section I; see Hamami (2018) for an extensive discussion.

[31] Wang made the following comment on Kurt Gödel: "In recent years set theory has become more and more specialized and removed from generally accessible conceptual problems. If G were young today, he would be unlikely to choose to specialize in set theory. He wants philosophy to be 'precise but not technical' and believes that highly specialized knowledge is not relevant to basic conceptual problems" (Wang 1987, 208). This is a danger for MP—that its tools become so complex that they turn into the object of study themselves (in place of the philosophical problems they were intended to solve).

Considerations within MP are responsive if they address a canonical philosophical problem or furnish a reaction to the exigencies of current philosophical debates. This is intimately connected with the very purpose of practicing MP. It happens that a philosopher, enchanted by the power of mathematical distinctions, gives up the philosophical goal and starts to practice mathematics. And so another field of knowledge profits from his talent—but it is no longer philosophy. Kazimierz Twardowski, the founder of the Lvov-Warsaw school (from which Alfred Tarski, among others, emerged), used the term "symbolomania" to refer to the placing of an exaggerated faith in formal tools, and this might be one of the more powerful temptations of mathematical philosophy: people fascinated by symbols "have an unfaltering belief in the infallibility of the symbolism they use (...). Symbols and operations performed on them, originally means to an end, become for them an end in themselves" (Twardowski, 1979, p. 4).

Of course, the line between the philosophical and the mathematical is by no means sharp, and we would not wish to define it that way here. One of the metaphysical reasons for the blurring of this border is that some philosophical and mathematical qualities—e.g., spatio-topological ones—are indeed one and the same.

Failure in respect of responsiveness can also be identified in another way: philosophy can only be responsive if it addresses traditional philosophical problems, or participates in current debates—two things that, of course, often coincide, and certainly complement and condition each other. Excessive focusing on the philosophical tradition makes us stop using arguments and cease debating, so that philosophical discourse is reduced to commentary on the classic literature. This danger has been referred to as *philosophical torpor* (see the contribution of Głowala to Skowron, Bigaj, Chrudzimski, Głowala, Król, Kuś, et al.,2019, pp. 280–281). Conversely, if we immerse ourselves too much in debating and exchanging arguments, we may lose sight of what we are talking about. Then, important problems are replaced by mere puzzles—the kind of problems that can be routinely solved with the tools we are already used to. (Głowala calls this kind of attitude *philosophical blindness*.) Both of these are incompatible with responsiveness.

3.3. The illuminating character of MP

One might say that philosophical findings count as illuminating if and when they shed new light on problems, opening up some novel perspective or other that significantly broadens our understanding. Consider two conceptual realms (see Skowron & Wójtowicz, 2021): a philosophical one P and a mathematical one M (where the latter could be any branch of mathematics). A mathematical philosopher starts with P, then notes certain similarities between P and M (based on common ideas), and, on that basis, constructs a mapping $\Phi: P \to M$. Here, we mean—informally—any kind of "embedding" of a philosophical problem within mathematics that is standardly performed (e.g., by paraphrasing, explicating, model-construction, computer simulation, etc.). The class of M will usually be much richer than our picture of P: i.e., there will be more fine-grained conceptual differences, and an established system of relations between the concepts. These might be exploited via a procedure of re-translating mathematical concepts into the philosophical domain, and transferring over the pre-existing relations from M to new relations in P. So, again informally, we will now have another mapping, Ψ:M→P*, where P* will be a philosophical domain enriched by the "retranslated" notions (which may also have inherited conceptual relations from M). This enrichment will spawn new ideas, new concepts, and new hypotheses. The procedure will be considered illuminating if the domain P has been significantly enriched: e.g., with new, previously unseen links, or novel concepts introduced to P. It can then additionally be asserted that this procedure has cast mathematical light on the latter domain (P), which in turn has led via a feedback effect to an enriched domain P*. M illuminates P, thereby actively enriching P with new content. Of course, the re-translation procedure is only possible in some cases: we cannot expect that every notion from, say, probability theory, will find a fruitful application in, for example, the theory of conditionals: there exist highly technical, mathematical notions without any natural philosophical interpretation or application in sight. Yet in many cases this retranslation can, as mathematics clearly shows, furnish genuinely new insights. The reason for this success is that some notions have a dual character, by virtue of being both mathematical and philosophical.

Fine's work is a case in point, enabling as it has a significant enrichment of part-whole theory via the use of topological ideas. Moreover, exhibiting the connection between mereological and topological qualities has led to the emergence of a new discipline: mereotopology (Pratt-Hartmann, 2007; Skowron, 2017). Mathematical notions, and results from the theory of Markov chains, are likewise "translated back". For instance, the notion of a process's *stopping time* might serve as an inspiration to introduce a notion of "practical probability".[32]

When we think of the illuminating character of MP, the notion of understanding (as distinct from explaining) comes to mind. For a long time, the notion of understanding was not given much attention in the philosophy of science. It is easily dismissed as a kind of "psychological side-effect". However, it is now receiving more and more interest as a putatively legitimate concept. See, for instance, the monographs (de Regt 2017) and (Khalifa 2017), and the edited collections (De Regt, Leonelli, Eigner 2009) and (Grimm, Baumberger, Ammon 2017). (Grimm (2018) offers a broader philosophical perspective). It is natural for scientists to demand better understanding, and talk of "understanding phenomena, mechanisms, causes, processes…" figures both in everyday scientific conversations and in scientific papers. Scientists want to understand—and not only create—formal models and predictions: "The grand idea (…) was that the world around them was something *that could* be *understood*, if one only took the trouble to observe it properly..." (Schrödinger 1956 (2014), p. 57). Enhancing scientific understanding (by providing tools, ideas, conceptual frameworks) is an important role of mathematics—apart from playing its role in mathematical modeling, predicting, constructing representations and providing explanations. Moreover, mathematics is not only a tool for arriving at a model of understanding, but also a place where

[32] Even if we are 99.999 percent sure that we will win some game, we would not pay $1,000 to participate in it if, on average, it takes 100 billion years to be completed (i.e. if this is the expected "absorption time" for the process). Thus, if this notion is to really be of practical value, we must have the feeling that it is possible to stop the process *soon*. (We thank an anonymous referee for this observation).

understanding shows up as an important factor. It is precisely understanding that allows mathematicians to distinguish between explanatory and non-explanatory proofs, while understanding is also the basis for the notion of depth in mathematics that is currently much discussed. (See the special issue of *Philosophia Mathematica*, 2015, 23(2)).

Philosophers who consider understanding to be a crucial epistemic virtue will accept only such undertakings as enhance it. It is therefore important to convince them that MP can indeed fulfil this role, and that an ideal structure of qualities and ideas underlies the formal investigations.

3.4. The promising character of MP

MP can be counted promising if its results have a chance of shaping the future development of philosophy. It is interesting, in this context, to examine the practice of MP from the perspective of cognitive science. More particularly, we can here make use of some general remarks of Hutchins (2012) about the creation of conceptual orders. By introducing mathematical concepts and mathematical accuracy into the network of philosophical concepts, the philosopher *de facto* adds to the order exhibited by philosophical concepts. Hence, practicing MP is a kind of transition from conceptual disorder to conceptual order. An important consequence of this increase in the organization of concepts is the fact that the conceptual coherence of the philosophical system increases as well, this being a consequence of the rigorous character of mathematics. Nevertheless, such coherence in respect of philosophical systems (and results) is not a common feature where philosophical practice is concerned: often, philosophers' conceptual systems are disparate, disordered, and incommensurable affairs. This is quite natural for a realm of what purports to be fundamental knowledge. It is worth recalling the words of Schopenhauer (1966, p. 95): "[W]e find philosophy to be a monster with many heads, each of which speaks a different language". Much the same holds, we think, for MP.

The philosophical contribution of MP is manifested in our increasing conceptual order through formalization and mathematical precision. However, loosely speaking, to make a further contribution we must also disrupt the order of concepts:

> The maintenance of order and the increase in order are not always good things. Human cognition moves through cycles of disorder and reorder on all time scales. In the conduct of a scientific investigation, accumulating disorder may lead to a productive conceptual reordering. (...) A jolt of unpredictability is sometimes needed to overcome stable but inadequate conceptual structures (Hutchins, 2012, p. 321).

Thus, mathematizing, formalizing, explicating, building models, running simulations, and all other activities responsible for increasing order in respect of philosophical concepts should be accompanied by others that destroy this same order. The certainty that we obtain as a result of building up our conceptual order, if it is not to lead to intellectual stagnation, must take on something of the uncertainty characteristic of more disparate states. Using information theory, we can say that when the level of entropy (construed as a measure of unpredictability) is too small, we should be sure to increase it in the next cognitive cycle, because if we do not, each subsequent intuition may provide us with less and less data (Hutchins, 2012, p. 321). The mathematical philosopher should, therefore, clarify his or her deliberations and guard against formal errors, but from time to time, for the sake of intellectual hygiene, he or she should destroy the developed order of concepts in a Socratic kind of way, just so that the certainty acquired does not turn into philosophical dogmatism.

Classical mereology affords a good example here. Mereologists have often wondered whether or not the part-whole relation is a transitive one (for an overview, see Pietruszczak, 2020). Mormann (2009) overturned this way of thinking, showing that if the relation is modelled using *subobjecthood* in the sense of category theory, then each category will possess its own mereology, and the general question of transitivity will be replaced by one pertaining to the overall character of the category structure in question. So even formal philosophy sometimes calls for thinking "outside of the box"! This sort of overcoming is obviously not fixed once and for all. Another way to break down the order of the classical system of mereology has been to confront it with the mereology of substances in chemistry. In this context, Rom Harré and Jean-Pierre Llored (2011, p. 75) propose that "(...) philosophers need to go back to laboratories of research to grasp what scientists are really doing with their new models and apparatus".

3.5. The relevance of MP

Loosely speaking, investigations in MP will count as relevant if based on a good match between philosophical and mathematical ideas (i.e. contents of ideas, these being concretized groups of qualities). Otherwise it can happen that mathematical ideas are imposed on philosophical notions in an artificial way. Aristotle was already complaining that "mathematics has come to be the whole of philosophy for modern thinkers, though they say that it should be studied for the sake of other things" (*Metaphysics*, 992a24-b9). It seems that in the 20th century the popularity of logical tools led to precisely this type of phenomenon: logicians started to see logical schemes and notions everywhere, notwithstanding the fact that they were engaged in analyzing schemes embedded in rich and highly interesting mathematical structures that were by no means exclusively of a logical type.[33] On the other hand, there is also a danger of overlaying philosophical and mathematical qualities with undue haste; sometimes such a procedure leads to interesting mathematical problems, but it can also often turn out to be mathematically incorrect or trivial, and a working mathematician may well consider it too speculative or confused.[34]

Ideal qualities, as we know, are arranged in groups, which are parts of larger wholes—i.e. ideas. The relevance of an undertaking in MP rests on the overlapping of the content of the

[33] Marquise (2020) describes how a categorical understanding of logic points to the rich mathematical structures contained in logic itself.

[34] Mac Lane (1939), in a review of Benedict Bornstein's philosophically erudite book *Geometrical Logic. The Structures of Thought and Space,* without investigating the philosophical purpose and metaphysical attitude of its author, characterizes the latter's work as a "grandiloquent, naive, and confused" endeavor. Mathematical philosophers are often cast as guilty of naivety.

respective philosophical and mathematical ideas. If we fail to base our deliberations on this, we lose contact with the subject we are investigating, and our investigations in this area lose their footing. One may be delighted by the sophistication of one's constructions, but if the results are based on a misguided identification, they have no chance of exerting long-term influence. One indication that our deliberations are not relevant is that we find we have chosen the wrong mathematical tools. Let us recall that the selection of the proper mathematical tools for a given issue will be based on this partial identity with respect to the content of the idea in question. If we view MP as a philosophically motivated practice of constructing mathematical models (i.e. done for the sake of philosophical issues), then relevance will be based on some relation of fit between the model and what is modeled (the target system). Williamson (2017, p. 160), convincingly defending the idea of modeling in philosophy, argues that "a model of something is a hypothetical example of it" (cf. the discussion of models and their relationship to their originals in Greif (2021)). The metaphysician is obliged to ask what the basis is of one phenomenon's being an example of another. Why does one model fit the picture we have, and another not? Our answer is clear: it is based on the fact that both fall under the same ideas.[35] Even so, we are not claiming that each and every mathematical philosopher is engaged in such modeling: we just wish to say that if a mathematical philosopher does this, then his or her deliberations will be relevant by virtue of being based on the content of ideas that are themselves so. The relationship of the model to what is modeled is not one of complete adequacy.[36] Often, in practice, it is enough that the model matches the core of what is being modeled. Nevertheless our thesis is that this fit is based precisely on falling under the same ideas.

Relevance, in consequence, can help render our findings non-accidental. It is based on appropriate relations obtaining within the contents of an idea. Indeed, phenomenologically speaking, ideas constitute a pre-configuring of all possibilities, and inhabit an *a priori* field of necessary co-existences. Thus, it is the proper recognition of the contents of an idea that imparts necessity to whatever discoveries we make.

3.6. The adequacy of MP

Where philosophical work is concerned, adequacy is closely related to relevance. For our considerations to be relevant, they must be based on an adequate recognition of the appropriate philosophical and mathematical group qualities. To be sure, one may accidentally alight upon a given quality and construct a good model on this basis, but philosophy offers us ways to refine this process of discovery: notably, that of eidetic intuition as described above. It is thanks to this—when carried out correctly—that we can reach an adequate grasp of our subject matter. Whereas relevance is based on an identity of contents independent of the

[35] Let us take the example mentioned above, of Kelly's topological epistemology, and ask why it is that, in the language of models, it is the topological model that fits the epistemological problems. It is because *verifiability* and topological *openness* turn out to be the same set of ideal qualities (providing that Kelly's intuitings are adequate). These sets of ideal qualities are then made concrete in the contents of certain philosophical and mathematical ideas, which turn out to be the very same ideas! For details, we refer the reader to the book by Kelly (1996), and to the accessible discussion of Kelly's results in Schulte and Juhl (1996).

[36] We thank an anonymous reviewer for pointing out this problem. Modeling is not a search for complete and perfect adequacy—if it were, there would be no room left for new insights.

human mind, adequacy is a property of the subject and his or her subjective experience, which only after assuming some form—be it linguistic, conceptual, or mathematical—can become intersubjectively accessible and communicable to others.

In connection with this, it has to be noted that *intuition* has hardly received a positive press, especially in naturalistically oriented philosophical communities. Yet depriving philosophical deliberation of metaphysical grounding is, we believe, a significant impoverishment. Of course, many arguments would need to be presented in order to fully rehabilitate it as a valid source of knowledge—something for which we lack space here. At the same time, it might be that we should think of mathematical philosophy not just as a scientific form of philosophy (cf. Leitgeb, 2013), but also as artistic—closer to a craft of sorts. Let us quote Williamson, who, though not explicitly invoking relevance or intuition, points to other related phenomena such as good judgment and experience which themselves also inform craft-based and artistic professions and practices:

> Another respect in which rigorous-minded philosophers may find the method of model-building alien is that selecting and interpreting models is an art—in science as well as in philosophy. It depends on good judgment, honed by experience. One must distinguish simplifications which abstract away inessential complications from those which abstract away crucial features of the phenomenon, and genuine insights from mere artefacts introduced for mathematical convenience (Williamson, 2017, p. 169).

The requirement for adequacy is certainly non-trivial where philosophical considerations are concerned: simply put, one can, in philosophy, be genuinely mistaken about matters of substance. By comparison, in the history of philosophy, such errors play a rather less prominent role. However, an excessive fixation on one's own results can lead to philosophical dogmatism and a conviction of infallibility—a common misconception on the part of metaphysicians. It is precisely the requirement of adequacy, however trivial it may seem, that can challenge and undermine any such ideas of philosophical infallibility. We cannot trust every sort of intuition, but as phenomenologists do believe that we can trust those of an eidetic kind.

Conclusion

From the phenomenological point of view, a mathematician looks out towards a horizon: that of ideal possibilities, metaphysically based on ideal complexes of qualities. The mathematician—but also the philosopher—examines qualities, and analyses the content of the relevant ideas. So the rationale behind MP is that mathematicians and philosophers often analyze the same ideas—which is why MP exists as an option and holds out its promise of efficacy. This is how it comes to be the case that the advanced findings of mathematics may potentially serve to enrich philosophical investigations, and it is also the reason why certain philosophical concerns and reflections can inspire mathematical deliberations—as was the case with Cantor, whose work on set theory was philosophically motivated. Philosophy and mathematics, then, are not entirely separate fields of thought: "A philosopher who has nothing to do with geometry is only half a philosopher, and a mathematician with no element of philosophy in him is only half a mathematician. These disciplines have estranged themselves from one another to the detriment of both" (Frege, 1979, p. 273).

Mathematical philosophy is often described simply as philosophy that involves the application of mathematical methods. Indeed, this is what mathematical philosophers do, every day. Nevertheless, the description fails to capture the larger ontological situation, or metaphysically ground such an approach to philosophy. Thus, amongst other things, it does not make any rational sense of why it is that mathematics can be applied with such success in philosophical contexts. Mathematical philosophy needs a metaphysical foundation. The theory of ideas and qualities proposed by Ingarden provides this. Without such grounds, mathematical philosophy would be groping around in a fog, and open to justified criticism on the part of non-analytical philosophers. Moreover, it must be assumed that it is not metaphysically neutral. It necessarily involves an implicit commitment either to platonism (cf. Król, 2015) or to other equally metaphysical perspectives, and as such cannot be thought of as simply an innocent application of mathematical methods themselves.

Wigner's famous essay discussed what he considered to be the "unreasonable effectiveness of mathematics in science" (Wigner, 1960). By contrast, we think it by no means surprising that something like this can be encountered at the border between mathematics and philosophy. However, nothing comes for free: to be effective, a mathematical philosophy must be correct, responsive, illuminating, promising, relevant and adequate. Otherwise it will be trivial, or blind, or empty.

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Bibliography

Aristotle, (1991). *Metaphysics*. In *The complete works of Aristotle*, ed. J. Barnes. Princeton University Press.

Baker, A. (2005). Are there genuine mathematical explanations of physical phenomena? *Mind, 114*(454), 223–238.

Baker, A. (2009). Mathematical explanation in science. *British Journal for the Philosophy of Science, 60*(3), 611–633.

Baker, A., & Colyvan, A. (2011). Indexing and mathematical explanation. *Philosophia Mathematica,19*, 232–224.

Balaguer, M. (1998). *Platonism and anti-platonism in mathematics*. Oxford University Press.

Brun, G. (2016). Explication as a method of conceptual re-engineering. *Erkenntnis, 81*(6), 1211–1241.

Carnap, R. (1937). *The logical syntax of language*. London: Kegan Paul.

Carnap, R. (1950). *Logical foundations of probability*. University of Chicago Press/Routledge and Kegan Paul.

Casari, E. (2000). On Husserl's theory of wholes and parts. *History and Philosophy of Logic, 21*(1), 1–43. https://doi.org/10.1080/01445340050044628

Davis, P. J., Hersh, R., & Marchisotto, E. A. (2012). *The mathematical experience, study edition*. Birkhäuser Boston. https://doi.org/10.1007/978-0-8176-8295-8

Detlefsen, M. (2005).Formalism. In S. Shapiro (Ed.), *The Oxford handbook of philosophy of mathematics and logic* (pp. 236–317), Oxford University Press.

Eilenberg, S., & Mac Lane, S. (1945). General theory of natural equivalences. *Transactions of the American Mathematical Society 58*, 231–294.

Fine, K. (1995). Part-whole. In B. Smith & D. W. Smith (Eds.), *The Cambridge companion to Husserl* (*Cambridge companions to philosophy*) (pp. 463–486). Cambridge University Press. https://doi.org/10.1017/CCOL0521430232.011

Fleck, L. (1979). *Genesis and development of a scientific fact*. (Originally published in German, 1935). The University of Chicago Press.

Frege, G. (1979). Sources of Knowledge of Mathematics and Natural Sciences, transl. P. Long & R. White. In *Gottlob Frege: Posthumous writings* (pp. 267–74). Chicago: University of Chicago Press.

Gödel, K. (1953/9). Is mathematics a syntax of language? In S. Feferman (Ed.), *Kurt Gödel: Collected works, vol. III* (pp. 334–363). Oxford University Press.

Gödel, K. (1964). What is Cantor's Continuum Problem? In P. Benacerraf and H. Putnam (eds.): *Philosophy of mathematics: Selected readings* (pp. 258–273). Englewood Cliffs: Prentice-Hall.

Greif, H. (2021). Exploring minds. Modes of modelling and simulation in artificial intelligence. *Perspectives on Science,* 29 (4): 409–435. https://doi.org/10.1162/posc_a_00377

Gruszczyński, R., & Varzi, A. (2015). Mereology then and now. *Logic and Logical Philosophy*, *24*(4), 409–427. http://dx.doi.org/10.12775/LLP.2015.024

Hales, T. C. (2001). The honeycomb conjecture. *Discrete and Computational Geometry, 25*, 1–22.

Hallett, M. (1984). *Cantorian set theory and limitation of size* (*Oxford logic guides 10*). Oxford: Clarendon Press.

Hamami, Y. (2018). Mathematical inference and logical inference. *The Review of Symbolic Logic, 11*(4), 665–704.

Hardy, G. H. (1929). Mathematical proof. *Mind* 38, 1–25.

Harré, R., & Llored, J.-P. (2011). Mereologies as the grammars of chemical discourses. *Foundations of Chemistry* 13, 63–76. https://doi.org/10.1007/s10698-011-9103-3

Hartimo, M.H. (2007). Towards completeness: Husserl on theories of manifolds 1890–1901. *Synthese 156*, 281–310. https://doi.org/10.1007/s11229-006-0008-y

Hartimo, M. H. (2010). (Ed.) *Phenomenology and mathematics* (*Phaenomenologica 195*). Springer Netherlands. https://doi.org/10.1007/978-90-481-3729-9

Hellman, G. (1989), *Mathematics without Numbers*, Oxford: Clarendon Press.

Husserl, E. (2001). *Logical investigations,vol. II*, trans. J. N. Findlay, ed. D. Moran. Routledge.

Hutchins, E. (2012). Concepts in practice as sources of order. *Mind, Culture, and Activity, 19*(3), 314–323, https://doi.org/10.1080/10749039.2012.694006

Ingarden, R. (1971), *U podstaw teorii poznania (Foundations of the theory of knowledge)*, PWN.

Ingarden, R. (2013). *Controversy over the existence of the world, vol. I*, trans. A. Szylewicz. Peter Lang.

Ingarden, R. (2016). *Controversy over the existence of the world, vol. II*, trans. A. Szylewicz. Peter Lang.

Kaczmarek, J. (2019). Ontology in Tractatus Logico-Philosophicus: A Topological Approach. In G. M. Mras, P. Weingartner & B. Ritter (Eds.) *Philosophy of Logic and Mathematics: Proceedings of the 41st International Ludwig Wittgenstein Symposium*, Berlin, Boston: De Gruyter, 2019, pp. 397–414. https://doi.org/10.1515/9783110657883-024

Kaufmann, S. (2004). Conditioning against the grain: Abduction and indicative conditionals. *Journal of Philosophical Logic*, *33*(6), 583–606.

Kaufmann, S. (2005). Conditional predictions: A probabilistic account. *Linguistics and Philosophy, 28*(2), 181–231

Kaufmann, S. (2009). Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic, 38*, 1–53.

Kaufmann, S. (2015). Conditionals, conditional probability, and conditionalization. In H. Zeevat & H. C. Schmitz (Eds.), *Bayesian natural language semantics and pragmatics*, (pp. 71–94). Springer.

Kelly, K. (1996). *The Logic of Reliable Inquiry*, Oxford University Press.

Król, Z. (2015). *Platonism and the development of mathematics. Infinity and geometry*, Wydawnictwo Instytutu Filozofii i Socjologii PAN.

Lange, M. (2017). *Because without Cause. Non-Causal Explanations in Science and Mathematics*. Oxford: Oxford University Press.

Leitgeb, H. (2011). Logic in general philosophy of science: Old things and new things. *Synthese*, *179*, 339–350. https://doi.org/10.1007/s11229-010-9776-5

Leitgeb, H. (2012) A probabilistic semantics for conditionals. Parts A and B. *The Review of Symbolic Logic, 5*(1), 26–84, 85–121.

Leitgeb, H. (2013). Scientific philosophy, mathematical philosophy, and all that. *Metaphilosophy*, *44*, 267–275. https://doi.org/10.1111/meta.12029

Lewis, D. (1986). Probabilities of conditionals and conditional probabilities. In D. Lewis, *Philosophical Papers*, *vol. II* (pp.133–156). Oxford University Press.

Mac Lane, S. (1939). Review of Benedykt Bornstein, *Geometrical logic. The structures of thought and space* (Bibliotheca Universitatis Liberae Polonae, ser. B, no. 8 (31), Wolna Wszechnica Polska, 1939),*The Journal of Symbolic Logic*, 4(3), 133–134.

Mac Lane, S. (1983). The health of mathematics. *The Mathematical Intelligencer5*, 53–56. https://doi.org/10.1007/BF03026510

Mac Lane, S. (1986). *Mathematics: Form and function*, Springer.

Maddy, P. (1990). *Realism in mathematics*, Oxford University Press, New York.

Marquis, J-P. (2020). Abstract logical structuralism. *Philosophical Problems in Science* (*Zagadnienia Filozoficzne w Nauce*) 69 (2020), 67–110. https://zfn.edu.pl/index.php/zfn/article/view/518/517

McGee V. (1989). Conditional probabilities and compounds of conditionals. *Philosophical Review*, *48*(4), 485–541.

Melia, J. (1998). Field's programme: Some interference, *Analysis, 58*(2), 63–71. https://doi.org/10.1093/analys/58.2.63

Monk, J. D. (1976). *Mathematical logic*. Springer New York. https://doi.org/10.1007/978-1-4684-9452-5

Mormann, T. (2009). Updating classical mereology. In C. Glymour, W. Wang, & D. Westerstahl (Eds.), *Proceedings of the 13th international conference on logic, methodology and philosophy of science, Beijing 2007* (pp. 326-343). King's College Publications.

Mormann, T. (2013). Topology as an issue for history of philosophy of science. InH. Andersen, D. Dieks, W. Gonzalez, T. Uebel, & G. Wheeler (Eds.), *New challenges to philosophy of science* (*The philosophy of science in a European perspective,* Vol 4). Springer. https://doi.org/10.1007/978-94-007-5845-2_34

Null, G. T. (1983). A first-order axiom system for non-universal part-whole and foundation relations. In L. Embree (Ed.), *Essays in Memory of Aron Gurwitsch* (pp. 463–483). Center for Advanced Research in Phenomenology.

Null, G. T. & Blecksmith, R. (1991). Matrix representation of Husserl's part-whole-foundation theory. *Notre Dame Journal of Formal Logic, 32*(1), 87–111.

Jackson, F. & Pettit P. (1990). Program explanation: a general perspective. Analysis. 50 (2), 107–117. https://doi.org/10.1093/analys/50.2.107

Pietruszczak, A. (2020). *Foundations of the theory of parthood. A study of mereology*. Springer. https://doi.org/10.1007/978-3-030-36533-2

Pincock, C. (2015). Abstract explanations in science. *British Journal for the Philosophy of Science*, 66(4), 857–882. https://doi.org/10.1093/bjps/axu016

Piwowarczyk, M. (2020). Roman Ingarden's early theory of the object. In W. Płotka & P. Eldridge (Eds.), *Early phenomenology in Central and Eastern Europe. Contributions to phenomenology* (*In cooperation with the Center for Advanced Research in Phenomenology*), vol. 113 (pp. 111–126). Springer. https://doi.org/10.1007/978-3-030-39623-7_7

Pratt-Hartmann, I. (2007). First-order mereotopology. In M. Aiello, I. Pratt-Hartmann, & J. van Benthem (Eds.), *Handbook of Spatial Logics* (pp. 13-97). Springer. https://doi.org/10.1007/978-1-4020-5587-4_2

Rota, G. C. (1991). Mathematics and the task of phenomenology. In T. M. Seebohm, D. Føllesdal, & J. N. Mohanty (Eds.), *Phenomenology and the formal sciences. Contributions to phenomenology (In cooperation with the Center for Advanced*

Research in Phenomenology), vol. 8 (pp. 133–138). Springer. https://doi.org/10.1007/978-94-011-2580-2_9

Rota, G.-C. (1992). Husserl. In M. Kac, G-C. Rota, & J., Schwartz (Eds.), *Discrete thoughts: Essays on mathematics, science, and philosophy* (2nd edition, corrected and edited by P. Renz). Birkhäuser.

Schopenhauer, A. (1966). *The world as will and representation, vol. 1*, trans. by E. F. J. Payne. Dover.

Schulte, O. and Juhl. C.(1996). Topology as epistemology. *The Monist* 79, no. 1:141–147. https://doi.org/10.5840/monist19967916

Simons, P. (2003). *Parts: A study in ontology*. Oxford University Press.

Skowron, B. (2015). Using mathematical modeling as an example of qualitative reasoning in metaphysics: A note on a defense of the theory of ideas. *Annals of Computer Science and Information Systems, 7*, 65–68. http://doi.org/10.15439/978-83-60810-78-1.

Skowron, B. (2017). Mereotopology. In H. Burkhardt, J. Seibt, G. Imaguire, & S. Gerogiorgakis (Eds.), *Handbook of Mereology* (pp. 354–361). Philosophia.

Skowron, B., & Wójtowicz, K. (2021). Throwing spatial light: On topological explanation in Gestalt psychology. *Phenomenology and the Cognitive Sciences* 20, 537–558, https://doi.org/10.1007/s11097-020-09691-1.

Skowron, B., Bigaj, T., Chrudzimski, A., Głowala, M., Król, Z., Kuś, M., et al (2019). An assessment of contemporary Polish ontology. In B. Skowron (Ed.), *Contemporary Polish ontology*. De Gruyter. https://doi.org/10.1515/9783110669411-015

Skowron, B., Wójtowicz, K., & Kaczmarek, J., (manuscript). *Towards a Topological Philosophy*.

Stalnaker, R. (1968). A Theory of Conditionals. *Studies in logical theory. American Philosophical Quarterly, monograph 2,* 98–112.

Stalnaker, R. (1970). Probability and conditionals. *Philosophy of Science, 37*(1), 64–80.

Stalnaker, R. & Thomason, R.H. (1970). A semantic analysis of conditional logic. *Theoria, 36*(1), 23–42.

Thom, R. (1971). “Modern” mathematics: An educational and philosophic error? *American Scientist, 59*(6), 695–699.

Tieszen, R. (1992). Kurt Gödel and phenomenology. *Philosophy of Science, 59*, 176–194.

Tieszen, R. (1995). Mathematics. In B. Smith & D. W. Smith (Eds.), *The Cambridge Companion to Husserl (Cambridge Companions to Philosophy)* (pp. 438-462). Cambridge University Press. https://doi.org/10.1017/CCOL0521430232.010

Tieszen, R. (1998). Gödel's path from the incompleteness theorems (1931) to phenomenology (1961), *The Bulletin of Symbolic Logic*, *4*, 181–203.

Tieszen, R. (2005). *Phenomenology, logic, and the philosophy of mathematics*. Cambridge University Press.

Trlifajová, K. (2022). Infinity and continuum in the alternative set theory. *European Journal for Philosophy of Science*, 2, 2–23. https://doi.org/10.1007/s13194-021-00429-7

Twardowski, K. (1979). Symbolomania and pragmatophobia. In J. Pelc (Ed.), *Semiotics in Poland 1984–1969. Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science), vol. 119*. Springer. https://doi.org/10.1007/978-94-009-9777-6_2

Wang, H. (1987). *Reflections on Kurt Gödel*, MIT Press.

Wigner, E. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications in Pure and Applied Mathematics, 13*(1), 1–14.

Williamson, T. (2017). Model-building in philosophy. In R. Blackford & D. Broderick (Eds.), *Philosophy's future: The problem of philosophical progress.* Wiley-Blackwell.

Wójtowicz, A., & Wójtowicz, K. (2021). A stochastic graphs semantics for conditionals. *Erkenntnis* 86, 1071–1105, https://doi.org/10.1007/s10670-019-00144-z

Wójtowicz A., & Wójtowicz, K. (2022). A graph model for probabilities of nested conditionals. *Linguistics and Philosophy* 45, 511–558, https://doi.org/10.1007/s10988-021-09324-z

Van Fraassen, B.C. (1976). Probabilities of conditionals. In W. L. Harper, R. Stalnaker, & G. Pearce (Eds.), *Foundations of probability theory, statistical inference, and statistical theories of science* (*The University of Western Ontario series in philosophy of science,* vol. 1) (pp. 261–308). D. Reidel.

Yablo, S. (2002). Abstract objects: A case study. *Nous, 36*(s1), 220–240. https://doi.org/10.1111/1468-0068.36.s1.7