Prolongments of *Ensaio*: Schrödinger Logics and Quasi-Set Theory

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Abstract

In his *Ensaio*, Newton da Costa advanced at least two interesting ideas to relate logic with quantum theory in a different direction than that of standard 'quantum logics', contributing to initiating the systematic investigations in the field of *non-reflexive* logics with the introduction of a system of logic where the standard notion of identity is not appliable to all objects, as some like Schrödinger believed with regards quantum elementary particles. In this chapter, we revise his system, relate the subject with philosophical and logical discussions about identity and individuation of quantum objects, and also show how they were extended in the directions of a group of higher-order logics and of a theory of *quasi-sets*. So, we show how the two challenges proposed by da Costa were solved, first with the development of a theory of quasi-sets and then by founding in such a theory a semantics for a special case of an *intensional* Schrödinger logic which encompasses da Costa's system and which copes with Dalla Chiara and Toraldo di Francia's claim that "microphysics is a world of intensions". Some further applications of these systems are also referred to, so as some philosophical ideas are further advanced. The References at the end provide material for more detailed analyses.

Keywords: Identity, indiscernibility, individuality, non-individuality, Schrödinger logics, non-reflexive logics, nonreflexive logics, quasi-set theory, quantum logics, Newton da Costa, non-standard semantics, quantum theories, Manin Problem.

Contents

It is beyond doubt that the question of 'sameness', of identity [for elementary particles], really and truly has no meaning.

Erwin Schrödinger

[One should] try to generalize the notion of set, for instance, building a theory of *quasi-sets* containing the usual sets as a special case ...

Newton da Costa

[m]icrophysics is a world of intensions

M. L. Dalla Chiara and G. Toraldo di Francia

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1 Introduction

In the above quotation, what kind of identity would Schrödinger make reference to? Impossible to say, because he was not explicit about that. But, probably, he was thinking in the intuitive concept surely we all share, namely, as philosophers use to say, *that thing* an object has that makes it the object it is and different from any *other* object, something which (as we suppose he believed) elementary particles lack.[1] Sometimes this is expressed by some intrinsic characteristic, sometimes it is thought of as a relation and when metaphysics is pushed a little, perhaps by some form of *susbtratum* underlying all properties an object can have and relations it can share with other objects. This informal notion of identity is linked to that of *individuation*; objects, such as the things in our surroundings and at our scale, do have an identity and this makes them *individuals*, unities that can be discerned in different contexts from any other thing, even if at least in principle.

But such an intuitive view is not much useful for foundational and logical analysis of the empirical domain. Really, we cannot accept the above de-

[1] We shall speak about 'objects' taking this word as neutral as we can, making reference to the entities coped with by physical theories, including their underlying mathematics and logic; so, numbers, for instance, are objects.

scription as a *definition* of identity since it is redundant and vague. Thus we need to go to logic. Doing that, we realize that there are different (and non-equivalent) ways to formalize a notion we can call *identity*, say 'first-order identity', 'higher-order identity', 'relative identity', 'necessary identity', and so on. Furthermore, even if we consider just first-order languages, we can conceptualize identity in different ways, as we shall see below. If this is really so, as we strongly believe that our argumentation below shows that it is, which one of these formalized notions captures the essence of the intuitive identity delineated above? We guess that no one of them. Intuitive identity remains intuitive, and the formalized versions become *formalized identities*. To acknowledge this distinction is important for our foundational purposes, as we shall see soon.

We call the Standard Theory of Identity (STI) that theory of identity formalized in standard classical logical systems; since, as we said above, there is no just one way to do that, we shall be in need of being more specific below. By 'standard mathematics' we mean that part of present-day mathematics that can be developed within a usual system of set theory, such as ZF, ZFC, NBG or KM, and 'classical logic' means *Logica Magna* [6, p.202,], [15, II, §2], that is, higher-order logic, perhaps involving set theory; when we will be restricted to first-order systems, we mention that explicitly. Another kind of consideration would be in need when we think of category theory, and we shall speak a little about this topic in the end.

This chapter is organized as follows. The aim of the paper is to present the prolonguements that were made from da Costa's 'Schrödinger Logic' and his suggestion of the development of a 'theory of quasi-sets', given in [15, pp.117ff]. But we also aim at to enlarge the discussion with more updated ideas, so our references not only mention most of these works but also aim to provide further readings. In particular, we not only extend da Costa's first-order logic to a higher-order system but also to an intensional higher-order logic with the aim of coping with the claim that in the quantum realm an intension can have more than one extension, something that challenges the standard theory of reference. But the main aim is that the chapter makes justice to the originality of da Costa's ideas.

2 Identity and the quanta

The basic idea was put above: *identity* is something that makes an object an *individual*. An individual can be said to be anything that satisfies the following three conditions: (1) it is an *unity*, a *one of a kind* (a person, a chair, a natural number, a real function), (2) it can be *discerned*, or *distinguished* (even if at least in principle) from any *other* individual, although a rather similar one, and (3) it can be *re-identified* as being *that* individual in different contexts; if we can name an individual, its name will act as a *rigid designator*, naming *the same* individual in all possible (and accessible) worlds.

The third condition makes clouds, portions of water, and elementary quantum systems *non-individuals* (see [67] for the first two, and [36], [59] for the later). Really, once we have one of them (for instance, an isolated cloud in the sky, a portion of water in a cup or an electron in an atom, we will be no more able to identify them when mixed in a swarm of similar things. The cloud, once merged with another cloud, the portion of water when mixed with other portions, and the electron, when expurged from the atom by ionization, cannot be identified anymore as being *that* thing we had before. Concerning 'macroscopic' objects, like persons and chairs, the re-identification comes with questioning too, as we know since antiquity. The problem of *transtemporal identity*, and in particular that of *personal identity* is as old as philosophy. We defend the view that for simplicity (and since it looks enough for usual practical and theoretical purposes – at least in some domains) we *postulate* that objects persist in time as individuals, suspending the judgment about *what* would be that makes the object the object we had previous contact with.[2] John Locke relegated such a thing to something "unknown" [64, chaps.23,27], but what would be not a substance [89], and David Hume said that we hold the transtemporal identity of a thing by habit [43, p.222], since (as we see it) there would be no 'logical reason' to sustain that an object, once leaving our ken and once we observe a quite similar object in a next opportunity, will conduce us to grant that it is the same object in the two apparitions. Similar views were held by Bertrand Russell [77, chap. 3] and, with regard to quantum particles, by Erwin Schrödinger:

> "When a familiar object re-enters our ken, it is usually recognized as a continuation of previous appearances, as being the same thing." [Emmending that] "The relative permanence of individual pieces of matter is the most momentous feature of both everyday life and scientific experience." [80, p.209]

As we see, he doesn't *affirm* that the individual endures or perdures,

[2] An additional philosophical discussion goes in the direction of to know whether an individual *endures* in time, being the same in all time-slices, or *perdures* in time, being the sum of all time-slices instances of the (supposed) individual. Both views have their defenders (see [5], [82], [65, chap.8]).

but he seems to suggest that *we think so*, perhaps by convenience. But, in what regards quantum objects, he is more incisive. Guessing that "states [of quantum systems] are well-defined individuals", he stresses that

> in favourable circumstances, long strings of successively occupied states may be produced (...) Such a string gives the impression of an identifiable individual, just as in the case of any object in our daily surrounding. [80, p.217, 218]

But, in the continuation, he goes in asserting that in experiments such as the observation of tracks in a cloud chamber, we usually say that the (continuous) tracks "are caused by the same particle", and that there is no reason to ban such a view (for scientific reasons), but that we should agree that

> the sameness of a particle is not an absolute concept. It has only a restricted significance and breaks down completely in some cases.

This is reinforced in several parts of his text: he insists that a quantum system (he speaks in terms of 'particles', but of course, this can be extended to the 'particles' in quantum field theories as well – field excitations) is not an individual, lacking identity. This is one of the quotations da Costa takes to motivate his argumentation for sustaining that any logical principle can be put within parentheses.

3 da Costa on logical laws

According to da Costa,

> "there is almost no logical principle that cannot be derogated, in the sense that there exists a reasonable logic in which it does not hold in general" [15, p.124]

This is his *norm of relativity*, and he exemplifies the possibility of violating the Principle of Contradiction with *paraconsistent logics*, of which he is one of the founding fathers and with relevant logics. The non universal validity of the Principle of the Excluded Middle is exemplified with *paracomplete logics* such as intuitionistic logic and many-valued logics. But, if we restrict the analysis to the three celebrated 'fundamental laws of reason' [15, pp.95ff], what can we say about the Principle of Identity?[3]

Logical laws, to da Costa, have their origin in their applications to the systematization of our experience. Since our present-day knowledge suggests that no logic can be *a priori* imposed to our sistematizations (*cf.* [15, p.124]), we are free to find non-classical systems and show how they can help us in the systematization processes. So, if we have theoretical or empirical reasons to question some logical principle, the reasonable attitude would be to present a logical system where the considered principle is violated in some way. This is what he did with respect to the Principle of Identity, and Schrödinger's view was taken as a motivation. His Schrödinger Logic will be recalled soon. But, before going that, and noticing that it is the notion of identity that will be put into parenthesis, let us comment a little bit about identity in classical contexts.

4 Identity in classical logic

In the beginnings of modern logic, Frege marked the way logic and mathematics would understand the concept of identity. After having proposed in his *Begriffsschrift* (1879) [35] that identity should be applied to *names*, in his *Über Sinn und Bedeutung* (*Sense and Reference*, 1892) [34] he turns to the application to *contents* (of names, say, which are objects). This is the view we accept today: when we say that $x = y$ is the case, then it is agreed that x and y have the same *referent*. Frege's classical example 'the morning star is identical to the evening star' is commonly cited as a paradigmatic example. Both descriptions refer to the same object, namely, the planet Venus. The negation of an identity statement is *difference*, written '$x \neq y$', and the *standard theory of identity* (STI) says that, in this case, *some distinctive property or relation exists*. This is quite relevant for the applications in the quantum domain, as we shall see soon. But notice that such a view moves away from any kind of 'transcendental individuality' (H. Post's words [73]), that is, any possibility of considering *substratum*, *haecceity*, or other forms of expressing that the identity of an object is given by something 'transcending'

[3]It should be remarked that these three 'principles' are not the only ones in classical logic that deserve attention. Really, we could add several other 'principles' (in reality, infinitely many) such as the double negation rule, compositionality, Peirce's law, De Morgan laws, reduction to the absurd, and so on. The mentioned three are important for historical reasons.

its properties or relations it shares with other objects.[4] This 'classical view' is known as *bundle theory of individuation* [65], and it is clear that there is a confusion between 'identity' and 'individuality', something that was put clear only a few years ago. Since this last point is relevant for our purposes here, let us say something about the mentioned distinction before turning to STI.

4.1 Identity, individuality, and individuation

Three notions are usually confounded and taken as equivalent: identity, individuality, and individuation (or 'isolation'). 'Identity' is a logical notion, given by some 'theory of identity', such as the STI we shall meet below. 'Individuality' is a metaphysical notion, ascribed by something we postulate and that serves to give us an account for "whatever it is that makes it the single object that it is – whatever it is that makes it *one* object, distinct from others, and the very object that it is as opposed to any other thing" [66, p.75]. A typical example is Leibniz's Principle of the Identity of Indiscernibles (PII), which states that it is sufficient for two objects to share all their properties and relations in order to be *the same* object.[5] In its turn, 'individuation' is an epistemological notion we develop to serve us in providing reasons for sustaining that some object possesses a relevant position for epistemological claims; for instance, we can suppose a quantum entity trapped in some trap device in our laboratory and then *we know* that it is there. Typical examples are the claims we have for considering the objects of our surroundings as *individuals* endowed with identity conditions, and trapped quantum objects as well.[6]

These distinctions, as mentioned, are relevant for the discussions on the logical foundations of quantum structures.[7] Really, a quantum object can be trapped (as done in several experiments), so (presumably) becomes isolated from others of the same kind, so that the (say) asymmetries and location of the laboratory provide the epistemological grounds for saying that we have a certain quantum object there. The metaphysical supposition that such a

[4] That quantum physics does not involve such forms of transcendental notions is also advocated by Teller in [87].

[5] Contrary to some such Muller [72], we regard PII as a metaphysical principle. Muller thinks that it is "a general truth about the universe".

[6] The reader can consult [51] for a discussion of why Hans Dehmelt's trapped positron named 'Priscilla' is not an individual.

[7] As Aerts and Pykacz suggest, we prefer to name 'quantum structures' the field usually termed 'quantum logic', since we agree that it is more accurate in giving a more precise account of what is going on [1].

move gives us its individuation is usually taken for granted; as Schrödinger says in the above quotation, such an assumption is "the most momentous feature of both everyday life and scientific experience". But, anyway, in regards to the identity of trapped quantum entities, such experiments should be still analysed further.[8]

The distinction between loss of identity and loss of individuality can then be explained. In the beginnings of quantum theory, Schrödinger and others such as Hermann Weyl, Max Born and others (see [36] for a wide discussion) spoke about the *loss of individuality* of quantum entities (quantum 'particles' in that time). The reason was that once we lose them for some reason, we cannot identify them anymore as being *that* entity we had before, as already said above. But taking into account what we have said above about these notions, the better would be to speak, as Schrödinger did, about the lack of identity of these entities; this was the direction da Costa has taken to motivate his logical system. Notice that the supposition that they have *lost* their identity is not totally correct, since a thing cannot lose what it doesn't have, as Schrödinger believed (and we agree with him). Thus, we call *non-individual* any object that fails to obey the standard theory of identity (STI). Despite being non-individuals, these entities can be isolated (individuated), as in the mentioned experiments of trapping quanta, and can also satisfy some principle of individuation. But, let us mark our position, the fact that we have an isolated quantum entity, a portion of water or a cloud, this fact does not make them individuals, entities endowed with identity. The physical reason is (mainly) condition (3) of our definition of an individual, namely, re-identifiability.

Other people have also questioned the sense of saying that quantum entities do have identity. One of the prominent ones was the Russian mathematician Yuri I. Manin. Due to its relevance, let us summarize it here.

4.2 Manin's Problem

In 1974, the American Mathematical Society sponsored a meeting directed to analyse the impacts and consequences of Hilbert's Mathematical Problems from 1900 to the date.[9] From that symposium, analyses of Hilbert's legacy

[8]Really, we can find in the web several 'pictures' of trapped quanta, but we know that such entities cannot be photographed due to their size. What we have in such 'pictures' is something like the 'single strontium atom' in the National Geographic site, or the 'single atoms' in the IBM film 'The boy and its atoms', etc. These are computational re-creations of quantum systems only.

[9]In the second International Congress of Mathematicians, Hilbert delivered a talk where he presented several problems he saw as a legate from the XIXth century to the century

were made and an updated list of mathematical problems has arisen. The mathematician Yuri I. Manin presented the first problem of the list, related to the foundations of mathematics. Manin's Problem, let us call it so, was concerning the necessity of searching for a 'new set theory', able to deal with collections of indiscernible quantum objects presented by quantum theory.[10] As he said when presenting his problem,

> "We should consider the possibilities of developing a totally new language to speak about infinity. [we recall that set theory is also known as the theory of the infinite] (...)
>
> I would like to point out that this [set theory] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things' ". [68]

The quotation clearly shows that he doesn't see a collection of quantum objects as forming a *set* in the standard ('Cantorian') sense, so that he uses quotation marks for expressing this. Let us recall that Georg Cantor himself 'defined' a set as

> [b]y an 'aggregate' (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought. These objects are called the 'elements' of M. ([10, p.85]; see [36, chap.6])

This intuitive account is compatible with Frege's later view on identity. Axiomatically (after Zermelo), the introduction of the Axiom of Extensionality establishes that two sets are identical (so they are not 'two' after all) if and only if they have the same elements, something that presupposes that the elements are themselves individuals, together with the axioms of the underlying logic (see below), characterize STI in extensional set theories. We shall know the consequences of this move quite soon.

which was starting soon. See [9], where Hilbert's paper is reproduced.

[10] By 'quantum theory' we wean both the non-relativistic versions of quantum mechanics and the quantum field theories; distinctions will be made when necessary.

Other authors have also questioned the use of standard sets to cope with collections of indiscernible quantum entities; for instance, Dalla Chiara and Toraldo di Francia expressed that

As we see, to sustain a *metaphysics of truly indiscernible things* apparently is something to be considered feasible.

5 STI

The standard theory of identity (STI) can be summarized as follows. In first-order languages, usually we take a binary predicate symbol '=' as primitive, subjected to two postulates, namely

1. $\forall x(x = x)$ (Reflevity of identity)

2. $\forall x \forall y(x = y \to (\alpha(x) \to \alpha(y)))$, where $\alpha(x)$ is a formula where x appears free and $\alpha(y)$ results from the first one by substituting x for y in some free ocurrences of x. (Substitutivity)

Then we can prove that also symmetry and transitivity hold [69]. In short, the relation of identity is a *congruence*;[11] mathematicians say that it is the *finest* congruence in the sense that any congruence extends identity. We can say this with other words and concepts. The standard interpretation of the identity symbol is the *diagonal* of the domain. Calling D the domain of an interpretation, then '=' corresponds to the set $\Delta_D = \{\langle a, a\rangle : a \in D\}$, also called the 'identity of D'. Thus, being R a congruence over D then, of course, $\Delta_D \subseteq R$.

In a first-order set theory, to these axioms, we add the axiom of extensionality, possibly adapted if there are atoms. In higher-order languages, we can define identity by the so-called Leibniz Law, namely, being x and y entities of type τ and being F a variable of type $\langle\tau\rangle$,[12]

$$x = y \coloneqq \forall F(F(x) \leftrightarrow F(y)). \tag{1}$$

[11]For the sake of completeness, we recall that being $\mathcal{A} = \langle D, R_i\rangle$, $i \in I$, a structure having D as its domain and being R_i n-ary relations on D, then a congruence is an equivalence relation R on D such that, being $x_i R x'_i$, then $R_i(x_1, \ldots, x_n) \leftrightarrow R_i(x'_1, \ldots, x'_n)$, that is, it is an equivalence relation that is compatible with the relations and operations of the structure. Being reflexive, it is obvious that $\Delta_D \subseteq R$.

[12]For instance, in second-order languages, x and y are individual variables and F is a variable for predicates of individuals.

This equation is a conjunction of two implications; one is the Principle of the Indiscernibility of Identicals, termed 'Leibniz Law' for some philosophers (we prefer to keep with the terminology we are using), that is,

$$x = y \to \forall F(F(x) \leftrightarrow F(y)), \tag{2}$$

and the Principle of the Identity of Indiscernibles, that is,

$$\forall F(F(x) \leftrightarrow F(y)) \to x = y. \tag{3}$$

Of course these expressions can be extended for relations other than unary relations (properties). Important to notice that Leibniz's Law links two notions: identity and indiscernibility, confusing them: identical things are indiscernible and indiscernible things are the very same thing. But, let us anticipate, in quantum physics we wish to have indiscernibility of quantum systems not implying their identity; at least this is what it seems Schrödinger believed.

We could extend the discussion of these principles, but what we have said is enough for the purposes of this chapter. Let us move.

6 Consequences of STI

Standard (extensional) set theories deal with *individuals*, that is, entities that can always, even if at least in principle, be distinguished from any other by some property. Really, given an object a whatever, we can form the unitary set $\{a\}$ (which is assumed to exist in almost all set theories) and define the property (formula with just one free variable) $I_a(x) \coloneqq x \in \{a\}$. This distinguishes a from any other entity of the universe since only a has this property. So, set theories are not completely 'neutral' as some philosophers tend to agree; see [38] for a discussion.[13]

Furthermore, if we consider *models* of a set theory such as the ZFC system (Zermelo-Fraenkel with the Axiom of Choice), we will be led to the cumulative hierarchy of sets (introduced by von Neumann), $\mathcal{V} = \langle V, E \rangle$, where V is given by transfinite recursion over the class On of the ordinals and E is a binary relation over V which is, for the inhabitants of the model,

[13] Timothy Williamson also claims (and we agree) that certain metalogical discussions require a metalanguage compatible with the logic [92]. By the way, this is also the opinion of Newton da Costa when he says that any logic would be provided with a semantics (even if an informal one) constructed in a realm compatible with its principles, say an intuitionistic semantics for intuitionist logic and so on.

the membership relation. If no atoms are considered,[14] then V is got this way:

$V_0 \coloneqq \emptyset$

$V_1 \coloneqq \mathcal{P}(V_0)$

$\vdots$

$V_\lambda \coloneqq \bigcup_{\beta<\lambda} V_\beta$ if λ is a limit ordinal, and

$V \coloneqq \bigcup_{\alpha\in On} V_\alpha$

All sets are elements of V, but V itself is not a set if ZFC is consistent; it is a *proper class*.[15] $\mathcal{V}$ is the structure where we interpret the language of ZFC, and E is the membership relation for the sets of the model (other requirements are also imposed, or got as a consequence), such as that V must be 'transitive', but they are not relevant here). When E is the membership relation restricted to the elements of D, we say that the model is *standard*, and we shall be restricted to this model, supposing it exists.[16] Among its elements it is On, the well-founded 'set' of all ordinals.

Inside the universe V, we can construct almost all mathematical structures such as groups, rings, fields, differentiable manifolds and practically all structures used in physics, such as geometries, Hilbert spaces and the like. Do these structures have something in common? The answer is interesting. Some of these structures are *rigid* in the sense that its only automorphism is the identity function, others are *non-rigid*, or *deformable*. The first fact is that the whole universe $\mathcal{V} = \langle V, \in\rangle$ is rigid [46, p.66]. This has consequences. The interesting one for the question of the notion of identity is that if $\mathfrak{A}$ is a non-rigid structure in V, it can be extended to a rigid one by adding new relations and operations to it. For instance, take the additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, +\rangle$. This structure is not rigid, since $h(x) = -x$ is a non-trivial automorphism. So, *inside* the structure, 2 and -2 are indiscernible since h is bijective, preserves the operation (that is, $h(a+b) = h(a) + h(b)$),

[14] Atoms are entities that are not set but can be elements of sets. It is usually supposed that they have no elements, but are distinct from the empty set. See [85].

[15] In the theory NBG (von Neumann, Bernays, and Gödel), there are sets and there are classes. Every set is a class, but not the other way around. There are classes, the *proper classes*, which are not sets. If we restrict NBG to sets only, it becomes equivalent to ZFC.

[16] It is consistent with ZFC, supposed consistent, that there are standard models of ZFC, but it is also consistent with ZFC (supposed consistent) that there are no such models.

and $h(2) = -2$). To rigidify $\mathcal{Z}$ it is enough to add the relation $<$ to it. Thus, we have the following facts:[17]

1. The universe $\mathcal{V}$ is rigid.

2. Any structure in V can be extended to a rigid structure.

3. Given a structure $\mathfrak{A}$, we say that the elements a and b of its domain are $\mathfrak{A}$-indiscernible if there exists an automorphism of the structure leading one of them in the another.

These facts show that inside a model of ZFC the most we can do with respect to indiscernibility, supposing we wish to express it for some entities, is to 'mimic' it inside a non-rigid structure, but *going outside* of it (that is, going to a rigid extension), we can realize that the supposed indiscernible elements are not indiscernible at all. Consequently, in a set theory such as ZFC, we cannot express *true* indiscernibility; ZFC, under STI, hence all standard mathematics, is a theory of individuals, as pointed out already.[18]

This is enough concerning standard mathematics and classical logic; we realize that they are, in a sense, 'Leibnizian', theories of individuals; let us emphasize this: within any model of a theory such as ZFC, it is always possible to discern between any two entities so that they act as individuals of our definition. Thus, if quantum theories provide motivation for questioning this fact, of course, we would be in need to change the mathematical basis in some way. Let us consider this point next.

7 Enter the quanta

As mentioned before, according to some scholars there is a sense in saying that quantum objects do not satisfy identity criteria or, at least, STI. They would be *non-individuals*, and despite can be isolated in some circumstances, they do not fulfil the (we suppose) very intuitive notion of an individual posed above.

But there are 'interpretations' (or 'theories') of the quantum formalism which say that elementary particles are individuals, having an identity in all situations. This is the case, for instance, with David Bohm's quantum mechanics. The *position* of a particle provides such a distinction from any

[17] For a more detailed analysis, the reader can see [60].

[18] Notice that this conclusion holds also when atoms are taken into account. The so-called Fraenkel-Mostowski permutation models [62] are constructed by a trick in making *distinct* atoms to look like indiscernible things.

other particle of the universe. But this has a cost, which we report as more problematic than the assumption that quantum objects are non-individuals, namely, the existence of a 'quantum potential' and the 'pilot wave' associated with the particle. In our opinion, from the metaphysical point of view, this is stranger than assuming that they lack identity conditions.

But let us turn to the 'standard' interpretation of quantum theories. By 'quantum physics' we mean both the orthodox (non-relativistic) quantum mechanics (QM) and the relativistic one, usually treated as a theory of quantum fields ('quantum field theory', QFT). Our argumentations independ on the particular version we can chose; 'quantum objects' can be both *particles* of QM and *field excitations* in QFT.

Quantum objects ('quantum entities' will be used alternatively), notwithstanding, can be seen as *non-individuals* in a sense (we shall be back to this point below, but see [59]); in certain situations, they cannot be discerned from any other quantum object by whatever means, so say the physical theory. Typical examples are bosons in a bosonic condensate. Even fermions, which obey Pauli's Exclusion Principle, so not being able to share all their quantum numbers, enter situations where we cannot say which is which; a typical case is of the two electrons of a helium atom in its fundamental state (less energy). We can even name the electrons, say 'Mike' and 'Ike' but, as remarked by Hermann Weyl, although his example is not precisely this one when referring to electrons in situations like the mentioned, said that "it is impossible [. . .] that one of them will always be able to say 'I am Mike', and the other 'I am Ike'. Even in principle, one cannot demand an alibi for the electron." [91, p.241].

Situations like these ones bring a challenge to the mathematical foundations of quantum theory if we are to relate it with a possible 'reality' of quantum entities.[19] In fact, although the physical theory (in this case, quantum theory) cannot discern quantum objects in the mentioned situations, its underlying mathematics does provide the means, since standard mathematics is, as told before a theory of individuals. On the contrary, collections o such entities, as recalled by Manin, should not be seen as forming *sets* of standard set theories, which are collections of *distinct* objects, according to Cantor's well-known 'definition' [10, p.85].[20] As he proposes, a *new* set

[19]Rigorously speaking, the quantum formalism (whatever it is) doesn't speak of something except states (of supposed quantum systems), observables and the like. The 'entities' to which the formalism would make reference to are given when we try to interpret it.

[20]Strictly speaking, this is not a definition of the concept of set. Usually, sets are characterized by the postulates we use. Cantor's account reads (in the English translation): "[b]y an 'aggregate'(*Menge*) we are to understand any collection into a whole (*Zusammen-*

theory was in order. Quasi-set theory, to be seen below, provides an answer to Manin's Problem.

8 da Costa's Schrödinger logic

In order to cope with the supposed failure of sameness or identity for quantum objects, da Costa proposed a two-sorted first-order two-sorted logic termer S [15, pp.117ss]. We reconstruct his system here; there are entities of two species, the first ones denoted by individual variables of the first species $x', x'', \ldots$, and the second ones denoted by individual variables of the second species $X', X'', \ldots$. The language uses the standard propositional connectives, quantifiers and auxiliary symbols. Furthermore, there are individual constants of both species, the symbol for equality and a non-empty family of predicate symbols of rank n for each $n > 0$.

The 'terms' are the variables and the constants; so there are terms of the first species (individual variables and individual constants of the first species) and of the second species (individual variables and constants of the second species). The formulas are defined in a standard way, with special care to identity, and respecting the nature of the terms; if P is a n-ary predicate symbol and if $t_1, \ldots, t_n$ denote terms, then $P(t_1, \ldots, t_n)$ is an atomic formula.[21] If t and u are terms of the second species, then expressions of the form $t = u$ are also atomic formulas; notice that this is not the case with first-order terms (at least one of them). So, $T = U$, with terms of the second species, is not a well-formed formula. From the atomic formulas, the 'general ones' are defined as usual. The notions of free and bound variable in a formula, so as of a free term for a certain variable, and so on, are also defined as usual.

The postulates of S are the following ones, and we use da Costa's numeration: being A, B, and C formulas,

I_1 $A \to (A \vee B)$

I_2 $(A \vee A) \to A$

I_3 $(A \to B) \to ((C \vee A) \to (C \vee B))$

I_4 $A, A \to B / B$

---

fassung su einem Ganzen) M of definite and separate objects m of our intuition or of our thought". An analysis of this definition can be found in [33, pp.4ff].

[21] He writes $Pt_1t_2 \ldots t_n$ for that.

The postulates for the quantifiers are

II$_1$ $\forall x A(x) \to A(t)$, where $A(x)$ is a formula, x an individual variable and t a term free for x in $A(x)$ *of the same species of* x.

II$_2$ $C \to A(x)/C \to \forall x A(x)$, if x doesn't appear free in the formula C.

Now the postulates for equality:

$=_1$ $\forall X'(X' = X')$

$=_2$ $U = V \to (A(U) \to A(V))$ with the standard restrictions, noticing that U and V are terms of the second species.

Other syntactic notions such as those of theorem, deduction and others are introduced accordingly. But the fundamental step remains in the semantics. Da Costa assumes a structure for the language of S on the following grounds: there are two non-empty sets D_1 and D_2 with the proviso that $D_2 \subseteq D_1$,[22] and for each n-ary predicate symbol of the language, a subset of D_1 is associated. Furthermore, the elements of D_1 correspond to the individual constants of the first species, while the elements of D_2 are in correspondence with the individual constants of the second species. Then, he makes the following remark: "naturally, to the equality symbol it is associated the equality relation in D_2" [15, p.119]. In other words, the language doesn't speak either of the equality or the difference of the elements of D_2. Our author also remarks that the standard semantic results are obtained without difficulty.

Then arises the fundamental remark, related to "philosophical difficulties" (*ibid.*), as the following ones. Since the relation of identity should make no sense for the elements of D_2, then D_1 should not be considered as a set of standard set theories. To surpass this difficulty, says da Costa, two possibilities are open: (1) to generalize the notion of set, say by elaborating a theory of *quasi-sets* (here he introduces this term) containing the standard sets as particular cases, and to found a semantics for S in such a theory; (2) to elaborate an *informal semantics*, "something imprecise", he says, grounded on the natural language, and taking into account the results of quantum mechanics.

Both directions were realized in the sequence. In 1990, this author constructed a theory of quasi-sets and delineated the 'informal semantics' for a generalization of the system S to a simple theory of types [47, 49]. We revise these works next.

[22] Da Costa writes '$\subset$' instead '$\subseteq$' with the same meaning.

9 Higher-order Schrödinger logics

Let us call '$S^{\omega}_{\equiv}$' a higher-order Schrödinger Logic involving the notion of indiscernibility (marked by the symbol '$\equiv$') elaborated as follows. We start by defining the set of types.

Definition 9.1 (Types) *The set of types as the set* Π *such that:*

(a) $i \in \Pi$ (i is the type of the *individuals*)

(b) if $\tau_1, \ldots, \tau_n$ belong to Π, then $\langle \tau_1, \ldots, \tau_n \rangle \in \Pi$.

(c) Nothing else is a type.

We admit that, in the language $\mathcal{L}^{\omega}_{\equiv}$ of $S^{\omega}_{\equiv}$, for each type $\tau \in \Pi$ ($\tau \neq i$), there exists a denumerably infinite set of variables $X^{\tau}_1, X^{\tau}_2, \ldots$ of type τ and a set of constants $A^{\tau}_1, A^{\tau}_2, \ldots$ of that type. When $\tau = i$, there are two sorts of terms: the m-terms and the m-terms. The former are the m-variables $x^i_1, x^i_2, \ldots$ and the m-constants $a^i_1, a^i_2, \ldots$, while the latter are the variables $X^i_1, X^i_2, \ldots$ and the constants $A^i_1, A^i_2, \ldots$, called m-variables and m-constants respectively. In other words, we have a two sorted language at the level of individuals. We use $U^{\tau}, V^{\tau}, \ldots$ and $u, v, \ldots$ as syntactic variables for terms of type τ (including the m-terms of type i) and for m-terms respectively; $U, V, \ldots$ are used as syntactic variables for any terms in general.

The definition *terms* of type τ and of *atomic formulas* are standard [42], but with respect to the predicate of identity, we require that only expressions of the form $U^{\tau} = V^{\tau}$ are atomic formulas; hence, expressions such as $U^{\tau} = u$, or $u = v$ etc. are meaningless. Consequently, as expected, we cannot talk about either the identity or the diversity of the objects denoted by the m-terms. The postulates of this logic are similar to those of the standard higher-order systems, including the axioms of Extensionality, Separation and Infinity (*loc. cit.*). The case of the Axiom of Choice will be discussed below.

We start by defining identity. Let U^{τ} and V^{τ} be terms of the same type τ which are not m-terms, and $F^{\langle \tau \rangle}$ a variable of type $\langle \tau \rangle$. Then

Definition 9.2 (Identity)

$$U^{\tau} = V^{\tau} := \forall F^{\langle \tau \rangle}(F^{\langle \tau \rangle}(U^{\tau}) \leftrightarrow F^{\langle \tau \rangle}(V^{\tau})). \tag{4}$$

This is *Leibniz's Law*; notice that identity is not defined for m-objects of type i, denoted by the m-terms. But, in order to involve also these entities, we introduce the following definition:

Definition 9.3 (Absolute Indistinguishability) *U^τ and V^τ be terms of the same type τ (including m-terms of type i), and $F^{\langle\tau\rangle}$ a variable of type $\langle\tau\rangle$, then*

$$U^\tau \equiv V^\tau := \forall F^{\langle\tau\rangle}(F^{\langle\tau\rangle}(U^\tau) \leftrightarrow F^{\langle\tau\rangle}(V^\tau))$$

If $U^\tau \equiv V^\tau$, we say that the entities denoted by U^τ and V^τ are *absolutely indistinguishable*. Note that the definition holds also for m-terms, since it does not exclude the possibility that $\tau = i$. Hence, by the definition of the atomic formulas, there is a sense in which, according to the canons of $S^\omega_\equiv$, the entities denoted by the m-terms may be 'absolutely indistinguishable', without being identical. But notice that the definiendum in both definitions are exactly the same: this needs explanation we shall give below (see 'Explanation' below). For now, it suffices to acknowledge that with respect to m-objects of type i, we can have $u \equiv v$, but not $u = v$.

From now on, sometimes we shall not indicate the types as superscripts, leaving to the context the distinction among the types, except when necessary.

As a consequence of the above definition, Leibniz's Law does not hold in general. In addition, let us remark that from the axiom and definition above, it follows that if U and V are terms of type $\langle\tau\rangle$, but not m-terms, then $U \equiv V$ is equivalent to $U = V$, that is to say, *identity* and *indistinguishability* are equivalent for those entities which are not m-terms. In other words, the the traditional theory of identity remains valid in the 'macroscopic world', that is, the domain(s) where the m-terms range over. The next definitions and postulates use a simplified notation when possible.

Definition 9.4 (Relative Indistinguishability) *If U and V are terms of type τ, F is a variable of type $\langle\tau\rangle$ and P_τ is a constant of type $\langle\langle\tau\rangle\rangle$, then*

$$U \equiv_{P_\tau} V := \forall F(P_\tau(F) \to (F(U) \leftrightarrow F(V))). \tag{5}$$

If $U \equiv_{P_\tau} V$, we say that U and V are indistinguishable *with respect to*, or *relative to* the attributes 'characterized' by P_τ. U and V being relatively indistinguishable means only that they agree with respect to some class of attributes, or *within a context*. It is interesting to note that the concept of relative indistinguishability can be formulated also within classical higher-order logic in exactly the same way. By using definition (5), we can formulate the concept of indistinguishable particles as used in quantum mechanics.[23]

[23] J. M. Jauch says that "[t]wo elementary particles are identical if they agree in all their intrinsic properties" [45, p.275]. That is, quantum mechanics' identity of elementary particles is simply indistinguishability relative to intrinsic properties (those that are independent of space and time).

This notion can be expressed in our formalism by using the above definition and considering a predicate I of type $\langle\langle i\rangle\rangle$ such that $I(F)$ says intuitively that F is an intrinsic property (whatever this means, since this is not important from the formal point of view). Then, to say that U and V are 'indistinguishable particles' in quantum mechanics means $U \equiv_I V$, but of course not that $U = V$. We shall return to these points in the last section.

The concepts of both absolute and relative indistinguishability can be related by means of the following result, which can be easily proved as a theorem of $S^{\omega}_{\equiv}$:

Theorem 9.1 *For U, V terms of type τ, P a variable of type $\langle\langle\tau\rangle\rangle$, and F being a variable of type $\langle\tau\rangle$,*

$$\forall P \forall F (P(F) \to (U \equiv_P V \leftrightarrow U \equiv V)$$

Let us call P a *context*. What the theorem is saying is that U and V (this is short for 'the objects denoted by U and V') are absolutely indistinguishable iff they are relatively indistinguishable in whatever context.

The axiomatics is given as follows. The first group of formulas are those of the propositional calculus, I_1 to I_4 as in da Costa's system, plus the adaptations of the remaining ones, as follows: let X and Y variables of any type τ and F a variable of type $\langle\tau\rangle$; then

(II_1) $\forall X \alpha(X) \to \alpha(Y)$, with standard restrictions, and

(II_2) $\beta \to \alpha(X)/\beta \to \forall X \alpha(X)$, idem.

We also make use, as it is common in type theory, of the following remaining postulates:

Let $X_1, \ldots, X_n$ variables of types $\tau_1, \ldots, \tau_n$ respectively, F and G variables of type $\langle\tau_1, \ldots, \tau_n\rangle$ and let H be of type $\langle\langle\tau_1, \ldots, \tau_n\rangle\rangle$. Then we have the following Axiom of Extensionality:

(III) $\forall A \forall F \forall G \forall X_1 \ldots \forall X_n \big((F(X_1, \ldots, X_n) \leftrightarrow G(X_1, \ldots, X_n))$
$\to (A(F) \to A(G))\big)$.

The consequent of the conditional could be substituted by $F = G$.

An Schema of Separation is also introduced, namely; being $X_1, \ldots, X_n$ variables of types $\tau_1, \ldots, \tau_n$ respectively and being F a variable of type $\langle\tau_1, \ldots, \tau_n\rangle$, let $\alpha(X_1, \ldots, X_n)$ be a formula containing the mentioned variables as free variables. Then each formula of the form below is an axiom:

(IV) $\exists F \forall X_1 \ldots \forall X_n (F(X_1, \ldots, X_n) \leftrightarrow \alpha(X_1, \ldots, X_n))$, which says that any formula with free variables can be substituted by a predicate. The usual axiom of choice can also be easily adapted to our case by using the relation of absolute indistinguishability instead of equality [42, p.156]. So, if F and F' are variables of type τ_1, G and L are variables of type τ_2, A and H are variables of type $\langle \tau_1, \tau_2 \rangle$, the following expression is an axiom of $S^{\omega}_{\equiv}$ (the *Axiom of Indistinguishable Choices*):

$$\text{(V)}\ \forall A \exists H (\forall F (\exists G A(F,G) \rightarrow \exists G (H(F,G) \wedge A(F,G))) \rightarrow \forall F \forall F' \forall G \forall L (H(F,G) \wedge H(F',L) \wedge F \equiv F' \rightarrow G \equiv L)).$$

In words, the 'function' H 'selects' indistinguishable objects from the collection of the images (by A) of indistinguishable objects (the F's). It is obvious that if we are not considering m-terms, the symbol '$=$' can replace '$\equiv$' (by the above results) and the above expression turns out to be equivalent to the axiom used in the standard simple theory of types.

Our last axiom is the Axiom of Infinite, which says that there exists an irreflexive, transitive and strongly connected relation on individuals; notice that there are no finite models for such a relation. Thus, let X^i, Y^i and Z^i variables for objects of 'the same kind', that is, either m-objects or m-objects. Then, using the type-notation this time,

$$\text{(VI)}\ \exists X^{\langle i,i \rangle} \big(\forall X^i (\neg X^{\langle i,i \rangle}(X^i, X^i) \wedge \forall X^i \exists Y^i (X^{\langle i,i \rangle}(X^i, Y^i)) \wedge \forall X^i \forall Y^i \forall Z^i (X^{\langle i,i \rangle}(X^i, Y^i) \wedge X^{\langle i,i \rangle}(Y^i, Z^i) \rightarrow X^{\langle i,i \rangle}(X^i, Z^i))\big).$$

It is possible to define a translation from the language of the simple theory of types to the 'macroscopic part' of the language of our system by supposing that all variables and constants that occur in the formulas are m-variables or m-constants. Then, we can prove the following theorem, states that all the results which can be obtained in the simple theory of types can also be obtained in $S^{\omega}_{\equiv}$.

Theorem 9.2 *Let α be a formula of the simple theory of types and α^{ω} its translation in the language of $S^{\omega}_{\equiv}$. Then, if α is a theorem of the simple theory of types, α^{ω} is a theorem of $S^{\omega}_{\equiv}$. So, all the mathematics that can be developed in the simple theory of types can also be developed in $S^{\omega}_{\equiv}$.*

Explanation In the above definitions (9.3) and (9.4), we notice that in both the definiendum is the same, hence the definiens would collapse in just one concept. This is partially true except for the fact that the variables range differently; while in the second there is no restriction to the application of the relation $\equiv$, except that the involved objects must be of the 'same kind', in the first one the equality symbol doesn't apply to m-objects of type i. This makes a huge difference between the two definitions. But a more faithful analysis can be provided semantically; as we shall see below, the domain of the m-objects of type i will be not a *set*, but a *quasi-set* of indiscernible entities. But we can turn to this point only after having had a look at the quasi-set theory.

9.1 Semantics

As it happens with standard higher-order logics, we may have *full* semantics and *Henkin* semantics [81]. In any one of them, we will face the same problems mentioned above regarding first-order Schrödinger logic: they are grounded on a standard set theory, so the notion of identity makes sense for all objects we consider, contrary to the spirit of Schrödinger logic.

But even so a 'weak' completeness theorem for $S^{\omega}_{\equiv}$ in the sense of Henkin [12, §54], [76, Chap.IV], grounded in a standard set theory such as the ZFC system, can be provided (see [47, 18, 36] for details). We shall just summarize the developments here, but the notation will be important for the understanding of the quasi-set notation to be seem below and for the continuation of the paper. In order to define a *frame* for the language of $\mathcal{L}^{\omega}_{\equiv}$, we chose $\mathbf{D}$ to be an infinite set such that $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ and $\mathbf{m} \cap \mathbf{M} = \emptyset$,[24] then a *frame* for $\mathcal{L}$ based on $\mathbf{D}$ is a function $\mathcal{M}$ whose domain in the set Π of types such that $\mathcal{M}(i) = \mathbf{D}$ and, for each type $\tau = \langle \tau_1, \ldots, \tau_n \rangle \in \Pi$, $\mathcal{M}(\tau) \subseteq \mathcal{P}(\mathcal{M}_{\tau_1}, \times \ldots \times \mathcal{M}_{\tau_n})$. If the inclusion in this last expression is replaced by the equality symbol, than the frame is *standard*.

If we write $\mathcal{M}_\tau$ instead of $\mathcal{M}(\tau)$, then the frame can be viewed as a family $(\mathcal{M}_\tau)_{\tau \in \Pi}$ of sets satisfying the above conditions. In what follows, we will refer to both this family and the set $\mathcal{F}_{\mathbf{D}} = \{X : \exists \tau \in \Pi \wedge X = \mathcal{M}(\tau)\}$ indifferently as *the frame* for $\mathcal{L}$ based on $\mathbf{D}$.

A *denotation* for $\mathcal{L}^{\omega}_{\equiv}$ based on $\mathbf{D}$ is a function ϕ whose domain is the set of constants of $\mathcal{L}$, defined as follows:

(i) $\phi(a^i_j) \in \mathbf{m}$, $j = 1, 2, \ldots$

(ii) $\phi(A^i_j) \in \mathbf{M}$, $j = 1, 2, \ldots$

[24] $\mathbf{D}$ is taken to be infinite since we intend to consider models also for the axiom of infinity.

(iii) $\phi(A^\tau_j) \in \mathcal{M}_\tau$ for every $\tau \neq i$, $j = 1, 2, \ldots$.

In particular, $\phi(=) \in \mathcal{M}_{\langle\tau,\tau\rangle}$, where the symbol '=' is written ambiguously to denote a predicate of type $\langle\tau, \tau\rangle$.

An *interpretation* for $\mathcal{L}^\omega_\equiv$ based on $\mathbf{D}$ as an ordered pair $\mathcal{A} = \langle(\mathcal{M}_\tau)_{\tau\in\Pi}, \phi\rangle$, where $(\mathcal{M}_\tau)_{\tau\in\Pi}$ and ϕ are as above. The interpretation is *principal* if the frame is standard and $\phi(=) = \Delta\mathcal{M}_\tau$, the diagonal of $\mathcal{M}_\tau$. A *valuation* for $\mathcal{L}^\omega_\equiv$ (based on $\mathcal{A}$) is a function ψ whose domain is the set of terms of $\mathcal{L}$ such that ψ is an extension to the set of terms of $\mathcal{L}$ of the denotation ϕ. In other words, ψ is defined as follows:

(i) $\psi(t) = \phi(t)$ if t is a constant

(ii) $\psi(x^i_j) \in \mathbf{m}$ for the m-variables $(j = 1, 2, \ldots)$

(iii) $\psi(X^i_j) \in \mathbf{M}$, for the m-variables $(j = 1, 2, \ldots)$

(iv) $\psi(X^\tau_j) \in \mathcal{M}_\tau$ for $\tau \neq i$ $(j = 1, 2, \ldots)$.

The definition of *satisfatibility*, that is, the concept of $\mathcal{A}, \psi \models A$, is defined by recursion on the length of the formula A as in the standard case. If $\mathcal{A}$ is an interpretation of $\mathcal{L}^\omega_\equiv$ based on $\mathbf{D}$, then

(i) $\mathcal{A}, \psi \models F(X^\tau_1, \ldots, X^\tau_n)$ iff $\langle\psi(X^\tau_1), \ldots, \psi(X^\tau_n)\rangle \in \psi(F)$, where F is a term of type $\langle\tau_1, \ldots, \tau_n\rangle$ and the X^τ_n are terms of type τ_j $(j = 1, \ldots, n)$.

(ii) $\mathcal{A}, \psi \models U = V$ iff $\langle\psi(U), \psi(V)\rangle \in \psi(=)$, where U and V are both terms of same type τ.

(iii) The satisfaction clauses for $\neg$, $\vee$ and $\forall$ are introduced as usual.

If A is an instance of the axioms of of $S^\omega_\equiv$, including Extensionality, Separation, Choice, and Infinity, let $\mathcal{A}$ bean interpretation for $\mathcal{L}$ based on $\mathbf{D}$ (as above), then $\mathcal{A}$ is an *sound interpretation* (or *appropriate*) for $\mathcal{L}$ iff $\mathcal{A}, \psi \models A$. In what follows, we will consider only *sound* interpretations.

A sound interpretation is *normal* iff $\mathcal{A}, \psi \models A$ where A is either an axiom of $S^\omega_\equiv$ or is derived by means of the inference rules from formulas $B_1, \ldots, B_n$ of $\mathcal{L}^\omega_\equiv$, and $\mathcal{A}, \psi \models B_j$, $j = 1, \ldots, n$. A normal interpretation for $\mathcal{L}$ which is not a principal interpretation is said to be a *secondary* interpretation.

A formula A is *true* with respect to an interpretation $\mathcal{A}$ iff $\mathcal{A}, \psi \models A$ for every valuation ψ based on $\mathcal{A}$. A is *valid* iff it is true with respect to all principal interpretations; A is *satisfiable* iff there exists a valuation ψ and a principal interpretation $\mathcal{A}$ such that $\mathcal{A}, \psi \models A$. A formula A is *secondarily valid* iff it is true under all normal interpretations. It is A is *secondarily satisfiable* iff there is a valuation ψ with respect to a normal interpretation $\mathcal{A}$ such that $\mathcal{A}, \psi \models A$.

The following results can be proved without difficulty: (1) A is valid iff $\neg A$ is not satisfiable; (2) A is secondarily valid iff $\neg A$ is not secondarily satisfiable; (3) A is satisfiable iff $\neg A$ is not valid; (4) A is secondarily satisfiable iff $\neg A$ is not secondarily valid and (5) A is valid (respect. secondarily valid) with respect to a normal interpretation iff its universal closure is valid (respect. secondarily valid) with respect to this interpretation.

If Γ is a set of formulas of $\mathcal{L}^{\omega}_{\equiv}$, then a *model* of Γ is a normal interpretation $\mathcal{A}$ such that $\mathcal{A}, \psi \models A$ for every formula $A \in \Gamma$. If $\mathcal{A}$ is a principal interpretation, we will talk of *principal models*, or of *secondary models* if $\mathcal{A}$ is a secondary interpretation.

The following terminology will be used below: $\Gamma \models A$ means that A holds in every model of Γ, and $\models A$ means that A is secondarily valid.

Then, the following results can be proven [12, *loc.cit.*]:

Theorem 9.3 (Soundness) *All theorems of $S^{\omega}_{\equiv}$ are secondarily valid. Hence, they are valid.*

That is, $\vdash A$ implies $\models A$; it is not difficult to generalize this result: $\Gamma \vdash A$ entails $\Gamma \models A$.

Lemma 9.1 (Lindenbaum) *Every consistent set Γ of closed formulas of $\mathcal{L}^{\omega}_{\equiv}$ can be extended to a maximal consistent class $\overline{\Gamma}$ of closed formulas of $\mathcal{L}$ (the concepts introduced here are the usual ones).*

Then, we can state a basic lemma, which is essential for the results of this section:

Lemma 9.2 (Basic Lemma) *If A is a closed formula of $\mathcal{L}^{\omega}_{\equiv}$ which is not a theorem, then there exists a normal interpretation whose domains $\mathcal{M}_\alpha$ are denumerably infinite, with respect to which $\neg A$ is valid.*
Proof: Let $\mathcal{L}'^{\omega}_{\equiv}$ be the language obtained by adding to $\mathcal{L}^{\omega}_{\equiv}$ the following list of symbols: (a) two disjointed denumerable infinite sets of new constants of type i, $w_1, w_2, \ldots$, and $W_1, W_2, \ldots$. We shall say that the first set is a new set of m-constants and the second one is a new set of m-constants. (b) for each $\tau \in \Pi$ ($\tau \neq i$), a denumerably infinite set of new constants of type τ: $W^{\tau}_1, W^{\tau}_2, \ldots$. We still suppose that there exists a fixed enumeration of the

closed formulas of $\mathcal{L}'$. Let H be a closed formula of $\mathcal{L}'^{\omega}_{\equiv}$ which is not a theorem of S^{ω}. Then we define recursively the classes Γ_j, $j = 1, 2, \ldots$ as follows:

a) $\Gamma_0 := \{\neg H\}$
b) If the $(n+1)$th closed formula of $\mathcal{L}'$ has the form $\forall X^{\tau} A$, and if the first new constant of type τ which does not occur either in A or in any member of Γ_n is W^{τ}_m, then Γ_{n+1} is Γ_n plus all expressions of the form

$$\mathsf{Subst}(W^{\tau}_m, X^{\tau}; A) \to \forall X^{\tau} A,$$

where $\mathsf{Subst}(W^{\tau}_m, X^{\tau}; A)$ is the result of substituting W^{τ}_m for all free occurrences of X^{τ} in A. Otherwise, Γ_{n+1} is Γ_n. We still remark that if X^{τ} is a m-variable of type i, then W^{τ}_m must be taken from the list $w_1, \ldots$ of new m-constants. As in Church,[25] *we may prove that the Γ_j ($j = 1, 2, \ldots$) are consistent. Then, we define $\Gamma := \bigcup_j \Gamma_j$ and let $\overline{\Gamma}$ be a maximal consistent class in the sense of Lindenbaum's lemma above.*

Now let $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ be a set as in the above definition of a frame for $\mathcal{L}$. Then we define a frame for $\mathcal{L}'$ based on $\mathbf{D'} = \mathbf{m'} \cup \mathbf{M'}$, where $m' := m \cup \{t_1, t_2, \ldots\}$ and $M' := M \cup \{t'_1, t'_2, \ldots\}$, with $\{t_1, t_2, \ldots\} \cap \{t'_1, t'_2, \ldots\} = \emptyset$, where both sets $\{t_1, t_2, \ldots\}$ and $\{t'_1, t'_2, \ldots\}$ are denumerably infinite. The frame is then defined as follows: (a) $\mathcal{M}_i := D'$, and (b) $\mathcal{M}_{\tau} := \{\phi(F) : F \text{ is a constant of} \mathcal{L}'\}$, and ϕ is an application whose domain is the set of constants of $\mathcal{L}'$ defined in the following way, where $j = 1, 2, \ldots$: (1) $\phi(a^i_j) \in m$; (2) $\phi(w_j = t_j$; (3) $\phi(A^i_j) \in M$; (d) $\phi(W^i_j) = t'_j$; (e) for each constant F of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle \neq i$, $\phi(F) := \{\langle \phi(T_1), \ldots, \phi(T_n) \rangle \in \mathcal{M}_{\tau_1} \times \cdots \times \mathcal{M}_{\tau_n} : F(T_1, \ldots, T_n) \in \overline{\Gamma}\}$, where the T_j are constants of types τ_j. Then, we can prove (1) that every $\mathcal{M}_{\tau}$ is denumerable and (2) that $\mathcal{A} = \langle (\mathcal{M}_{\tau})_{\tau \in \Pi}, \phi \rangle$ is a model for every formula A of $\mathcal{L}$. Really, if $X_1, X_2, \ldots, X_n$ are variables occurring in A of types $\tau_1, \tau_2, \ldots$ respectively, then $\mathcal{A}, \phi \models A$ if and only if the formulas obtained by replacing $T_1, T_2, \ldots$ for all occurrences of $X_1, X_2, \ldots$ in A belong to $\overline{\Gamma}$. So, let $\neg H$ be such a formula. Then it belongs to $\overline{\Gamma}$ since this set is complete and by hypothesis H is not a theorem. Hence, $\neg H$ is valid with respect to $\mathcal{A}$ above, and this proves the lemma. ▌

By using this lemma, we can prove our main result:

Theorem 9.4 (Henkin Completeness) *Every formula of $S^{\omega}_{\equiv}$ which is secondarily valid is a theorem.*
Proof: Let A be a secondarily valid formula of $\mathcal{L}^{\omega}_{\equiv}$ and let H be its universal

[25] *Ibid.*, pp. 311-12.

closure. Then, by the above Lemma, H is secondarily valid. But, in this case, H is true with respects to all sound interpretations, hence $\neg H$ is not secondarily satisfiable, which entails that there exists an interpretation relative to which $\neg H$ is valid. So, by the Basic Lemma, H is a theorem, hence A is a theorem. ▮

In other words, $\models A$ implies $\vdash A$. In general, if Γ is a set of closed formulas of $\mathcal{L}^{\omega}_{\equiv}$ which is not inconsistent, then $\Gamma \models A$ implies $\Gamma \vdash A$, that is, if A holds in every model of Γ, then A is derivable from the formulas of Γ.

In the above, in taking $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ as a *set*, we keep the semantics, subject to the same problems already alluded to with respect to the first-order systems. Of course, from the point of view of $S^{\omega}_{\equiv}$, $\mathbf{m}$ should be not considered as a set, since in principle the relation of equality cannot be applied to its elements. So the problem remains of basing a semantics for Schrödinger logics on quasi-set theory. We shall solve this problem below, but in connection with a modified version of the system presented above. Before we do that, let us reinforce some points regarding this semantics.

In order to surpass these 'classical' deficiencies of standard semantics to capture the spirit of Schrödinger logics, we shall show how a semantics, grounded in the theory of quasi-sets, can be developed. But, first, we need to have an idea of how such a theory works.

10 Quasi-set theory

As seen above, the construction of a theory which generalises standard set theory and which would be able to deal with collections of entities such as the *quanta* which in certain situations can be considered as *absolutely indiscernible* is something regarded as important. Quasi-set theory (the name was suggested by da Costa) aims to be one such a theory. The first version of the theory was developed in [47] (see [49]) and was further ameliorated and improved, gaining several versions, and several people have contributed to that.[26] Here we shall describe the main traits of the theory $\mathfrak{Q}$ as we see it today, but without the formal details (for that, see [26]).

Of course, we wish to preserve a standard set theory inside $\mathfrak{Q}$. We have chosen the ZFA (Zermelo-Fraenkel set theory with atoms) system for that,

[26] I would like to mention Adonai Sant'Anna and Aurelio Sartorelli from the Federal University of Paraná, Jonas R. B. Arenhart from the Federal University of Santa Catarina, Federico Holik from the National University of La Plata, Argentina, and Eliza Wajch, from the Siedlce University of Natural Sciences and Humanities, Poland.

although we could base the theory on a different ground, such as the NBG system or other.[27] So, ZFA is the core of the theory, and within this core, we can develop all standard mathematical concepts such as ordinals and cardinals. The atoms of ZFA are represented in $\mathfrak{Q}$ by a monadic predicate M, and we call them M-atoms. The entities represented by the M-atoms sometimes will be termed 'M-objects'. The novelty is that the theory encompasses another kind of atoms, the m-atoms, which in the intended interpretation would play the role of quantum entities; to these entities, the standard notion of identity does not apply, and this is done by assuming that expressions of the form '$x = y$' are not well-formed if either x or y denote an 'm-object'. So, the theory goes in the direction pointed out by Schrödinger.

Quasi-sets are objects that are neither m-atoms nor M-atoms. Their elements may be either kind of atoms and also other quasi-sets; a version of the Axiom of Regularity is used to avoid that a quasi-set can be an element of itself. Some quasi-sets do not involve m-atoms in their transitive closure, that is, they are built within the 'classical' part of $\mathfrak{Q}$, and are termed *sets*, being copies of the ZFA sets. If the M-atoms are also dropped out, then we get a version of 'pure' ZFC. The unary primitive predicates m, M, and Z cope with m-objects, M-objects and sets respectively. Two binary primitive predicates are $\equiv$ (indistinguishability, or indiscernibility), and $\in$ (membership) also make part of the language, so that '$x \equiv y$' means that x is indistinguishable (or indiscernible) from y and '$x \in y$' means that x is an element of y. Furthermore, there is still a unary primitive functional symbol, qc such that $qc(x)$ is a term which stands for 'the *quasi-cardinal* of x', informally standing for the number of elements it has. Formulas are defined as usual, and the postulates provide the details of the theory.

Given a formula $\varphi(x)$ of the language, the collection $[x : \varphi(x)]$ is called a *quasi-class*; we deserve the usual '$\{$' and '$\}$' for the case of sets. Given a quasi-set q and $x \in q$, we define the 'singleton' of x (relative to q) as the qset $[x]_q := [y \in q : y \equiv x]$, that is, the quasi-set of the indiscernible from x that belong to q; of course, its quasi-cardinal can be greater than one. If such a quasi-cardinal is precisely one, we call it the *strong singleton* of x and denote it by $[\![x]\!]_q$; the details of how to derive the existence of such quasi-sets are being omitted.

Important to realize that in having a quasi-cardinal, the elements of the quasi-set can continue to be indiscernible; nothing implies that they can be 'counted' by standard means (that is, by means of bijections, which need

[27] In [53], we have developed a paraconsistent version of the theory; the main consequences will be mentioned *en passant* below.

identity for defining them). Identity (symbolized by the equality symbol '$=$') is not a primitive notion, but a concept of *extensional identity*, '$=_E$', is defined this way:

$$x =_E y := (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee \\ (M(x) \wedge M(y) \wedge \forall z(x \in z \leftrightarrow y \in z)). \quad (6)$$

The reader could think that it would be more convenient to restrict the extensional identity to sets and M-objects only. This of course could be done but brings difficulties for expressing certain things, as in defining certain frames at section (12), as we shall mention there. But we think that the above definition can be used; *when* two quasi-sets do have the *same* elements, they are extensionally identical, endpoint, yet we possibly never know when this happens.

It can be proven that this identity has all the usual properties of standard identity of ZFA for the objects it applies to. Notice that '$=_E$' does not hold if at least one of the involved terms is an m-atom. So, if we interpret the m-atoms as denoting quantum elementary systems, we are within Schrödinger's realm.

The relation '$\equiv$' has all the properties of an equivalence relation (reflexive, symmetric and transitive), but it is not a congruence; in fact, it does not preserve membership: if $x \in y$ and $x' \equiv x$, we cannot prove that $x' \in y$. So, $\equiv$ and standard identity ($=$) are different notions since the former applies to all entities in the universe of quasi-sets while the last one (in the form '$=_E$') applies only to sets and the M-atoms.

Postulates similar to those of ZFA are given, say a Scheme of Separation, union, power, etc. The null quasi-set turns out to be a set and it is unique, represented by '$\emptyset$'. For 'classical entities' (either M-atoms or sets), an Axiom of Extensionality holds, but when m-atoms are also involved, we cannot state it in its usual form, so the theory postulates a Weak Extensionality Axiom, which says (with the due definitions and existential postulates) that quasi-sets comprising 'the same quantities' (in terms of quasi-cardinals) of elements *of the same sort* are indistinguishable. Thus we can treat formally two sulfuric acid molecules as indiscernible, yet not identical: $H_2SO_4 \equiv H_2SO_4$ and the quasi-set having only two of such molecules as elements has quasi-cardinal two. So, the quasi-set can have a quasi-cardinal even if its elements cannot be discerned from one another.

The elements of a quasi-set can be distinguished in 'kinds' by some property, as in physics we distinguish among electrons, protons and neutrons. What imports is not their identities, but their *kinds* and *quantities*, as when we consider a sulfuric acid molecule; so, in informal parlance, we can pose a finite quasi-set as something like the tuple

$$q = \langle k_1, k_2, \ldots; \lambda_1, \lambda_2, \ldots \rangle, \tag{7}$$

where the k's indicate the kinds and the λ's the quasi-cardinals of each kind. Thus, H_2SO_4 turns out to be something like $\langle \mathrm{H}, \mathrm{S}, \mathrm{O}; 2, 1, 4 \rangle$, which emphasizes just the kinds and quantities, and not the nature of the involved entities.[28]

We can construct (in the metamathematics) a *universe of quasi-sets* $\mathcal{Q}$ by transfinite recursion over the class On of ordinals as follows: $Q_0 := m \cup M$, where m and M are disjoint collections of atoms, $Q_1 := \mathcal{P}(Q_0)$, ..., $Q_\lambda := \bigcup_{\beta<\lambda} Q_\beta$ if λ is a limit ordinal, and finally $\mathcal{Q} := \bigcup_{\alpha \in On} Q_\alpha$. This structure is not rigid, since the identity function cannot be defined for all quasi-sets of the universe (due to the presence of the m-atoms) and of course the quasi-function (see below) that leads an element in an indistinguishable one is a nontrivial automorphism.

We can defined a version of an 'ordered pair' as follows: given a and b in a quasi-set q, define $\langle a, b \rangle_z := [[a]_z, [a, b]_z]_{\mathcal{P}(z)}$, where $[a]_z$ and $[a, b]_z$ come from the 'pair axiom'.[29] By means of this definition, we can define binary and n-ary 'quasi'-relations and 'quasi'-functions (q-function). Interesting that a q-function does not distinguish between its arguments (or values) when there are indistinguishable m-atoms involved; so, the definition says that indistinguishable things are lead in indistinguishable things. By flexibilizing the idea we can also define q-injections, q-surjections and q-bijections.

The theory has also a version of the axiom of choice we call *Axiom of Quasi-Choice*, which informally read as follows: given a qset x, non-empty and formed by disjoint and non-empty quasi-sets, there exists a quasi-set u such that given an element v of x and an element $t \in v$, there exists a qset s which is a sub-quasi-set of the qset of the indiscernibles from t that belong to x with quasi-cardinal one and whose intersection with u is indiscernible from its intersection with v. This last affirmative says that the only element of s is indiscernible from t, but of course, we cannot state that it is t itself. Obviously, if no m-atoms are involved, this axiom is equivalent

[28] It should be remarked that Hermann Weyl has called our attention to precisely this point, positing that in quantum physics what imports is an 'ordered decomposition' emphasizing precisely the kinds and quantities only; see [90, App.B], [36], [48]. Just to remark, this was also the idea underlying the birth of modern chemistry with Boyle, Hooker, and mainly Dalton, to whom the atoms of a given element are indiscernible – see [24].

[29] An alternative definition could be $\langle a, b \rangle_z := [[\![a]\!]_z, [\![a]\!]_z \cup [\![b]\!]_z]_{\mathcal{P}(z)}$.

to the standard one in ZFA. In other words, the quasi-set u is formed by selecting one element indiscernible from some element of each element of x. Other formulations could of course be given.

One interesting result is a theorem which asserts that 'permutations are not regarded as observable', a central thing in quantum mechanics. In this theory this needs to be introduced by a postulate, the Indistinguishability Postulate, which read as follows: for all vectors $|\psi\rangle$, all operators $\hat{A}$ and all particle label permutation operator P, we have that $\langle\psi|\hat{A}|\psi\rangle = \langle P\psi|\hat{A}|P\psi\rangle$, that is, the expectation value of the measurement of an observable A (represented by the self-adjoint operator $\hat{A}$) for the system in the state $|\psi\rangle$ is the same before and after the action of the permutation operator P. In other words, permutations are not observable. In $\mathfrak{Q}$, we have a theorem which says that given a quasi-set q, if $x \in q$, $y \equiv x$ being $y \in q'$ where $q \subseteq q'$, but $y \notin q$, then $(x \setminus [\![x]\!]_q) \cup [\![y]\!]_{q'} \equiv q$. In words, we are 'exchanging' an indistinguishable from x by an indistinguishable from y and the resulting quasi-set remains indistinguishable from the original one. This is of course a version of the Indistinguishability Postulate and does not need to be introduced by force (as a postulate), resulting in $\mathfrak{Q}$ from the assumed indistinguishability of the involved elements.

11 An intensional Schrödinger logic

As we have seen before, da Costa claimed that a 'right' semantics for his Schrödinger logic would be constructed in a theory that generalizes the usual notion of set, enabling the existence of entities to which the standard notion of identity does not apply. We have the theory $\mathfrak{Q}$ now; what can we achieve in using it to provide the mathematical tools for a semantics for his logic? The answer was given not specifically for da Costa's system, but for a more general one, constructed to cope with the quantum fact that, as we have seen, an intension can have more than one extension. So, an *intensional* Schrödinger Logic of order ω was elaborated in the directions given by D. Gallin [37]. The main characteristics of the system are as follows; details in [50], [19], [36]. From now on, 'qset' will stand for 'quasi-set'.

In quasi-set theory a separation schema is assumed, so we may talk of the sub-qsets of a given qset, and in considering the intended interpretation of 'pure' qsets as collections of quanta, we may talk of sub-collections of indistinguishable objects obeying certain conditions. Consider the following example: suppose we are considering the four absolutely indiscernible neutrons of the nucleus of a ^{7}Li atom. We may suppose that there are six sub-collections with two of them each; call P the property which expresses that. Intuitively speaking (by applying elementary mathematics), consider the predicate 'x belongs to a sub-collection with 2 elements each'. If we intend to specify the extension of the predicate P above, according to standard semantics, what sub-collection should we choose? There is no criterion of choice. We express this by saying that the extension of P is not well defined, an idea already expressed by Dalla Chiara and Toraldo di Francia [23], as we have seen before. As they say, speaking of electrons,

> take the spin. We can choose a z-axis and state how many electrons have $s_z = +1/2$ and how many have $s_z = -1/2$ [they are referring to the values of spin in the z direction]. But we could instead refer to the x-axis, or the y-axis, or any other direction, obtaining different sets of quantum states, all having the same cardinality. We thus arrive at a situation, which is usually believed to be impossible in classical semantics: different extensions can correspond to one and the same intension. Of course, the reverse situation of one and the same extension corresponding to different intensions is trivially possible, as in classical semantics (for instance, instead of giving the mass of a particle, one could give its rest energy).
>
> What kind of entities are the properties that constitute a particular intension? Are they linguistic or extralinguistic entities? As is well known, this is a tremendous problem of classical semantics, but it seems to us that it does not represent a peculiar difficulty of microphysics. [23]

It is important to note the sense in which we are using the expression 'not well defined'. We are supposing that there are four neutrons and that we may think of a collection with two of them. The problem is that we cannot offer a suitable way of choosing among the six possibilities we have. No one can do it on the basis of physics. There are two options: either we admit that 'reality' is hidden behind a veil,[30] so that the predicate P (here only roughly defined) 'to be a neutron of the nucleus of a ^{7}Li atom which belongs to a collection with two of them' *has* an extension, although we cannot specify it precisely, say, by ostension, pointing to a collection and saying 'This one!'. The another possibility, which we regard as more adequate, is to leave the extension underdetermined in the sense that *whatever* collection with two

[30] This idea resembles a point made by B. D'Espagnat [28, Chap.9].

neutrons works for all purposes. This is what Dalla Chiara and Toraldo di Francia had in mind, and can be achieved by a suitable *quasi-set semantics*.

Let us recall that these remarks pose a fundamental difference between quasi-sets and fuzzy sets;[31] in short, in the case of the latter [88], the elements do have an identity, but we do not know precisely *where* they are. Concerning quasi-sets, the borderlines of the collections are well defined in principle (actually, all we have is a cardinal – it's quasi-cardinal), but the elements don't have an identity. The nucleus of the lithium atom has (in principle) borderlines: either a neutron belongs to it or not, but the problem is that we cannot specify by ostension the neutrons themselves.

So, it is in this sense that we say that predicates like P do not have well-defined extensions: every qset of a certain class of indiscernible qsets with the same quasi-cardinality may be considered as their extension as well. Using a terminology which shall be developed below, such predicates may be seen as *relations-in-intension*. Next, we shall try to make sense of these claims from a formal point of view.

Let us call $S^{\omega}\mathcal{I}$ a higher-order modal logic (simple theory of types with modalities) described as follows. We shall begin by modifying the already defined set of types as follows. We shall let the set of *types* be the smallest collection Π such that: (a) $e_1, e_2 \in \Pi$, and (b) if $\tau_1, \ldots, \tau_n \in \Pi$, then $\langle \tau_1, \ldots, \tau_n \rangle \in \Pi$. Here, e_1 and e_2 are taken to be the types of the *individuals*; the objects of type e_1 are again called m-objects and thought of as representing quanta. Once again following Schrödinger, we will suppose that the concept of identity cannot be applied to them. The language of the system $S^{\omega}\mathcal{I}$ contains the usual connectives (we suppose that '$\neg$' and '$\rightarrow$' are the primitive ones, while the others are defined as usual), the symbol of equality $=$ holds among objects other than individuals of type e_1, auxiliary symbols and quantifiers ('$\forall$' is the primitive and '$\exists$' is defined in the standard way) and the necessity operator '$\Box$'. With respect to variables and constants, for each type $\tau \in \Pi$ there exists a denumerably infinite collection of variables $X_1^{\tau}, X_2^{\tau}, \ldots$ of type τ and a (possibly empty) set of constants $A_1^{\tau}, A_2^{\tau}, \ldots$ of that type; we use X^{τ}, Y^{τ}, C^{τ} and D^{τ} perhaps with subscripts as meta-variables for variables and constants of type τ respectively.

The *terms* of type τ are the variables and the constants of that type; so, we have in particular individual terms of type e_1 and individual terms of type e_2. We use U^{τ}, V^{τ}, perhaps with subscripts, as syntactical variables for terms of type τ. The atomic formulas are defined in the usual way: if U^{τ} is a

[31] So as to Dalla Chiara and Toraldo di Francia's *quasets*, which were also proposed to cope with quantum entities but preserve the identity for all elements; see [23], [21], [36].

term of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ and $U^{\tau_1}, \ldots, U^{\tau_n}$ are terms of types $\tau_1, \ldots, \tau_n$ respectively, then $U^\tau(U^{\tau_1}, \ldots, U^{\tau_n})$ is an atomic formula; so is $U^\tau = V^\tau$ if τ is not of type e_1 (the formal details can be completed without difficulty). Then, once again, the language does not permit us to talk either about the identity or the diversity of the individuals of type e_1. The other formulas are defined as usual. A formula containing at least $U^{\tau_1}, \ldots, U^{\tau_n}$ as free variables sometimes shall be written $F(U^{\tau_1}, \ldots, U^{\tau_n})$.

11.1 The Theory $S^\omega\mathcal{I}$

Since we are dealing with two kinds of basic types, it would be adequate to make explicit the postulates in order to provide the right intuitions about the system. The postulates of $S^\omega\mathcal{I}$ (axiom scheme and inference rules) are as follows:

(A1) A, where A comes from a tautology in ‘’¬ and ‘→’ by uniform substitution of formulas of $S^\omega\mathcal{I}$ for the variables.

(A2) $\forall X^\tau(A \to B) \to (A \to \forall X^\tau B)$, where X^τ does not occur free in A.

(A3) $\forall X^\tau A(X^\tau) \to A(U^\tau)$ where U^τ is a term free for X^τ in $A(X^\tau)$ and of the same type of X^τ.

(A4) $X^{e_2} = X^{e_2}$

(A5) $X^{e_2} = Y^{e_2} \to \Box(X^{e_2} = Y^{e_2})$

(A6) $\Box(U^\tau = V^\tau) \to (A(U^\tau) \to A(V^\tau))$, where U^τ and V^τ are free for X^τ in $A(X^\tau)$.

(A5) $\Box A \to A$

(A6) $\Box(A \to B) \to (\Box A \to \Box B)$

(A7) $\Diamond A \to \Box\Diamond A$

(R1) From A and $A \to B$ to infer B

(R2) From A to infer $\forall X^\tau A$

(R3) From A to infer $\Box A$

The usual syntactical notions are defined in the standard way, such as the concept of deduction ($\vdash$), formal theorem of $S^\omega\mathcal{I}$, the concept of consequence of a set of formulas, and so on. A set Σ of formulas is *consistent* if and only if there exists some formula of the language which is not derivable from Σ in $S^\omega\mathcal{I}$.

11.2 Comprehension and Other Axioms

Our logic can be extended to a system which encompasses all the instances of the following *Comprehension Schema*, where $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ is a type and X^τ is the first variable of type τ which does not occur free in the formula $F(X^{\tau_1}, \ldots, X^{\tau_n})$:

$$\exists X^\tau \Box \forall X^{\tau_1} \ldots \forall X^{\tau_n} (X^\tau(X^{\tau_1}, \ldots, X^{\tau_n}) \leftrightarrow F(X^{\tau_1}, \ldots, X^{\tau_n})). \qquad (8)$$

This schema, which is valid in all standard models of $S^\omega\mathcal{I}$ (see below), formalizes the principle that every formula $F(X^{\tau_1}, \ldots, X^{\tau_n})$ with free variables determines a relation-in-intension (a predicate). In considering a g-model (as in the next section) $\mathcal{A} = \langle (\mathcal{M}_\tau)_{\tau \in \Pi}, \rho \rangle$ for $S^\omega\mathcal{I}$, if $U^{\tau_1}, \ldots, U^{\tau_n}$ are respectively elements of $\mathcal{M}_{\tau_1}, \ldots, \mathcal{M}_{\tau_n}$, the predicate F defined by

$$F(i) \coloneqq \{(X^{\tau_1}, \ldots, X^{\tau_n}) : \mathcal{A}; i; f, U^{\tau_1}, \ldots, U^{\tau_n} \ \mathsf{sat}\ F(X^{\tau_1}, \ldots, X^{\tau_n})\} \qquad (9)$$

for all $i \in I$ and assignment $f \in As(\mathcal{A})$ belongs to $\mathcal{M}_\tau$. Consequently, the g-model is also a g-model for $S^\omega\mathcal{I}$ plus the Comprehension Schema, and the completeness theorem is also true for this extended logic.

The *Principle of Extensional Comprehension*, which says that every formula with free variables determines an (extensional) n-ary relation can also be formulated in the language of $S^\omega\mathcal{I}$, as can the axioms of infinity and choice, although they are not important to us here.

As is the case in Gallin's intensional system, the Principle of Extensional Comprehension can be proved to be independent of the axiomatics of $S_\omega\mathcal{I}$ plus the Comprehension Schema. That is, there are g-models of $S^\omega\mathcal{I}$ plus Comprehension in which the extensional comprehension principle fails.

12 Generalized Quasi-Set Semantics

All that follows is developed within the quasi-set theory $\mathcal{Q}$ presented in the previous chapter. When we speak about *sets, mappings* and other concepts which resemble the standard set-theoretical ones, they should be understood as defined in the 'standard part' of quasi-set theory, that is, in the 'copy' of

ZFU we can define within $\mathfrak{Q}$. Let $\mathbf{D} = \langle \mathbf{m}, \mathbf{M}\rangle$ be an ordered pair where $\mathbf{m} \neq \emptyset$ is a finite pure qset (that is, a finite quasi-set which has only m-atoms as elements) and $\mathbf{M} \neq \emptyset$ is a set (in the sense that it satisfies the predicate Z of the language of $\mathcal{Q}$). Furthermore, we suppose that I is a non-empty set (whose elements are called an *index* or *state of affairs*).[32]

By a *frame* for $S^\omega\mathcal{I}$ based on D and I we mean an indexed family of qsets $(\mathcal{M}_\tau)_{\tau\in\Pi}$ where:[33]

(i) $\mathcal{M}_{e_1} =_E \mathbf{m}$

(ii) $\mathcal{M}_{e_2} =_E \mathbf{M}$

(iii) For each $\tau = \langle \tau_1, \ldots, \tau_n\rangle \in \Pi$, $\mathcal{M}_\tau$ is a non-empty subqset of $[\mathcal{P}(\mathcal{M}_{\tau_1} \times \cdots \times \mathcal{M}_{\tau_n})]^I$.

If the equality holds in (iii), then the frame is *standard*. By a *general model (g-model* for short) for $S^\omega\mathcal{I}$ based on $\mathbf{D}$ and I we understand an ordered pair

$$\mathcal{A} =_E \langle (\mathcal{M}_\tau)_{\tau\in\Pi}, \phi\rangle \tag{10}$$

such that :

(i) $(\mathcal{M}_\tau)_{\tau\in\Pi}$ is a frame for $S^\omega\mathcal{I}$ based on $\mathbf{D}$ and I

(ii) ϕ is a quasi-function which assigns to each constant C^τ an element of $\mathcal{M}_\tau$. Then, in particular $\phi(C^{e_1}) \in \mathbf{m}$ and $\phi(C^{e_2}) \in \mathbf{M}$.

A *standard model* for $S^\omega\mathcal{I}$ is a g-model whose frame is standard.[34] Before introducing other semantic concepts, let us consider some examples which illustrate the peculiarities of such a semantics, as mentioned at the beginning of this section. The first two examples below show that the classical intensional case remains valid when the entities are not individuals of type e_1. The last two exemplify the specific case of Schrödinger logics.

Example 12.1 *Let us consider a constant C^{e_2}. Since $\mathcal{M}_{e_2} =_E \mathbf{M}$, then $\phi(C^{e_2}) \in \mathbf{M}$, as remarked above. This intuitively means that a constant of*

[32] Adapting Montague's approach to intensional logic, we may suppose that I is the Cartesian product $W \times T$ where W is a (quasi-)set of possible worlds and T is a totally ordered set of instants of time. See [39, chap.2].

[33] Here is where the remark made just after the definition of extensional identity (6) is used with quasi-sets.

[34] These definitions are obvious adaptations in the language of quasi-sets of those presented by D. Gallin [37].

type e_2 'names' an element of a standard set (that is, $\mathbf{M}$ *is a set, according to the above definition of frame). This is not of course surprising, since the given constant is a 'classical' one.*

Example 12.2 *Now we shall consider a constant $C^{\langle e_2 \rangle}$ of type $\langle e_2 \rangle$, which stands for a 'property' of entities of type e_2. In this case, according to the above definition,*

$$\mathcal{M}_{\langle e_2 \rangle} \subseteq [\mathcal{P}(\mathcal{M}_{e_2})]^I =_E [\mathcal{P}(\mathbf{M})]^I.$$

Then, $\mathcal{M}_{\langle e_2 \rangle}$ is a class of (quasi-)functions from I to $\mathcal{P}(\mathbf{M})$, also according to the classical case. This is also not surprising, since the chosen constant is also 'classical'.

The next two examples will give us a better idea involving intensions as previously discussed.

Example 12.3 *Let us take a constant C^{e_1}. In this case, $\mathcal{M}_{e_1} =_E \mathbf{m}$, and then $\phi(C^{e_1}) \in m$, that is to say, that constant 'names' an m-atom. Since the m-atoms can be supposed to be indistinguishable, they cannot be individuated, counted etc., and so the denotation of C^{e_1} must be ambiguous. We can say that a constant of type e_1 plays the role of a generalized noun, or g-noun for short.*

The commitment to quasi-set theory needs to be made clear. It is precisely by considering such a mathematical framework that we can express the idea that a certain constant of the language does not stand for a name of a well defined object of the domain. As in the case of electrons, it acts as a kind of an *ambiguous* or *sortal* constant, since the entities to which it refers cannot be identified without ambiguity. So, by a *sortal constant*, or a *g-noun*, we mean a constant that refers (ambiguously) to an element of a certain class of indistinguishable objects, or objects given as *sorts* of a certain kind. We will pursue this idea below.[35]

Example 12.4 *We now consider a constant $C^{\langle e_1 \rangle}$ of type $\langle e_1 \rangle$, which should stand for a 'property' of entities of type e_1. In this case, according to the above definition,*

$$\mathcal{M}_{\langle e_1 \rangle} \subseteq [\mathcal{P}(\mathcal{M}_{e_1})]^I =_E [\mathcal{P}(\mathbf{m})]^I.$$

[35] Our use of the term 'sortal' is according to the 'essence criteria' put by F. Feldman [32], namely, "[a] sortal predicate, on this view, is one that gives a suitably 'substantial' answer to a question of the form 'what is x?' A sortal expresses the 'nature' or 'essence' of the things to which it truly applies."

Then, $\mathcal{M}_{\langle e_1\rangle}$ is a class of (quasi-)functions from I to $\mathcal{P}(\mathbf{m})$. If $\mathbf{m}$ is a pure qset whose elements are all indistinguishable from one another, then the denotation quasi-function does not distinguish between qsets in $\mathcal{P}(\mathbf{m})$, in the sense that whatever element of a class of indiscernible qsets acts as the 'extension' of the predicate $C^{\langle e_1\rangle}$ as well. This interpretation accommodates the intuitive idea that a predicate such as 'to have spin UP in the x direction' does not have a well defined extension, as required in Dalla Chiara and Toraldo di Francia's analysis.

This last example shows how to consider *relations-in-intension* of sort $U^{\langle e_1\rangle}$ within our formalism. This should not be taken to suggest that we are strongly committed to intensional issues only. As we hope to have made clear, our predicates *do have* extensions, but they are not well defined in the sense outlined earlier. Since the m-atoms have no proper names, the terms of type e_1 have no precise denotation; they refer ambiguously to an arbitrary element of a certain class of the domain, which may be characterized by the particular chosen constant. We may properly say that such constants do not represent anything in particular: they lack a (precise, well defined) referent, although they have a sense as constants, namely, the sense ascribed by the kind (sort) of entities they stand for (neutrons, say). The same holds for constants of type $\langle e_1\rangle$ and for whatever constant of type $\tau = \langle\tau_1,\ldots,\tau_n\rangle$ where at least one of the τ_i is obtained (recursively) from e_1. As we have said, the relationship with sortal logics will be mentioned later.

Coming back to the formal details, we consider as the set of all *assignments over* a g-model $\mathcal{A}$, denoted by $As(\mathcal{A})$, the set of all q-functions f on the set of variables of $S^\omega\mathcal{I}$ such that $f(X^\tau) \in \mathcal{M}_\tau$, for every variable X^τ of type τ. For any $f \in As(\mathcal{A})$, we denote by $\overline{f}$ the extension of f to the set of all constants, defined by

$$\overline{f}(C^\tau) =_E \rho(C^\tau) \in \mathcal{M}_\tau.$$

If $i \in I$ and $f \in As(\mathcal{A})$, then the notion of *satisfaction*, in symbols,

$$\mathcal{A}, i, f \ \mathsf{sat}\ A \tag{11}$$

is defined by recursion on the length of the formula A as follows:

(i) $\mathcal{A}, i, f \ \mathsf{sat}\ U^\tau(U^{\tau_1},\ldots,U^{\tau_n})$ iff $\langle\overline{f}(U^{\tau_1}),\ldots,\overline{f}(U^{\tau_n})\rangle \in \overline{f}(U^\tau)(i)$

(ii) $\mathcal{A}, i, f \ \mathsf{sat}\ U^\tau = V^\tau$ iff $\langle\overline{f}(U^\tau),\overline{f}(V^\tau)\rangle \in \Delta_\equiv(\tau)$ where $\Delta_\equiv(\tau)$ is the 'pseudo-diagonal' of $\mathcal{M}_\tau$, which may be defined in quasi-set theory as the subqset of $\mathcal{M}_\tau \times \mathcal{M}_\tau$ whose elements are indistinguishable from one another (when $\tau \neq e_1$, this qset is the diagonal of $\mathcal{M}_\tau$ in the standard sense).

(iii) $\mathcal{A}, i, f$ sat $\Box A$ iff $\mathcal{A}, j, f$ sat A for every $j \in I$

(iv) The usual clauses for $\neg$, $\rightarrow$ and $\forall$

A formula A is *true* in a g-model $\mathcal{A}$ (denoted $\models_{\mathcal{A}} A$) iff $\mathcal{A}, i, f$ sat A for every $i \in I$ and $f \in As(\mathcal{A})$. A set Σ of formulas of $S_\omega\mathcal{I}$ is *g-satisfiable* in $S^\omega\mathcal{I}$ iff for some g-model $\mathcal{A}$, index i and assignment f, $\mathcal{A}, i, f$ sat A for all $A \in \Sigma$. A formula A is a *g-semantic consequence* of a set Γ of formulas, and we write $\Gamma \models_g A$, iff $\mathcal{A}, i, f$ sat A for $i \in I$, $f \in As(\mathcal{A})$ and g-model $\mathcal{A}$ whenever $\mathcal{A}, i, f$ sat B for every formula $B \in \Gamma$. If $\Gamma = \emptyset$, we write $\models_g A$ and say that A is *g-valid* in $S_\omega\mathcal{I}$.

In the next section we present an axiom system for $S^\omega\mathcal{I}$ and prove a generalized completeness theorem for this logic.

12.1 Soundness and Generalized Completeness

The Soundness Theorem for $S^\omega\mathcal{I}$ is formulated as follows. If $\vdash A$ in $S^\omega\mathcal{I}$, then $\models_g A$ in $S_\omega\mathcal{I}$. This result implies that if $\Gamma \vdash A$, then $\Gamma \models_g A$ and that if a set of formulas Σ is g-satisfiable in $S^\omega\mathcal{I}$, then Σ is consistent.

The proof is obtained by showing that all the axioms of $S^\omega\mathcal{I}$ are g-valid and that the inference rules preserve g-validity. This follows from the fact that if $\mathcal{A}$ is a g-model for $S_\omega\mathcal{I}$ and the term U^τ is free for the variable X^τ in $A(X^\tau)$, then for every $i \in I$ and $f \in As(\mathcal{A})$, it follows[36] that

$$\mathcal{A}; i; f, \overline{f}(C^\tau) \ \mathsf{sat}\ A(X^\tau) \text{ iff } \mathcal{A}, i, f \ \mathsf{sat}\ A(C^\tau) \tag{12}$$

where the terminology has an obvious meaning.

The generalized completeness theorem for $S^\omega\mathcal{I}$ is the converse of the above result; it is sufficient to prove that Σ is consistent iff Σ is g-satisfiable. The implication from right to left is straightforward, so we shall consider only the implication from left to right.

To begin with, let us assume that the consistent set Σ omits infinitely many variables of each type, that is, there are infinitely many variables of each type which do not occur in any formula of Σ. Then there exists[37] a sequence $\overline{\Sigma} = (\overline{\Sigma}_i)_{i \in \omega}$ of sets of formulas such that:

[36] As in Gallin *op. cit.*

[37] The existence of such a sequence can be proved by adapting the method presented in Gallin *ibid.*

(i) $\Sigma \subseteq \overline{\Sigma}_0$

(ii) For each $i \in \omega$, $\overline{\Sigma}_i$ is a maximal consistent set of formulas in $S_\omega \mathcal{I}$.

(iii) For each $i \in \omega$ and each formula $B(X^\tau)$, $\exists X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for some variable Y^τ which is free for X^τ in $B(X^\tau)$.

(iv) For each $i \in \omega$ and each formula B, we have $\Diamond B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for some $j \in \omega$.

(v) For each $i \in \omega$ and each formula $B(X^\tau)$, we have $\forall X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for every variable Y^τ which is free for X^τ in $B(X^\tau)$.

(vi)For each $i \in \omega$ and each formula B, we have $\Box B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for every $j \in \omega$.

The g-model relative to which the formulas of Σ are g-satisfiable can be described as follows. Firstly, we consider an equivalence relation on the collection Tr_τ of terms of $S^\omega \mathcal{I}$ of type τ such that U^τ is equivalent to V^τ if and only if $\Box(U^\tau = V^\tau) \in \overline{\Sigma}_i$ if $\tau \neq e_1$ and, if $\tau = e_1$, then U^{e_1} is equivalent to V^{e_1} (in this case we write $U^{e_1} \equiv V^{e_1}$) iff for every formula F that belongs to $\overline{\Sigma}_i$, it follows that $F[U^{e_1}/V^{e_1}]$ also belongs to $\overline{\Sigma}_i$. In other words, U^{e_1} is equivalent to V^{e_1} iff U^{e_1} and V^{e_1} can be replaced by one another in all their occurrences in any predicate in such a way that the resulting formulas are necessarily equivalent.

The defined relation does not depend on $i \in \omega$. Then, by recursion on the type τ, we define a set $\mathcal{M}_\tau$ and a mapping ϕ_τ from the set of terms of type τ into $\mathcal{M}_\tau$ such that:

(i) ϕ_τ is onto $\mathcal{M}_\tau$

(ii) $\phi_\tau(U^\tau) \equiv \rho_\tau(V^\tau)$ iff $U^\tau \simeq V^\tau$

First, let $\mathcal{M}_{e_i}$ be the quotient set $Tr_{e_i}/\simeq$ $(i = 1, 2)$, that is, $\mathbf{m} =_E Tr_{e_1}/\simeq$ and $\mathbf{M} =_E Tr_{e_2}/\simeq$, and $\phi_{e_i}(U^{e_i}) =_E [U^{e_i}]_\simeq$ (these are the equivalence classes of U^{e_i} by the relation $\simeq$). Then, by supposing that $\mathcal{M}_{\tau_\tau}$ and ϕ_{τ_τ} have been defined for $\tau < n$, we define the mapping ϕ_τ from Tr_τ into $[\mathcal{P}(\mathcal{M}_{\tau_0} \times \cdots \times \mathcal{M}_{\tau_{n-1}})]^\omega$, where $\tau = \langle \tau_0, \ldots, \tau_{n-1} \rangle$, as follows:

$$\langle \phi_0(U_0^{\tau_0}), \ldots, \phi_{n-1}(U_{n-1}^{\tau_{n-1}}) \rangle \in \phi_\tau(U^\tau)(i) \tag{13}$$

if and only if the formula $U^\tau(U^{\tau_0}, \ldots, U^{\tau_{n-1}})$ belongs to $\overline{\Sigma}_i$. If we let $\mathcal{M}_\tau$ be the range of ϕ_τ, then the conditions 1 and 2 above are met. The g-model based on $\mathbf{D} =_E \mathbf{m} \cup \mathbf{M}$ and index set $I =_E \omega$ is then the ordered pair $\mathcal{A} = \langle (\mathcal{M}_\tau)_{\tau \in \Pi}, \phi \rangle$, where $\phi(C^\tau) = \phi_\tau(C^\tau)$ for every constant C^τ. So, by induction on the length of the formula A, we may prove in the same way as posed by Gallin for the 'classical case', that

$$\mathcal{A}, i, \mu \ \mathsf{sat} \ A \text{ iff } A \in \overline{\Sigma}_i \tag{14}$$

for every $i \in I$, where $\mu \in As(\mathcal{M})$. In the case where $i = 0$ and $\mu = f$, we obtain the desired result.

13 A glimpse on other applications of $\mathfrak{Q}$

The theory of quasi-sets, as surely understood, was created with motivation in quantum theories. Despite as a formal theory it can have other models, this is the main one, the *intended* one. So, let us make some comments on the applications of this theory to such a field.

13.1 Quantum statistics

In 1997 and 1999, Adonai Sant'Anna, A. Volkov and I have shown how the so-called quantum statistics can be obtained 'naturally' in $\mathfrak{Q}$ once indistinguishability is assumed "right at the start" [73] and not made *a posteriori* as is usual when we ascribe identity to the entities, say by labelling them and then postulating that permutations do not conduce to distinct measurement results [78], [79] (see also [36]). The idea is that in having a quasi-set with N indiscernible objects, and given that they can be distributed in K 'states', then the computation of how many ways this can be done conduces 'directly' to the formula of the Bose-Einstein statistics,

$$\frac{(N+K-1)!}{N!(N-K)!}, \tag{15}$$

and, in the case that the states can be occupied by just 0 or 1 objects, we get the formula that corresponds to the Fermi-Dirac statistics,

$$\frac{K!}{N!(N-K)!}. \tag{16}$$

Thus, the theory which considers that indistinguishability is assumed 'right from the start'[38] we can avoid the need of introducing a strong postulate called Indistinguishability Postulate, which roughly says that permutations of indiscernible quanta do not conduce to different results in the

[38] This is how Heinz Post has suggested considering quantum entities to be [73].

measurements [74, 75]. But this is not the only gain: we gain a way to express a *metaphysics of non-individuals* [59], something we repute to be closer to what quantum mechanics says.[39]

13.2 Non-relativistic quantum mechanics

The standard formulation of orthodox (non-relativistic) quantum theory (QM) is by means of Hilbert spaces.[40] Except for a few exceptions ([44] is one of them), the mention of 'quantum systems' is omitted at all. The formalism speaks of states and observables, and despite one regard that states are states of *something*, such a 'something' does not appear in the axioms and plays no role in the postulates except in the usual reference that "for each quantum system we associate a separable complex Hilbert space. . . " or something like that. But we can suppose that we have a sufficiently well-defined 'quantum language', let us call it $\mathcal{L}_{QM}$, and so we can ask for a formal semantics for such a language [22, 23].[41] The question is: what kind of structure is one adequate for $\mathcal{L}_{QM}$ in order to preserve the minimum of fidelity to the theory?

Of course, we have here a similar problem than that of the semantics for Schrödinger logics. In the spirit of the seminal work by McKinsey, Sugar and Suppes in the axiomatization of classical particle mechanics, where a set P of 'particles' was added to the structure (see [86]), we think that we would be in need of considering a quasi-set of quantum systems in the structure too. So, in [56], we introduced the following structure, build in $\mathfrak{Q}$:

$$\mathscr{Q} = \langle Q, \{H_i\}, \{\hat{A}_{ij}\}, \{U_{ik}\}, \mathcal{B}(\mathbb{R})\rangle_{(i\in I, j\in J, k\in K)} \tag{17}$$

where Q is a quasi-set of quantum systems, the H_i are separable complex Hilbert spaces, the $\hat{A}_{ij}$ are self-adjoint operators on the relevant Hilbert spaces, the U_{ik} are unitary operators also over the chosen Hilbert spaces, and $\mathcal{B}(\mathbb{R})$ is a collection of Borelian sets over the set of the real numbers. Suitable postulates are associated to these notions [56, p.106ff].

Furthermore, to each quantum system $s \in Q$ we associate a 4-uple of the form $\sigma = \langle \mathbb{E}^4, \psi(\mathbf{x}, t), \Delta, P\rangle$, where $\mathbb{E}^4$ is the Galilean space-time, $\psi(\mathbf{x}, t)$ is a function over $\mathbb{E}^4$ called the wave-function, $\Delta \in \mathcal{B}(\mathbb{R})$ is a Borelian and P is a

[39] Of course we know that there are plenty of interpretations of quantum formalism, some of them accepting that quantum entities can be viewed as individuals. But it is a question of opinion to say that the *better one* is that which consider them as non-individuals. Really how individuals can obey quantum statistics?

[40] Although there are several other ways to get 'the same theory'; see [84].

[41] The informal semantics is given by the quantum theory itself.

function defined on $\mathcal{H}_i \times \{\hat{A}_{ij}\} \times \mathcal{B}(\mathbb{R})$ and assuming values in $[0,1] \subseteq \mathbb{R}$, so that $P(\psi, \hat{A}, \Delta)$ is the probabioity that the measurement of the observable represented by $\hat{A}$ for the system in the state $\psi(\mathbf{x}, t)$ lies in Δ.

The relevant fact is that with this move we are explicitly introducing the quasi-set of quantum systems in the formalism, and since Q is a quasi-set, we can regard these systems as indistinguishable without the need of assuming mathematical tricks such as some kind of indistinguishability of permutations postulates. The details obviously, cannot be provided here.[42]

13.3 Quantum field theory

Orthodox quantum mechanics does not deal with the physical assumed facts that quantum entities can be created and annihilated, so as deal with a fixed number of quantum systems each time. In order to cope with a variable number of systems, we need to go to relativistic quantum mechanics, which is a theory of quantum fields. In this direction, a first step was advanced in [36, chap.9], but was pushed by Domenech and Holik in their [29] (see also [31] and [30]). The details scape the finalities of this chapter, but we remark that 'quasi-Hilbert-spaces' were defined where the use of labels such as in $x_1, x_2, \ldots$ can be dispensed with and quantum mechanics can be constructed in such framework, based on suitable Fock spaces so that bosonic and fermionic states can be defined. As Domenech, Holik and Krause say in their conclusions (adapting the notation),

> We have shown that it is possible to construct the quantum mechanical formalism for indistinguishable particles without labelling them in any step. To do so, we have built a vector space with an inner product, the $\mathfrak{Q}$-space, using the non-classical part of $\mathfrak{Q}$, the generalization of ZFA, to deal with indistinguishable elements. Vectors in $\mathfrak{Q}$-space refer only to occupation numbers and permutations operators act as the identity operator, reflecting in the formalism the fact of unobservability of permutations, already expressed in terms of the formalism of $\mathfrak{Q}$.
>
> We have also argued that it is useful to represent operators (which are intended to represent observable quantities) as combinations of creator and annihilation operators, in order to avoid particle

[42] So, our approach is different from that of da Costa and Doria [13, chap.3], for they assume a structure built in ZFC and take a *set* to stand for the collection of quantum entities; due to the possible indiscernibility of them, of course, a quasi-set would be recommended instead.

> indexation in the expression of observable quantities. We have shown that creation and annihilation operators which act on $\mathfrak{Q}$-space can be constructed. We have proved that they obey the usual commutation and anti- commutation relations for bosons and fermions respectively, and this means that our construction is equivalent to that of the Fock-space formulation of quantum mechanics. Thus, using the results reviewed in [Sect. 4 of the paper], this implies that we can recover the n-particles wave equation using $\mathfrak{Q}$-spaces in the same way as in the standard theory. Though both formulations are equivalent 'for all practical purposes', when subjected to careful analysis, the conceptual difference turns very important. Our construction avoids the LTPSF[43] by constructing the state spaces using $\mathfrak{Q}$, a theory which can deal with truly indistinguishable entities, and so, it gives an alternative (and radical) answer to the problems posed in [75], so as (we guess) answers Manin's problem posed in [mentioned above].

13.4 Paraconsistency

As is well known, paraconsistent logics can be used to base inconsistent but non-trivial theories. That is, we may have theses of the form $\alpha \wedge \neg\alpha$ without turning all the well-formed sentences of the theory's language into a theorem.[44]

The problem with paraconsistent logics, in our opinion, is regarding the notion of negation: what does it mean '$\neg\alpha$' when α is a formula? Of course, you could say that the postulates provide an operational definition, and this is true. But, as da Costa likes to say, any logic should be provided also with an intuitive semantics, one that explains the intuitive meaning of the logic notions. In our opinion, the best way to achieve a view on paraconsistent negation (p.n.) is by using the ancient Square of Oppositions; sub-contrary propositions can both be true, although they cannot be false, and they are not 'contradictory' as in classical logic, that is, one being the contradictory of another, thus resulting in that if one of them is true, the other is false and reciprocally. This is of course not what happens with quantum mechanics. 'Quantum negation' is not contradictoriness, as several philoso-

[43] [This is short for 'label tensor product space formalism', or the standard formalism where quantum systems are first labelled and then assumed that a permutation of labels does not conduce to different results of measurement.]

[44] For a general history of paraconsistency and some of the main systems, see [40]. The reader can also consult [20].

phers and quantum logicians have enlightened; see [4] for a discussion and references. Thus, it seems that this can be a good starting point for a possible understanding of p.n.. In what respects quasi-set theory, by taking a paraconsistent calculus as underlying logic, we can create a *paraconsistent quasi-set theory* we denote by $\mathfrak{Q}_P$. In this theory, we can give an interpretation to quantum situations such as that where the two electrons of a helium atom, whose states are entangled so that the individual quantum systems can not be analysed separately. But, even so, they can be discerned by an irreflexive but symmetric relation; philosophers say that they are *weakly discernible* [71]. How would be possible to understand a situation where we cannot discern the electrons and even so they are discerned by such a relation?[45] Paraconsistent quasi-set theory offers a solution; as pointed out in [53],

> We can say that they [that is, quantum systems such as two electrons in an helium atom] are distinguishable by the irreflexive relation: "x has opposite direction of spin to y". This relation distinguishes the two fermions, so, $x \not\equiv y$ in $\mathfrak{Q}_P$. But $\mathfrak{Q}_P$ has all theorems of the paraconsistent calculus $\mathscr{C}_1$,[46] in particular, $\beta^o, \alpha \to \beta \vdash \neg\beta \to \neg\alpha$.[47] Furthermore, axiom ($\in$2) states that $x = y \to x \equiv y$ so, since $x \equiv y$ is not well-behaved, $\mathfrak{Q}_P \nvdash x \not\equiv y \to x \neq y$. This result may be interpreted as follows. The fact that two fermions are weakly discernible does not entail that they are distinct *individuals*.[48] So, although counting as more than one, they may continue to be indiscernible in a sense, the sense according to which we don't have any criterion to say which is which. Really, since $x \equiv y$ is ill-behaved, and since $\neg$ is the paraconsistent negation, the formula $(x \equiv y) \wedge x \not\equiv y$, being a theorem of $\mathfrak{Q}_P$, does not trivialize the system and seems to be in complete agreement with some claims posed by quantum physics, namely, that the two electrons of a He atom in its fun-

[45] Notice that if they are discernible, according to the canons of classical logic, they are *different* and so there exists a (monadic) property obeyed by just one of the entities; but is the entities cannot be discerned, we would have a contradiction in classical logic, but not in paraconsistent logics. We remark that there are also explanations out of the paraconsistent setting, say by assuming that the systems are not individuals.

[46] [This is the first paraconsistent calculus in da Costa's hierarchy $\mathscr{C}_0$ (which is the classical calculus), $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_\omega$ – see [20].]

[47] [β^o means that β is 'well-behaved', that is, obeys the laws of classical logic. The contrapositive doesn't hold in general in the calculus $\mathscr{C}_1$.]

[48] [Italics were added here.]

> damental state are discernible for there exists such an irreflexive and symmetric relation, but continue to be indiscernible for they cannot be identified as individuals (in the standard sense).

Let us reinforce the result. In classical logic, once we admit that '$\equiv$' stands for any equivalence relation, we have $\vdash x \not\equiv y \to x \neq y$, but this cannot be so in quantum mechanics when '$\equiv$' is the indiscernibility relation, due to the assumption that quantum systems can be indiscernible but not identical. In $\mathfrak{Q}_P$ we can surpass this difficulty.

You could say that the above conclusion holds also in standard mathematics; if '$\equiv$' denotes an equivalence relation whatever (or a congruence), then we may have $x \equiv y$ without being $x = y$. This is true, but the situation with $\mathfrak{Q}_P$ is different. Really, in this case, the identity relation doesn't hold for all objects and, in the 'classical case', when we say that x and y are in the equivalence relation R, they still keep their identities, are still *different* objects (except, of course, it they are identical). Thus, using $\mathfrak{Q}_P$, we find a way to accept that quantum objects may be indiscernible and even so remain *paraconsistently discernible*, that is, discerned by a paraconsistent negation, and being able to form a collection (a quasi-set) with cardinal greater than one.

13.5 Quantum mereology

In $\mathfrak{Q}$, the M-atoms and the m-atoms do not speak to one another. But of course it would be interesting to find a way to say that M-objects can be 'formed' by m-objects in the same sense that we accept that macroscopic things, such as my pen, are formed by microscopic entities. This suggests the development of a mereology suitable for quantum things, which we call *quantum mereology*. Some hints about such a construction were given in [52, 54], but the reader can see also [26]. Let us summarize here the ideas.

Mereology was originally developed by S. Leśniewicz in the first decades of the XXth century, out of any set theory, just as a formal system. Later, Nelson Goodman, Leonard, and Tasrki developed mereologies incorporating a set theory. In standard mereology (a treatise is [83], but see also [54]), the basic notions are the binary relations '*part of*' ($a < b$), '*proper part of*' ($a \ll b$), '*overlapping*', '*disjointness*', '*sum or fusion*' (the way parts sum (form) wholes), etc. Among the standard presuppositions is Mereological Extensionality,[49] which says that wholes are identical when having identical parts. This poses a challenge to the finding of a quantum mereology since,

[49] There are also *intensional mereologies*, but we do not consider them here; see [83].

as we know already, in quantum physics the exchange of a 'part' by an indiscernible one does not produce any physical difference in the whole. Furthermore, in standard mereologies, given a 'whole', we can, in principle, analyse anyone of its parts in separate from the others, and this is not so in quantum theory. Really, given a Bell entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1^A\psi_2^B\rangle \pm |\psi_2^A\psi_1^B\rangle)$, we can say that we have a totality formed by two quantum systems but which cannot be 'separated' in its components. So, the problem of finding a 'right' mereology suitable for quantum entities is still open to discussion.

In $\mathfrak{Q}$, we can mimic such a quantum situation as follows. Suppose a quasi-set q whose elements are two indiscernible m-objects, let us call them a and b; thus the quasi-cardinal of q is two, but we cannot discern between a and b. These labels are just what Toraldo di Francia called *mock names* (see [23]). Notice that these labels do not identify the entities as *rigid designators*, hence the epithet 'mock'. So, our quasi-set can be taken as a possible extension of the predicate 'to be in the Bell state given above'. We have a whole but cannot divide it (physicists say that such a state cannot be 'factored') in sub-systems and consider these sub-systems separately; this is typical of quantum mechanics.

Furthermore, there is still another challenge, and let us suppose that the Bell state above stands for the state of the two electrons (then we take the minus sign in the sum) of a helium atom in its fundamental (less energy) state. We can ionize the atom, so realizing one (perhaps both of them) of the electrons, getting a positive ion. Later, we can 'recupere' one electron getting a neutral atom again. What is the difference between the first and the second neutral atoms or between the realized and the recuperated electron? Of course, there are no differences, but in a certain sense, we are 'exchanging' parts of a whole and *nothing happens*. It seems clear that standard mereologies cannot cope with *quantum* parts and wholes. These issues were discussed in [52, 54, 55]. A more detailed analysis is being given in [61], still in preparation, where a quantum mereology is being constructed based on the theory $\mathfrak{Q}$.

13.6 Category theory

Intuitively speaking, the purpose of employing universes in the foundations of set theory is that we can perform set-theoretical operations widely and still remain inside the universe. As is known already, we cannot do it in theories such as ZFC, as Russell's paradox clearly shows. It may be thought that we could use classes as well, say in a theory like NBG, but yet in this case categories such as **Grp** (groups), **Ring**, **Set** and other would be proper classes, the problem is that being classes, they cannot be members of other classes, so no much gain would be obtained. The use of universes has been considered one of the best options; another one would be the assumption of the existence of strongly inaccessible cardinals — but this turns out to be equivalent to the use of universes. Fred Muller has still another approach; in using an extension of W. Ackermann's set theory, he shows how category theory can be defined in his theory ARC ('Ackermann plus Regularity plus Choice') [70].

We employ the resources of the theory of quasi-sets $\mathfrak{Q}$ form just sketching the category **QSet** in a quite similar way that the category **Set** is obtained from, say, the ZFC set-theory by adding universes to it [41], [7]. Since further developments are available for this study, in the sense that we may be interested in considering an universe as a starting point for other large universes (see [63, pp. 259 and 262] for the justification of Grotendieck's use of universes instead of NBG), we have opted for strengthening $\mathfrak{Q}$ with special universes, the *quasi-Ehresmann-Dedecker universes*. A more detailed discussion on the category **QSet** is given in [26].

Ehresmann-Dedecker universes were introduced in [16] (but see also [14], [27], [8]) and generalize the concept of Sonner-Grothendieck universes [41], [7], being more adapted for dealing with atoms, which is the case of $\mathfrak{Q}$. We could exclude the axiom of regularity, but here we shall keep with it. We base much of our definitions and theorems in [8].

Definition 13.1 (qED universe) *A non-empty quasi-set $\mathcal{U}$ is a quasi-Ehresmann-Dedecker universe (qED for short) if the following conditions are obeyed:*

(1) $x \in \mathcal{U} \to \mathcal{P}(x) \in \mathcal{U}$

(2) $x \in \mathcal{U} \to [x]_{\mathcal{U}} \in \mathcal{U}$

(3) $x, y \in \mathcal{U} \to x \times y \in \mathcal{U}$

(4) If $(x_i)_{i \in I}$ is a family of quasi-sets such that $x_i \in \mathcal{U}$ for every $i \in I \in \mathcal{U}$ and being I a "classical" quasi-set (that is, a quasi-set obeying the predicate Z), then $\bigcup_{i \in I} x_i \in \mathcal{U}$.

In (2), $[x]_{\mathcal{U}}$ stands for the quasi-set of all indistinguishable from x that belong to the universe. Recall that it is called the *singleton* of x relative to $\mathcal{U}$. It should be remarked that this 'singleton' may contain more than one element, that is, its quasi-cardinal may be greater than one. Intuitively speaking, indistinguishable objects cannot be distinguished by any means, but in $\mathfrak{Q}$ we may form collections (quasi-sets) of them with cardinals different from 1. In other words, indistinguishability doesn't collapse in identity, as in 'Leibnizian' theories such as ZFC and others grounded on classical logic, as we have seen already.

Theorem 13.1 *If $\mathcal{U}$ is a qED universe and $x, y \in \mathcal{U}$, then $x \cup y$, $[x, y]_{\mathcal{U}}$, and $\langle x, y\rangle_{\mathcal{U}}$ belong to $\mathcal{U}$.*
Proof: The proof is similar, but with adaptations for the use of quasi-sets, to those presented in [11, Teo.58, p.42]. ▌

Axiom AqED — *Every quasi-set is an element of a qED universe.*

Definition 13.2 $\mathfrak{Q}^{\star} := \mathfrak{Q} + AqED$

In words, $\mathfrak{Q}^{\star}$ is the theory got from $\mathfrak{Q}$ by adding the axiom **AqED** to it. It is similar to the theory got from adding Grothendieck's universe axiom "For each set x, there exists a universe $\mathcal{U}$ such that $x \in \mathcal{U}$".

Theorem 13.2 $\mathfrak{Q}^{\star} \vdash \text{Cons}(\mathfrak{Q})$
Proof: It is a homework to check that all postulates of $\mathfrak{Q}$ are satisfied in any qED universe $\mathcal{U}$. ▌

Definition 13.3 *Let $\mathcal{U}$ be a qED. We call $\mathcal{U}$-**qset** any element of $\mathcal{U}$. A $\mathcal{U}$-**qclass** is a subset of $\mathcal{U}$. A $\mathcal{U}$-**proper qclass** is a $\mathcal{U}$-qclass that is not an $\mathcal{U}$-qset.*

Definition 13.4 *A $\mathcal{U}$-**small category** is a category $\mathbb{C}$ such that $\mathsf{ob}(\mathbb{C})$ and $\mathsf{mor}(\mathbb{C})$ are $\mathcal{U}$-qsets, and it is $\mathcal{U}$-**large** otherwise.*

Definition 13.5 *We call **QSet** is the category of all quasi-sets of $\mathfrak{Q}$.*

The objects of **QSet** are the quasi-sets and the morphisms are the *quasi-functions* between quasi-sets. We recall once more that the elements of a quasi-set may be indistinguishable from one each other and even so the cardinal of the collection (termed its *quasi-cardinal*) is not identical to 1, as would be the case in 'standard' set theories such as ZFC, NBG, KM or other, where the standard theory of identity implies that there cannot exist indistinguishable but not identical objects (these theories are, in a sense, *Leibnizian*). Thus, taking two quasi-sets containing sub-collections of indistinguishable objects, a quasi-function takes indistinguishable objects in the first and associate to them indistinguishable elements of the other. In symbols, being A and B the quasi-sets and q the quasi-function, we have that

$$(\forall x \forall x' \in A)(\exists y \forall y' \in B)(\langle x, y\rangle \in q \wedge \langle x', y'\rangle \in q \wedge x' \equiv x \to y \equiv y'),$$

where $\equiv$ is the relation of indistinguishability, which applies also to quasi-sets and in particular to quasi-functions.

We can prove that the composition $\circ$ of quasi-functions (morphisms) is associative and that for each object A there exists an 'identity' morphism $A \xrightarrow{1_A} A$, that is, a quasi-function whose domain and co-domain are both A itself, such that for any other morphisms $B \xrightarrow{f} A$ and $A \xrightarrow{h} C$, we have that $1_A \circ f \equiv f$ and $h \circ 1_A \equiv h$. This characterizes the category **Qset**; see also [26].

Since the collection of all quasi-sets and the collection of all quasi-functions are not quasi-sets, but things similar to proper classes, we can state the following result:

Theorem 13.3 *The category* **QSet** *is a large category.*

The theorem shows that **QSet** plays, relatively to quasi-set theory $\mathfrak{Q}$ the role played by **Set** relatively to, say, ZFC. In other words, in $\mathfrak{Q}^\star$ we can develop the category **QSet** as suggested in [25].

14 Concluding remarks

In this chapter we have seen how da Costa's ideas on non-reflexive logics were extended to higher-order systems and to a theory of quasi-sets, so as some hints of the applications of such logics to quantum theories.

Further readings are [2], [3], [26], [57], [58], [59]. Additional works by da Costa (with F. A. Doria) on the foundations of physics can be seen in [13, 17].

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