Understanding Time Reversal in Quantum Mechanics: A Full Derivation

Shan Gao
Research Center for Philosophy of Science and Technology,
Shanxi University, Taiyuan 030006, P. R. China
E-mail: gaoshan2017@sxu.edu.cn.

August 2, 2022

Abstract

Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal in quantum mechanics has been with us since Wigner's first presentation. In this paper, I propose a new approach to solving this puzzle. First, I argue that the standard account of time reversal can be derived from the requirement that the continuity equation in quantum mechanics is time reversal invariant. Next, I analyze the physical meaning of the continuity equation and explain why it should be time reversal invariant. Finally, I discuss how this new analysis help solve the puzzle of time reversal in quantum mechanics.

Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation in quantum mechanics? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal has been with us since Wigner's (1931) first presentation, although some progress has been made to solve it recently (see, e.g. Roberts, 2017, 2020; Callender, 2021). According to some authors, time reversal "can involve nothing whatsoever other than reversing the velocities of the particles" (Albert 2000, p.20), and "It does not make sense to time-reverse a truly instantaneous state of a system" (Callender, 2000). While according to others (Earman, 2002; Malament, 2004; Roberts, 2017), this is not the case. In this paper, I will propose a new approach to solving this puzzle of time reversal. I will first give a full derivation of the standard account of time reversal in quantum mechanics based on the requirement that the continuity equation is time reversal invariant. Then, I will analyze the physical meaning of the continuity equation and explain why it should be time reversal invariant. Finally, I will discuss how the new analysis may help solve the puzzle of time reversal in quantum mechanics. In particular, I will explain why it makes sense to time-reverse certain instantaneous quantities such as momentum and spin.

Consider the Schrödinger equation for a spin-0 quantum system in an external scalar potential:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)]\psi(\mathbf{r},t), \tag{1}$$

where $\hbar$ is Planck's constant divided by 2π, $\psi(\mathbf{r},t)$ is the wave function of the system, m is the mass of the system, and $V(\mathbf{r},t)$ is an external scalar potential. From this equation we can derive the continuity equation:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{2}$$

where $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and $\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}[\psi^*(\mathbf{r},t)\nabla\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\nabla\psi^*(\mathbf{r},t)]$ are probability density and probability current density, respectively.

Now I will show how the standard account of time reversal in quantum mechanics can be derived based on the requirement that the continuity equation is time reversal invariant. First, it can be argued that time reversal does not change the probability density. From a physical point of view, the probability density of finding a particle in certain position in space does not depend on the direction of time. Moreover, from a mathematical point of view, it can be proved that any transformation of $\rho(\mathbf{r},t)$, $F(\rho(\mathbf{r},t))$, which satisfies the nomalized condition $\int F(\rho(\mathbf{r},t))d\mathbf{r} = 1$ for any $\rho(\mathbf{r},t)$, must be an identity transformation.[1] Then, we have $T\rho(\mathbf{r},t) = \rho(\mathbf{r},-t)$, where T is the time reversal operator. The time reversal invariance of the continuity equation (2) further requires that $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$.

By writing the wave function in the polar form $\psi = Re^{iS/\hbar}$, where R and S are real functions, we can obtain the following relation:

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{m}\rho(\mathbf{r},t)\nabla S(\mathbf{r},t). \tag{3}$$

By using the transformation rules for $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$, we have $TS(\mathbf{r},t) = -S(\mathbf{r},-t)+C_0$, where C_0 is a real constant. Then we can obtain the standard antiunitary transformation rule for the wave function: $T\psi(\mathbf{r},t) = \psi^*(\mathbf{r},-t)$ when ignoring an overall constant phase. Based on this transformation rule for the wave function, we can derive the transformation rule for every observable from its definition (or its operation on the wave function). For example, for position $\mathbf{r}$, we have $T\mathbf{r}T^{-1} = \mathbf{r}$, and for momentum $\mathbf{p} = -i\hbar\nabla$, we have $T\mathbf{p}T^{-1} = -\mathbf{p}$, and for angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, we have $T\mathbf{L}T^{-1} = -\mathbf{L}$.

[1] I thank Phil Pearle and Rodi Tumulka for showing me a proof of this result.

In addition, by analyzing the probability current acceleration:

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = \frac{1}{m}[\nabla Q(\mathbf{r},t) - \nabla V(\mathbf{r},t)], \tag{4}$$

where $\mathbf{v}(\mathbf{r},t) = \frac{\mathbf{j}(\mathbf{r},t)}{\rho(\mathbf{r},t)}$ is the local velocity for the probability current, and $Q(\mathbf{r},t) = \frac{\hbar^2}{2m}\frac{\nabla^2 R(\mathbf{r},t)}{R(\mathbf{r},t)}$, we can obtain the transformation rule for the scalar potential: $TV(\mathbf{r},t) = V(\mathbf{r},-t)$. Notably this transformation rule applies to the electric scalar potential $T\phi(\mathbf{r},t) = \phi(\mathbf{r},-t)$. Using the definition $\mathbf{E} = -\nabla\phi$, we can obtain the transformation rule for the electric field $T\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},-t)$. Furthermore, by analyzing the continous equation for a charged system in an electromagnetic field, we can also obtain the transformation rules for the magnetic potentials and fields. The probability current for a spin-0 system with mass m and charge Q in an external electromagnetic field is

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{m}\rho(\mathbf{r},t)[\nabla S(\mathbf{r},t) - Q\mathbf{A}(\mathbf{r},t)], \tag{5}$$

where $\mathbf{A}(\mathbf{r},t)$ is the magnetic vector potential. Then $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$ leads to $T\mathbf{A}(\mathbf{r},t) = -\mathbf{A}(\mathbf{r},-t)$. Using the definition $\mathbf{B} = \nabla \times \mathbf{A}$, we can obtain the transformation rule for the magnetic field $T\mathbf{B}(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},-t)$.

Lastly, we can also obtain the time reversal transformation rule for spin in a similar way. The probability current for a spin-s system with mass m and charge Q and magnetic moment μ_s in an external electromagnetic field is

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{2m}\rho(\mathbf{r},t)[(\psi^*(\mathbf{r},t)\mathbf{p}\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\mathbf{p}\psi^*(\mathbf{r},t)) - 2Q\mathbf{A}(\mathbf{r},t)] + \frac{\mu_s}{s}\frac{\nabla \times (\psi^*(\mathbf{r},t)\mathbf{S}\psi(\mathbf{r},t))}{\psi^*(\mathbf{r},t)\psi(\mathbf{r},t)}, \tag{6}$$

where $\mathbf{S}$ is the spin operator. Then $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$ leads to $T\mathbf{S}(\mathbf{r},t) = -\mathbf{S}(\mathbf{r},-t)$. Based on the transformation rules for spin and the wave function, we can also derive the result $T^2 = -I$ for spin-1/2 systems.

The above analysis provides a full derivation of the standard time reversal transformation rules in quantum mechanics. Based on this analysis, we can confirm that the Schrödinger equation is time reversal invariant as usually thought. This analysis can be extended to relativistic quantum mechanics and quantum field theory.

Now we need to justify the time reversal invariance of the continuity equation. The continuity equation in quantum mechanics is a local and stronger form of the probability conservation law. A weak version of the probability conservation law states that the total probability of finding a particle in the whole space is one. The continuity equation says that when the local probability density changes continuously or the probability current is continuous, the increase/decrease of the probability in a volume is equal to the net probability that flows into/out the volume. It is reasonable to assume that the probability conservation law, either weak or strong, is valid under time reversal (and other symmetrical transformations). For example, if $\mathbf{j}$ is not time-reversed either as ρ and thus the continuity equation is not invariant under time reversal, then in the time-reversed world when the net probability current flows into a volume, the probability in the volume will not increase but decrease, which is not reasonable.[2]

There have been also worries about the physical meaning and measurability of $\mathbf{j}$ in the continuity equation (see, e.g. Sakurai, 1996).[3] I think protective measurement may provide a further insight here. According to the principle of protective measurement (Aharonov and Vaidman, 1993; Aharonov, Anandan and Vaidman, 1993; Gao, 2015; Piacentini et al, 2017), when the wave function of a single quantum system is known, one can measure both ρ and $\mathbf{j}$ by a series of protective measurements on the system. Moreover, when assuming the psi-ontic view (Pusey, Barrett and Rudolph, 2012), ρ and $\mathbf{j}$, when multiplied by the mass and charge of the system, can be explained as the mass and charge density and current density (Gao, 2017). Then the continuity equation can also be explained as the local form of the conservation law for mass and charge. This will further justify its validity.

Finally, let us see in what sense the above analysis provides an intelligible way to understanding time reversal in quantum mechanics. According to the above analysis, the time reversal invariance of the continuity equation, along with the reliable transformation rule for the density $T\rho(\mathbf{r},t) = \rho(\mathbf{r},-t)$, leads to the transformation rule for the current density $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$. This is consistent with our intuition that time reversal reverses the direction of a current. Note that the current density can be written as $\mathbf{j}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$, where $\mathbf{v}(\mathbf{r},t) = \frac{1}{m}\nabla S(\mathbf{r},t)$ is the local velocity associated with the current. Then, why time reversal involves complex conjugation is because the phase of the wave function is the integral of the current velocity and time reversal reversing the velocity as argued above amounts to taking the complex conjugation of the wave function. Several authors have given a similar account (Earman, 2002; Sebens, 2015; Callender, 2021). Moreover, why time reversal reverses momentum, spin, and magnetic fields is because these quantities are related to the current density or velocity in a certain way.

However, there is an important point which needs to be emphasized

[2] In this case, energy will be negative in the time-reversed world, which seems to be also a very serious issue.

[3] Sakurai wrote, "we would like to caution the reader against a too literal interpretation of j as ρ times the velocity defined at every point in space, because a simultaneous precision measurement of position and velocity would necessarily violate the uncertainty principle." (Sakurai, 1996, p.102-3)

here. It is that the above current velocity, unlike the velocity in Newtonian mechanics, is not defined as the rate of change of some instantaneous configurational quantity. This means that one cannot directly determine the transformation rule for the current velocity by its definition, which is different from the situation in Newtonian mechanics. This is also the reason why we resort to the time reversal invariance of the continuity equation to derive the transformation rule for the current density.[4]

The above analysis may also help settle the controversy on the meaning of time reversal in quantum mechanics. As noted before, it has been debated whether an instantaneous quantity should be changed by time reversal. According to some authors, it does not make sense to time-reverse a truly instantaneous quantity (Callender, 2000), and time reversal can involve nothing other than reversing the rate of change of instantaneous quantities such as velocities of particles (Albert 2000). The above analysis seems to disfavor this nonstandard view of time reversal. According to this view, time reversal will keep both ρ and $\mathbf{j}$ in the continuity equation unchanged, and thus the continuity equation is not time reversal invariant. In other words, this nonstandard view is inconsistent with the local form of conservation laws for probability, mass and charge. However, this is not reasonable as argued before, since in the time-reversed world, when the net probability (or mass/charge) current flows into a volume, the probability (or mass/charge) in the volume does not increase but decrease.

Then, what is wrong with the nonstandard view of time reversal? It seems reasonable to require that an instantaneous quantity should not be reversed by time reversal, and only the rate of change of something invariant by time reversal should be reversed by time reversal. But when the rate of change is produced and determined by an instantaneous quantity, it seems more reasonable to assume that this instantaneous quantity is also reversed by time reversal (see also Earman, 2002).[5] If not, then the reversed rate of change cannot be explained in the time-reversed world, and the corresponding law will also be violated. The nonstandard view's violation of the continuity equation is just such a case. By the continuity equation, the rate of change of density is produced and determined by the current density. If the rate of change of density is reversed by time reversal, while the current density is not reversed by time reversal, then the rate of change of density cannot be explained in the time-reversed world, and the corresponding con-

[4]Note that even though in the de Broglie-Bohm theory we can determine the transformation rule for the velocity of a Bohmian particle by its definition, which is assumed to be equal to the current velocity, we still need to resort to the time reversal invariance of the guiding equation to derive the standard transformation rule for the wave function. Then, why not directly assume the time reversal invariance of the Schrödinger equation? In my view, the de Broglie-Bohm theory does not help much in solving the puzzle of time reversal in quantum mechanics (cf. Allori et al, 2008).

[5]Earman wrote, "If X is produced by Y and Y is 'turned around' by time reversal, then X must also be affected by time reversal." (Earman, 2002, p.247)

servation law will be also violated; when the net probability (or mass/charge) current flows into a volume, the probability (or mass/charge) in the volume does not increase but decrease. This analysis applies to momentum, spin, and magnetic fields, since they appear in the current density and produce the rate of change of density. Note, however, that if an instantaneous quantity does not produce the rate of change of something invariant by time reversal, then it is indeed reasonable to assume that this instantaneous quantity should be not reversed by time reversal.

To sum up, I have argued that by analyzing the continuity equation, the standard account of time reversal in quantum mechanics can be derived. Moreover, the meaning of time reversal may be clarified by analyzing the relationship between the rate of change and the instantaneous quantity which produces it. This analysis provides a new way to solve the puzzle of time reversal in quantum mechanics. It remains to be seen whether this solution is complete and fully satisfying.

References

[1] Aharonov, Y., Anandan, J. and Vaidman, L. (1993). Meaning of the wave function. Phys. Rev. A 47, 4616.

[2] Aharonov, Y. and Vaidman, L. (1993). Measurement of the Schrödinger wave of a single particle, Physics Letters A 178, 38.

[3] Albert, D. Z. (2000). Time and Chance. Cambridge, MA: Harvard University Press.

[4] Allori, V., S. Goldstein, R. Tumulka, and N. Zanghì (2008). On the common structure of Bohmian mechanics and the Ghirardi-Rimini-Weber theory, British Journal for the Philosophy of Science 59 (3), 353. Section 4.2.

[5] Callender, C. (2000). Is Time 'Handed' in a Quantum World?, Proceedings of the Aristotelian Society, 100 (3): 247-269.

[6] Callender, C. (2021). Quantum Mechanics: Keeping It Real? The British Journal for the Philosophy of Science. https://doi.org/10.1086/715032

[7] Earman, J. (2002). What time reversal is and why it matters. International Studies in the Philosophy of Science 16(3): 245-264.

[8] Gao, S. (ed.) (2015). Protective Measurement and Quantum Reality: Toward a New Understanding of Quantum Mechanics. Cambridge: Cambridge University Press.

[9] Gao, S. (2017). The Meaning of the Wave Function: In Search of the Ontology of Quantum Mechanics. Cambridge: Cambridge University Press.

[10] Malament, D. B. (2004). On the time reversal invariance of classical electromagnetic theory, Studies in History and Philosophy of Modern Physics 35: 295-315.

[11] Piacentini, F. et al. (2017). Determining the quantum expectation value by measuring a single photon. Nature Phys. 13, 1191.

[12] Pusey, M., Barrett, J. and Rudolph, T. (2012). On the reality of the quantum state. Nature Phys. 8, 475-478.

[13] Roberts, B. (2017). Three myths about time reversal in quantum theory. Philosophy of Science 84, 315-334.

[14] Roberts, B. (2020). Time Reversal. The Routledge Companion to the Philosophy of Physics, Eleanor Knox and Alistair Wilson (eds).

[15] Sakurai, J. J. (1994). Modern Quantum Mechanics, Revised Edition. Reading, MA: Addison Wesley.

[16] Sebens, C. (2015). Quantum Mechanics as Classical Physics. Philosophy of Science, 82(2): 266-291.

[17] Wigner, E. P. (1931). Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, New York: Academic Press (1959).