Quantifying over indiscernibles

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Abstract

One of the main criticisms of the theory of collections of indiscernible objects is that once we quantify over one of them, we are quantifying over all of them since they cannot be discerned from one another. In this way, we would call the collapse of quantifiers: 'There exists one x such as P' would entail 'All x are P'. In this paper we argue that there are situations (quantum theory is the sample case) where we do refer to a certain quantum entity, saying that it has a certain property, even without committing all other indistinguishable entities with the considered property. Mathematically, within the realm of the theory of quasi-sets $\mathfrak{Q}$, we can give sense to this claim. We show that the above-mentioned 'collapse of quantifiers' depends on the interpretation of the quantifiers and on the mathematical background where they are ranging. In this way, we hope to strengthen the idea that quantification over indiscernibles, in particular in the quantum domain, does not conform with quantification in the standard sense of classical logic.

Keywords: quantification, quantum logic, indiscernibility, identity, in- discernible objects.

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1 Introduction

The quantum realm, where quantum theories reign, is the land of indiscernible things.[1] Thus, this field offers a nice place for motivating the discussion we intend to promote and shall be used to exemplify our claims, although the logical discussion is not restricted to the quantum world.

Physicists say that all electrons are 'identical', meaning indiscernible from one another with respect to intrinsic properties (those that are independent of the state of the system), and so are all protons, all atoms of the same substance, and this goes at least to some molecules.[2] We still don't know until what stage these results go, since 'quantum phenomena have been observed in systems that almost could be called 'classical'. But at least for the 'micro objects', it can be taken for granted that in certain situations we cannot discern among those belonging to a certain collection.[3]

The consideration of indiscernible entities poses a problem to any attempt to discuss logical matters. The literature is abundant in pointing to the differences among 'classical connectives' and their quantum correspondents, so as about the validity of some 'classical' rules (such as the Lindenbaum property, the full Theorem of Deduction, etc.; see [1, 7] and the references therein). A particular case concerns quantification and this is one of the topics that has not received the deserved attention until now (some few exceptions being mentioned

[1] It is disputable whether we really can, as Bohr seems to have proposed, divide the world into 'micro' and 'macro' parts. Apparently, quantum mechanics would be a 'totalitarian theory' (to use Leggett's expression) which applies not only to electrons and protons but also to macroscopic objects; see [14]. But here we shall be concerned, in the examples mentioned, with the microscopic level only. We assume, as usual, that 'macroscopic' objects obey classical logic, although the frontier is known to be fuzzy.

[2] We keep the words 'identical', 'identity', etc. to be used in a logical and mathematical context meaning the same. Today experiments have shown that also 'big' molecules such as C_{60} and C_{70} present a 'quantum behaviour', say in the two-slits experiment [20].

[3] I shall leave the discussion about Bohmian mechanics, which encompasses an ontology similar to classical physics, out of this paper and assume the standards in quantum theories. It is disputable whether Bohmian's positions can be known. As far as I am concerned, they remain hidden.

below, such as [4, 5, §5]). In general, quantum logicians speak less about quantification than about the propositional level, but it is to quantification that we restrict our analysis here. The basic question can be put this way: consider a collection A (we shall not refer to 'sets' for the reason to be mentioned soon) of indiscernible objects and let F be a property that applies to them. Then it is supposed that if F applies to one of these objects, due to their indiscernibility, it should apply to any other of them as well, at least this is what it seems. So, if $\exists x F(x)$, we ought to conclude that $\forall x F(x)$. This of course would cause a collapse of quantifiers and brings a problem for the quantification in non-reflexive logics. As da Costa and Bueno say [5],

> In order to quantify over each object in the domain, such objects need to be distinguishable from one another. But this presupposes that identity can be applied to these objects so that quantification ranges over distinct objects rather than the same object again and again. Given the identification of 'each' and 'all' in reflexive logics [logics where identity applies to all objects], the latter presupposes that the objects that are quantified over have well-defined identity conditions. However, this assumption need not hold in the cases of logics, such as non-reflexive ones, in which the principle of identity is restricted. In fact, if we are unable to speak of the identity of certain objects, we cannot speak of these objects being different from one another either, given that difference involves the negation of identity.

The problem is with the assumption that indiscernibles are 'the same', as the above quotation seems to suggest. Quantum objects are absolutely indiscernible in certain situations, as in a bosonic condensate, and even so, they are not the same entity. But OK, you can say that we cannot ascribe to one of them certain characteristics not shared by the others. So, let us go to another relevant case involving fermions. As is known, fermions cannot have all the same quantum numbers, so even in the 'worst' situation, they do present a difference. A typical example is that of the two electrons of a helium atom in the fundamental state. The state of the system with the two electrons is given by a state vector of the form

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}\big(|\psi_1, \uparrow\rangle |\psi_2, \downarrow\rangle - |\psi_2, \uparrow\rangle |\psi_1, \downarrow\rangle \big) \quad (1)$$

where the labels '1' and '2' label the electrons and the arrows indicate the direction of the spin in a given direction; the vector is chosen so that a permutation of these labels conduce to $-|\psi_{12}\rangle$, but the probabilities are the same since $| |\psi_{12}\rangle |^2 = | - |\psi_{12}\rangle |^2$.

But the problem is different. Even presenting a difference with respect to their spins, we are unable to identify which electron has, say, spin UP. The most we can say is that *one of them* has this characteristic while the other does not; no more information is available. But we of course (at least informally) can quantify the electrons and say things like "there exists one electron (of the atom, etc.) with spin UP in the given direction", and this of course does not entail that both of them have such a value of spin. So, contrarying da Costa and Bueno, we agree that we really are unable to speak of the identity of the electrons, but even so, we *can* say that they have some difference. Quantum objects are not classical objects and behave differently.

da Costa and Bueno's claim is valid in the standard frameworks where identity is defined by means of indiscernibility and holds in full, but not in all domains. In quantifying over indiscernibles, we of course are not quantifying over the same object "again and again" as they suggest. In fact, we can agree that indiscernible things are not the same thing; what we need is just to find a framework where this can be expressed, thus of course we need to go out of standard logical settings. Thus, let us fix a point here: when we say that a collection of objects comprises indiscernibles, not always we are thinking of things like the bosons in a bosonic condensate; it may be the case that we are speaking of things such as fermions in a situation like that of the electrons of a helium atom: they can present a difference, but we cannot say which is which. So, we cannot say that $\exists x P(x) \to \forall x P(x)$ even in the context of non-reflexive logics, as the above interpretation shows (just take $P(x)$ meaning that x is an electron of a He atom with spin UP in a given direction).

From a mathematical perspective, we need to give life to these claims, more specifically, to the possibility of finding a mathematical framework where we can express a situation where a certain collection of objects has as elements indiscernible things but so that there is not just one thing. The way to express that is by means of cardinalities; the collection of the electrons of the He atom is two, even if we cannot distinguish its elements. Even a bosonic condensate can be 'weight' so that we can have an idea of the quantify of elements it comprises, although absolutely indiscernible. *Quasi-set theory* provides such a framework, as we shall mention below. There, as we shall see, we really can quantify indiscernible things without the collapse of quantifiers.

2 A glimpse on the theory of quasi-sets

Let us name '$\mathfrak{Q}$' a theory which generalises standard set theory and which would be able to deal with collections of entities such as the *quanta* which in certain situations can be considered as *absolutely indiscernible*. Of course, we wish to preserve a standard set theory inside $\mathfrak{Q}$, so ZFA (Zermelo-Fraenkel set theory with atoms) system is the 'core' of the theory, although one could base the theory on a different ground, such as the NBG system or other. So, ZFA enables us to develop all standard mathematical concepts such as ordinals and cardinals. The atoms of ZFA are represented in $\mathfrak{Q}$ by a monadic predicate M, and we call them M-atoms. The entities represented by the M-atoms sometimes will be termed 'M-objects'. The novelty is that the theory encompasses another kind of atoms, the m-atoms, which in the intended interpretation would play the role of quantum entities; to these entities, it is supposed that the standard notion of identity does not apply, and this is done by assuming that expressions of the form '$x = y$' are not well-formed if either x or y denote an 'm-object'. So, the theory goes in the direction pointed out by Schrödinger, who has claimed that "the notion of sameness, of identity, really and truly has no meaning [to the elementary particles]" ([17, p.122]; see [10] for a discussion).

Quasi-sets are objects that are neither m-atoms nor M-atoms. Their elements may be either kind of atoms and also other quasi-sets; a version of the Axiom of Regularity is used to avoid that a quasi-set can be an element of itself. Some quasi-sets do not involve m-atoms in their transitive closure, that is, they are built within the 'classical' part of $\mathfrak{Q}$, and are termed *sets*, being copies of the ZFA sets. If the M-atoms are also dropped out, then we get a version of 'pure' ZFC. The unary primitive predicates m, M, and Z cope with m-objects, M-objects and sets respectively. Two binary primitive predicates are $\equiv$ (indistinguishability, or indiscernibility), and $\in$ (membership) also make part of the language, so that '$x \equiv y$' means that x is indistinguishable (or indiscernible) from y and '$x \in y$' means that x is an element of y. Furthermore, there is still a unary primitive functional symbol, qc such that $qc(x)$ is a term which stands for 'the *quasi-cardinal* of x', informally standing for the number of elements it has. Formulas are defined as usual, and the postulates provide the details of the theory.

Given a formula $\varphi(x)$ of the language, the collection $[x : \varphi(x)]$ is called a *quasi-class*; we deserve the usual '{' and '}' for the case of sets. Given a quasi-set q and $x \in q$, we define the 'singleton' of x (relative to q) as the qset $[x]_q := [y \in q : y \equiv x]$, that is, the quasi-set of the indiscernible from x that belong to q; of course, its quasi-cardinal can be greater than one. If such a quasi-cardinal is precisely one, we call it the *strong singleton* of x and denote it by $[\![x]\!]_q$; the details of how to derive the existence of such quasi-sets are omitted.

Important to realize that in having a quasi-cardinal, the elements of the quasi-set can continue to be indiscernible; nothing implies that they can be 'counted' by standard means (that is, by means of bijections, which need identity for defining them). Identity (symbolized by the equality symbol '$=$') is not a primitive notion, but a concept of *extensional identity*, '$=_E$', is defined this way:

$$x =_E y := (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee \\ (M(x) \wedge M(y) \wedge \forall z(x \in z \leftrightarrow y \in z)). \tag{2}$$

The reader could think that it would be more convenient to restrict the extensional identity to sets and M-objects only. This of course could be done but brings difficulties for expressing certain things, as in defining certain frames at section (**??**), as we shall mention there. But we think that the above definition can be used; *when* two quasi-sets do have the *same* elements, they are extensionally identical, endpoint, yet we possibly never know when this happens.

It can be proven that this identity has all the usual properties of the standard identity of ZFA for the objects it applies to. Notice that '$=_E$' does not hold if at least one of the involved terms is an m-atom. So, if we interpret the m-atoms as denoting quantum elementary systems, we are within Schrödinger's realm. Thus, such an interpretation provides a model for us to show that $\exists x P(x) \to \forall x P(x)$ does not hold in $\mathfrak{Q}$.

The relation '$\equiv$' has all the properties of an equivalence relation (reflexive, symmetric and transitive), but it is not a congruence; in fact, it does not preserve membership: if $x \in y$ and $x' \equiv x$, we cannot prove that $x' \in y$. So, $\equiv$ and standard identity ($=$) are different notions since the former applies to all entities in the universe of quasi-sets while the last one (in the form '$=_E$') applies only to sets and the M-atoms.

Postulates similar to those of ZFA are given, say a Scheme of Separation, union, power, etc. The null quasi-set turns out to be a set and it is unique, represented by '$\emptyset$'. For 'classical entities' (either M-atoms or sets), an Axiom of Extensionality holds, but when m-atoms are also involved, we cannot state it in its usual form, so the theory postulates a Weak Extensionality Axiom, which says (with the due definitions and existential postulates) that quasi-sets comprising 'the same quantities' (in terms of quasi-cardinals) of elements *of the same sort* are indistinguishable. Thus we can treat formally two sulfuric acid molecules as indiscernible, yet not identical: $H_2SO_4 \equiv H_2SO_4$ and the quasi-set having only two of such molecules as elements has quasi-cardinal two. So, the quasi-set can have a quasi-cardinal even if its elements cannot be discerned from one another.

The elements of a quasi-set can be distinguished in 'kinds' by some property, as in physics we distinguish among electrons, protons and neutrons. What imports is not their identities, but their *kinds* and *quantities*, as when we consider a sulfuric acid molecule; so, in informal parlance, we can pose a finite quasi-set as something like the tuple

$$q = \langle k_1, k_2, \ldots ; \lambda_1, \lambda_2, \ldots \rangle, \tag{3}$$

where the k's indicate the kinds and the λ's the quasi-cardinals of each kind. Thus, H_2SO_4 turns out to be something like $\langle \mathrm{H}, \mathrm{S}, \mathrm{O}; 2, 1, 4 \rangle$, which emphasizes just the kinds and quantities, and not the nature of the involved entities.[4]

We can construct (in the metamathematics) a *universe of quasi-sets* $\mathcal{Q}$ by transfinite recursion over the class On of ordinals as follows: $Q_0 := m \cup M$, where m and M are disjoint collections of atoms, $Q_1 := \mathcal{P}(Q_0)$, ..., $Q_\lambda := \bigcup_{\beta<\lambda} Q_\beta$ if λ is a limit ordinal, and finally $\mathcal{Q} := \bigcup_{\alpha \in On} Q_\alpha$. This structure is not rigid, since the identity function cannot be defined for all quasi-sets of the universe (due to the presence of the m-atoms) and of course, the quasi-function (see below) that leads an element into an indistinguishable one is a nontrivial automorphism.

We can defined a version of an 'ordered pair' as follows: given a and b in a quasi-set q, define $\langle a, b\rangle_z := [[a]_z, [a, b]_z]_{\mathcal{P}(z)}$, where $[a]_z$ and $[a, b]_z$ come from the 'pair axiom'.[5] By means of this definition, we can define binary and n-ary 'quasi'-relations and 'quasi'-functions (q-function). Interesting that a q-function does not distinguish between its arguments (or values) when there are indistinguishable m-atoms involved; so, the definition says that indistinguishable things are lead to indistinguishable things. By softening the idea we can also define q-injections, q-surjections and q-bijections.

The theory has also a version of the axiom of choice we call *Axiom of Quasi-Choice*, which informally read as follows: given a qset x, non-empty and formed by disjoint and non-empty quasi-sets, there exists a quasi-set u such that given an element v of x and an element $t \in v$, there exists a qset s which is a sub-quasi-set of the qset of the indiscernibles from t that belong to x with quasi-cardinal one and whose intersection with u is indiscernible from its intersection with v. This last affirmative says that the only element of s is indiscernible from t, but of course, we cannot state that it is t itself. Obviously, if no m-atoms are involved, this axiom is equivalent to the standard one in ZFA. In other words, the quasi-set u is formed by selecting one el-

[4]It should be remarked that Hermann Weyl has called our attention to precisely this point, positing that in quantum physics what imports is an 'ordered decomposition' emphasizing precisely the kinds and quantities only; see [19, App.B], [10], [13]. Just to remark, this was also the idea underlying the birth of modern chemistry with Boyle, Hooker, and mainly Dalton, to whom the atoms of a given element are indiscernible – see [9].

[5]An alternative definition could be $\langle a, b\rangle_z := [[\![a]\!]_z, [\![a]\!]_z \cup [\![b]\!]_z]_{\mathcal{P}(z)}$.

ement indiscernible from some element of each element of x. Other formulations could of course be given.

One interesting result is a theorem which asserts that 'permutations are not regarded as observable', a central thing in quantum mechanics. In this theory this needs to be introduced by a postulate, the Indistinguishability Postulate, which read as follows: for all vectors $|\psi\rangle$, all operators $\hat{A}$ and all particle label permutation operator P, we have that $\langle\psi|\hat{A}|\psi\rangle = \langle P\psi|\hat{A}|P\psi\rangle$, that is, the expectation value of the measurement of an observable A (represented by the self-adjoint operator $\hat{A}$) for the system in the state $|\psi\rangle$ is the same before and after the action of the permutation operator P. In other words, permutations are not observable. In $\mathfrak{Q}$, we have a theorem which says that given a quasi-set q, if $x \in q$, $y \equiv x$ being $y \in q'$ where $q \subseteq q'$, but $y \notin q$, then $(x \setminus [\![x]\!]_q) \cup [\![y]\!]_{q'} \equiv q$. In words, we are 'exchanging' an indistinguishable from x by an indistinguishable from y and the resulting quasi-set remains indistinguishable from the original one. This is of course a version of the Indistinguishability Postulate and does not need to be introduced by force (as a postulate), resulting in $\mathfrak{Q}$ from the assumed indistinguishability of the involved elements.

In the theory of quasi-sets, we can do the following move. Suppose we have a quasi-set q with n indiscernible elements. Define a subquasi-set q' with $n-1$ of them, which is possible from the axioms; just consider the quasi-set $q \setminus [\![a]\!]_q$, where $a \in q$ and $[\![a]\!]_q$ is the *strong singleton* of a relative to q.[6] Then we can say that there is one element of q that has a characteristic not shared by the others, namely, to belong to the complement of q' relative to q. But due to their indiscernibility, we cannot point to one of them and say 'this one'.

3 Still on quantification over indiscernibles

So, quantification over collections of indiscernible objects demands a different logic, and a different way to look at the quantifiers. The

[6]In this case, since all elements of q are indiscernible, the n strong singletons are also indiscernible, but the postulates grant that there are n of them. The notation '$[\![a]\!]_q$' stands for a quasi-set with cardinal one and whose only element is indiscernible from a, but the theory is unable to prove that it is really a since there is no identity holding among them.

first question would be this one: why quasi-sets, that is, why go out of classical logic? The answer comes from the interpretation of the word ‘semantics’ as used in this context. We suppose that quantum mechanics has a certain underlying logic, where the meaning of the quantifiers is syntactically given, but we are trying to provide them with appropriate semantics, in the sense of considering what they say about an intended domain. Surely we can apply also to quantum languages what da Costa et al. said in general about semantics for non-classical logics (and we agree with them):

> …it is [important] to note that a set theoretical semantics [that is, a semantics grounded on a standard concept of set] for a non-classical logic (…) being constructed within classical set theory, (…) reveals itself, from the philosophical perspective, completely *unsatisfactory* (my emphasis). [6]

They continue

> [m]ore generally, the usual set-theoretical semantics, given the way it is articulated at present, depends on its underlying set theory: if one changes such a theory, the semantics itself is, ipso facto, changed. In particular, the same is the case for Tarski’s definition of truth.

We could add ‘…and, of course, the meaning of the quantifiers’ to the last expression. As we see, the choice of metamathematics is extremely relevant for formal semantic discussions and, further, a logical system (and this can of course be extended to scientific theories) cannot be understood out of (at least) an intended interpretation. Thus, in order to provide the right semantics for any reasonable quantum theory, we should accompany its account to indiscernibility, which is impossible to do within a standard framework.[7]

[7]We shall not enter in this important topic here; the resume is that standard mathematical frameworks, as we shall comment a little bit below, work with individuals, that is, with entities endowed with identity conditions, given by the standard theory of identity, which of course is not the case of quantum systems. Notice that one of the main characteristics of an individual is its re-identifiability, that it, it can be recognized later as being *that* individual from before; it is clear that this does not happen with quantum objects.

Just to state the problem with more precision, let L be a quite simple pure quantificational first-order language where $x_1, x_2, \ldots$ name the individual variables and P stands for a unary predicate.[8] Let $\mathfrak{A} = \langle D, I \rangle$ be a structure for L and let us suppose further that D is a quasi-set of indiscernible elements, that is, $s \equiv t$ for any $s, t \in D$, where '$\equiv$' is the indiscernibility relation, which has the properties of an equivalence relation. As we see, it is not enough to assume a collection of indiscernible entities, but it is necessary to say where such collections are living, and surely the place is not a standard set theory.

Our aim is to find conditions for both

(1). $\mathfrak{A} \models \exists x_i P(x_1)$, and

(2). $\mathfrak{A} \models \forall x_i P(x_1)$

and analyse if they collapse into one another. We remark that the two above conditions can be rewritten respectively as

1. $\mathfrak{Q} \vdash (\mathfrak{A} \models \exists x_i P(x_1))$, and
2. $\mathfrak{Q} \vdash (\mathfrak{A} \models \forall x_i P(x_1))$

meaning that (1) and (2) must be proven within $\mathfrak{Q}$. Let's recall that this is a typical way to indicate that what we need is a metamathematical proof, as for instance, it happens in Tarski's truth conditions, which depends on the metamathematics, as already indicated.

If D is a standard set (say of a theory such as ZFC), then the conditions for 'satisfaction', 'truth', 'model' and others are the usual ones [12]. The most interesting case is when D comprises only indiscernible elements (say a collection of bosons in the same quantum state).[9] In this case, we make use as a variable assignment a quasi-function s

[8] In a 'pure' quantificational language, the only terms are the individual variables. We don't lose generality in assuming this. Of course, we can generalize our simplified example; see [18].

[9] A Bose-Einstein Condensate is a typical case, obtained when certain atoms or other quantum entities are cooled to quite closer the absolute zero. As far as the temperature decreases, the wavefunctions of the elements become so that at that temperature they behave as a single thing, not presenting any differences at all. Although they are not the same entity, they cannot be discerned by any means. See [12].

that assigns elements of D to the individual variables. Before continuing, an important remark is in order here.

In $\mathfrak{Q}$, when indiscernible elements appear in the quasi-sets, we don't have a way to distinguish among the arguments and among the values for defining a function in the standard sense. So, the notion of quasi-function is introduced in order to cope with this. Intuitively, a quasi-function (q-function) between qsets A and B associates elements of B to the elements of A in such a way that if the elements in the domain are indiscernible, then their images are also indiscernible. More formally, if f is a q-function and $\langle a, b\rangle \in f$, $a' \equiv a$ and $\langle a', b'\rangle \in f$, then $b' \equiv b$. Of course that when the only element indiscernible from some object is the object itself (as in standard set theories), the definition coincides with the standard one. Let us further indicate that $s[x/d]$ says that an object referred to as d was assigned to the variable x. In $\mathfrak{Q}$, let us insist, if indiscernible elements are being considered, d doesn't designate a well-determined object, but an object whatever of a certain sub-collection of the domain (*one* electron, say, without specification of which one), which (of course!) cannot be discerned from those which are indiscernible from it. But, as it seems clear, the assignment s does not attribute every object (taken from a collection of indiscernibles) to x.[10] A further remark for clarifying something. We have said in the last footnote that the symbols of L obey classical logic. We cannot confound language with metalanguage of course. The theory $\mathfrak{Q}$ involves a 'classical' part where all standard mathematics can be performed. So, if the reader is questioning the assertion we made, she can consider that L is described in this 'classical' part of $\mathfrak{Q}$.

With these ideas in mind, we can state the notion of satisfaction. As in standard semantics, we associate a subset of D to the predicate P, let us call it S, that is, $I(P) \coloneqq S \subseteq D$.[11] The atomic formula $P(x)$,

[10]Notice that the assignment s is attributing $d \in D$ to x. Since every variable is indiscernible only from itself –we assume that the elements of the language obey classical logic–, there is just an element being associated with it, namely, d. The interesting fact is that we cannot put our finger over the element designed by d and say 'this one'; it is simply 'd', and we cannot identify it.

[11]Another subtlety: if D comprises indiscernible elements, we cannot characterize S except by indicating that its elements belong to D (or that are of the 'kind' of the elements of D), and its cardinality. Whatever 'other' subqset of D with the

for x indicating an individual variable, is satisfied in the structure $\mathfrak{A} = \langle D, I \rangle$ iff $s(x) \in S$ (that is, $d \in S$). Let us use again our preferred example to provide a motivation: suppose again a Sodium atom and take D as the qset of all its electrons. Let us define P as indicating that something satisfies P iff it is an electron of the orbital 2p of the atom. Then S is the qset of these electrons, and $\mathfrak{A} \models P(x)$ iff $s(x) \in S$, that is, iff the assignment given to x is an electron of the orbital 2p. Remember that the distribution of the electrons in the orbitals is made according to the electrons's quantum numbers n, l, ml and m_s (spin) **[?]**. The differences among these quantum numbers do not make the electrons different in the 'classical' sense, since it is impossible to know which group of electrons are in each orbital, something that *in principle* should be done in a classical setting. So, by considering the qset of the eleven electrons, we can take a formula which specifies the suitable quantum numbers for a certain orbital, say 2p, and by the Separation Schema of $\mathfrak{Q}$ we select the desired subqset with six electrons. This strategy can be generalized for other similar cases so that, going back to the generalities of logic, it makes sense to reason as if there are conditions for knowing whether the elements $s(x_i)$ fulfil adequate conditions to be members of the suitable qset, although we continue unable to know which elements would be them.

Of course, you can ask some questions, let us anticipate one of them. Question: how can we know that $s(x) \in S$ and not in the (say) 3s orbital? The answer goes as follows. It depends on the q-function s and the formula that defines the property ascribed by the predicate P. Such a formula says that the associated electron has quantum numbers typical of those that belong to the 2p orbital. Chemistry teaches us that there are at most six of them; so, in speaking of d, we just assume that it designates one, without the need of specifying which one.[12] Notice again that physics says that the electrons in the different orbitals have different quantum numbers. But this does not enable us to identify them except by these bunches of proprieties (that give

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same cardinality than S (that is, whatever subqset indiscernible from S) would act as the extension of P as well. This is an interesting trait of quantum mechanics: it subverts the standard meaning of intensions and extensions, enabling that one intension may have 'different' extensions; see [8].

[12] We could use Hilbert's ϵ symbol for that: $\epsilon x P(x)$ stands for *one* electron with such and such quantum numbers. See below footnote 20.

us the quantum numbers), but never individually. So, we are unable to say which electron is d, but just that it has some quantum numbers characteristics of those electrons in the required orbital. If, dear reader, you question this, are you able to distinguish from the others, say, the only electron in the orbital 3s? I think that we both should agree that this discourse is faithful to quantum physics.

Concerning this last point, we should be aware that the orbitals are not empty places to be fulfilled by electrons as if they were like a shelf in a library; about this, it would be interesting to pay attention to the remark made by Bruce Mahan:

> A word of caution concerning the interpretation of (...) "feeding" electrons into orbitals is in order. While it is useful to describe atoms qualitatively by saying that there are electrons "in" certain orbitals, and while it is sometimes helpful to think of atoms as being built up by "placing" electrons into a set of vacant orbitals, this language must not be taken too literally. The orbitals of an atom are not a permanent set of "boxes" rigidly placed on an energy scale (...). When we say an electron is "in an orbital" we are saying only that an electron is behaving in a certain manner, and in this sense an orbital exists physically only if an electron is "in" it. Moreover, each atom and ion has a unique set of energy levels determined by its nuclear charge and number of electrons. Consequently, the energy associated with a given orbital depends on what other orbitals are occupied, and is not the same for all atoms. [**?**, p.453]

Thus, we do not pick electrons and destine them to the orbitals, but the theory gives the conditions for the electrons to behave this or that way so that the orbitals are being 'fulfilled' (pardon for the expression) by the electrons according to the rules of chemistry. This is what we need to assume regarding quantum theories; when we say that there is one quantum so and so, we are just providing conditions for something of a kind to be so and so, and no identification would be advanced. This meshes with quantification.

4 Semantic conditions for quantification

Let us be a little bit more general now. Our object language is now a standard one for pure quantificational first-order logic [18]. Given an atomic formula[13] of the form $P^n(x_1, \ldots, x_i, \ldots, x_n)$ where P^n is an n-ary predicate and the free variables are among the x_k. In addition, let $\mathfrak{A} = \langle D, I \rangle$ be a structure for our language, where D is, again, a quasi-set of indiscernible elements,[14] and s is an assignment q-function for the individual variables. The valuation function I associates to the predicate P^n a subqset of D^n, regarding the ways $\mathfrak{Q}$ expresses that.[15] Notice again that, by the definition of quasi-function, just one element of D is associated with each variable;[16] you can express that by saying that the image of x is the element of a strong singleton $[\![d]\!]_D$, which has a cardinal one and whose only element is indiscernible from d.[17]

Definition 1 *Let $I(P^n)$ be the n-place q-relation that the interpretation associates to the predicate P^n. Then the sequence $(b_1, \ldots, b_i, \ldots, b_n, \ldots)$ of elements of D satisfies the atomic formula $P^n(x_1, \ldots, x_i, \ldots, x_n)$ iff $\langle s(x_1), \ldots, s(x_n) \rangle \in I(P^n)$. The sequence doesn't satisfy the formula otherwise.*

[13] Shall be enough to analyse this case, the most general ones can be got from extending the reasoning to formulas in general in the usual sense.

[14] It could be not necessarily so. We could assume that D comprises sub-collections of indiscernible elements which are discernible from the elements of another collection of indiscernible elements, and even that there could exist 'classical' elements in D, that is, elements which obey the rules of classical logic. In more precise terms, D is a quasi-set as described in $\mathfrak{Q}$.

[15] Since ordered tuples need to be understood adequately within the scope of $\mathfrak{Q}$; the details are not relevant here and the reader can reason as usual in standard set theories.

[16] Remember again what was said above that the elements of L obey classical logic. So, a variable x is an individual and the q-function s associates to it just one element $d \in D$, yet 'hidden', that is, yet we may be unable to discern it from others.

[17] Given a qset A and an element $x \in A$, we can form the 'unitary qset' of x as being the qset of all elements of A that are indiscernible from x. It is denoted $[x]_A$ and may have more than one element (that is, its cardinal, or 'quasi-cardinal', may be greater than one). We also call the 'strong singleton' of x the qset $[\![x]\!]_D$, which is a subqset of $[x]_A$ and has a quasi-cardinal one (for a proof of its existence, see [10, chap.7]).

The definition is precisely the standard one [15, p.48], but its interpretation is not (so as the interpretation of the quantifiers cannot be 'classical'). The important remark is to remember that $s(x_j)$ is not specifying a well-defined element of the domain, so things are left open in the specification of a which-is-which element; the only thing that is relevant is its kind (electron, proton, etc.). But a further remark is in order here, similar to that already made before, which we repute as necessary to fix the ideas. We know that the interpretation q-function associates to P^n a subqset $I(P^n) \subseteq D^n$ of the domain. The question is the following. Since the elements of D may be all indiscernible from one another, how can we grant that the tuple $\langle s(x_1), \ldots, s(x_n) \rangle$ belongs to $I(P^n)$? An analogy helps also here, and again we use our example of a Sodium atom. As we know, all the eleven electrons of the atom are indiscernible from the point of view of the theory (they have the same intrinsic properties) but they have different quantum numbers since they are fermions. So, the tuple will be in $I(P^n)$ if the elements $s(x_j)$ fulfil the conditions that specify the subqset $I(P^n)$ as already remarked above. This is in conformity with the way physicists seem to reason.

A predicate P^n that obeys this definition is invariant by exchange of indiscernibles, since if it holds, say, for one electron of the 2p level, it holds for all of them (in 2p). In other words, the atomic formula continues to be true in the structure even if $s(x_j)$ is exchanged by $s(x_k)$, provided that the corresponding elements are in the same orbital. This is similar to the Indistinguishability Postulate mentioned below. So, it seems that the substitution by indiscernibles would state a congruence in the sense that if a certain formula (in a simplified version) $\varphi(x)$ holds for some $d \in S \subseteq D$, it would hold for every $d' \equiv d$ as well.

But, before believing in this saying, two remarks are in order. Despite the similarities with the classical case, as remarked above, this is a different thing in the quantum domain. In quantum physics, one of the fundamental assumptions is the Indistinguishability Postulate (IP) [16], which says (roughly speaking) that the expected value of the measurement of an observable is the same before and after a permutation of indiscernibles [10]. What we have in (IP) is the accordance between results of measurement, and not a congruence strictly speaking. The second and more important remark is that the relation of indiscernibility of $\mathfrak{Q}$ is not a congruence, and this is precisely what distinguishes it from identity. So, we need to check if the formula we are using enables the substitution for indiscernibles; following Bourbaki (in the terminology only), let us call them *transportable formulas* [3]. Thus, we postulate that the formulas that quantum physics accepts as relevant for such issues are transportable in this sense (this can be sustained as a postulate for now, but of course demands a discussion which we intend to do in another work). But let me explain a little by showing that there are formulas that are not transportable, such as the relation of membership, which is not invariant by substitution of indiscernibles. In other words, there is nothing in the theory that grants that $x \in y$ and $z \equiv x$ entail $z \in y$. The proof is straightforward within $\mathfrak{Q}$ and roughly speaking goes as follows. Being x and z indiscernible elements of a qset D, we can form the strong singletons of them, namely, the qsets $[\![x]\!]_D$ and $[\![z]\!]_D$ having both quasi-cardinal 1 and comprising just one element indiscernible of (respectively) x and z. These qsets are also indiscernible from one another since their elements are indiscernible (this is entailed by the Axiom of (Weak) Extensionality of $\mathfrak{Q}$), but the theory doesn't entail that they are the same (identical), for this requires identity, and in $\mathfrak{Q}$, identity lacks a sense for some objects, as you surely know already.[18] Now we can answer our first question.

Definition 2 *We say that* $\mathfrak{A} \models \exists x_i P^n(x_1, \ldots, x_i, \ldots, x_n)$ *iff there exists an element* $s(x_i)$ *so that the sequence* $(b_1, \ldots, s(x_i), \ldots, b_n, \ldots)$ *of elements of* S *satisfies* $P^n(x_1, \ldots, x_i, \ldots, x_n)$*.*

Notice, again, that the definition is similar to the standard one in classical logic, but its interpretation is not. Really, the reader could reason as follows, thus accompanying Bueno's claim: if the formula is satisfied by some $s(x_i)$, it would be satisfied by every $s(x_j) \equiv s(x_i)$. We remark that this could be the case if '$\equiv$' were a congruence, but it is not. So, the invariance by substitution of indiscernibles doesn't hold for any predicate you chose but only for those that conduce to transportable formulas, so this conclusion cannot be assumed without qualification.

[18] These objects are term m-atoms; for them, expressions like $x = y$ are not well formed formulas when x or y stand for an m-atom.

In fact, recall once more, $s(x_i)$ is not a proper name for an object of the domain, since there are no proper names in the quantum realm; no label can act as a rigid designator [8], [10, p.224]. The reader should think of $s(x_i)$ (let us call it 'd') as a parameter, designating an element of a certain sort (an electron, say) without any commitment to identifying it. In classical first order logic, when we state that $P(d) \rightarrow \exists x P(x)$, d is a term whatever, free for x in $P(x)$,[19] so d of course designates a specific element of the domain (since in standard semantics all of them do have identity). But in $\mathfrak{Q}$ things are in general not so in the following sense: although there can be some element obeying the formula, not always we can identify it, name it; we just know that there is (at least) one, but not which one. As said earlier, d doesn't name (necessarily) a specific object, but *some* object in the domain.[20]

The condition for the universal quantifier can also be stated in the same terms, namely,

Definition 3 *We have that* $\mathfrak{A} \models \forall x_i P^n(x_1, \ldots, x_i, \ldots, x_n)$ *if and only if every element* $s(x_i)$*is so that the sequence* $(b_1, \ldots, s(x_i), \ldots, b_n)$ *satisfies* $P^n(x_1, \ldots, x_i, \ldots, x_n)$.

Notice (again!) that in saying (in the metalanguage) 'for every element $s(x_i)$', we are not conferring identity conditions to the element designated by it; this element is an element whatever of a subcollection of indiscernibles in the domain. If we were in a standard set theory, then of course we could reason as if this condition (being obeyed) would entail that the formula is satisfied for this, for that, for that one etc. elements of the domain, that is the 'for all' could be transformed in 'each one', something that apparently entails identity. This could not be different, since we are in a standard set theory where identity makes sense to every object. But we are not there: we are in $\mathfrak{Q}$, and here things act differently, as for sure the reader has understood already.[21]

[19] In our pure logic, we state this theorem as $P(y) \rightarrow \exists x P(x)$, where y is a variable distinct from x.

[20] Using Hilbert's epsilon symbol [2], perhaps we could write it as $\epsilon x P(x)$, but this needs to be analysed, for by obvious reasons (the lack of identity) the schema of extensionality $\forall x(A(x) \leftrightarrow B(x)) \rightarrow \epsilon x A(x) = \epsilon x B(x)$ doesn't hold.

[21] By the way, this shows that the criticisms advanced by da Costa and Bueno to

5 Summing up

We have seen above that the claim that the quantifiers collapse if indiscernible elements are considered cannot be sustained due (basically) to the following reasons:

1. The first remark is that you cannot even properly discuss any claim about indiscernible things (but see below) within a 'standard' framework such as the ZFC (or ZFA) system, for there are no indiscernibles there.[22] In such frameworks, there may exist only 'indiscernibles' made ad hoc as elements related by some equivalence relation. But this is just a trick to make them appear indiscernible. Furthermore, some 'indiscernibles', such as Ramsey's or Silver's [11, pp.298ff], are invoked in a different notion than that we are dealing with here. Notice that in such a framework, every object has identity; Leibniz's Principle of the Identity of the Indiscernibles holds in some way in classical logic and standard mathematics (see [10]).

2. Secondly, the theory $\mathfrak{Q}$ shows that we can vindicate mathematically something that is assumed in most interpretations of quantum theories, namely, as we have exemplified above, that we can speak (and make sense!) of one quantum object in a certain situation without providing it with an identity (but just a 'mock one', a 'false' identity as when we get a quantum trapped in some device. Momentarily, we can say that it 'has identity', but it loses such

the semantics of 'non-reflexive logics' [5] should be put within parentheses since they ground their semantic reasoning in a standard set theory, contrary to what is required. The right semantics should be constructed for instance in $\mathfrak{Q}$ according to their own requirements that a semantics would reflect the aims and presuppositions of the logic [5]. Non-reflexive logics are non-classical systems that depart from classical logic with respect to the theory of identity. A 'right' semantics should be developed within a set theory where identity is limited, as $\mathfrak{Q}$ does.

[22] Well, you could argue that in ZFA (the Zermelo-Fraenkel theory with atoms, entities that are not set but can be elements of sets) any permutation of atoms conduces to an automorphism of the whole universe. This strategy is useful for the construction of 'permutation models', where (inside the model) distinct atoms are made indiscernible by nontrivial automorphisms. Good, but this doesn't entail that the atoms cannot be discerned; in fact, in the whole universe of ZFA, given any atom a, we can always form the unitary set $\{a\}$ and distinguish it from any other element of the universe by the property 'to belong to the singleton of a', as seen already.

'identity' as soon is leaves the apparatus; so, it cannot be taken as an individual and its 'identity' is just a fake one). So, 'There exists x so-and-so' means precisely this: there exists something so-and-so, and we can reason with it without the necessity of identifying it. In the same vein, 'For all x so-and-so' means that every element of a certain domain is so-and-so, and this can be done without any reference to their identities.

3. Thirdly, the above discussion, conduced in an adequate mathematical framework, shows that the claim that 'for all' is equivalent to 'for each' (this meaning 'for this, for that, for that other, etc.' implying that we are able to provide an identification of the elements) is a false claim. A further example; in a BEC (Bose-Einstein Condensate), it is said that all elements (atoms, molecules, whatever form the BEC) behave as if they were just one thing (the 'big wave') [12]. Of course, we can speak this way and we really understand what it means, but we have no way to grasp the elements of the BEC one by one to fulfil the hypothesis of identification.

So, we can respond (and agree) with Bueno's claim that the relationships between quantification and identity should be not only formal [4] but we depart from him in that we should look to the meaning of quantification by precisely claiming that the understanding of quantification over a domain D means precisely this: 'For all x something' means exactly that for every $x \in D$ something happens, and we do not need to identify them; by the way, why should we? If I say that the COVID vaccine is available for anyone in a certain city, I don't need to name the inhabitants one by one. In the same vein, 'Exists x something' claims that the sub-collection of D of the objects that 'something' is non-empty, and we of course do not need to say which ones are them; if someone took the vaccine, then someone took the vaccine, endpoint. The identifications, of course, can be made, at least in principle, within a standard setting, but not in all domains. Quantum physics is a testimony that this is the case, and the theory of quasi-sets provides the stuff for describing such a case.

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