Born rule from counting states

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I give a very simple derivation of the Born rule by counting states from a continuous basis.

More precisely, I show that in a continuous basis, the contributing basis vectors are present in a state vector with real and equal coefficients, but they are distributed with variable density among the eigenspaces of the observable. Counting the contributing basis vectors while taking their density into account gives the Born rule without making other assumptions. This works only if the basis is continuous, but all known physically realistic measurements involve a continuous basis, because they involve the positions of the particles.

The continuous basis is not unique, and for subsystems it depends on the observable.

But for the entire universe, there are continuous bases that give the Born rule for all measurements, because all measurements reduce to distinguishing macroscopic pointer states, and macroscopic observations commute. This allows for the possibility of an ontic basis for the entire universe.

In the wavefunctional formulation, the basis can be chosen to consist of classical field configurations, and the coefficients $\Psi[\phi]$ can be made real by absorbing them into a global U(1) gauge.

This suggests an interpretation of the wavefunction as a nonuniform distribution of classical states. For the many-worlds interpretation, this result gives the Born rule from micro-branch counting.

Keywords: Born rule; state counting; Everett's interpretation; many-worlds interpretation; branch counting.

I. INTRODUCTION

In quantum mechanics, the Born rule prescribes the probability that the outcome of a quantum measurement is the eigenvalue λ_j of the observable is

$$\operatorname{Prob}(\lambda_j) = \langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle, \tag{1}$$

where the unit vector $|\psi\rangle$ represents the state of the observed system right before the measurement, and $\widehat{\mathsf{P}}_j$ is the projector on the eigenspace corresponding to λ_j.

The *projection postulate* states that $|\psi\rangle$ projects onto one of the eigenspaces $\widehat{\mathsf{P}}_j$ with the probability from (1).

von Neumann expressed already in 1927 the desirability of having a derivation of the Born rule "from empirical facts or fundamental probability-theoretic assumptions, *i.e.*, an inductive justification" [24]. Gleason's theorem shows that any countably additive probability measure on closed subspaces of a Hilbert space $\mathcal{H}$, $\dim\mathcal{H} > 2$, has the form $\operatorname{tr}(\widehat{\mathsf{P}}\widehat{\rho})$, where $\widehat{\mathsf{P}}$ is the projector on the subspace and $\widehat{\rho}$ is a density operator [13]. If the state is represented by $\widehat{\rho}$, this can be interpreted as the Born rule. Gleason's theorem is very important, in showing that if there is a probability rule, it should have the form of the Born rule. But it does not say that the density operator of the observed system is the same $\widehat{\rho}$, how the probabilities arise in the first place, and what they are about [9]. For example, it is unable to convert the amplitudes of the branches in the many-worlds interpretation (MWI) [7, 10, 22, 26] into actual probabilities. For this reason, the search for a proof of the Born rule continues.

There are numerous proposals to derive the Born rule. Earlier attempts to derive it from more basic principles include [12], [14], and others [11]. Such approaches based on a frequency operator were accused of circularity [5, 6]. Other proposals, in relation to MWI, are based on many-minds [1], decision theory [8, 18, 25] (also accused of circularity in [2, 3]), envariance [27] (accused of circularity in [19]), measure of existence [21] *etc.* For a review see [23]. The necessity to obtain the Born rule in MWI by branch counting was advocated in [17].

In this article I adhere to the following guideline:

Goal 1. Ideally, the Born rule should be obtained in the old-fashioned way, as *the ratio of the number of favorable outcomes to the total number of possible outcomes.*

I show that, in a continuous basis, it is possible to express the state vector as a linear combination of basis vectors of equal norm, but distributed unevenly. Then the probability density can be understood as a distribution of "classical" states relatively to that basis (Fig. 1).

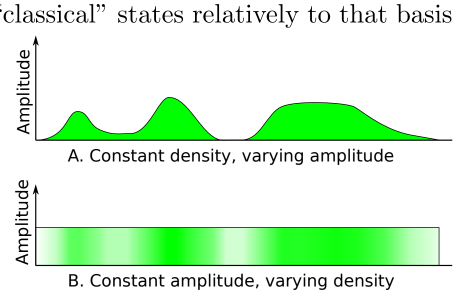

FIG. 1. **The Born rule from counting basis states.**
A. The usual interpretation of a wavefunction as a linear combination of basis state vectors of different norms.
B. The interpretation of the wavefunction in terms of equal norm basis state vectors, but with inhomogeneous density.

In Sec. §II I prove the main result. In Sec. §III I discuss its interpretation, how it makes possible the existence of a "classical" or ontic basis for the entire universe, how the wavefunction becomes real, and how this yields probabilities in the many-worlds interpretation.

II. COUNTING STATES

Before proving the main result, let us motivate it. Consider a state vector of the form

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |\phi_k\rangle. \tag{2}$$

where $(|\phi_k\rangle)_{k\in\{1,\dots,n\}}$ are orthonormal vectors from $\mathcal{H}$. Then, if every $|\phi_k\rangle$ is an eigenvector of the operator $\widehat{\mathsf{A}}$ representing the observable, the Born rule simply coincides with the counting of basis states:

$$\begin{aligned}\langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle &= \frac{1}{n}\left(\sum_{k=1}^{n}\langle\phi_k|\right)\left(\widehat{\mathsf{P}}_j\sum_{k=1}^{n}|\phi_k\rangle\right) \\ &= \frac{1}{n}\sum_{|\phi_k\rangle\in\widehat{\mathsf{P}}_j\mathcal{H}}\langle\phi_k|\phi_k\rangle = \frac{n_j}{n},\end{aligned} \tag{3}$$

where $\widehat{\mathsf{P}}_j$ is the projector of the eigenspace corresponding to the eigenvalue λ_j, and n_j is the number of basis vectors $|\phi_k\rangle$ that are eigenvectors for λ_j. This would satisfy Goal 1, but the state vectors of the form (2) are very special. In other words, this does not seem to work in general.

Interestingly, in the continuous case, the basis vectors can be distributed with nonuniform density, making it possible for the continuous version of eq. (2) to apply to any state vector. This motivates the following results.

Let $\mathcal{C}$ be a topological manifold with a measure μ on its σ-algebra, and $\mathcal{H} := L^2(\mathcal{C},\mu,\mathbb{C})$ the Hilbert space of square-integrable complex functions on $\mathcal{C}$. Let $(|\phi\rangle)_{\phi\in\mathcal{C}}$ be (formally) an "orthonormal" basis of $\mathcal{H}$, *i.e.*

$$\int_{\mathcal{C}} \langle\phi|\phi'\rangle\psi(\phi')d\widetilde{\mu}(\phi') = \psi(\phi) \tag{4}$$

for any square-integrable function $\psi\in\mathcal{H}$.

Without loss of generality, for any given state vector $|\psi\rangle$ so that $|\langle\phi|\psi\rangle|$ is μ-measurable, we can assume that $\langle\phi|\psi\rangle\in\mathbb{R}$ for all ϕ. If not, substitute the basis by $|\phi\rangle\mapsto e^{i\alpha(\phi)}|\phi\rangle$, where $\alpha(\phi)$ is the phase in the polar form of $\langle\phi|\psi\rangle$, for all $\phi\in\mathcal{C}$.

Theorem 1. *The state vector* $|\psi\rangle$ *has the form*

$$|\psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}(\phi), \tag{5}$$

where $\alpha:\mathcal{C}\to\mathbb{R}$, *and* $\widetilde{\mu}$ *is a measure on* $\mathcal{C}$ *specifying the density of the basis vectors* $(e^{i\alpha(\phi)}|\phi\rangle)_{\phi\in\mathcal{C}}$.

If $\mathcal{C}'\subseteq\mathcal{C}$ *is* μ*-measurable,*

$$\left|\int_{\mathcal{C}'} |\phi\rangle d\widetilde{\mu}(\phi)\right|^2 = \int_{\mathcal{C}'} r^2(\phi)d\mu(\phi). \tag{6}$$

Proof. Let $r(\phi) := |\langle\phi|\psi\rangle|$. Then, $r\in L^2(\mathcal{C},\mu,\mathbb{R})$ is a real non-negative square-integrable function, and

$$|\psi\rangle = \int_{\mathcal{C}} r(\phi)|\phi\rangle d\mu(\phi). \tag{7}$$

The measure $d\widetilde{\mu}(\phi) := r(\phi)d\mu(\phi)$ satisfies eq. (5):

$$|\psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}(\phi). \tag{8}$$

Since $r(\phi)$ is μ-measurable, the measure $\widetilde{\mu}$ is absolutely continuous with respect to μ.

Then,

$$\begin{aligned}\left|\int_{\mathcal{C}'} |\phi\rangle d\widetilde{\mu}(\phi)\right|^2 &= \left(\int_{\mathcal{C}'}\langle\phi|d\widetilde{\mu}(\phi)\right)\left(\int_{\mathcal{C}'}|\phi'\rangle d\widetilde{\mu}(\phi')\right) \\ &= \int_{\mathcal{C}'}\left(\int_{\mathcal{C}'}\langle\phi|\phi'\rangle d\widetilde{\mu}(\phi')\right)d\widetilde{\mu}(\phi) \\ &= \int_{\mathcal{C}'}\left(\int_{\mathcal{C}'}\langle\phi|\phi'\rangle r(\phi')d\mu(\phi')\right)d\widetilde{\mu}(\phi) \\ &= \int_{\mathcal{C}'} r(\phi)d\widetilde{\mu}(\phi) = \int_{\mathcal{C}'} r^2(\phi)d\mu(\phi).\end{aligned} \tag{9}$$

This proves eq. (6). □

Therefore, the density $\widetilde{\mu}$ of the basis states corresponds to the Born rule, according to Goal 1.

This suggest an interpretation of the wavefunction that will be discussed in the next Section.

III. INTERPRETATION OF THE WAVEFUNCTION

For any physically realistic quantum measurement there is a continuous basis in which the observable is diagonal, as required by Theorem 1. Even for a single particle in nonrelativistic quantum mechanics, the Hilbert space is infinite-dimensional, and admits continuous bases, *e.g.* the position basis. In general, measurements reduce to position measurements. For example, to measure the spin of a particle, the Stern-Gerlach device uses a magnetic field to entangle the spin with the position, and then the position is recorded. From the position, the spin is inferred. The pointer of the measuring device indicates the result by its position or displays a result that depends on a position. For a photographic plate we read the position where the particle hit it *etc.* In practice, these are not exact measurements of position, but of regions of space of positive measure. Therefore,

Observation 1. All measurements satisfy, in practice, the conditions from Theorem 1.

Subsystems admit observables that cannot be diagonalized simultaneously, so the continuous basis depends on the observable.

However, every measurement ultimately becomes a direct observation of a macro-state, the state of the pointer of the measuring device. So every measurement reduces to distinguishing macro-states. *Macro-states* are represented by subspaces of the form $\widehat{\mathsf{P}}_\alpha\mathcal{H}$, where $(\widehat{\mathsf{P}}_\alpha)_{\alpha\in\mathcal{A}}$ is a set of commuting projectors on $\mathcal{H}$, so that $[\widehat{\mathsf{P}}_\alpha,\widehat{\mathsf{P}}_\beta]=0$

for any $\alpha \neq \beta \in \mathcal{A}$, and $\bigoplus_{\alpha\in\mathcal{A}} \widehat{\mathsf{P}}_\alpha \mathcal{H} = \mathcal{H}$. Since ultimately every measurement translates into an observation represented by the macro projectors, there is a universal continuous basis for all measurements, which diagonalizes all macro projectors.

Observation 2. For the entire universe, there is a universal continuous basis compatible with macro-states.

Since different measurement settings ultimately translate to distinguishing macro-states defined by the same set of macro projectors, the ontic basis is consistent with any observable we measure for the subsystem.

Therefore, this universal basis can be taken as representing "classical states", which may be called *ontic states*. Theorem 1 allows us to interpret the Born rule for any measurement as "counting" such ontic states.

It may seem too much to count states of the entire universe just to account for the probabilities of the measurement of a single particle. But in fact we always do this, because the observed particle can be entangled with any other system in the universe. The usual separation between the observed system and the rest of the universe that enters in our theoretical description is an idealization that may make us not the forest for the trees. Then,

Observation 3. The state of the universe is not a set of independent states of subsystems, but as a single state.

But what are these ontic states? Since each particle is represented on a Hilbert space of wavefunctions that have, among their degrees of freedom, the positions, which play a role in any measurement, and also form a continuous basis, it may be tempting to interpret the ontic states as position eigenstates. But we know that in fact the world is not described by nonrelativistic quantum mechanics, but by quantum field theory.

A unique basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ that really is ontic or classical is possible in quantum field theory. In the Schrödinger wavefunctional formulation of quantum field theory [15, 16], $\mathcal{C}$ becomes the configuration space of classical fields, and the Schrödinger *wavefunctional*

$$\Psi[\phi] := \langle\phi|\Psi\rangle \tag{10}$$

replaces the nonrelativistic wavefunction. Here, ϕ stands for a collection of classical fields, $\phi = (\phi_1, \ldots, \phi_n)$.

It is unusual to interpret quantum mechanics in this way. For some reason, the nonrelativistic framework is widely used in the foundations of quantum mechanics, as a benchmark to test various interpretations.

The wavefunctional formulation represents quantum states in terms of classical field states, in the sense that the wavefunctional is a complex functional defined on the configuration space of classical fields. The usual Fock representation can be obtained from the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ [15]. The Fock representation can then be used to interpret the quantum fields in terms of more commonly used nonrelativistic quantum mechanical wavefunctions and operators. But this is a departure from the more foundational description provided by quantum fields.

We never observe individual particles directly, but only macro-states. Macro-states are imported from the classical theory, and they are appropriate, because at the macro level the world looks classical. All macro projectors $(\widehat{\mathsf{P}}_\alpha)_{\alpha\in\mathcal{A}}$ commute.

Observation 4. At any instant, at the macro level, a classical world in the classical state ϕ looks the same as a quantum world in the quantum state $|\phi\rangle$ or a linear combinations of such states from the same macro-state.

And indeed, it took us a very long time to realize that our world is not classical, but quantum.

Therefore, it makes sense to assume that states of the form $|\phi\rangle$ belong to macro-states, *i.e.* for every $|\phi\rangle$ there is a macro-state $\widehat{\mathsf{P}}_\alpha\mathcal{H}$ so that $|\phi\rangle \in \widehat{\mathsf{P}}_\alpha\mathcal{H}$.

But if we look back at eq. (5), we recall that its form is based on absorbing the phase factor in the vector by substituting $|\phi\rangle \mapsto e^{i\alpha[\phi]}|\phi\rangle$, done just before stating Theorem 1. This substitution depends on the state $|\Psi\rangle$, in particular $\alpha[\phi]$ changes in time. So we cannot simply interpret $|\Psi\rangle$ directly as a set of classical states distributed according to the density from eq. (5).

But the phase change $|\phi\rangle \mapsto e^{i\alpha[\phi]}|\phi\rangle$ can be identified with an U(1) gauge transformation of the classical field, $\phi \mapsto e^{i\alpha[\phi]}\phi$, so that

$$e^{i\alpha[\phi]}|\phi\rangle \equiv |e^{i\alpha[\phi]}\phi\rangle. \tag{11}$$

This makes sense because (1) multiplying a state vector with a phase factor changes the vector, but not the physical (quantum) state it represents, and (2) an U(1) gauge transformation of a classical field represents the same physical (classical) state.

Charged and spinor fields, and electromagnetic potentials, admit an U(1) symmetry, so we can apply this treatment to these classical fields.

Then, $\Psi[\phi]$ *can be made real by changing the global* U(1) *gauge of the basis of classical fields*, and the wavefunctional $|\Psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}[\phi]$ can be interpreted directly as a set of gauged classical fields distributed according to a density functional.

The gauge transformation depends on the state $|\Psi\rangle$, so it also changes in time. Then,

Observation 5. The wavefunctional $\Psi[\phi]$ can be understood as a set of classical fields with different densities and gauges that change in time according to the Schrödinger equation.

There are several benefits in using this interpretation of the wavefunctional as starting point in the investigations of foundations of quantum theory. It is more foundational, since quantum field theory is more foundational than nonrelativistic quantum mechanics. It has embedded an ontology. Each state $|\phi\rangle$ corresponds to a set of fields defined on the 3d-space, not on the configuration space. These fields are the *local beables*, whose necessity was advocated by Bell [4]. The Born rule can be interpreted in terms of such ontic states. The foundational literature is overwhelmingly dominated by the nonrelativistic framework, and still does not have all the answers. Basing the foundational research on the wavefunctional may give a better perspective.

A state does not consist of a single ontic state, but of a set of such states (Observation 5). The projection postulate should not be understood as collapsing the system to a basis state $|\phi\rangle$, no measurement can extract the complete information about the state of the entire universe. All ontic states making $\Psi[\phi]$ belonging to the resulting macro-state $\widehat{\mathsf{P}}_\alpha\mathcal{H}$, and only them, should remain after the projection is invoked.

But if decoherence makes the components of $\Psi[\phi]$ corresponding to different macro-states no longer interfere, there is no need to invoke the projection postulate, and we can adopt the *many-worlds interpretation* (MWI). However, "naively" counting the worlds or macro-branches gives the correct probabilities only if the state has the form (2) in the eigenbasis of the observable.

Observation 6. Counting micro-branches that correspond to the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ gives the correct probabilities in MWI, in accord with Goal 1 (even if they may interfere in the future, unlike the macro-branches). Moreover, since each micro-branch consists of classical fields ϕ, and since these are the local beables, it becomes justified to count each micro-branch as a world [20].

Observation 7. We should also include quantum gravity in our foundational investigations of quantum theory. In background-free approaches to quantum gravity, it becomes impossible to interpret physically all linear combinations as superpositions, because states in which the geometry of space is different cannot be superposed, so the ontic states dissociate automatically [20]. They can reassociate, unless the separation reaches the macro level. This provides an additional justification for the many-worlds interpretation (in a revised form).

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