The Π-Theorem as a Guide to Quantity Symmetries and the Argument Against Absolutism*

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Abstract

In this paper a symmetry argument against quantity absolutism is amended. Rather than arguing against the fundamentality of intrinsic quantities on the basis of transformations of basic quantities, a class of symmetries defined by the Π-theorem is used. This theorem is a fundamental result of dimensional analysis and shows that all unit-invariant equations which adequately represent physical systems can be put into the form of a function of dimensionless quantities. Quantity transformations that leave those dimensionless quantities invariant are empirical and dynamical symmetries. The proposed symmetries of the original argument fail to be both dynamical and empirical symmetries and are open to counterexamples. The amendment of the original argument requires consideration of the relationships between quantity dimensions. The discussion raises a pertinent issue: what is the modal status of the constants of nature which figure in the laws? Two positions, constant necessitism and constant contingentism, are introduced and their relationships to absolutism and comparativism undergo preliminary investigation. It is argued that the absolutist can only reject the amended symmetry argument by accepting constant necessitism. I argue that the truth of an epistemically open empirical hypothesis would make the acceptance of constant necessitism costly: together they entail that the facts are nomically necessary.

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1 Introduction

There is an old question which has recently gained renewed and generalized attention; most famous is Poincaré's statement of this question regarding space:

> Suppose that in one night all the dimensions of the universe became a thousand times larger. The world will remain *similar* to itself, if we give the word *similitude* the meaning it has in the third book of Euclid. Only, what was formerly a metre long will now measure a kilometre, and what was a millimetre long will become a metre. The bed in which I went to sleep and my body itself will have grown in the same proportion. when I wake in the morning what will be my feeling in face of such an astonishing transformation? Well, I shall not notice anything at all. The most exact measures will be incapable of revealing anything of this tremendous change, since the yard measures I shall use will have varied in exactly the same proportions as the objects I shall attempt to measure. In reality the change only exists for those who argue as if space were absolute. (Poincaré, 1914, 94)

It is apparent that such considerations generalize to other quantity dimensions beyond the spatial ones. There is an apparent paradox: Everywhere in the laws of physics it appears that solutions depend on the absolute values of quantities. Yet, there is also an intuition behind thought experiments like Poincaré's: if all quantities of a kind were scaled by the same factor, *including those of the relevant measurement standards*, that world would be in every way empirically indistinguishable from the actual world. This paper provides a reconciliation of the absolutist form of the laws and comparativist intuitions about measurement.

A case which has recently captured the attention of some philosophers: would it make a difference if all the masses doubled overnight? The answer turns on a metaphysical debate regarding quantity absolutism and quantity comparativism:

> (Absolutism) Intrinsic quantities are fundamental, qualitative properties, quan-

tity relations supervene on them.[1]

(Comparativism) Quantity relations are at least as fundamental as intrinsic quantities and do not supervene on them.[2]

Intrinsic quantities are determinate properties of particular physical objects and not relations. We think of them as having an *essentially* monadic logical form.[3] An object's property of being two kilograms in mass is intrinsic. Alternatively, the comparativist grounds the object's being two kilograms in mass relationally: the object stands in a relation of being twice as massive as, say, some standard kilogram in Paris. The comparativist holds that these relations are not grounded in intrinsic quantities, but are (relatively) fundamental.

Central to the debate is a symmetry argument against absolutism.[4] Such arguments have a general form: some supposed fundamental feature of reality, F, varies under some symmetry transformation; so F is not a fundamental feature of reality. In this case the supposed fundamentals are intrinsic or absolute quantities. The comparativist argues that there is a class of symmetries that leave quantity ratios invariant while varying intrinsic quantities. If this is the case, a supervenience principle follows:

(Comparativist Supervenience) No change in intrinsic quantity Q of object O

[1] Maybe this is better put in terms of "dependence", "determination", or "grounding" (see discussion in: McKenzie, 2014, 2020; Sider, 2020). I am not here concerned with whatever the proper relation between fundamentals and non-fundamentals is, just that there is some distinction to be drawn which at least implies a supervenience relation describable with possible world semantics.

[2] Here I focus on arguments against the fundamentality of intrinsic quantities. This corresponds to the weak absolutism and weak comparativism described in Martens (2021). Some, including Dasgupta (2016), have presented the argument against absolutism in eliminativist terms. The argument is taken to show that intrinsic quantities comprise surplus structure which ought to be eliminated from our ontology. See Ismael and van Fraassen (2003) and Dasgupta (2016) for accounts of such symmetry arguments. See Martens (2018) for an argument against mass eliminativism. See Sider (2020) and Wolff (2020, chap. 8) for accounts of the absolutist-comparativist dispute in terms of fundamentality.

[3] For an example of a relatively standard metaphysical account of intrinsicality, see Langton and Lewis (1998). See Sider (1996) for a distinction between metaphysical and syntactic criteria of intrinsicality and a discussion of the latter type.

[4] Martens (2021, 2523) usefully distinguishes three approaches to the debate regarding the empirical adequacy of comparativism which are present in the literature: (1) the symmetry approach, (2) the detectability approach, and (3) the possibility-checking (i.e. possible worlds) approach. He only discusses (1) in passing and shows that insofar as (2) is useful it is equivalent to (3). Further, I take (1) to be equivalent to (3). I understand possible world semantics to provide a model theory for discussing the symmetries of physical equations. However, as will be made evident, I believe the symmetry approach in is some ways more illuminating and useful for some of the unsettled modality questions (see section 5).

without some change in relation R between Q and Q' of some O'.[5]

The relevant symmetries are *physical* symmetries which map physical systems to physically indistinguishable systems. Well known examples of such symmetry arguments include the argument against absolute velocities due to velocity boost symmetries and arguments against absolute space due to translation and rotation symmetries.

Dasgupta (2013) has influentially levied such a symmetry argument against the fundamentality of intrinsic quantities. The argument depends on a notion of mass doubling as a transformation that doubles the mass of every massive object in some physical system and *leaves everything else unchanged*. This *ceteris paribus* condition requires mass doubling be a "full symmetry", meaning a *dynamical* and *empirical* symmetry. Dynamical symmetries map nomically possible systems to nomically possible systems. Empirical symmetries map systems to observationally indistinguishable systems. The argument runs so: Mass doubling is a full symmetry. Intrinsic mass quantities vary under this full symmetry. Properties that vary under full symmetries are not fundamental. Therefore, intrinsic mass quantities are not (relatively) fundamental.

Baker (2020) and Martens (2018, 2021) have shown that this argument is unsound because its first premise is false. Counterexamples, such as a two body system in which a projectile escapes a planet, show that Dasgupta's *ceteris paribus* clause is untenable. Either the mass doubling transformation changes the empirical situation because the projectile fails to escape, or the projectile's trajectory breaks the laws. Mass doubling is not a full symmetry. Further, any other global transformation which acts on a single basic quantity dimension is not a full symmetry.[6] By contrast, I refer to any quantity transformation that leaves both the laws and the observable situation unchanged a *full* quantity symmetry. That all basic quantity

[5]The comparativist likely is committed to more than this, but this minimalist principle excludes absolutist possibilities, e.g. mass doubling. Dasgupta puts this principle somewhat differently. For him the principle justified by these global quantity symmetries is a global supervenience principle. The important thing is whatever set of facts are more fundamental are not explained by the other set of facts (Dasgupta, 2013, 108-9). Dasgupta actually makes the case that his "pluralistic" grounding is less demanding than the individualistic condition I am stipulating.

[6]I assume throughout this paper that mass, length, and time are the basic quantity dimensions of mechanics. This is a standard convention but is inessential to the argument—any adequate basis will do.

transformations fail to be full quantity symmetries will be made clear by consideration of the Π-theorem.

The argument against absolutism can be rehabilitated by showing that there is a class of full quantity symmetries that shows that intrinsic quantities are not fundamental. This class of symmetries is characterized by Edgar Buckingham's (1914) Π-theorem, a foundational result of dimensional analysis. This theorem establishes a general form of physical equations which is invariant under both representational unit transformations and ontic quantity symmetries. The general structure of these quantity symmetries has implications for the nomological role of the dimensional constants. I will argue that whether these quantity symmetries are accepted as *dynamical symmetries* depends on whether or not the values of physical constants (e.g. the gravitational constant) are fixed by the laws or are instead contingent. The absolutist's escape route from the amended symmetry argument requires that the values of physical constants are nomically necessary. I argue that this is a costly move, conditional on the truth of an open physical hypothesis.

2 The Quantity Calculus and the Argument Against Absolutism

The argument against absolutism requires the existence of symmetries that are ontic counterparts to a class of broadly accepted representational quantity symmetries, unit transformations. It is necessary that these ontic quantity symmetries are both dynamical symmetries and empirical symmetries. The relationship between these two classes of symmetries will be made clear by an explanation of the quantity calculus.[7]

As terminology varies I will establish my vocabulary with a tripartite distinction:

(Quantity) A property of a physical object that is representable by class of a

[7]See de Boer (1995) for a history of the development of the quantity calculus. See JCGM (2012) for the contemporary metrological standard, which I am broadly in line with.

number-unit pairs, usually by a number multiplied by a unit (e.g. my quantity of height is approximately 1.854 meters or $1.854 \times \mathrm{m}$);

(Quantity Dimension) A collection of quantities which are all representable by the same set of units, i.e. commensurable (e.g. my quantity of height, yours, the length of route 66, an Angstrom);

(Unit) A standard magnitude of quantity in some dimension whose assignment to the numerical representation 1 induces numerical values to all quantities in that dimension (e.g. the standard lengths defined by the meter stick, the foot of Julius Caesar, the distance a beam of light travels in a vacuum in one second).

I have defined these terms circularly—My aim here is not an analysis but a specification of their relations so as to avoid confusion.

The representation of any quantity as a product of a number (synonymously: value, magnitude, measure)[8] and a unit informs us that these quantities of concern exist on a ratio scale. Further we keep track of the units of some derivative quantity by not only performing algebraic operations on the numerical representations of quantities but also on their units, e.g. $\frac{5 \times meters}{2 \times seconds} = 2.5 \times \frac{meters}{seconds}$. We will see that the algebra of units obeys a necessary condition on the wellformedness of physical equations—dimensional homogeneity. This necessary condition has to do with the dimensions of which each unit instantiates, e.g. the dimensions of force: $[N] = [dyn] = \mathrm{MLT}^{-2}$. Complex, derivative dimensions are constructed from products of powers of basic dimensions, usually M, mass, L, length, and

[8]This may seem odd to some. Russell (1903) roughly used magnitude for what I here call quantity, but his usage is still consistent with that of the literature since—at least the physics literature with which I will largely be concerned—the phrases "the magnitude of a quantity", "the value of a quantity", "the measure of a quantity" are all felicitous and equivalent. They all refer to its numerical extent relative to some defined unit. See Berberan-Santos and Pogliani (1999) for a useful discussion and formalism. One major caveat is that "magnitude" is sometimes contrasted with "value" or "measure" as referring to the objective, unit-independent extent or size of a quantity, as is done in the definition of "unit" above (for a metaphysics agnostic definition of unit-independent magnitudes see Tal, 2021). Context will make it clear when the "magnitude" or "value" of a quantity is meant in a unit-relative or unit-free way.

T, duration.[9] Any quantity has a dimensionality or dimension, $[Q] = \mathrm{D}$, which can be multiplied and divided arbitrarily, e.g. $[Q_1 \times Q_2] = [Q_1] \times [Q_2] = \mathrm{D}_1 \times \mathrm{D}_2$.[10] Consider $[F] = [m] \times [a] = \mathrm{MLT}^{-2}$. However, only quantities of like dimensions can be summed or subtracted. In other words, if $k_1Q_1 + k_2Q_2 = Q_3$ is coherent, then $[Q_1] = [Q_2] = [Q_3]$. That the terms of a physical equation must have equal dimension is *dimensional homogeneity*.[11] Intuitively, it makes no sense to add a length to a force, etc. The dimensionality of a dimensionless quantity (i.e. a number) is $[1]$, which is the identity—for a quantity Q of arbitrary dimension $[Q] \times [1] = [Q]$.[12] The product of a quantity of some dimension and another of inverse dimension is dimensionless: $[Q] \times [Q]^{-1} = [1]$.

Equations are mere representations of relations between quantities, which are themselves "worldly".[13] Relations between quantities are physical systems, and equations are their representative counterparts, mathematical models. Quantities are either represented by variables associated with dimensions or numbers associated with units. Whether or not the units or dimensions of the quantities in some equation are literally represented, the structure of a dimensional system (or, derivatively, that of a unit system) determines the possible forms of any quantitative equation. As quantitative equations represent physical systems, the structure of a dimensional system determines what relations among the quantities themselves are possible: the algebra of dimensions mirrors the algebra of quantities. If such a dimensional system is at hand, any definable system of units on those quantity dimensions is *coherent*. That we are dealing with such dimensional or unit systems is a foundational assumption of dimensional analysis and the source of its utility. As Sterrett has it:

[9] Outside of mechanics, additional dimensions for electrical charge, C and for temperature, Θ, are introduced. I will only deal with mechanics in this paper for simplicity.

[10] The square brackets denote the dimensionality extraction function. Basic dimensions will be denoted by un-italicized letters, like L for length (and 1 in the case of numbers). Products of powers of these basic dimensions are the values of the $[X]$ function. Tolman (1917) and Dewar (Forthcoming) give more sophisticated accounts of the operations of the quantity calculus.

[11] We owe this formulation of the principle to Fourier (1878), see De Clark (2017).

[12] Quantity dimensions form an Abelian group. The formal properties of dimensions deserves a much more thorough discussion. See Raposo (2018, 2019) for some details and a fiber bundle model.

[13] My usage here is at odds with Martens (2021), for whom "quantity" refers to the representation and not the physical property.

> Thus, if it is known that the system of units is coherent, it follows that the numerical relation has the same form as the fundamental [dimensional] relation. *The form of the numerical equation can be known independently* of actually using units and numerical expressions to express the quantities and then deriving the numerical equation from the quantity equation—so long as the requirement that the system of units is coherent is met. (Sterrett, 2009, 806)

This is what generates the representational-ontic symmetry duality described below, which is essential to my argument (see 4.1).

The numerical representations of quantities are determined by the system of units used. We understand a unit system as a collection of maps from physical quantities to numerical representations. Each particular unit system partitions the physical quantities into equivalence classes independent of the particular homomorphism it adopts, e.g. mass-in-grams vs mass-in-kilograms. These unit systems are related by two kinds of isomorphisms—those that act on the quantities themselves and those that act on the unit system mappings. Distinguishing the quantities from their representatives, we can define two classes of symmetry transformations:

> (Representational Symmetries) Transformations on the assignment of numerical representatives to quantities that leaves the quantities and their ratios unchanged, e.g. unit system transformations.

> (Ontic Symmetries) Transformations on the quantities themselves which change the numerical representatives of quantities for any given unit system, leaving their ratios unchanged, e.g. universal velocity boosts.[14]

Representational symmetries are transformations of mere representation. Ontic symmetries are transformations of physical systems.

[14] The ontic-representational distinction corresponds to the active-passive distinction that some may be familiar with.

The comparativist holds that quantity ratios are more fundamental than intrinsic mass quantities, owing to their invariance under *ontic* scale transformations. The (naive) comparativist argues that scale transformations of basic quantity dimensions, like mass doubling, are full symmetries:

> (Comparativist Commitment) Basic quantity (ontic) scale transformations are full symmetries.[15]

The absolutist rejoinder shows that basic mass doubling cannot meet both criteria required of a full symmetry, so it is important to distinguish the two conditions:

> (Empirical Symmetry) An empirical symmetry is a map from one physical system to another that leaves unchanged all observable phenomena, i.e. it takes a system and generates an observationally indistinguishable system.

> (Dynamical Symmetry) A dynamical symmetry is a map from one lawful physical system to another lawful physical system, i.e. a transformation that leaves the laws invariant.[16]

Again, a full symmetry, one that justifies the variance-to-unreality inference used by the comparativist, *must* be both dynamical and empirical.

An explicit version of the argument against absolutism can be stated:

(1) If quantity Q is variant under some full symmetry then Q is not fundamental.

[15] The range of anti-absolutist views includes more than just comparativism. To accept that these scale transformations are full symmetries only requires the denial of intrinsic mass quantity *quiddities*. The denial of quiddities can be accommodated by multiple views. Most weakly it implies a sophisticated substantivalism (Wolff, 2020). More strongly there would be no quiddities if there were no intrinsic quantities at all—or at least no objective facts about them, as in a relationalist view (Dasgupta, 2020). There are also a variety of comparativisms on offer, as developed by Martens (2017, 2018, 2020).

[16] In this context, a criterion of a dynamical symmetry is that that the application of transformation to a system *commutes* with the lawful time evolution of the system. See Ismael and van Fraassen (2003), Roberts (2008), and Wigner (1979) for discussions of the relation between these two classes of symmetries.

(2) Mass doubling is an empirical symmetry: If all of the mass quantities were doubled there would be no observable difference.[17]

(3) Mass doubling is a dynamical symmetry.

(4) Mass quantities are variant under a full symmetry. (2, 3)

(5) Mass quantities are not fundamental. (1, 4)

Premise (1) of this argument is a form of the more general variance-to-nonfundamentality (or unreality, or non-objectivity) principle commonly accepted by physicists and philosophers alike. Premise (3) is a posit that there is a comparativist paraphrase of the laws that is genuinely Newtonian but is indifferent to mass doublings. The problem raised against this argument is the inconsistency of (2) and (3). Generally counterexamples to this argument are taken to target the empirical symmetry premise, (2), but my presentation below will focus on how the counterexamples can be used to bring more focus to (3). By showing that the problem with the argument against absolutism is the misclassification of a basic quantity transformation as a full symmetry, I provide an argument against absolutism that is immune to counterexample.

3 Baker's Counter Example

Baker (2020) presents a counter example to comparativism, showing that mass doubling is not a full symmetry.[18] Consider a two body system: a projectile traveling with velocity v_{pro} away from a planet's surface. From Newton's laws we can derive an equation for the critical

[17]This argument is meant to directly parallel arguments against the existence of absolute velocity, see Dasgupta (2013, 2016). Crucially this argument depends on absolute mass, and some class of physical quantities more generally, not being observable. Roberts (2008) and Dasgupta (2016) cash this out in terms of the impossibility of constructing absolute quantity detectors. Both parties to the debate tend to accept that absolute quantities like mass are not directly observable. For criticism of the detectability interpretation see Martens (2021, 2540-44). In light of this we might drop the observable adjective and say that empirical symmetries leave all the qualitative facts unchanged (see Russell, 2014).

[18]Martens (2018, 2021) has shown this to be only one instance of a broader class of counterexamples in which two or more particles either collide or escape each other, depending on their absolute masses. These are all equivalent for my purposes.

escape velocity such that if $v_{pro} > v_{escape}$, the projectile will escape the orbit of the planet:

$$\text{(Escape Velocity) } v_{escape} = \sqrt{\frac{2GM}{R}},$$

where G is the gravitational constant, M is the mass of the planet, and R is its radius. Note that the equation for the escape velocity depends only on the mass of the planet and not the projectile mass. On Earth, the M and v_{pro} are such that the projectile escapes. In the mass doubled counterpart system, the planet Pandora's mass is such that the projectile does not escape. The sticking point is that the comparativist sees initial states of the Earth and Pandora systems as *empirically equivalent*. So the comparativist cannot hold that the initial state of the two body system has a unique future as determined by the laws.

Baker presents this counterexample to comparativism as showing that comparativism introduces indeterminism into deterministic systems. A different presentation will better serve our purposes: the counterexample generates an inconsistent triad. The three inconsistent propositions are:

(a) The initial states of the Earth and Pandora systems are indistinguishable;

(b) The final states of the Earth and Pandora systems are indistinguishable;

(c) The dynamics are left invariant by the transformation that maps the Earth system to the Pandora system.

The first two propositions follow from mass doubling being an empirical symmetry, the third from mass doubling being a dynamical symmetry. If mass doubling is an empirical symmetry, then it cannnot be a dynamical symmetry: the trajectory required is inconsistent with the escape velocity equation. If mass doubling is a dynamical symmetry it cannot be an empirical one: either the initial state or final state of the system must be changed for the position of the projectile to match the Earth case at the opposite temporal state (final or initial) while having a trajectory consistent with the laws. Generally the issue has been characterized as one of empirical adequacy or indistinguishability, this presentation highlights the sometimes implicit assumption that the relevant empirical symmetries are a subset of the set of dynamical symmetries.[19] Mass doubling cannot be both an empirical and a dynamical symmetry, so it is not a full symmetry.

4 The Argument Against Absolutism Redux: Lessons from Dimensional Analysis

Dimensional analysis depends on a number of basic (though not totally uncontroversial) assumptions. An account of these assumptions provides a route to a proof of the Π-theorem. The first assumption is that all of the basic quantities that figure in equations which are representationally adequate have a ratio scale structure. This means that all quantities of a basic dimension can be related by a scalar multiplication operation of the form $f : x \mapsto \mathbb{R}x$. Mass, length, and time all share this structure and so are suited to form the dimensional basis for mechanics. This can be attributed to the fundamental idea that these are all extensive quantities, i.e. the magnitude of a whole is an additive function of the magnitudes of its parts.[20] By treating these quantities as *basic* we are treating them as building blocks from which all other quantities are defined.

In a "complete" system of units (or dimensions) the derived quantities inherit some properties from the basic quantities which define them, i.e. they too exist on a ratio scale. The ratio scale structure of the mechanical quantities defines a group of unit transformations:

> (Unit Transformation) For any quantity $Q = V \times U$, there is a class of maps $Q \mapsto Q'$, $U \mapsto U'$, $V \mapsto V'$, such that $U' = xU$, $V' = x^{-1}V$, $Q' = Q$, where $x \in \mathbb{R}^+$.

[19] Martens makes this explicit. The dynamical condition is pronounced even in the guise of the possibility checking approach: "Comparativism should provide at least one metaphysically distinct (and dynamically allowed) possible world for each empirically distinct possible world allowed by absolutism. If the metaphysically distinct worlds that comparativism acknowledges fail to differentiate between those distinct empirical possibilities, then comparativism is wrong. If, on the other hand, the set of all the metaphysically distinct possible worlds acknowledged and *dynamically allowed by comparativism* contains all the empirically distinct possible worlds (that are *dynamically allowed by absolutism*), then we may opt for comparativism over absolutism based on an Occamist norm." (Martens, 2021, 2524, my emphasis)

[20] I here ignore any distinction between additivity and "proper" extensivity, cf. Perry (2015).

An example: the representational Q-transformation $10 \times$ kilogram $\iff$ $10,000 \times$ gram. It involves a U-transformation, $1 \times$ kilogram $\iff$ $1000 \times$ gram, and a V-transformation, $10 \iff \frac{1}{1000} \times 10,000$. While there are an indefinite number of representations of a quantity in some dimension owing to the indefinite number of reference units, all of the representations are equivalent under the group action.

Insofar as we take quantities and equations of quantities to be describing physical phenomena and not our measurement standards we require all "objective" quantity equations to be invariant under unit transformations. Given that these unit transformations are multiplicative in nature it is intuitive that quantities defined by products and divisions (and iterations thereof, i.e. powers) of the basic quantities will also be so invariant. As for equations, it is also relatively intuitive that equations only involving terms of like dimension will be invariant under unit transformations. The equation $Z = X + Y$ remains true under transformations of the form $Z \mapsto cZ$, $X \mapsto aX$, $Y \mapsto bY$, in cases in which $a = b = c$, i.e. $Z = X + Y \iff aZ = aX + aY$. These are cases in which all three quantities share the same dimension and so the same unit transformation factor. If, say, Y was of different dimension such that for some unit transformation $a = c \neq b$, then $aZ = aX + bY$ would not remain true in any case in which $aY' \neq bY$, where $Y' = Z - X$ as defined in the original units. This violation of dimensional homogeneity does not guarantee variance under *all* unit transformations, there may be some unit transformations of some equations in which it just happens that $a = b$ even though $[X] \neq [Y]$.

4.1 From the Representational to the Ontic

Here I describe the main metaphysical move made by my usage of the Π-theorem. For those interested in proofs of the theorem, see appendices A and B and the references therein. Those uninterested in any technical detail (or those who take the active-passive duality as a matter of course) can skip to 4.2 and 4.3 for the solution of the escape velocity case and a summary of the amended argument against absolutism. In order to provide proofs of the

Π-theorem, we must first prove *Bridgman's Lemma*: that all derivative quantities take the form of product of powers of the basic quantities. I do not provide a proof here,[21] but will briefly describe the reasoning and importance of the lemma.

The task of the lemma is to show that the derived quantities in a coherent systems of units have an essential form. It is established that dimensionally homogeneous equations are unit independent (see above). We take as a constraint on the form of a derived quantity that it is a function of basic quantities. Further, the *defining* equation is itself unit invariant, e.g. $F = kma$. That these defining equations only take the form of products of powers of the basic quantities (with a numerical scale factor, k) is Bridgman's Lemma. Bridgman's proof of the lemma is presented as an analytic elaboration of "our" requirement that relative magnitudes have absolute significance—independent of numerical representation, i.e. units. For Bridgman (1931, 21) this naturally follows from an operationalist point of view: the measurement of relative magnitudes is first and foremost a comparison of bodies which could not be affected by a change in our operational standards. Setting aside operationalism, we take it as an assumption that there is some unit transformation invariant relation underlying the comparative measurement of basic quantities.[22] From this it can be proved that all derived quantities must defined as powers of products of the basic quantity units: $Q_{mechanics} = km^x l^y t^z$. This defines a complete dimensional system: $[Q_{mechanics}] = [m^x][l^y][t^z]$. So much for Bridgman's Lemma.

The Π-theorem, in a nutshell, states that any adequate physical equation that describes a system can be put into Ur-Equation form:[23]

(Ur-Equation) $\psi(\Pi_1, \Pi_2, \ldots, \Pi_n) = 0$,

where the Π-terms are dimensionless derived quantities—products of powers of basic quantities—

[21] Readers can consult Bridgman (1931) and Berberan-Santos and Pogliani (1999) for proofs of the lemma.

[22] Perhaps we do so on the basis of the necessity of objective communication, see Roberts (2008). Even the absolutist will accept the invariance of quantity ratios of like dimension under unit transformations.

[23] This is my terminology. Sterrett (2017) calls this "The Reduced Relation Equation of 1914". Others sometimes refer to this equation as the Π-theorem itself, but I think it is more proper to consider the theorem the claim that any complete physical equation can be put in this form.

adequate to describe the system and ψ is some arbitrary function.The Π-theorem gets its name from the form of the functions that define the Π-terms: $\Pi = k \prod_i^n Q_i^{a_i}$, as established by Bridgman's Lemma.

I will distinguish two version of the theorem not often distinguished. Usually authors have one interpretation or another of the result,[24] all agree that there is an important sense in which the result is about mathematical structure. The question is the proper location of that structure: is this a result of the algebra of quantities or of the numbers which measure them? For the representational proof, the invariance of the numbers which measure quantities is essential the to the resulting Π-theorem. For the ontic proof, it is rather the quantities themselves and their dimensional properties which are essential to the result. My argument here is that these two proofs can indeed be seen as mere differences in "interpretation" such that both readings of the transformations described—unit transformations and ontic scale transformations—are available.[25] Further, I argue that a commitment to the representational theorem and some minimal assumptions regarding measurement entail a commitment to the ontic theorem.[26]

We first reconsider the nature of the unit transformation discussed above. Let us first specify a neutral conception of an equation between numerical and the quantitative. We take

[24] For example, Bridgman (1931) takes the formalist approach indicated by the representational interpretation of the theorem, while Buckingham (1914) takes on (somewhat reluctantly) the metaphysical significance of the ontic interpretation, as does Tolman (1915), more enthusiastically. For discussions of the history of the theorem, including priority disputes see Pobedrya and Georgievskii (2006) and Sterrett (2005, 2017) and their references. Gibbings (1982, 2011) gives a typology of proofs and his own metaphysical account. See also Walter (1990) and Mitchell (2019) on the historical metaphysical dispute regarding dimensions—see Skow (2017) for a contemporary discussion.

[25] Sterrett (2009) has brought it to my attention that Maxwell (2002) also noted this ambiguity in the interpretation of physical equations. My understanding of these equations as *quantity equations* is in line with Sterrett's view and the account of Lodge (1888). Accepting quantity equations means accepting the application of mathematical operations to quantities. This avoids the awkward work around of Maxwell who avoids the supposed inapplicability of algebra to physical quantities by converting between numbers which can be so manipulated and proper quantities via the introduction and elimination of units—Bridgman (1931) takes this implicit constraint to be the total significance of dimensions. For more on the "double interpretation of physical equations" see de Courtenay (2015) and Mitchell (2019).

[26] That, therefore, the denial of the ontic symmetry transformations would force the absolutist to reject the representational symmetry transformations (unit transformations) is argued by Wolff (2020, 149-150). This is a difficult and involved argument so I will not press the point here. Further, I discuss a different loophole that the absolutist may take in section 5.

an equation to represent relations between quantities either directly or indirectly, in either case we take the representatives which figure in an equation to have the canonical form of a numerical value multiplied by a unit quantity: $Q = V \times U$. If we take the representation to be direct, then we take the dimension associated with the unit to be constitutive of Q such that the principle of dimensional homogeneity is a metaphysical instantiation of Leibniz' Law.[27] Alternatively, we take the dimension of the unit to be inessential to the representation[28] and merely a bookkeeping device which reminds us of the conventionally decided *rules* which correspond to the principle of dimensional homogeneity. Under either interpretation of units, they are taken to represent a member of a group of homomorphic maps from quantities to numbers which represent the magnitude of the quantity, here represented by $V \in \mathbb{R}$.[29] That the units of some dimension form a group is simply another way of saying that unit transformations are symmetries of the form $U_{trans} : V \mapsto V'$. For the represntationalist or conventionalist, this is a direct numerical transformation and the new units associated with V' merely indicate a different standard for measuring Q.

For the metaphysician there is another set of symmetries that share the form $V \mapsto V'$ with U_{trans}. These symmetries are transformations of the quantity itself $Q \mapsto Q'$, which is defined as an automorphism of the quantity dimension: $Q, Q' \in D$. These ontic transformations can change the appropriate numerical representation of a quantity while leaving the units they are described with invariant. The unit map is preserved under the transformation $U_{map} \circ Q = V \to U_{map} \circ Q' = V'$.[30] This provides another set of transformations under which physical equations may be invariant: quantity dimension transformations.

The Π-theorem provides a bridge from the invariance of physical equations under unit

[27] Here only the dimension associated with the unit, but not the unit itself is essential to the quantity. Here a unit is just another quantity.

[28] The *unit* is of course essential to the *representation*—a change of units constitutes a change of representation.

[29] Or, if we commit to abstract or maybe only hypothetical quanitty magnitudes (which then mediate the application of value-unit representations to concrete quantities), unit systems are *isomorphic* maps (see Wolff, 2020; Tal, 2021). This makes no difference to my arguments.

[30] Note that the value of the unit quantity $U_{map} \circ Q = 1$ will change $V \neq V'$. This subtlety is (as far as I can tell) largely irrelevant for what follows, but see Wolff (2020, 151-153) for a discussion of quantity dimension translations vs quantity dimension dilations.

transformations to the invariance of physical systems under quantity transformations. That any physical system can be represented by an equation of dimensionless quantities, is the crux of the revised argument against absolutism. All symmetries of an Ur-Equation representation of a system are dual. On the one hand we have the representational symmetries accepted by all parties—unit transformations. These change the numerical values associated with constituent dimensional quantities but they leave the dimensionless Π-terms unchanged. This requires us to understand the Π-terms as providing a semantic link between an equation and a system itself: the Π-terms represent the quantity relations of the system that have absolute significance, their values have unit-independent meaning. As shown above, there is also a class of unit-independent, ontic symmetries which act on the constituent dimensional quantities of Π-terms—these symmetries act on the system's quantities themselves. The class of ontic physical symmetries which leave the Π-terms invariant are the class of *non-trivial empirical* symmetries. For this reason the Ur-Equation provides a well-tuned representation of physical systems—its formalism is coordinated to the physical structure of systems without excess representation. A change in the value of a Π-term necessarily represents a change in the physical system, while a change in the value of a constituent dimensional quantity may be an artifact of a purely representational change, like a change of units systems.[31]

4.2 Symmetries Defined by the Π-theorem: The Escape Velocity Case

Recall that one historical aim of this theorem was ultimately to provide a standard for scale models in aeronautics.[32] This theorem provides a condition that must be met for one

[31] Sterett's analogy between Buckingham's theorem and Wittgenstein's *Tractatus* has greatly clarified my thinking on this point. We may consider the dimensional quantities as atomic objects and the dimensionless Π-terms as propositions about the relations of these objects (atomic facts). As it were, the world consists of facts and not things; the Π -terms are accordingly isomorphic to the physical facts while the dimensional quantities fail to represent in isolation. ψ represents higher-order propositions which are decomposable into relations, here dynamical rather than logical, between the basic propositions, Π-terms. *The equation itself serves as a model of the system.* See especially the diagrams on pages 225 and 227 of Sterrett (2005).

[32] See Sterrett (2005) for more on the historical development of the formal results of dimensional analysis.

physical system to serve as a model of another, i.e. the theorem defines empirical symmetries for physical systems.[33] We can describe two systems S and S':

$$S: \psi(\Pi_1, \Pi_2, \ldots, \Pi_i) = 0$$

$$S': \psi'(\Pi'_1, \Pi'_2, \ldots, \Pi'_i) = 0$$

S and S' are empirically indistinguishable if and only if the values of the dimensionless Π-terms are invariant, i.e. $\Pi_i = \Pi'_i$, under some transformation of the basic dimensional quantities which compose the Π-terms.

In our escaping projectile case $\Pi = \sqrt{\frac{2GM}{rv^2_{pro}}}$ and ψ is a function that yields the Ur-Equation form $\Pi - 1 + \epsilon = 0$.[34] The ratio between the projectile's escape velocity and its actual velocity is conserved across symmetry transformations that leave Π invariant at approximately 1: $\frac{v_{ecscape}}{v_{pro}} + \epsilon = 1$. If two systems are to be dynamically similar and share the same ψ, it must be the case that the numerical values of the Π-terms are equivalent between the two systems. ψ is a high level description of the dynamics of the system: it constrains solutions to the equations of motion to those in which the projectile *just* has the velocity necessary to escape the planet's orbit.

Buckingham's argument is that changes in the basic quantity *dimensions* will leave the Π-terms unchanged in the transformation $f: S \mapsto S'$, because they are *dimensionless*. If all of the operands and the values of ψ and ψ' are identical, then the functions must be the same.[35] This identity signifies a symmetry. In the cases we are concerned with ψ and ψ' stand in for the *dynamical laws*. This makes good on an assumption made by comparativism,

[33] It should be noted that the notion of "physically similar systems" that the Π-theorem allows us to formalize is more fine-grained and sophisticated than the standard of empirical symmetry I am considering here. Besides dynamical similarity, there is e.g. geometrical similarity and kinematic similarity. Philosophers concerned with symmetries would do well to consider physical similarity, see Sterrett (2009, 2017).

[34] ϵ is simply a small constant added so we can deal with equalities rather than inequalities since strictly speaking the escape situation requires $\Pi < 1$.

[35] As stated this is an invalid inference. Consider the operations of addition and multiplication which have the same value, 4, with the same operands 2 and 2. The argument relies on the operands being more fine-grained than the values of Π. If we distinguish different instances of the Π-terms by the values of their constituent basic quantities, then we can claim that ψ is identical to ψ' iff for every instance of a set of Π-terms related by basic quantity symmetries (i.e. of the same value) they yield the same value. The dynamical laws are shielded from non-empirical differences in quantity values.

that the relevant empirical symmetries of a system are a subset of its dynamical symmetries and are hence full symmetries. A formerly problematic principle is justified: measurable quantities must be invariant under dynamical symmetries.[36] This licenses the inference from the existence of a class of empirical quantity symmetries to the existence of a class of *full* quantity symmetries.

We can generate a full quantity symmetry by: (i) arbitrarily transforming any of the basic quantity dimensions; (ii) adjusting the derivative quantities according to their dimensional composition, in line with Bridgman's Lemma, which will keep the values of the Π-terms are invariant.

Now we can return to the escape velocity case and show that a *full* mass doubling symmetry, which involves more than doubling the masses of *objects*, does not generate indeterminism or violate the laws. Consider again a situation in which the projectile escapes, $v_{pro} = \sqrt{\frac{2GM}{r}}$.[37] In step (i) of the transformation, as mass is a basic quantity dimension, we apply an arbitrary ratio transformation: $m_i \to 2m_i$ for i massive objects. We can describe the transformed situation thus: $v_{escape} = \sqrt{\frac{4GM}{r}}$, so v_{pro} is now insufficient to escape, *but we do not stop here.*

In step (ii) we change one of the derived quantities in order to preserve the relevant Π-term. The Π-term is $\Pi = \sqrt{\frac{2GM}{rv_{pro}^2}}$, or $\Pi = \frac{v_{escape}}{v_{pro}}$, and the derived quantity to be transformed is the gravitational constant G, whose dimensions $\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$define the compensating transformation the value as a halving, according to dimensional homogeneity.

Here's an explicit derivation modeled on Bridgman (1916): Let's define G as the product of a dimensionless number γ and its dimensions $\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$ (in abstraction from any particular units). If we define mass doubling as operating directly on the dimension, then

[36] This measurability-invariance-principle is the puzzle that is taken up by Roberts (2008). Roberts denies that the principle is analytic and I agree. The synthetic principles are work here are dimensional homogeneity and Bridgman's Lemma. I think it is plausible that these are equivalent or closely related to the publicity principle Roberts proposes. Note that—with Roberts—I take this to also provide an explanation of Earman's (1989) prescription that geometrical symmetries should not exceed dynamical ones in a "well-tuned" theory.

[37] From here on I drop the ϵ.

$\mathrm{M}' = 2\mathrm{M}$. So then the new gravitational constant G' equals $\gamma \mathrm{L}^3 \mathrm{M}'^{-1} \mathrm{T}^{-2}$, and by substitution $G' = \frac{1}{2}\gamma \mathrm{L}^3 \mathrm{M}^{-1} \mathrm{T}^{-2}$. Therefore $G' = \frac{1}{2}G$. Another way to understand this induced transformation of G is that G *has* (inverse) mass; its dimensionality has negative exponents in the mass dimension, so if all quantities with mass dimensions double, those with inverse mass dimension will halve. Under the completed symmetry transformation,

$$v_{escape} = \sqrt{\frac{4G'M}{r}},$$

where $G' = \frac{1}{2}G$. The original empirical situation in which the projectile escapes is preserved:

$$v_{escape} = v_{pro} = \sqrt{\frac{2GM}{r}},$$

and

$$\Pi = \frac{v_{escape}}{v_{pro}} \approx 1.$$

That the Π-terms are invariant under some transformation of quantity dimensions is *Buckingham's Criterion* for a full quantity symmetry:

> (Buckingham's Criterion) Only those quantity transformations which preserve the values of Π-terms that represent a physical system are full symmetries of that system.

If the absolutist is committed to the principle that physical equations are unit invariant (representational symmetries) and some fundamental principles of dimensional analysis, they are committed to the Π-theorem. This in turn commits them to ontic quantity symmetries, which, if they accept the validity of variance to unreality (or non-fundamentality) inferences, provides a decisive argument against their absolutism.[38]

[38] Wolff (2020) comes to a similar conclusion, though by way of measurement theory rather than dimensional analysis. Roberts (2016) also responds to the counterexample to comparativism much the same as I do, but works on the basis of a less general principle than the Π-theorem. See also Dewar (Forthcoming)—it is not clear to me whether or not his group-theoretic sophisticated absolutism is equivalent to group theoretical

Put somewhat more simply: a quantity transformation is a full symmetry if and only if it leaves the ratios of all quantities sharing some dimension invariant according to their exponent in that dimension.

4.3 Executive Summary of the Amended Argument Against Absolutism

I here present an amended symmetry argument against quantity absolutism:

(1) For any supposed fundamental property F, if F varies under a full symmetry, then F is not fundamental. (variance-to-unreality inference)

(2) Mass doubling is a full symmetry. (naive comparativist commitment)

(3) Intrinsic mass quantities vary under mass doubling. (definition of mass doubling)

Therefore, intrinsic mass quantities are not fundamental. (1, 2, 3)

Premise (2) was falsified. A full symmetry is a transformation that is both a dynamical and an empirical symmetry. It was established that mass doubling cannot be both. I amend the argument by substituting (2) and (3) with (2*) and (3*):

(2*) There are a class of *full* quantity symmetries defined by the Π-theorem, one of which, full mass doubling, doubles the masses and halves the gravitational constant. (Buckingham's Criterion)

(3*) Intrinsic mass quantities vary under full mass doubling. (definition of full mass doubling)

Note the generality of the result: any quantity which is not dimensionless, is not fundamental.[39]

presentations of the Π-theorem and the results of dimensional analysis, compare Corrsin (1951); Boyling (1979); Curtis et al. (1982); Raposo (2018, 2019).

[39] This avoids the "pushing-the-bump-under-the-carpet" objection that can be made against other comparativisms, see Martens (2020, 15).

The argument for the pivotal amended premise (2*) is the establishment of Buckingham's Criterion for a general quantity symmetry. The first part of the establishment of Buckingham's Criterion is to provide a general form for physical equations, the Ur-Equation. Its generality is justified by the assumption of dimensional homogeneity and the completeness, or unit invariance, of the physical equations in question. These are undeniable, at least for the equations we call physical laws.[40] Bridgman's Lemma tells us the form of the quantities, Π-functions, that figure in the Ur-Equation. These are measurable quantities, dimensionless products of powers of basic quantities. With this all in place, the Π-theorem can be be proved. As the Ur-Equations which represent systems embed dynamical equations, i.e. equations of motion, quantity transformations which leave their ψ-functions invariant are by definition dynamical symmetries.[41] As only Π-terms figure in these equations, such a dynamical symmetry must leave their values invariant as well. All absolutely significant quantities are Π-terms, therefore any transformation that leaves the Π-terms invariant is an empirical symmetry. So we have an intersection of the dynamical and empirical quantity symmetries of a system. These are symmetries in which individual quantities may be transformed according to their ratio structure, and constraints defined by the preservation of the systems dynamics according to general law will induce transformations on other quantities such that the empirical situation, specified by Π-terms, remains invariant.

It is important to clarify the results of this argument against absolutism. This symmetry argument, impervious to the sorts of counterexamples to Dasgupta's symmetry argument against absolutism, has a more modest aim than its predecessor. It shows that absolute quantities are not *more fundamental* than quantity relations. However, if one adopts a strong, Ockhamist variance-to-unreality principle, one may use the Buckingham class of symmetries to argue for the *non-existence* of absolute quantities and the absolute fundamentality of

[40] But see Grozier (2020) for some of the issues regarding "unit-invariance". A much deeper account of the nature of dimensions and their relation to the laws and the significance of dimensionless quantities is needed.

[41] For example, similitude methods exploiting the results of the Π-theorem are widely used in fluid mechanics, where analytical methods are intractable. Examples of such derivations of dynamical equations can be found in textbooks like Gibbings (2011).

quantity relations. For Dasgupta, these relations are between actual bodies, but one may instead adopt a quasi-Platonist structuralist view in which the fundamental quantity relations are between unworldly quantity magnitudes or second order properties which physical quantity relations somehow participate in (a *gross* gloss of Wolff, 2020). As I see it, there is no reason not to extend the argument in either of these directions: weak comparativism (Martens), relationalist comparativism (Dasgupta), and structuralism (Wolff) are all consistent with my argument as it stands.[42] As it stands my argument is not an argument *for* a comparativist account of quantity, but rather propaedeutic to one. Further commitments are needed to have a complete metaphysics of quantity.

This account of the quantity symmetries takes into account *inter*-quantity relations. Quantity symmetries generally require the transformation of multiple quantities, though they may transform only a single *basic* quantity *dimension*. By ignoring inter-quantity relations, the contemporary debate has been built on a fallacious assumption—a primary target of one of Galileo's two new sciences:

> Only by a miracle could nature form a horse the size of twenty horses, or a giant ten times the height of a man—unless she greatly altered the proportions of the members, especially those of the skeleton, thickening the bones far beyond their ordinary symmetry.
>
> Similarly, to believe that in artificial machines the large and small are equally practicable and durable is a manifest error. Thus, for example, small spire, little columns, and other solid shapes can be safely extended or heightened without risk of breaking them, whereas very large ones will go to pieces at any adverse accident, or for no more cause than that of their own weight. (Galilei, 1638, 14)

[42] One may worry that my appeal to a constant (G) is in tension with Dasgupta-style relationalist comparativism in which the fundamental quantity relations are between *bodies*. This is not the case. The constants can be understood as highly complex relations between bodies or regions of empty space—they are distinctive in that they are, in some sense, the same relation everywhere.

5 The Nomological Role of Constants

There is one lingering issue. I cannot hope to settle it here, but I'd like to open this vista for surveying. The account of dimensional analysis above gives no special role to the constants of nature, particularly the gravitational constant. The constants are merely parameters of equations and are available to be transformed by quantity symmetries. Indeed, they are only special in the sense that they are most apt to be manipulated in quantity symmetries as they describe the coupling of various logically independent basic quantity dimensions.

Let me clarify what I mean by "constants of nature". Johnson (2018) distinguishes three kinds of quantities called "constants": scale factors, system-dependent parameters, and system-independent parameters. The system-independent parameters are *universal* constants and are my concern here. Scale factors are mere numerical artifacts that can be inserted or removed from equations at will by unit changes.[43] System-dependent parameters on the other hand are quantities that correspond to aspects of particular physical systems. For example, the density of a fluid ρ may be defined as the ratio of its mass and volume $\rho = m/V$. For the treatment of some particular fluid, like an idealized incompressible fluid, this quantity may indeed remain constant, but its value differs for different fluids.

Among system *independent* parameters there are two philosophically relevant subkinds of constants of *nature*.[44] We distinguish: properties of the fundamental particles (e.g. m_p, m_e, e) and properties of the fundamental fields (e.g. c, h, G). My concern here is solely the third class of constants, the interaction constants which describe various sorts of fields.[45] Given

[43] E.g. the 4π factors that appear and disappear in electromagnetism (Maxwell's equations) depending on whether one is working in rationalized or unrationalized unit systems (see Silsbee, 1962).

[44] I assimilate Johnson's third category, numerical artifacts, to scale factors. A core example is Avogadro's constant N_A. N.B.: This is contrary to the usual conception of Avogadro's constant as a constant of nature and a mole as a unit of "amount of substance" that can be found in the SI unit system (BIPM, 2019). See Johansson (2011) for some common sense dissent.

[45] There is a somewhat similar delineation of the constants given by Lévy-Leblond (1977, 2019). Lévy-Leblond distinguishes class B constants which describe general classes of phenomena (e.g. electromagnetic) and truly universal class C constants. Johnson (2018) finds the distinction unfounded. As Lévy-Leblond holds that the classification of the constants is context dependent, it makes no difference to me here whether or not e.g. c is considered as a mere electromagnetic constant or a universal constant (describing the causal structure of spacetime). All of the constants I am concerned with here may be considered class C constants.

that the debate between the comparativist and the absolutist has concerned the possibility of changing the basic quantities, i.e. constants describing fundamental properties of the fundamental particles we can understand the question raised here as: Do transformations of the particle constants induce transformations in the interaction constants?

However, interaction constants seem to play a more significant role in all physical laws: their values seem *constitutive* of the laws. It seems that if the gravitational constant or any other constant of nature is changed, then *the laws* have changed.[46] This would mean that there is some discrepancy, though unobservable, between the two escape velocity cases, vindicating the absolutist—full mass doubling would fail to be a *dynamical* symmetry. The question is then whether the values of the constants determine the nomically possible worlds.[47] Broadly, there are two views one can have towards the gravitational constant in particular and interaction constants which appear in the laws in general:

> (Constant Contingentism) The magnitudes of the dimensional constants are independent of the laws and depend on non-nomic quantity regularities—they vary across nomically possible worlds.[48]

> (Constant Necessitism) The magnitudes of the dimensional constants are fundamental and necessary across nomically possible worlds. These values constrain non-nomic regularities.[49]

Contingentism naturally pairs with comparativism. For the contingentist comparativist, it is only the general, dimensional relation between the constants and dimensional quantities

[46] I will not here consider "singularity" limits where the constants are taken to be infinity or 0. We take such cases, e.g. $G \to 0$, to represent the absence of the relevant physics which tells in favor of their necessity to the law. Another case, $c \to \infty$, represents the classical limit of relativity. I believe that these cases are different in nature from a mere doubling, etc. of the magnitudes of the constants. See Lévy-Leblond (2019) for an introduction to some of the relevant issues.

[47] I set aside issues regarding spatiotemporal variations of the constants in a single universe. See Barrow (2004) and Barrow and Webb (2005) for accessible introductions.

[48] This view has been floated in Ehrenfest-Afanassjewa (1916, 1926) and Nordström (1914) and has recently come under criticism by Martens (2020).

[49] This is to be distinguished from Dahan's (2020) view of the constants as (defeasible) *identifiers* of universal laws—Dahan makes this point herself. Though necessitism is consistent with the idea that constants "baptize" universal laws, it is independent of it.

that constitutes the laws, the actual magnitude of dimensional constants is irrelevant. Some pioneers of dimensional analysis thought of the constants as properties of the environment, the gravitational constant and the permittivity constant were thought to be properties of empty space.[50] Though one may not need to accept this sort of *ontological* grounding for physical constants: Bridgman (1916) held that the constants are *conventional* conversion factors. One of the most intuitive cases for constant contingentism is the fact that we take constants to be *measured* quantities.[51]

Similarly, absolutism naturally pairs with necessitism.[52] It is nomically impossible that the gravitational force be stronger than it is. We might understand the role of the constants in this way: Rather than have the gravitational constant as a parameter in functions that represent gravitational systems, $f(G, x, y, z \ldots)$, the gravitational constant is an essential constituent of that function which represents the dynamics of gravitational systems, $f_G(x, y, z \ldots)$. One natural understanding of necessitist absolutism is that it leads to a total determination of the facts of the world by the laws—a theory of everything would have no terms left to be determined by experiment:

> One plausible view of the Universe, is that there is one and only one way for the constants and laws of Nature to be... The values of the constants of Nature are thus a jigsaw puzzle with only one solution and this solution is completely specified by the one true theory of Nature. If this were true then it would make no more sense to talk about other hypothetical universes in which the constants

[50]For example, Mercadier and Vaschy, see De Clark (2017, 312-19).

[51]Recent changes in SI units aside. That the values of constants are there treated as *defined* is to be understood as a conventional fiction. The high-precision measurement of the constants are to be cordoned off to the special science of metrology, while those measured values are taken as analytic truths by the rest of the sciences (see Petley, 1983). It is merely cognitive division of labor—we do not, in the same practice, design the tools that we use.

[52]Both of these "natural" pairings are superior to their mixed counterparts: necessitist comparativism and contingentist absolutism. These mixed views entail a mismatch between the metaphysics of quantities and the metaphysics of the laws in a way that generates unsynchronized changes—both violate some of our modal scruples. I will not provide arguments here. For similar reasons I will resist the deflationist move of holding contingentism and necessitism to merely generate different gradations of nomological necessity. The evaluation of counterfactuals (and their contrast with counterlegals) is central to scientific practice and we should hope for a univocal standard.

> of Nature take different values than it would make sense to talk of square circles. There simply could not be other worlds. (Barrow, 2004, 178)[53]

One ought not be mislead by Barrow's comparison to square circles; what we are and have been concerned with throughout this work is nomological or natural necessity and not any broader sense of metaphysical or logical necessity. The view expressed here, then, is that constant necessitism leads to necessitarianism or strong determinism: that everything that is true is (nomically) necessarily so. A sketch of the argument for this: Assuming the world is lawful, the combination of the necessity of the laws, including the values of the constants, and the absolute significance of intrinsic quantities, including the constants, would entail a strict, two-way supervenience relation between the laws and the quantities they govern. As changes in the laws are, by assumption, impossible, so too are changes in the quantities.[54]

That sketch is not likely to be convincing. Let me quickly elaborate on what is meant by strong determinism and on the case for its entailment by the conjunction of quantity absolutism and constant necessitism. Chen (2022) has recently given a definition of "strong determinism" and has laid out some of its consequences. Strong determinism is logically stronger than determinism and logically weaker than superdeterminism, which comes up in the foundations of quantum mechanics literature. I adopt this definition from Chen:

> (Strong Determinism) A world is strongly deterministic if its fundamental laws are compatible with only one possible world.

This view is compatible with the leading accounts of physical laws. As Chen has it, there are a couple of toy models in which strong determinism can be shown to hold, but only

[53]Note: Barrow himself makes the case for the opposing, contingentist conception of the constants—though it seems that this goes along with a (metaphysical) contingentism regarding the laws as well.

[54]This presentation of the these two positions is intended as a clarification of what has been at stake in debates concerning the comparativist reformulation of the laws. The laws seem intuitively to refer to absolute quantities—the escape velocity equation does not (explicitly) refer to any mass other than that of the planet. Starting with Dasgupta (2013) and continuing with (Baker, 2020, 83-92) and (Sider, 2020, 145-50), many different formulations of the comparativist Newtonian laws have been proposed and criticized. To debate whether or not there is a coherent comparativist statement of the laws just is to debate the merits of constant contingentism.

one realistic physical theory which is stongly deterministic: the Everettian Wentaculus (see also Chen, 2023).[55] If I can make the case that strong determinism is entailed by quantity absolutism and constant necessitism then I will have shown that strong determinism is easier to get than Chen makes it seem—it will be theory *independent* and follow from *metaphysical* theses.

I know of no way to make this case. In order to argue that constant necessitism implies strong determinism, I must substitute a metaphysical thesis for an empirical conjecture. Rather than involve absolutism in the argument directly, I only hold that a strategy to avoid the ammended argument against absolutism given above is to adopt constant necessitism, so a commitment to absolutism weakly implies a commitment to constant necessitism (until another solution is found). The empirical conjecture is one that naturally comes out of a tradition of taking the magnitudes of the constants of nature to determine many important facts about our world—upper and lower bounds on planet size, the existence of habitable planets, the existence of life, etc.—of which Barrow was an heir (e.g. Weisskopf, 1975; Press et al., 1983). The conjecture is this:

> (Constant Determinism) As our understanding of physical theory increases we will increasingly find that the magnitudes of the constants determine more and more of the facts of our universe: in the limit the constants will be sufficient to determine the initial conditions of the universe.

This increase of theoretical understanding may include, for example, the discovery of further fundamental constants related to novel forces or the unification of multiple constants in a more fundamental theory. The argument then is that the conjunction of constant necessitism and constant determinism implies strong determinism (in the sense of Chen, 2022). While this seems to be alternative route to strong determinism than Chen countenances, it is clearly not theory-independent and so is a matter ultimately for empirical investigation rather than a metaphysical thesis.

[55] I am not dealing with quantum mechanics here, so I will say nothing about this case.

While constant determinism is endorsed by some impressive figures, its plausibility is questionable and its empirical confirmation a long ways away. That said, *if* it is accepted, then my argument against absolutism can ultimately be understood as a *reductio*, with the consequences of constant necessitism (with the assumed fact of constant determinism) being the *absurdum*.[56] It is worth noting, however, that this result is not regarded as absurd by some: Einstein, who held a version of (nomological) Spinozism,[57] seems to have to taken constant necessitism and constant determinism as a package deal.

> Of course, I cannot prove this. But I cannot imagine a unified and reasonable theory which explicitly contains a number which the whim of the Creator might just as well have chosen differently, whereby a qualitatively different lawfulness of the world would have resulted.
>
> Or one could put it like this: A theory which in its fundamental equations explicitly contains a non-basic [i.e. non-numerical] constant would have to be somehow constructed from bits and pieces which are logically independent of each other; but I am confident that this world is not such that so ugly a construction is needed for its theoretical comprehension. (Einstein to Rosenthal-Schneider 1945, in Rosenthal-Schneider, 1980, 37-8)

The apparent arbitrariness in values of the constants to which the contingentist is committed, that of the *real*, fundamental constants and not merely *apparent*, eliminable ones, troubled Einstein.[58] Whatever one thinks of the metaphysical possibility of strong determinism, Einstein's expression of it puts the contingentist comparativist on notice: The comparativist ought to endeavor to show that the arbitrariness of the connections between logically distinct quantity dimensions is not so ugly a construction after all.

Let me say a bit in defense of a contingentist comparativism that points the way to the

[56]For some of the problems with strong determinism see Chen (2022).
[57]See Paty (1986)
[58]Einstein's categorization differs from the one used here. For him G is apparent while α is real. This is largely unimportant for the immediate philosophical point discussed here.

work still to be done to fully flesh out what the comparativist's commitments are. The contingentist comparativist does not think that "anything goes" with respect to the real, fundamental, physical constants; there is a feature of them that is nomically necessary. What remains invariant under the comparativist symmetries is the algebraic relation between the constants and the other parameters in the laws. Their relation is encoded by their *relative* dimensionality. With $[G] = \mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$, as required by the dimensional homogeneity of Newton's law of gravitation, $F = \frac{GMm}{r^2}$, it is nomically necessary that G scales inversely with M and cubicly with L. This is independent on any changes of convention regarding units or the basic quantity dimensions—the relation holds if force is treated as basic and mass as derived.[59] It is a contingent matter of fact what magnitudes quantities have, including constants. It is matter of convention which units we use to measure them and (maybe) which dimensions we stipulate as basic. The dimensional (i.e. scaling) relation between different quantities is *nomically necessary*.[60] A full account of comparativism must find some way for accounting for these inter-quantity relations and their boundedness by physical law.

Something more can be said to sharpen this problem and to present an initial answer to it. Some, like Sider (2020) and Baker (2013), seem to hold that the value of some constant as *essential* to that constant to the extent that they introduce units into the laws by their interpretation. Therefore, comparativist and absolutist alike have a problem retaining the representational symmetry of unit system changes. Sider attempts to get around this problem by relativizing the laws to a choice of representation functions (unit systems), but this only forces the (mixed) absolutist into a constant contingency or into the naive absolutism—where unit transformations fill to be symmetries, an absurd conclusion.[61] Baker argues that the comparativist may posit fundamental mixed relations among quantities of different dimensions in order to avoid the positing of numerical constants, but in doing so the compar-

[59] On the scope and limitations of conventionality in dimensional systems see Palacios (1964); Johnson (2018).

[60] I cannot discuss this material fully here, but it may be worth comparing the results here to the discussion in Duff (2014). Compare also recent discussions in Grozier (2020) and Riordan (2015) on the question of the fundamental constants and dimensionality.

[61] I've avoided discussion of this issue, see Eddon (2013) and Wolff (2020) for discussions of this view.

ativist loses any parsimony advantage over the absolutist, sacrificing the original motivation for the comparativist.

This would seem to be an ugly construction indeed. Let me suggest that the notion of a constant used here causes no such problem. First let me simply reiterate that on the contingentist conception of the constants the nature of the constants is entirely *unit-free*, my understanding here is of the (field) constants as dimensional *quantities*. I have not endeavored to give a full account of the constants and their ontology, however positing them as something like fundamental mixed relations between quantity *dimensions* seems right.[62] Though there has not been much philosophical work on the nature of the constants, recently Jacobs (2021, chapter 6) has, by way of giving a very similar solution to the escape velocity case as that above, offered an account of G as a fundamental mixed relation that is part of the structure of Newtonian Gravity. Still we can ask where G ought to be placed in the structure of Newtonian Gravity.[63] The suggestion here is that G, *qua* dimensional relation, plays a *nomological* role, hence no violation of ontological parsimony. The *dimensions* of G fix the functional form of the gravitational force law. This generalizes to a principle that every ineliminable constant of mixed dimension determines the functional form of a fundamental law. It is precisely these constants that tell us how otherwise logically independent quantity dimensions come together to form a physical universe. Rather than "number[s] which the whim of the Creator might just as well have chosen differently" that "would have to be

[62] Here fundamental is only relative to the ontology of quantity dimensions—quantity dimensions themselves may depend on something more fundamental. That said, quantity dimensions seem like natural candidates for the fundamental ontology of physics. There are different sorts of realist views we can have about quantity dimensions, as discussed in Skow (2017). We could alternatively be irrealists and conventionalist regarding quantity dimensions and therefore regard the constants, *qua* fundamental mixed relations, as mere conceptual guides to thinking about the relations between measured quantities (Bridgman, 1931). I believe the argument against absolutism can run regardless of the metaphysics of dimension—this is the point of the two different proofs of the Π-theorem—but I will defend a moderate quantity dimension realism in further work.

[63] More recently Jacobs (Forthcoming) has made the case that G is part of the kinematic structure of Newtonian gravitation and so my full mass doubling (or what Jacobs calls an "inclusive active mass scaling") is not quite a dynamical symmetry, but rather a similarity. I will not adjudicate this issue here. I have already alluded to the fact that similarities seem to be more general than the sorts of symmetries philosophers have so far been concerned with and that the Π-theorem is a guide to similarities, some of which correspond to dynamical and empirical symmetries (see Buckingham, 1914; Sterrett, 2009). I locate G in the *nomology* of Newtonian gravitation and hence in its dynamics.

somehow constructed from bits and pieces which are logically independent of each other" (as Einstein would have it), we have a picture of the constants as construction principles which glue together the fundamentals of creation into a coherent and lawful world.

The comparativist may take inspiration from the example provided by Plato's *Timaeus*:

> So if the body of the universe were to have come to be as a two dimensional plane, a single middle term would have sufficed to bind together its conjoining terms with itself. As it was, however, the universe was to be a solid, and solids are never joined together by just one middle term but always by two. Hence the god set water and air between fire and earth, and made them as proportionate to one another as was possible, so that what fire is to air, air is to water, and what air is to water, water is to earth. He then bound them together and thus he constructed the visible and tangible universe. This is the reason why these four particular constituents were used to beget the body of the world, making it a symphony of proportion. They bestowed friendship upon it, so that, having come together into a unity with itself, it could not be undone by anyone but the one who had bound it together. (Plato, 1997, 1237-1238)

6 Conclusion

This paper presents an amendment to the symmetry argument against quantity absolutism. Rather than requiring that any universal scale transformations of the quantities of some basic dimension are empirical and dynamical symmetries, the argument against absolutism depends only on those symmetries defined by the Π-theorem. These symmetries may involve scale transformations of basic quantities, but they also involve transformations of derived quantities, most notably the physical constants. The symmetries defined by the Π-theorem are transformations that leave the dimensionless quantity ratios which describe some system invariant—these transformations are both empirical and dynamical symmetries.

The transformation of the constants in some symmetries defined by the Π-theorem raises the question of their modal status. On the one hand is constant contingentism, which states that the laws and the relations of the basic quantities determine the values of the constants—their values can vary in nomically possible worlds, supervening on variations of the relations of the basic quantities. On the other hand is constant necessitism, which states that the values of the constants are fixed across nomically possible worlds and are fundamental—their values and dimensions fix the laws and the relations between the basic quantities. My purpose has been to introduce the debate and set some of its terms. Though I give reason to prefer constant contingentism to constant necessitism, not least of all its superior fit with quantity comparativism, the discussion here is not conclusive.

Appendix A Proof of the Representational Π-theorem

This proof proceeds on an understanding of equations as relations between numbers or representations of numbers (i.e. variables) and unit transformations as transformations of numbers. The structure of the proof is the same in both versions. The fundamental assumption being that we are only dealing with unit invariant or dimensionally homogeneous equations. The ratio scaling symmetries of basic quantities will propagate to derived quantities according to Bridgman's lemma. This allows any physical equation equation to be recast enitrely in terms of dimensionless derived quantities, by way of the reduction of superfluous variables in the expression of some equation.

A review of the requisite assumptions:

(0) *Zeroth assumption*: Any equation describing a physical system can be represented by some function of numbers which represent quantities set equal to zero:

$$f(Q_1, Q_2, \dots Q_N) = 0.$$

(1) *First assumption*: We are only concerned with "complete" equations whose algebraic form is unit-invariant. For such equations there is a class of representations:

$$f'({Q'}_1, {Q'}_2, \dots {Q'}_N) = 0,$$

where $x_i Q_i = {Q'}_i$ and the unit transformation factors $x_i \in \mathbb{R}^+$.

(2) *Second assumption*: If the equation describing the system is unit invariant, then the numbers representing derivative quantities are unit-transformed by transformation factors that can be defined as products of powers of the unit-transformation factors of the numerical representations of the constituent basic quantities.

These are the fundamental assumptions of dimensional analysis; to give them up would be to forgo many important patterns of physical reasoning and would threaten the marriage of measurement and number.

The proof proceeds:[64] We define the number of derivative quantities, r, as the difference in the total set of quantities describing the phenomena, N, and the subset of basic quantities, n: $r = N - n$. We can understand the n basic dimensions to serve as a *reduction base* for the original description of the system by N quantities. If r is non-zero, then the reduction exists, and with Bridgman's lemma, we can define the relations between the derivative and basic transformation factors with a set of r equations:

$$\left\{\begin{array}{c} x_{n+1} = x_1^{a_{n+1,1}} x_2^{a_{n+1,2}} \dots x_n^{a_{n+1,n}} \\ \\ x_{n+2} = x_2^{a_{n+2,1}} x_2^{a_{n+2,2}} \dots x_n^{a_{n+2,n}} \\ \vdots \\ x_{n+r} = x_1^{a_{n+r,1}} x_2^{a_{n+r,2}} \dots x_n^{a_{n+r,n}} \end{array}\right\},$$

where the exponents $a_{i,j}$ are defined by the relation $Q_i \propto Q_j^{a_{i,j}}$, with $i = n+1, n+2, \dots, n+r$ and $j = 1, 2, \dots, n$.[65] These $x_{n+1}, \dots, x_{n+r}$ factors are the numerical scale factors for unit transformations of the numerical representations of the derived quantities. The values of these factors depend on the unit transformations on the basic quantities and the defined relationships between the derived and basic quantity representatives.

From this equation set, we define r unitless Π-terms, eliminating all of the transformation factors and involving all of the relevant representations of quantities:

$$\left\{\begin{array}{c} \Pi_{n+1} = \frac{Q'_{n+1}}{Q'^{a_{1,1}}_1 Q'^{a_{1,2}}_2 \dots Q'^{a_{1,n}}_n} = \frac{Q_{n+1}}{Q^{a_{1,1}}_1 Q^{a_{1,2}}_2 \dots Q^{a_{1,n}}_n} \\ \\ \Pi_{n+2} = \frac{Q'_{n+2}}{Q'^{a_{2,1}}_1 Q'^{a_{2,2}}_2 \dots Q'^{a_{2,n}}_n} = \frac{Q_{n+2}}{Q^{a_{2,1}}_1 Q^{a_{2,2}}_2 \dots Q^{a_{2,n}}_n} \\ \vdots \\ \Pi_{n+r} = \frac{Q'_{n+r}}{Q'^{a_{r,1}}_1 Q'^{a_{r,2}}_2 \dots Q'^{a_{r,n}}_n} = \frac{Q_{n+r}}{Q^{a_{r,1}}_1 Q^{a_{r,2}}_2 \dots Q^{a_{r,n}}_n} \end{array}\right\}.$$

Since the Π-terms are unit transformation invariant, the Π-terms are equivalently defined out of the Qs and the Q's—and encode the essential relations between derived and basic

[64] This presentation of the proof is based on Ehrenfest-Afanassjewa (1916).
[65] The exponent will be zero for all irrelevant quantities Q_j.

quantity measurements. This means that any equation describing the measurement results of a physical system can be recast in Ur-Equation form:

$$f(\Pi_1, \Pi_2, \ldots, \Pi_r) = 0.$$

This is the (representational) Π-theorem. It was derived from three assumptions: (0) that numerical representations of physical systems exist which can be described as a function of numbers set equal to zero; (1) There is a subset of such numerical equations that are unit invariant; (2) Bridgman's lemma, i.e. the numerical measures of derived quantities are products of powers of the numerical measures of basic quantities.

Appendix B Proof of the Ontic Π-theorem

This proof understands equations to directly represent quantity relations themselves and proceeds by considerations of ontic transformations of quantity dimensions rather than unit transformations. The relation between the two was more fully analyzed in 4.1, but the outline bares repeating. What is meant by calling this proof "ontic"? Of course, in neither the ontic or in the representational case are the *syntactic* objects which make up equation tokens taken to be quantities or numbers (rather they are variables and numerals). The ontic-representational distinction is this: either equations represent relations between quantities which are properties of physical systems or they represent relations between numbers which *measure* quantities according to some externally defined convention. Given the assumption of faithful measurement conventions, conclusions drawn under interpretations of the latter kind entail counterpart conclusions under interpretations of the former kind. This is to say: the two interpretations are in an important sense interchangeable, if numbers can measure quantities at all.

We begin again with a generalized functional form of a complete (i.e. unit-invariant) equation describing a physical system:[66]

[66] This presentation of the proof is based on Gibbings (1982, 2011).

$$f(Q_1, Q_2, \dots Q_N) = 0 = f({Q'}_1, Q_2, \dots, {Q'}_N),$$

where each Q_i is a quantity composed of a dimensionless number and a unit quantity, $Q_i = V_i U_i$, and the primed quantities are related by dimensionless transformation factors x_i. Here we abstract from the (conventional) determinancy of "value" and the "unit" of some quantity to its magnitude, M, and dimension, D, where each unit transformed quantity counterpart is identical in these respects: $Q_i, {Q'}_i, {Q''}_i, \dots = M_i$ and $[Q_i], [{Q'}_i], [{Q''}_i] = D_i$. This abstraction serves us with a *unit-free* representation of the quantities, much like tensor calculus allows us *coordinate-free* representations of spacetime—this makes it clear that we are dealing with the ontic quantities and not their mere representations.[67]

Given Bridgman's lemma, we can define each quantity as products of powers of the basic quantities, $Q_1, Q_2, \dots, Q_n$:

$$\left\{ \begin{array}{c} Q_1 = Q_1^{a_{1,1}} Q_2^{a_{1,2}} \dots Q_n^{a_{1,n}} \\ \\ Q_2 = Q_1^{a_{2,1}} Q_2^{a_{2,2}} \dots Q_n^{a_{2,n}} \\ \vdots \\ Q_N = Q_1^{a_{N,1}} Q_2^{a_{N,2}} \dots Q_n^{a_{N,n}} \end{array} \right\}.$$

Since we are dealing with a coherent dimensional system, the same construction applies to quantity dimensions themselves; they can be defined in terms of basic quantity dimensions, $D_1, D_2, \dots, D_n$:

$$\left\{ \begin{array}{c} [Q_1] = D_1^{a_{1,1}} D_2^{a_{1,2}} \dots D_n^{a_{1,n}} \\ \\ [Q_2] = D_1^{a_{2,1}} D_2^{a_{2,2}} \dots D_n^{a_{2,n}} \\ \vdots \\ [Q_N] = D_1^{a_{N,1}} D_2^{a_{N,2}} \dots D_n^{a_{N,n}} \end{array} \right\}.$$

[67] Note that this is merely a presentational move, the "representational" proof given above proceeds in a unit fixed representation, but defines transformations and relations which are invariant under any unit standard. There is an important sense in which these two approaches are equivalent, compare Wallace (2019) and Wolff (2020, chap. 9).

Now we take a quantity Q_i from a subset $Q_i \subset Q_N$ of quantities such that some of Q_i's basic quantity exponents, $a_{i,j}$, are zero, meaning that their dimension does not require all D_n and divide through each row of the quantitative matrix by Q_i so as to cancel its dimension $[Q_i]$ in all the other quantities. As this elimination process iterates, we will be left with dimensionless quantities. For the first dimension D_1 and each Q_j, $j \neq i$:

$$\frac{Q_j^{a_{i,1}}}{Q_i^{a_{j,1}}} \rightarrow \frac{D_1^{a_{j,1}a_{i,1}}}{D_1^{a_{i,1}a_{j,1}}} = 1.$$

The division procedure described above guarantees that the power of the dimension in the numerator and the denominator is equal, hence the dimension is eliminated in the quotient quantity. This creates the functional, complete equation:

$$f(\frac{Q_1^{a_{i,1}}}{Q_i^{a_{1,1}}}, \ldots, \frac{Q_N^{a_{i,1}}}{Q_i^{a_{N,1}}}) = 0.$$

Successive cancellations up to D_n for all Q_i lead to all dimensions being eliminated and so all quantities in the function are dimensionless Π-terms of the same form as those defined in the last subsection:

$$\left\{ \begin{array}{l} \Pi_{n+1} = \frac{Q'_{n+1}}{Q'^{a_{1,1}}_1 Q'^{a_{1,2}}_2 \ldots Q'^{a_{1,n}}_n} = \frac{Q_{n+1}}{Q_1^{a_{1,1}} Q_2^{a_{1,2}} \ldots Q_n^{a_{1,n}}} \\ \\ \Pi_{n+2} = \frac{Q'_{n+2}}{Q'^{a_{2,1}}_1 Q'^{a_{2,2}}_2 \ldots Q'^{a_{2,n}}_n} = \frac{Q_{n+2}}{Q_1^{a_{2,1}} Q_2^{a_{2,2}} \ldots Q_n^{a_{2,n}}} \\ \vdots \\ \Pi_{n+r} = \frac{Q'_{n+r}}{Q'^{a_{r,1}}_1 Q'^{a_{r,2}}_2 \ldots Q'^{a_{r,n}}_n} = \frac{Q_{n+r}}{Q_1^{a_{r,1}} Q_2^{a_{r,2}} \ldots Q_n^{a_{r,n}}} \end{array} \right\},$$

yielding a proof of the (ontic) Π-theorem:

$$f(\Pi_1, \Pi_2, \ldots, \Pi_r) = 0.$$

The divisional procedure of eliminating dimensions highlights an important aspect of the Π-theorem. It provides a definitive answer to the number of variables and the number of dimensionless groups required to describe some system. As indicated above there are $r = N - n$ dimensionless groups of variables, Π-terms, necessary to describe a system of

N quantities formed by n basic dimensions. The removal of each dimension is associated with the addition of a variable to each Π-term, yielding a number of $n+1$ variables per Π-term.[68]

[68]For details and exceptions see Gibbings (2011, 59-61).

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