Length Abstraction in Euclidean Geometry

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Abstract

I define abstract lengths in Euclidean geometry, by introducing an abstraction axiom: $\lambda(a, b) = \lambda(c, d) \leftrightarrow ab \equiv cd$. By geometric constructions and explicit definitions, one may define the *Length structure*: $\mathbb{L} = (\mathbb{L}, \oplus, \preceq, \bullet)$, "instantiated by Euclidean geometry", so to speak. I define the notion of a "(continuous) positive extensive quantity" and prove that $\mathbb{L}$ is such a (continuous) positive extensive quantity. The main results given provide the general characterization of $\mathbb{L}$ and its symmetry group (the multiplicative group of the positive reals); along with the relevant mathematical relationships between (abstract) lengths and *coordinate* lengths (relative to a coordinate system); and also between lengths, measurement scales and units for length.

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1 Introduction

Science is built up by assuming certain quantities (functions, usually) exist, and then a scientific theory is given by a list of sentences asserting that certain general (lawlike even) relations hold between them. For example, $pV = Nk_BT$ and $\nabla \bullet \mathbf{E} = \rho$ and the like. It's an over-arching methodological rule that quantities be *measurable*:

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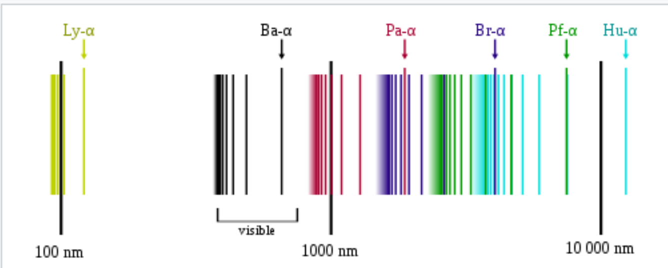
> It is an important principle of physics that no quantity should be introduced into the theory which cannot, at least in principle, be measured. Newton's Laws involve not only the concepts of velocity and acceleration, which can be measured by measuring distances and times, but also the new concepts of mass and force. To give the laws a physical meaning, we have, therefore, to show that these are measurable quantities. (Kibble & Berkshire (1996): §1.3, p. 8)

The outcome of such measurements (experiments) are then called "*measurement reports*", like "the period of Halley's comet, on average, is 76 years", or "the temperature of the cosmic microwave background is 2.7 K" or "the rest mass of an electron is 9.1×10^{-31} kg" and so on.[1]

The theory is tested by comparing its concrete predictions with this list of measurement reports. For example, in the late 19th century, atomic spectroscopy revealed the curious "quantization" or "discreteness" of the emission lines of light from hydrogen atoms:[2]

Figure 1: Hydrogen Emission Series



The spectral series of hydrogen, on a logarithmic scale.

Soon it was guessed, first by Johann Balmer (1885), and generalized by Johannes Rydberg (1888), that the observed discrete numerical wavelengths can be classified by a pair of integers, satisfying what is now called the Rydberg formula:[3]

$$\frac{1}{\lambda_{nm}} = R_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right) \tag{1}$$

[1] A list of measurement reports corresponds to a *relational database*. For example, the database of planetary distances and planetary period measurements which Kepler took to corroborate his three laws, or the database of LHC CERN measurements which verified the existence of the Higgs particle, or the database for the LIGO experiment which verified the existence of gravitational waves, and so on, are large relational databases stating that certain quantities take certain values (and usually, along with an error estimate).

[2] The observed spectrum is a kind of *experimental* "photograph" of the underlying electronic orbitals.

[3] The quantity $R_H = \frac{\mu e^4}{8(\epsilon_0)^2 h^3 c} \approx 1.09678 \times 10^7\ \mathrm{m}^{-1}$ is called Rydberg's constant (for hydrogen), where μ is the "reduced mass" of the electron relative to the proton. The quantity $R_\infty = \frac{m_e e^4}{8(\epsilon_0)^2 h^3 c}$ is the limiting case where we pretend the nucleus is very massive compared to the electron mass m_e.

We now have a general law. One can then compare the predicted wavelengths from the law with the observed wavelengths of spectral lines, corresponding to the Lyman series ($n = 1$), Balmer series ($n = 2$), etc. The currently most accurate data for hydrogen emission lines is the following:[4]

Table 1: Observed wavelengths (nm) of Hydrogen emission lines

Series name	n	$m = n+1$	$m = n+2$	$m = n+3$	$m = n+4$	$m = n+5$	Converges to
Lyman	1	121.57	102.57	97.254	94.974	93.780	91.175
Balmer	2	656.3	486.1	434.0	410.2	397.0	364.6
Paschen	3	1875	1282	1094	1005	954.6	820.4
Brackett	4	4051	2625	2166	1944	1817	1458
Pfund	5	7460	4654	3741	3297	3039	2279
Humphreys	6	12370	7503	5908	5129	4673	3282

This is a relational database, of the kind that computers and programming languages are very happy to deal with. In Python or `R`, called a `data.frame`. It is a finite syntactical object, consisting in a finite number of measurement reports, the numerals appearing being fixed point decimals, standing for their usual real number referents.

We can then plug in the relevant n, m parameters and the value of R_H to obtain the predicted wavelengths:

Table 2: Predicted wavelengths (nm) of Hydrogen emission lines

Series name	n	$m = n+1$	$m = n+2$	$m = n+3$	$m = n+4$	$m = n+5$
Lyman	1	121.57	102.57	97.255	94.975	93.781
Balmer	2	656.5	486.3	434.2	410.3	397.1
Paschen	3	1876	1282	1094	1005	954.9
Brackett	4	4052	2626	2166	1945	1818
Pfund	5	7460	4654	3741	3297	3039
Humphreys	6	12372	7503	5908	5129	4673

As the reader can see, the agreement between the experimental data and prediction is extremely close.

In this article, though, I am only interested in *length*. Very simple examples of measurement reports, involving length, would be:

(1) The length (height) of Nelson's Column = 5159 cm.
(2) The length (height) of Nelson's Column = 2031 inch.

[4] See Wiese & Fuhr (2009); Kramida et al. (2010).

Figure 2: Nelson's Column

As one can see from the photo, the column looks about 20 or 25 times as high as a person. Assuming people are around 2 m height, Nelson's Column must be something like 40 m to 50 m in height. In fact, it's about 52 m.

Since my undergraduate and graduate physics days, it has always been somewhat puzzling to me what expressions like "1 cm" and "1 inch" and "52 m" *refer* to. Not numbers, to be sure. But not physical objects, either. It also seems clear that the linguistic expression "5159 cm" must refer to the real number 5159 "multiplied" by whatever the expression "1 cm" refers to—but what is this "multiplication", exactly?[5]

I wish to explain what these mean. More generally, I wish to explain what "*the length of* ..." means, by working up from geometry as our background and introducing lengths as abstract entities, introduced by an abstraction principle over the geometrical equivalence relation of congruence ("having the same length as"). Thereby, in the process, we may explain how the above measurement reports are to be analysed.

Towards this end, in this paper, I intend to establish six main results, described in a moment, which I hope clarify these questions.

We shall adopt the following basic assumptions.

[5]As it turns out, it is a form of scalar multiplication by a real, just as one has, e.g., $3\mathbf{v} + 7\mathbf{w}$ in a vector space.

(1) We assume as primitives unary predicates $\texttt{point}(p)$ and $\texttt{length}(l)$, with the set $\mathbb{P}$ defined as $\{p \mid \texttt{point}(p)\}$ and the set $\mathbb{L}$ defined as $\{l \mid \texttt{length}(l)\}$.

(2) We assume as geometrical primitives a 3-place predicate $\texttt{Bet}$, where $\texttt{Bet}(a, b, c)$ means: "the point b lies on a straight line (inclusively) between the points a and c", and a 4-place predicate $\equiv$, where $ab \equiv cd$ means: "the segment ab is as long as the segment cd is" (or, equivalently, "the segments ab and cd are congruent")

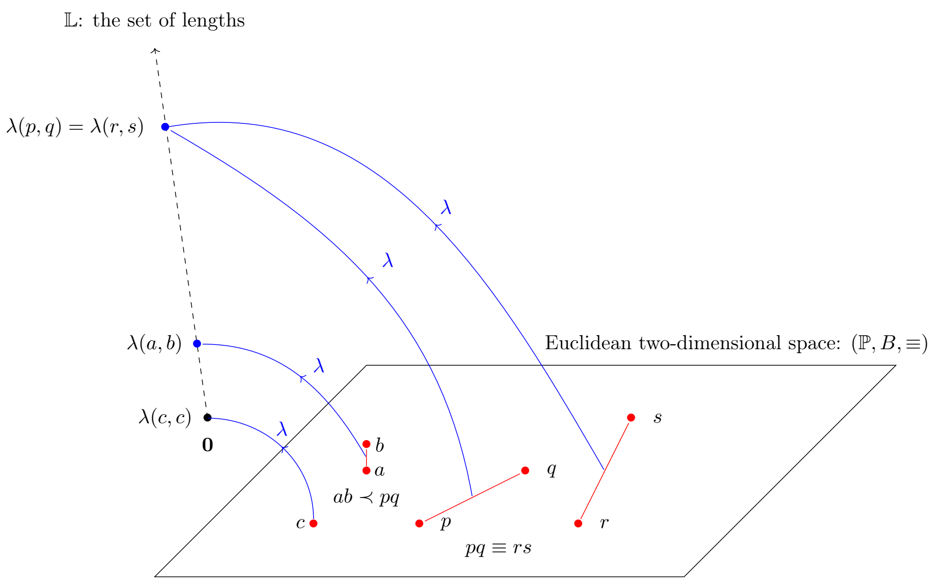
(3) We assume that the system $(\mathbb{P}, B, \equiv)$ of points, equipped with the geometrical betweenness relation B and the geometrical congruence relation $\equiv$, satisfies the axioms of two-dimensional (synthetic) Euclidean geometry, which we denote $\mathsf{EG}(2)$ below (The axioms are given in §3.1. Here B is the physical extension of $\texttt{Bet}$. And $\equiv$ is the physical extension of $\equiv$.) So, we assume $(\mathbb{P}, B, \equiv)$ is a (full) model of $\mathsf{EG}(2)$.

(4) We assume that, on the points, is defined a binary function λ, mapping a pair (a, b) of points (i.e., a segment) to a *length*:

$$\lambda : \mathbb{P}^2 \to \mathbb{L} \tag{2}$$

The figure below indicates our informal picture of the length function λ:

Figure 3: Lengths of Segments



Each segment (which can be identified with the ordered pair of its endpoints) is mapped by λ to a length in $\mathbb{L}$. We assume that this function λ satisfies the "length abstraction axioms":

$$(L1) \quad (\forall a, b, c, d \in \mathbb{P}) \, (\lambda(a,b) = \lambda(c,d) \leftrightarrow ab \equiv cd) \tag{3}$$

$$(L2) \quad (\forall l \in \mathbb{L}) \, (\exists a, b \in \mathbb{P}) \, (l = \lambda(a,b)) \tag{4}$$

We do not assume that a length is a *number*. Rather, a length $\lambda(a,b)$ is a *sui generis* abstract object associated with a segment ab between two points a and b. Indeed, for a null segment cc, we have: $\lambda(c,c) = \mathbf{0}$, a zero length, which is not a number. Given that $\lambda(a,b)$ is not a number, we need to figure out how $\lambda(a,b)$ is connected to, or represented by, a number. To this end, we aim to understand how to connect these *abstract lengths*, and *coordinate lengths* (i.e., real numbers)—which are the objects that physicists and engineers and architects measure and write down in measurement databases.

Given a Cartesian coordinate system $\Phi : \mathbb{P} \to \mathbb{R}^2$, one may define the *coordinate length* $\Delta_\Phi(a,b)$ of the segment ab. One may also define a "connecting function" $h_\Phi : \mathbb{L} \to \mathbb{R}^+$ such that, for all points $a, b \in \mathbb{P}$:[6]

$$h_\Phi(\overbrace{\lambda(a,b)}^{\text{length}}) = \overbrace{\Delta_\Phi(a,b)}^{\text{coordinate length}} \tag{5}$$

This function thus relates the (abstract) *lengths* and *coordinate lengths*, according to Φ. For example, such a function will relate the length (i.e., height) of Nelson's Column to the number 51.59.

We next define, using geometrical methods in Euclidean geometry, a length addition operation ($\oplus$), a linear order ($\preceq$), and a scalar multiplication by non-negative reals ($\bullet$). Assembled together, we obtain a certain structure, the *Length quantity*:[7]

$$\mathbb{L} := (\mathbb{L}, \oplus, \preceq, \bullet). \tag{6}$$

(Here, I follow the usual mathematical practice of conflating the name of a *structure* with the name of its underlying *carrier set*.)

Separately, we shall define the notion of a (continuous) "*extensive quantity*" and a "*positive extensive quantity*". In particular, a *measurement scale* is defined simply be an isomorphism from such a quantity to the *positive cone* of the linearly ordered vector space $(\mathbb{R}, +, \leq, \bullet)$.

Our first result is that, for any Cartesian coordinate system Φ,

[6] In this paper, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid 0 \leq x\}$ and so includes 0. If I wish to remove 0, I write: $\mathbb{R}^+ - \{0\}$.

[7] There is a minor conflation, in all measurement theory literature between the notion of *Length*—as a quantity or quantitative property—and the notion of specific or individual lengths, like 3 cm and 25 m. I shall simply acquiesce in this, but sometimes capitalize, using "Length" to mean the quantitative property as a whole, and "length" to mean a specific element of Length. Some authors, and this is standard in physics, instead say that *Length* is a "*quantitative property*" (e.g., Eddon (2013)) or say that *Length* is a "*physical attribute*". On our view, *Length* is (the positive cone of) a linearly ordered vector space, and specific lengths are "vectors" in that space, and units are basis vectors.

(Theorem 1) $\quad h_\Phi$ is a measurement scale for $\mathbb{L}$.

And then, from Theorem 1, it follows that:

(Theorem 2) $\quad \mathbb{L}$ is a positive extensive quantity.

Third, we establish, using the theory of extensive quantities, that, given any length unit $\mathbf{u} \in \mathbb{L}$ (i.e., a positive length $\mathbf{u}$ not equal to $\mathbf{0}$), for all any $a, b \in \mathbb{P}$:

(Theorem 3) $\quad \lambda(a,b) = \|\lambda(a,b)\|_{\mathbf{u}} \bullet \mathbf{u}$

Next, we establish the following connections between coordinate lengths and lengths:

(Theorem 4) $\quad \Delta_\Phi(a,b) = \|\lambda(a,b)\|_{\mathbf{u}_\Phi}$
(Theorem 5) $\quad \lambda(a,b) = \Delta_\Phi(a,b) \bullet \mathbf{u}_\Phi$

Finally, we obtain a description of the *automorphism group* of the*Length*quantity:

(Theorem 6) $\quad \mathrm{Aut}(\mathbb{L}) \cong (\mathbb{R}^+ - \{0\}, \times)$.

So, $\mathrm{Aut}(\mathbb{L})$ is the multiplicative group of positive reals.
These six theorems will appear bundled together in §4.

2 A Simple Theory of (Continuous) Extensive Quantity

The theory (more exactly, a few definitions, followed by lemmas) given in this section pertains to *continuous extensive quantities*. Indeed, Nature seems to provide us with*Mass* (M), *Length* (L) and *Time* (T), and these seem to be precisely such continuous extensive quantities. This is, of course, an empirical claim, and is subject to revision.[8]

Our first empirical, or scientific, assumption is that Euclidean geometry EG holds on the system of points with respect to primitive betweenness and congruence relations.[9] And our second assumption is that lengths are "implicitly defined" by the length abstraction principle. We can define an addition operation on the lengths, an ordering of the lengths, and a scalar multiplication of the lengths by non-negative reals, and prove that the length structure obtained is indeed a (continuous, positive) extensive quantity. But I shall generally drop the qualifier "continuous" below.

[8]However, one notes that Nature also provides us with electrical charge, which is discrete, and seems to be also extensive in a certain sense (charges, in some sense, can be "added"; there is a zero charge; it is ordered; and one has a scalar multiplication by *integers*). This structure, to speak abstractly, is therefore related to the ordered group of integers $(\mathbb{Z}, +, \leq, \bullet)$, equipped with a scalar multiplication $\bullet$ by integers. Note that the group $(\mathbb{Z}, +)$ has only two automorphisms (i.e., $z \mapsto z$ and $z \mapsto -z$), and the ordered group $(\mathbb{Z}, +, \leq)$ is rigid, and hence its symmetry group is trivial. On the other hand, the symmetry group for *Length* $(\mathbb{L})$, as we shall see, turns out to be $(\mathbb{R}^+ - \{0\}, \times)$, the multiplicative group of positive reals. As I understand it, in the recent monograph Wolff (2020), Joanna Wolff suggests that quantities are structures whose symmetry groups are Archimedean groups. I'm unsure how this applies to charge, since its symmetry group is trivial (it has one element: the identity map). But I am possibly not understanding the proposal.

[9]The relevant axioms for EG are given below, in §3.1.

Definition 1 (Extensive quantity)**.** A *(continuous) extensive quantity*

$$\mathcal{E} = (E, \oplus, \preceq, \bullet) \tag{7}$$

is defined to be a *one-dimensional linearly ordered vector space* over $\mathbb{R}$. Any element of E is also called a *specific, or individual, quantity* (in $\mathcal{E}$).

As mentioned above, we drop the qualifier "continuous".[10]

Lemma 1. If $\mathcal{E}$ is an extensive quantity, then $\mathbf{0}$ is the unique $q \in E$ such that, for all quantities $q' \in E$, $q' + \mathbf{0} = q'$. (I.e., the quantity $\mathbf{0}$ is the "zero vector" in $\mathcal{E}$.)

Definition 2 (Standard coordinate extensive quantity)**.** We define the linearly ordered one-dimensional vector space:[11]

$$\mathcal{E}_0 \quad := \quad (\mathbb{R}, +, \leq, \bullet). \tag{8}$$

We call $\mathcal{E}_0$ "*the standard coordinate extensive quantity*".

Lemma 2. All extensive quantities are isomorphic to $\mathcal{E}_0$.

Proof. Suppose V and V' are vector spaces over $\mathbb{R}$ of dimension n, with n a positive integer. It is a theorem of linear algebra that V is isomorphic to V'.[12] Consequently (we ignore the ordering for a moment), if $\mathcal{E}$ and $\mathcal{E}'$ are vector spaces of dimension 1, then they are isomorphic (as vector spaces), and in this case, isomorphic to $\mathbb{R}$.

Next, it is also a theorem of linear algebra that there exists exactly one linear ordering on the vector space $\mathbb{R}$. Let $(V, \preceq)$ and $(V, \preceq')$ be linearly ordered one-dimensional vector space over $\mathbb{R}$. Let u be a basis for V. Define relations R, R' on $\mathbb{R}$ by the following: for any reals, x, y, $xRy := x \bullet u \preceq y \bullet u$ and $xR'y := x \bullet u \preceq' y \bullet u$. Then $(\mathbb{R}, R)$ and $(\mathbb{R}, R')$ are *ordered fields*. Hence, $R = R'$. By cancellation, $\preceq = \preceq'$. Consequently, if $\mathcal{E}$ and $\mathcal{E}'$ are ordered vector spaces of dimension 1, then they are mutually isomorphic as ordered vector spaces. $\mathcal{E}_0$ is a linearly ordered vector space of dimension one. Hence, all extensive quantities are isomorphic to $\mathcal{E}_0$.

□

Definition 3 (Unit)**.** Let $\mathcal{E} = (E, \oplus, \preceq, \bullet)$ an extensive quantity. Then any element $\mathbf{u} \in E$ with $\mathbf{0} \prec \mathbf{u}$ is called a "*unit*".

[10] The reason it may be called "continuous" is that the ordering reduct $(E, \preceq)$ of an extensive quantity, as defined, is isomorphic to the standard continuous ordering $(\mathbb{R}, \leq)$ of the real numbers. In other words, $(E, \preceq)$ is a separable, order-complete DLOWE (dense linear order without endpoints). It is a classic result of order theory (Cantor's Isomorphism Theorem) that any countable DLOWE is isomorphic to $(\mathbb{Q}, \leq)$. And it is a classic result of order theory that, up to isomorphism, there is exactly one such separable, order-complete DLOWE: it is isomorphic to $(\mathbb{R}, \leq)$.

[11] $+$ here is the vector addition, $\leq$ is the linear order, and $\bullet$ is the scalar multiplication.

[12] This is quite a simple result, and holds for any base field F: if V is a vector space over F of dimension n, then $V \cong F^n$. See, e.g., Dummit & Foote (2004): 411 (Theorem 6).

Any unit is nothing more than a *positive basis vector* in the ordered vector space.

Definition 4 (Measurement scale)**.** A *measurement scale* for an extensive quantity $\mathcal{E} = (E, \oplus, \preceq, \bullet)$ is an *isomorphism*

$$h : \mathcal{E} \to \mathcal{E}_0 \tag{9}$$

In particular, if $h : \mathcal{E} \to \mathcal{E}_0$ is an isomorphism, the following *isomorphism conditions* hold:[13]

(1) $h : E \to \mathbb{R}$ is a bijection.
(2) For any $q_1, q_2 \in E$: $h(q_1 \oplus q_2) = h(q_1) + h(q_2)$.
(3) For any $q_1, q_2 \in E$: $q_1 \preceq q_2 \leftrightarrow h(q_1) \leq h(q_2)$.
(4) For any $q \in E$, any $x \in \mathbb{R}$: $h(x \bullet q) = x \cdot h(q)$.

Definition 5. Meas.Scale($\mathcal{E}$) is the class of measurement scales on a fixed extensive quantity $\mathcal{E}$.

Lemma 3 (Closure)**.** Let $h : \mathcal{E} \to \mathcal{E}_0$ be a measurement scale. Let $c > 0$ be a real. Define $h' : \mathcal{E} \to \mathcal{E}_0$ by: for all $q \in E$, $h'(q) = c \cdot h(q)$. Then $h' : \mathcal{E} \to \mathcal{E}_0$ is a measurement scale.

Proof. Multiplication of reals by a fixed non-zero real yields a bijection, which implies that $h' : E \to \mathbb{R}$ is a bijection. Then what I like to call an "equation stream" as follows verifies the isomorphism condition for h' wrt $\oplus$:

$$\begin{aligned}
h'(q_1 \oplus q_2) &= c \cdot h(q_1 \oplus q_2)) && (10)\\
&= h(c \bullet (q_1 \oplus q_2)) && (11)\\
&= h(c \bullet q_1 \oplus c \bullet q_2) && (12)\\
&= h(c \bullet q_1) + h(c \bullet q_2) && (13)\\
&= c \cdot h(q_1) + c \cdot h(q_2) && (14)\\
&= h'(q_1) + h'(q_2) && (15)
\end{aligned}$$

A similar equation stream verifies the isomorphism conditions for h' wrt $\preceq$ and $\bullet$. $\square$

So, the class Meas.Scale($\mathcal{E}$) of measurement scales on a fixed extensive quantity $\mathcal{E}$ is closed under multiplication by a positive real.

Lemma 4 (Uniqueness Theorem for Scales)**.** Let $\mathcal{E}$ be a given extensive quantity. Let $h, h' \in$ Meas.Scale($\mathcal{E}$). I.e.,

[13] The smaller dot symbol "$\cdot$" here, in the term "$x \cdot h(q)$" below, is simply the multiplication of real numbers, inside the underlying field, $\mathbb{R}$. I use it because the term "$xh(q)$" might be a bit confusing.

$$h, h' : \mathcal{E} \to \mathcal{E}_0 \tag{16}$$

are measurement scales for $\mathcal{E}$. Then there exists a constant $c \in \mathbb{R}$ with $c > 0$ such that, for all quantities $q \in E$,

$$h'(q) = c \cdot h(q) \tag{17}$$

Proof. Let $h, h' : \mathcal{E} \to \mathcal{E}_0$ be our given isomorphisms. We define $f := h' \circ h^{-1}$. Then

$$f : \mathcal{E}_0 \to \mathcal{E}_0 \tag{18}$$

is an *automorphism* of $\mathcal{E}_0 = (\mathbb{R}, +, \leq, \bullet)$. So, f is a bijection $\mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, we have:

$$f(x) \leq f(y) \leftrightarrow x \leq y \tag{19}$$
$$f(x + y) = f(x) + f(y) \tag{20}$$

The first equation implies that f is *monotonic.* The second equation is Cauchy's Additive Functional Equation. It is a famous theorem of analysis that if $f : \mathbb{R} \to \mathbb{R}$ is monotonic and f satisfies Cauchy's Additive Functional Equation, then $f(x) = c \cdot x$, for some $c > 0$ (i.e., $c = f(1)$).

Hence, there is a $c > 0$ such that, for all $x \in \mathbb{R}$, we have:

$$f(x) = c \cdot x \tag{21}$$

Thus,

$$(h' \circ h^{-1})(x) = h'(h^{-1}(x)) = c \cdot y \tag{22}$$

So, letting $q = h^{-1}(x)$, we have:

$$h'(q) = c \cdot h(q) \tag{23}$$

as claimed.

$\square$

Lemma 5. Let $h : \mathcal{E} \to \mathcal{E}_0$ be a measurement scale for an extensive quantity $\mathcal{E}$. There is a unique $q \in E$ such that $h(q) = 1$. Moreover, $\mathbf{0} \prec q$.

Proof. Since h is an isomorphism, we have a bijection $h : E \to \mathbb{R}$. Since h is a surjection, we have $h(q) = 1$, for some $q \in E$; and since h is an injection, this q is unique. Obviously $0 < 1$. Since h is an isomorphism, it follows that $h^{-1}(0) \prec h^{-1}(1)$. Hence, $\mathbf{0} \prec q$. □

Lemma 6. Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be any unit. Let $q \in E$. Then there is unique $x \in \mathbb{R}$ such that:

$$q = x \bullet \mathbf{u} \tag{24}$$

Proof. Since $\mathcal{E}$ is an extensive quantity, we have an isomorphism $\mathcal{E} \overset{h}{\cong} (\mathbb{R}, +, \leq, \bullet)$. Let $\mathbf{u}$ be any unit. Let $q \in E$.

Given h, let $w = h(\mathbf{u})$ and let $y = h(q)$. Since $\mathbf{u}$ is a unit, $\mathbf{0} \prec \mathbf{u}$, and hence, by the isomorphism, $0 < h(\mathbf{u})$. So, $0 < w$ and we can divide by w. Let

$$x := \frac{y}{w} \qquad \left(= \frac{h(q)}{h(\mathbf{u})}\right) \tag{25}$$

So, $wx = y$. I.e., $xh(\mathbf{u}) = h(q)$. Since h is an isomorphism, it preserves scalar multiplication $\bullet$. So, $h(x \bullet \mathbf{u}) = h(q)$. And therefore, by injectivity of h, we have: $x \bullet \mathbf{u} = q$, as claimed.

□

Definition 6. Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. Let $q \in E$. Then $\|q\|_{\mathbf{u}}$ is defined to be the unique $x \in \mathbb{R}$ such that $q = x \bullet \mathbf{u}$.

Definition 7 (Magnitude of a quantity)**.** Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. The real number $\|q\|_{\mathbf{u}}$ is called "*the magnitude of the quantity q, with respect to* $\mathbf{u}$".

Lemma 7 (Magnitude lemma)**.** Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. Let $q \in E$. Then

$$q = \|q\|_{\mathbf{u}} \bullet \mathbf{u}. \tag{26}$$

Lemma 8 (Invariance under change of unit)**.** Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$, $\mathbf{u}'$ be units for $\mathcal{E}$. Let $q \in E$. Then

$$\|q\|_{\mathbf{u}} \bullet \mathbf{u} = \|q\|_{\mathbf{u}'} \bullet \mathbf{u}' \tag{27}$$

Proof. This is immediate from The Magnitude Lemma. □

Lemma 9 (Division)**.** Let $\mathcal{E}$ be an extensive quantity. Let $q_1, q_2 \in E$ be quantities where $q_2 \neq \mathbf{0}$. Let $\mathbf{u}, \mathbf{u}'$ be units. Then:

$$\frac{\|q_1\|_{\mathbf{u}}}{\|q_2\|_{\mathbf{u}}} = \frac{\|q_1\|_{\mathbf{u}'}}{\|q_2\|_{\mathbf{u}'}} \tag{28}$$

Proof. Since $\mathbf{u}$, $\mathbf{u}'$ are both units, there is unique $x > 0$ such that:

$$\mathbf{u}' = x \bullet \mathbf{u} \tag{29}$$

Taking the component magnitudes relative to the units, we have:

$$\|q_1\|_{\mathbf{u}} \bullet \mathbf{u} = \|q_1\|_{\mathbf{u}'} \bullet \mathbf{u}' = x\|q_1\|_{\mathbf{u}'} \bullet \mathbf{u} \tag{30}$$
$$\|q_2\|_{\mathbf{u}} \bullet \mathbf{u} = \|q_2\|_{\mathbf{u}'} \bullet \mathbf{u}' = x\|q_2\|_{\mathbf{u}'} \bullet \mathbf{u} \tag{31}$$

And thus (recall that $\mathcal{E}$ is a vector space, and $\mathbf{u}$ is a basis vector)

$$\|q_1\|_{\mathbf{u}} = x\|q_1\|_{\mathbf{u}'} \tag{32}$$
$$\|q_2\|_{\mathbf{u}} = x\|q_2\|_{\mathbf{u}'} \tag{33}$$

The magnitudes $\|q_2\|_{\mathbf{u}}$ and $\|q_2\|_{\mathbf{u}'}$ are non-zero, by hypothesis. Dividing, we have:

$$\frac{\|q_1\|_{\mathbf{u}}}{\|q_2\|_{\mathbf{u}}} = \frac{\|q_1\|_{\mathbf{u}'}}{\|q_2\|_{\mathbf{u}'}} \tag{34}$$

□

Definition 8 (Division)**.** Let $\mathcal{E}$ be an extensive quantity. Let $q_1, q_2 \in E$ be quantities where $q_2 \neq \mathbf{0}$. We define:

$$\frac{q_1}{q_2} := \frac{\|q_1\|_{\mathbf{u}}}{\|q_2\|_{\mathbf{u}}} \tag{35}$$

where $\mathbf{u}$ is any unit.

This definition yields a unique result (i.e., is independent of the unit chosen) by the previous lemma.

Definition 9 (Unit of a scale)**.** Let $h : \mathcal{E} \to \mathcal{E}_0$ be a measurement scale for an extensive quantity $\mathcal{E}$. The unique $q \in E$ such that $h(q) = 1$ is called "*the unit of the measurement scale* h". This unit is denoted $\mathbf{1}_h$.

Lemma 10. *$q = \mathbf{1}_h$ if and only if $h(q) = 1$.*

Proof. Let $q = \mathbf{1}_h$. Then $h(q) = h(\mathbf{1}_h) = 1$ by the above definition. Conversely, suppose $h(q) = 1$. Let $\mathbf{1}_h$ be the unit of h. Thus, $h(\mathbf{1}_h) = 1$. This implies: $h(q) = h(\mathbf{1}_h)$. Then, by injectivity of h, we have: $q = \mathbf{1}_h$. □

Lemma 11. *Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. Let measurement scales h, h' be such that $h(\mathbf{u}) = 1$ and $h'(\mathbf{u}) = 1$. Then $h = h'$.*

Proof. Let $q \in E$. We have unique x such that $q = x \bullet \mathbf{u}$. So, we infer

$$h(q) = h(x \bullet \mathbf{u}) = xh(\mathbf{u}) = x \quad \text{and} \quad h'(q) = h'(x \bullet \mathbf{u}) = xh'(\mathbf{u}) = x \tag{36}$$

So, for all $q \in E$, $h(q) = h'(q)$. Thus, $h = h'$.

□

Lemma 12. Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. Then there is a unique measurement scale h such that $h(\mathbf{u}) = 1$.

Proof. Suppose $\mathcal{E}$ is an extensive quantity and $\mathbf{u}$ a unit. Since $\mathcal{E}$ is an extensive quantity, we have an isomorphism $\mathcal{E} \overset{h'}{\cong} (\mathbb{R}, +, \leq, \bullet)$. Let $\mathbf{1}_{h'}$ be its unit. So, $h'(\mathbf{1}_{h'}) = 1$. Let $x = \|\mathbf{u}\|_{\mathbf{1}_{h'}}$. So,

$$\mathbf{u} = x \bullet \mathbf{1}_{h'} \tag{37}$$

We define h as follows:

$$h(q) = \frac{1}{x} h'(q) \tag{38}$$

Now h is a measurement scale, since $x \neq 0$.

$$h(\mathbf{u}) = h(x \bullet \mathbf{1}_{h'}) = xh(\mathbf{1}_{h'}) = x\frac{1}{x}h'(\mathbf{1}_{h'}) = h'(\mathbf{1}_{h'}) = 1 \tag{39}$$

This measurement scale is unique by the previous lemma.

□

Definition 10. Let $\mathcal{E}$ be an extensive quantity. Let $\mathbf{u}$ be a unit. The unique measurement scale h such that $h(\mathbf{u}) = 1$ is denoted: $h_{\mathbf{u}}$. This is called "*the measurement scale for the unit* $\mathbf{u}$".

Lemma 13 (Scale invariance)**.** Let $\mathcal{E}$ be an extensive quantity and let $h, h' : \mathcal{E} \to \mathcal{E}_0$ be measurement scales for $\mathcal{E}$. Then:

$$\|q\|_{\mathbf{1}_{h'}} \bullet \mathbf{1}_{h'} = \|q\|_{\mathbf{1}_h} \bullet \mathbf{1}_h. \tag{40}$$

Proof. This is immediate from The Magnitude Lemma. □

Lemma 14. Let $\mathcal{E}$ be an extensive quantity. Let $\pi : \mathcal{E} \to \mathcal{E}$ be an automophism of $\mathcal{E}$. Then there is a unique $c > 0$, such that, for all $q \in E$,

$$\pi(q) = c \bullet q. \tag{41}$$

Proof. Let $\pi : \mathcal{E}$ be an automophism of $\mathcal{E}$. Because $\mathcal{E}$ is isomorphic to $\mathcal{E}_0$, it follows that any automorphism $\pi : \mathcal{E}$ can be factored as:

$$\pi = (h')^{-1} \circ h. \tag{42}$$

But, we know that there exists $c > 0$ such that, for all $q \in E$, $((h')^{-1} \circ h)(q) = c \bullet q$, with $c > 0$. Hence, for any automorphism π, there is a unique $c > 0$, such that, for all $q \in E$, $\pi(q) = c \bullet q$. □

Lemma 15 (Automorphism group)**.** Let $\mathcal{E}$ be an extensive quantity. Then:

$$\mathrm{Aut}(\mathcal{E}) \cong (\mathbb{R}^+ - \{0\}, \times). \tag{43}$$

Proof. Let $\pi : \mathcal{E}$ be an automophism of $\mathcal{E}$. By Lemma 14, there is a bijection $c \mapsto \pi_c$, between the automorphisms $\mathrm{Aut}(\mathcal{E})$ and the positive reals $\mathbb{R}^+ - \{0\}$. Let π_c be the automorphism corresponding to $c > 0$.

By defining composition $\circ$ of automorphisms in the obvious way, it follows that, for any $c_1, c_2 \in \mathbb{R}^+ - \{0\}$,

$$\pi_{c_1} \circ \pi_{c_2} = \pi_{c_1 \times c_2}. \tag{44}$$

Hence, $\mathrm{Aut}(\mathcal{E}) \cong (\mathbb{R}^+ - \{0\}, \times)$. □

Next, we introduce the derivative notion of a *positive extensive quantity.* This requires the mathematical notions of a *cone*, and *positive cone* in an ordered vector space.

Definition 11 (Cone)**.** Let $V = (V, \oplus, \bullet)$ be a vector space over $\mathbb{R}$. A subset $C \subseteq V$ is called a *cone* if it is closed under multiplication by a positive real: i.e., for all $x > 0$, $x \bullet C \subseteq C$. A cone is called *pointed* if it contains the origin. A cone C is called *convex* if it is closed under vector addition: $C \oplus C \subseteq C$. Let $(V, \preceq)$ be an ordered vector space. The subset $V^+ = \{v \in V \mid \mathbf{0} \preceq v\}$ is a pointed convex cone with vertex $\mathbf{0}$. V^+ is called the *positive cone* of V. It is denoted by $\mathtt{PosCone}(V)$.

Fortunately, we are dealing with a very simple one-dimensional vector space over $\mathbb{R}$, unique up to isomorphism. I.e., $\mathcal{E}_0$. In this case, we do not have to deal with all the complexities of the theory of cones and positive cones, aside from this simple case.

Definition 12. Let $\mathbb{R}^+ := \{x \in \mathbb{R} \mid 0 \leq x\}$. Then $\mathtt{PosCone}(\mathcal{E}_0) := (\mathbb{R}^+, +, \leq, \bullet)$.

Definition 13 (Standard coordinate positive extensive quantity)**.** We define $\mathcal{E}_0^+$ to be $\mathtt{PosCone}(\mathcal{E}_0)$. We call $\mathcal{E}_0^+$ "*the standard coordinate positive extensive quantity*".

Definition 14 (Positive extensive quantity)**.** A *(continuous) positive extensive quantity* $\mathcal{E}^+$ is a structure isomorphic to $\mathcal{E}_0^+$. Below we shall sometimes call this a CPEQ.

Definition 15 (Measurement scale)**.** A *measurement scale* for a (continuous) positive extensive quantity $\mathcal{E}^+$ is an isomorphism

$$h : \mathcal{E}^+ \to \mathcal{E}_0^+ \tag{45}$$

Definition 16 (Unit)**.** A *unit* for a (continuous) positive extensive quantity $\mathcal{E}^+$ is an element $\mathbf{u}$ such that $\mathbf{0} \prec \mathbf{u}$.

Then, Lemmas 1–15 also hold for (continuous) positive extensive quantities, with suitable adjustments.

3 Length Abstraction in Euclidean Geometry

What is the point of all these definitions and lemmas in §2, you ask?

First, it seems to be true that the base quantities, M, L, T, in physics all seem to be CPEQs: continuous positive extensive quantities. I.e., (isomorphic to) the positive cone of the linearly ordered vector space over $\mathbb{R}$. This explains why we see the following relationship between a specific length, its magnitude relative to a unit and the unit itself (and likewise for a specific temporal duration or a specific mass):

$$\text{The Magnitude Lemma} \quad q = \|q\|_{\mathbf{u}} \cdot \mathbf{u} \tag{46}$$

E.g.,

$$5\,\text{cm} = \|5\,\text{cm}\|_{\mathbf{1}_{\text{cm}}} \cdot \mathbf{1}_{\text{cm}} \tag{47}$$

$$3\,\text{kg} = \|3\,\text{kg}\|_{\mathbf{1}_{\text{kg}}} \cdot \mathbf{1}_{\text{kg}} \tag{48}$$

And relationships between different units, such as:

$$\mathbf{1}_{\text{cm}} = 0.01 \cdot \mathbf{1}_{\text{m}} \tag{49}$$

$$\|\mathbf{1}_{\text{cm}}\|_{\mathbf{1}_{\text{m}}} = 0.01 \tag{50}$$

Which then give, for example:

$$\|5\,\text{cm}\|_{\mathbf{1}_{\text{m}}} = 0.05 \tag{51}$$

$$5\,\text{cm} = \|5\,\text{cm}\|_{\mathbf{1}_{\text{cm}}} \cdot \mathbf{1}_{\text{cm}} = \|5\,\text{cm}\|_{\mathbf{1}_{\text{m}}} \cdot \mathbf{1}_{\text{m}} = 0.05 \cdot \mathbf{1}_{\text{m}} \tag{52}$$

But we have, as yet, not precisely defined how the system of lengths behaves or even what it is. The idea next developed is that we shall begin with the axioms for Euclidean geometry, expressed synthetically. We then use *abstraction axioms* to define the system

$\mathbb{L}$ of abstract lengths, with its operation $\oplus$, its relation $\preceq$ and its scalar multiplication $\bullet$ (by positive reals). And from this, we can prove that the result system is a (continuous) positive extensive quantity.

We shall begin with two dimensional synthetic Euclidean geometry $\mathsf{EG}(2)$ and suitable abstraction axioms for lengths:

$$(L) \quad \lambda(a,b) = \lambda(c,d) \leftrightarrow ab \equiv cd \tag{53}$$

Taking (L) as the basic axiom from which one introduces length has been mentioned several times before. An example, from Paolo Mancosu's monograph *Abstraction and Infinity*:

> Consider the notion of equality of segments in Euclid. Starting from the congruence relation between segments, a contemporary mathematician might naturally introduce length using a definition by abstraction such as '$\lambda(a) = \lambda(b)$ iff a is congruent to b' (with the option of explicitly defining length by means of equivalence classes or other devices or simply accepting lengths as new entities, as Peano does). But Euclid does not do this and he simply says, in common notion 4, that two segments are equal if they 'coincide with one another' ('Things which coincide with one another are equal to one another'). Then in the midst of the proof of proposition I.4 we find the converse being implicitly used for segments ('if two segments are equal they coincide with one another'). Is the notion of equality of segments taken to be primitive or is it introduced by abstraction (for it is not defined explicitly)? If we exclude the former case then, if there is a definition by abstraction of equality of segments, it is at best implicit, for what we are originally given is not a definition introduced by an 'if and only if' (and a fortiori not a definition by abstraction). Moreover, there is no mention of the class of segments that have in common the property of being congruent (which in amore contemporary setting could be used to define $\lambda(a)$, namely the length of a). (Mancosu (2016): 23)

Mancosu also mentions multiple authors from the period roughly 1850 through to 1910 at least *saying* rather similar things about the use of abstraction, for the case of length. I emphasize "saying" though. Because what is striking, however, is that, aside from these many mentions of the relevant abstraction principle (for length), it seems that nowhere has the resulting theory been worked out explicitly. Mancosu mentions, "...accepting lengths as new entities, as Peano does as Peano does", but I'm not directly familiar with the specific work in mind. So, modulo that, this approach has not been worked out for *lengths*. Here, the aim is to work out that missing theory.

The method of definition by abstraction—upon discovering an equivalence relation, $\sim$, considering the equivalence classes: these will, by construction, satisfy $[a] = [b] \leftrightarrow a \sim b$—is perfectly routine in modern mathematics, although it was novel when versions of it appeared in the works of Cantor, Dedekind and Frege. In particular, Frege's analysis of "natural number" is grounded in the equivalence relation of *equinumerosity of finite sets*.[14]

[14]See Frege (1884). Frege's overall project did flounder, as Bertrand Russell discovered an inconsistency

Likewise, Cantor's analysis of cardinals and ordinals. And the standard construction of the integers and the rationals, starting with the natural numbers, equally proceeds via abstraction with respect to an equivalence relation.[15]

We too will work with an abstraction axiom: (L). Using this, and a separate axiom (stating that, for each length l, there exist points a, b such that $l = \lambda(a, b)$), we may, using various explicit definitions, *define* the "*Length* quantity" (or, equivalently, the "*Length* attribute", or the "*Length* property"):

$$\mathbb{L} = (\mathbb{L}, \oplus, \preceq, \bullet) \tag{54}$$

Then we shall prove a main theorem, Theorem 2, that $\mathbb{L}$ is a positive extensive quantity. We can then piggyback on the results in §2 to see how (abstract) lengths, coordinate systems, numerical coordinate lengths, and units are all inter-related.

In §3.1 and §3.2, I need to make use of a series of established results concerning the axiomatization of synthetic geometry and representation theorems related to synthetic geometry. Especially the following three: Lemma 18, Lemma 19 and Lemma 23. Lack of space prevents providing proofs of these. Instead, I provide guidance to the literature.

3.1 Axioms

Our approach is axiomatic.

> In constructing an axiomatic theory T, we usually make use of other axiomatic theories which are *presupposed* in the following sense: all the primitive notions in the presupposed theory are included in the system of primitive notions of T, and all the axioms of those theories are included in the axiom system of T. Mathematical theories presuppose as a rule mathematical logic and usually also set theory (to a larger or smaller extent). In developing geometry in this book we presuppose mathematical logic, set theory and the arithmetic of the real numbers (which can either be treated as an independent theory or can be constructed as a portion of set theory). An axiomatic treatment of these theories can be found in various special works. (Borsuk & Szmielew (1960): 6–7)

I intend to do precisely the same. But in the times when the idea of a scientific theory as an axiomatic system has fallen out of fashion, it's worth including the details of these

in May 1901 and informed Frege about it in 1902. Thereupon Frege gave up, unable to properly locate the source of the problem. But Frege's analysis of what natural numbers are was not the real culprit. Rather it was Frege's background analysis of "sets, classes and concepts" which contained an inconsistency: essentially, Basic Law V: $\epsilon(X) = \epsilon(Y) \to X = Y$. This is inconsistent. But if we drop this and use a different abstraction axiom, $\texttt{card}(X) = \texttt{card}(Y) \to X \cong Y$, now often called "Hume's Principle" (HP), everything works and one can prove the axioms of second-order arithmetic from (HP) using second-order logic. For detailed presentations of the precise derivation of second-order arithmetic from Frege's "logicist" assumptions, see Heck (2011) and Zalta (2021).

[15] Such matters relating to cardinals, ordinals and the construction of the number systems, are generally explained, in some detail, in a standard set theory textbook, such as Halmos (1974) or Drake & Singh (1996).

presuppositions. It's not in fact difficult to in fact state what the presupposed set theory is, as follows:[16]

Definition 17 (Ambient set theory)**.** We adopt an ambient set theory, with urelements (or atoms), which we'll call AM ("applied mathematics"). It is three-sorted, $\{\texttt{atom}, \texttt{class}, \texttt{global}\}$, where the sorts $\texttt{atom}$ and $\texttt{class}$ are assumed disjoint and exhaustive, and both of these are treated as *subsorts* of $\texttt{global}$. In the strictest syntactic sense, the sort $\texttt{atom}$ has variables a_i; the sort $\texttt{class}$ has variables X_i; and the sort $\texttt{global}$ has variables x_i. But we shall violate this stricture almost immediately (e.g., using p, q, r, s, a, b, c, d as variables for points (atoms), using $l, l_1, l_2, \dots$ are variables for lengths (also atoms), and various other dedicated variables, such as, e.g., "Φ", and "Φ'" for coordinate systems). The single primitive is the binary membership predicate $\in$. We assume that all standard pure mathematical notions have been defined inside AM, via some standard implementation.[17]

A suitable axiom system AM is:[18]

Partition	$\texttt{atom}(x) \leftrightarrow \neg\texttt{class}(x)$
Atoms	$\texttt{atom}(x) \to (\texttt{empty}(x) \wedge \texttt{El}(x))$
Extensionality	$\forall x\, (x \in X \leftrightarrow x \in Y) \to X = Y$
Class Comprehension	$\exists X\, \forall x\, (x \in X \leftrightarrow (\texttt{El}(x) \wedge \varphi(x)))$
Pairing	$x, y \in U \to \{x, y\} \in U$
Union	$x \in U \to \bigcup x \in U$
Power	$x \in U \to \mathcal{P}(x) \in U$
Infinity	$(\exists x \in U)\, (\varnothing \in x \wedge (\forall w \in x)\, (w^+ \in x))$
Replacement	$(\texttt{Fun}(F) \wedge \texttt{Dom}(F) \in U) \to \texttt{Ran}(F) \in U$
Choice	$(\forall y \in X)\, (\texttt{set}(y) \wedge y \neq \varnothing) \to (\exists F : X \to \bigcup X)\, (\forall y \in X)\, (F(y) \in y)$

where we have adopted the following definitions:

$\texttt{atom}(x) := \exists a\, (a = X)$	"x is an atom".
$\texttt{class}(x) := \exists X\, (x = X)$	"x is a class".
$\texttt{El}(x) := \exists X\, (x \in X)$	"x is an element".
$\texttt{set}(x) := \texttt{class}(x) \wedge \texttt{El}(x)$	"x is a set".
$\texttt{empty}(x) := \neg\exists y\, (y \in x)$	"x is empty".
$U := \{x \mid x = x\}$	"the universal class"

Definition 18 (Signature and definitions)**.** The signature σ (i.e., for applied mathematics) we assume is given by the following primitives:

[16] For the curious reader not so closely familiar with how these definitions and reductions work, I would very *very* strongly recommend Suppes (1960), Halmos (1974), Drake & Singh (1996) or Machover (1996) (and for the approach we sketch, Rubin (1967), which unfortunately, is out of print). There are more advanced set theory textbooks, like Jech (2002) or Kunen (1980). But these are not the place to start, as they are too advanced.

[17] E.g., singleton $\{x\}$, subset $A \subseteq B$, union $A \cup B$, intersection $A \cap B$, pair $\{x, y\}$, ordered pair $\langle x, y\rangle$, Cartesian product $A \times B$, power set $\mathcal{P}(A)$, being a relation or a function $\texttt{Rel}(R)$ or $\texttt{Fun}(R)$, the domain $\texttt{Dom}(R)$ and range $\texttt{Ran}(R)$ of a relation and a function, the usual number systems, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, sequences $(a_i)_{i\in I}$, and so on.

[18] With a small adjustment, this is the version of Morse-Kelley class theory with urelements given in Rubin (1967).

$$\sigma = \{\overbrace{\texttt{point}, \texttt{Bet}, \equiv,}^{\text{geometrical}} \ \texttt{length}, \quad \overbrace{\lambda}^{\text{length function}} \quad , \in\} \tag{55}$$

The sorts of these primitive symbols are declared as follows:[19]

point	::	atom ⇒ bool
length	::	atom ⇒ bool
Bet	::	atom ⇒ atom ⇒ atom ⇒ bool
$\equiv$	::	atom ⇒ atom ⇒ atom ⇒ atom ⇒ bool
λ	::	atom ⇒ atom ⇒ atom
$\in$	::	global ⇒ global ⇒ bool

We introduce the following explicit definitions:[20]

$\texttt{co}_1(a, b, c)$	:=	$\texttt{Bet}(a, b, c) \vee \texttt{Bet}(b, c, a) \vee \texttt{Bet}(c, a, b)$	a, b, c are collinear
$\texttt{co}_2(a, b, c, d)$	:=	$\exists p\, [(\texttt{co}_1(p, a, b) \wedge \texttt{co}_1(p, c, d)) \vee (\texttt{co}_1(p, a, c) \wedge \texttt{co}_1(p, b, d)) \vee (\texttt{co}_1(p, a, d) \wedge \texttt{co}_1(p, b, c))]$	a, b, c, d lie in the same plane
$\texttt{cnfg}(a, b, c)$	:=	$a \neq b \wedge \texttt{co}_1(a, b, c)$	a, b, c is a configuration
$\mathbb{P}$	:=	$\{p \mid \texttt{point}(p)\}$	the set of points
$\ell(a, b)$	:=	$\{c \in \mathbb{P} \mid \texttt{co}_1(a, b, c)\}$	set of points collinear with a, b
$\mathbb{L}$	:=	$\{l \mid \texttt{length}(l)\}$	the set of lengths

Definition 19. $L(\sigma)$ is the formalized language built up over the signature (and sorts). It is a three-sorted first-order language.

Definition 20. The axioms of synthetic two-dimensional Euclidean geometry $\mathsf{EG}(2)$ in $L(\sigma)$ are the following eleven (the variables "p", "q" are so on are ranging over points):

E1.	Bet-Identity	$\texttt{Bet}(p, q, p) \to p = q$.
E2.	$\equiv$-Identity	$pq \equiv rr \to p = q$.
E3.	$\equiv$-Transitivity	$pq \equiv rs \wedge pq \equiv tu \to rs \equiv tu$.
E4.	$\equiv$-Reflexivity	$pq \equiv qp$.
E5.	$\equiv$-Extension	$\exists r\, (\texttt{Bet}(p, q, r) \wedge qr \equiv su)$.
E6.	Pasch	$\texttt{Bet}(p, q, r) \wedge \texttt{Bet}(s, u, r) \to \exists x\, (\texttt{Bet}(q, x, s) \wedge \texttt{Bet}(u, x, p))$.
E7.	Euclid	$\texttt{Bet}(p, q, t) \wedge \texttt{Bet}(r, q, s) \wedge p \neq q \to (\exists x, y)\, (\texttt{Bet}(p, r, x) \wedge \texttt{Bet}(p, s, y) \wedge \texttt{Bet}(x, t, y))$.
E8.	5-Segment	$(p \neq q \wedge \texttt{Bet}(p, q, r) \wedge \texttt{Bet}(p', q', r') \wedge pq \equiv p'q' \wedge qr \equiv q'r' \wedge ps \equiv p's' \wedge qs \equiv q's') \to rs \equiv r's'$.
E9.	Lower Dimension	There exists three points which are not $\texttt{co}_1$.
E10.	Upper Dimension	Any four points are $\texttt{co}_2$.
E11.	Continuity Axiom	$\exists r\, (\forall p \in X_1)\, (\forall q \in X_2)\, \texttt{Bet}(r, p, q) \to \exists s\, (\forall p \in X_1)\, (\forall q \in X_2)\, \texttt{Bet}(p, s, q)$

[19] This is an Isabelle-style notation. Isabelle is a higher-order logic (HOL) theorem proving assistant and automated prover designed by Lawrence Paulson in the late 1980s in Cambridge. See Wenzel et al. (2020) for the current Isabelle user's manual. In this notation, "Bet :: atom ⇒ atom ⇒ atom ⇒ bool" is a metatheory claim, indicating that Bet is a three-place predicate on atoms; and "λ :: atom ⇒ atom ⇒ atom" likewise indicates that λ is a binary function symbol on atoms.

[20] See Field (1980): 53, footnote. The precise definitions of the predicates $\texttt{co}_n$, expressing points being in the same n-dimensional subspace, are given in Szczerba & Tarski (1979): 190. (Szczerba & Tarski call these predicates $\mathbf{L}_n$.) The definition is recursive: for $n > 1$, each $\texttt{co}_n$ is defined in terms of the previous ones. These definitions are due to Kordos (1969).

See Tarski (1959), pp. 19–20, for a formulation of the first-order two-dimensional theory, with twelve axioms and one axiom scheme (for continuity); and Tarski & Givant (1999) for a simplification down to ten axioms and one axiom scheme (for continuity). The above axiom system is the second-order, two-dimensional theory: i.e., the single Continuity Axiom is the second-order axiom which quantifies over *sets* of points. Of course, this is second-order *relative* to points; relative to sets & classes, it is *first-order*, and our whole theory is indeed a first-order theory in a first-order language $L(\sigma)$, with $\in$ included amongst the primitives.

Definition 21 (Length abstraction axioms)**.** The axioms for length abstraction are:

$$(L1) \quad (\forall a, b, c, d \in \mathbb{P})\,(\lambda(a, b) = \lambda(c, d) \leftrightarrow ab \equiv cd). \tag{56}$$

$$(L2) \quad (\forall l \in \mathbb{L})\,(\exists a, b \in \mathbb{P})\,(l = \lambda(a, b)). \tag{57}$$

Definition 22 (Typing axioms)**.** To ensure our theory knows what kinds of entities $\mathbb{P}, \mathbb{L}, B, \equiv$ and λ are we need to tell it, explicitly. This requires eight "typing axioms":

(T1) $\mathbb{P} \subseteq \mathsf{Atom}$
(T2) $\mathbb{L} \subseteq \mathsf{Atom}$
(T3) $\mathbb{P} \cap \mathbb{L} = \varnothing$
(T4) $\mathtt{set}(\mathbb{P})$
(T5) $\mathtt{set}(\mathbb{L})$.
(T6) $B \subseteq \mathbb{P}^3$
(T7) $\equiv \subseteq \mathbb{P}^4$
(T8) $\lambda : \mathbb{P}^2 \to \mathbb{L}$

(Strictly speaking, since we have a function symbol "λ", we need to tell the theory how calculate what the value of "$\lambda(x, y)$" is when either x or y are *not* points. A standard convention would be, e.g., $\lambda(x, y) = 0$. Since we are not trying to put this into a theorem prover, I feel I can fuzz over such arcane details, and we leave it to common sense never to state a result where "λ" gets applied to something not a point.)

Definition 23 (Axioms)**.** Our overall non-logical axioms are:

(1) The axioms of AM (ambient set (class) theory).
(2) The axioms of $\mathsf{EG}(2)$ (geometry).
(3) The typing axioms (T1)–(T8).
(4) The axioms for length abstraction, $(L1)$ and $(L2)$.

We let $\mathsf{EG}^+(2)$ be the set of these axioms.

The results we give below are stated and verified semi-formally; but they can, in principle, be proved *inside* $\mathsf{EG}^+(2)$.

Lemma 16 (Symmetry of λ)**.** $\lambda(a, b) = \lambda(b, a)$.

Proof. We have: $ab \equiv ba$ (by E4). By $(L1)$, $\lambda(a, b) = \lambda(b, a)$. $\square$

Lemma 17 (Null segments are congruent)**.** $\lambda(p,p) = \lambda(q,q)$.

Proof. Consider points p, q. From E5, there exists a point r such that $\texttt{Bet}(p,p,r)$ and $pr \equiv qq$. And hence, from E2, $p = r$. And so $pp \equiv qq$. By $(L1)$, $\lambda(p,p) = \lambda(q,q)$. $\square$

Definition 24 (Definition of zero abstract length)**.** $\mathbf{0} := \lambda(p,p)$.

This definition is independent of the point p, by the previous lemma.

3.2 Representation

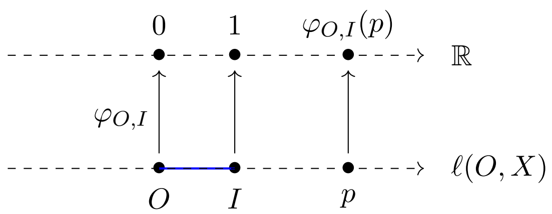
Lemma 18 (Representation theorem for lines)**.** Let $O, I \in \mathbb{P}$ be distinct points and let $\ell = \ell(O,I)$ be the line containing them. We define on $\ell(O,I)$ an addition operation $+$, a multiplication operation $\times$ and an order $\leq$. Then there exists a unique isomorphism:[21]

$$\varphi_{O,I} : (\ell, O, I, +, \times, \leq) \to (\mathbb{R}, 0, 1, +, \times, \leq) \tag{58}$$

(I have "overloaded" the symbols pertaining to operations on the line ℓ, and the symbols pertaining to operations in the ordered field of reals. Hopefully context always disambiguates.)

We may indicate this isomorphism like this:

Figure 4: Line representation: $\varphi_{O,I}$



The isomorphism is called "*the local coordinate system*" on the line $\ell(O,I)$. Following Burgess (Burgess & Rosen (1997): 107), we may call the two points O, I "benchmarks".

Definition 25. For $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u} \in \mathbb{R}^2$, we define:[22]

$$\begin{aligned} \Delta_2(\mathbf{x},\mathbf{y}) &:= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} && (59)\\ B_{\mathbb{R}^2}(\mathbf{x},\mathbf{y},\mathbf{z}) &:= (\exists\lambda \in [0,1])\,(\Phi(q) - \Phi(p) = \lambda(\Phi(r) - \Phi(p))) && (60)\\ \mathbf{xy} \equiv_{\mathbb{R}^2} \mathbf{zu} &:= \Delta_2(\mathbf{x},\mathbf{y}) = \Delta_2(\mathbf{z},\mathbf{u}) && (61) \end{aligned}$$

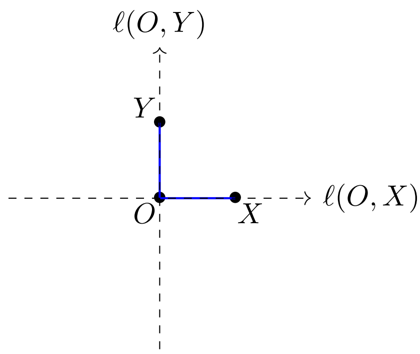
We say that the system $(\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$ is "*the standard Euclidean coordinate structure*".

[21] Aside from the part dealing with $\leq$, the most detailed proof of this I know of can be extracted from Bennett (1995), which is a descendant of proofs going back to Hilbert (1899) and Veblen (1904). The (short) proof of Theorem 1 of Tarski (1959) provides the required definition of $\leq$ on a line.

[22] $\mathbf{x} = (x_1, x_2)$ etc.

Definition 26. Let O, X, Y be three points in $\mathbb{P}$. Then we say that O, X, Y is a *Euclidean 2-frame* in $\mathbb{P}$ if $O \neq X$, $O \neq Y$, $X \neq Y$, and $OX \equiv OY$ and $OX \perp OY$. Again, following Burgess, we can call these three points "benchmarks".

Figure 5: Euclidean 2-frame



Lemma 19 (Representation theorem for $\mathsf{EG}(2)$)**.** Let O, X, Y be a Euclidean 2-frame. There exists a unique bijection:[23]

$$\Phi : \mathbb{P} \to \mathbb{R}^2 \tag{62}$$

such that:

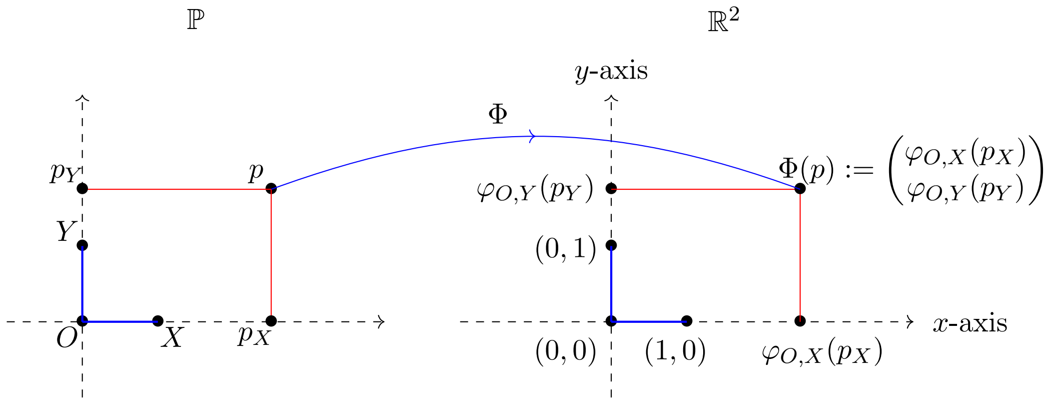
(1) $\Phi(O) = (0,0)$ and $\Phi(X) = (0,1)$ and $\Phi(Y) = (1,0)$
(2) For any $p, q, r \in \mathbb{P}$: $\mathtt{Bet}(p,q,r)$ iff $B_{\mathbb{R}^2}(\Phi(p), \Phi(q), \Phi(r))$
(3) For any $p, q, r, s \in \mathbb{P}$: $pq \equiv rs$ iff $\Phi(p)\Phi(q) \equiv_{\mathbb{R}^2} \Phi(r)\Phi(s)$.

Definition 27 (Cartesian coordinate system)**.** Such a mapping Φ as given above is called a *Cartesian coordinate system.* It is an isomorphism from $(\mathbb{P}, B, \equiv)$ to the standard coordinate structure $(\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$

The basic gist of the proof of the representation theorem is provided in the diagram:

[23] The proof of this is sketched in Theorem 1 in Tarski (1959). In fact Tarski proves a slightly different result, concerning the first-order theory which I call $\mathsf{EG}_0(2)$. Tarski shows that, for any model $M \models \mathsf{EG}_0(2)$ there is a real-closed field F such that M is isomorphic to "the two-dimensional Cartesian coordinate space over F". For any *full* model of the second-order theory $\mathsf{EG}(2)$, this field is forced to be $\mathbb{R}$.

Figure 6: Coordinate system Φ (based on benchmarks O, X, Y)



Given a point $p \in \mathbb{P}$, we first project it onto the axes $\ell(O, X)$ and $\ell(O, Y)$ by lines parallel to the axes. This yields two unique points, p_X and p_Y. We then define $\Phi(p)$ to be the pair of the local coordinates on the axes for these two points. I.e.,

$$\Phi(p) := \begin{pmatrix} \varphi_{O,X}(p_X) \\ \varphi_{O,Y}(p_Y) \end{pmatrix} \tag{63}$$

One must then prove that this function $\Phi : \mathbb{P} \to \mathbb{R}^2$ is a bijection and that it yields the three required conditions:

(1) $\Phi(O) = (0, 0)$ and $\Phi(X) = (0, 1)$ and $\Phi(Y) = (1, 0)$
(2) For any $p, q, r \in \mathbb{P}$: $\texttt{Bet}(p, q, r)$ iff $B_{\mathbb{R}^2}(\Phi(p), \Phi(q), \Phi(r))$
(3) For any $p, q, r, s \in \mathbb{P}$: $pq \equiv rs$ iff $\Phi(p)\Phi(q) \equiv_{\mathbb{R}^2} \Phi(r)\Phi(s)$.

One can do this (it does require a lot of detailed calculation).

The representation theorem is equivalent to stating that $(\mathbb{P}, B, \equiv)$ is a full model of $\mathsf{EG}(2)$ if and only if $(\mathbb{P}, B, \equiv) \cong (\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$.

Lemma 20. Given the construction of the points p_X and p_Y, we see that

$$p \in \ell(O, X) \quad \leftrightarrow \quad p_Y = O \tag{64}$$
$$p \in \ell(O, Y) \quad \leftrightarrow \quad p_X = O \tag{65}$$

Lemma 21. From the above lemma and the definition of Φ, we see that

$$p \in \ell(O, X) \quad \leftrightarrow \quad \Phi(p) = \begin{pmatrix} \varphi_{O,X}(p_X) \\ 0 \end{pmatrix} \tag{66}$$
$$p \in \ell(O, Y) \quad \leftrightarrow \quad \Phi(p) = \begin{pmatrix} 0 \\ \varphi_{O,Y}(p_Y) \end{pmatrix} \tag{67}$$

Definition 28 (Unit length of a coordinate system)**.** Let a Cartesian coordinate chart, $\Phi : \mathbb{P} \to \mathbb{R}^2$ be given, with benchmarks O, X, Y. We define:

$$\mathbf{u}_\Phi := \lambda(O, X). \tag{68}$$

This is called "*the distinguished unit of the coordinate system* Φ".

Lemma 22. $\lambda(a, b) = \mathbf{u}_\Phi$ iff $ab \equiv OX$.

Proof. By the abstraction principle, $\lambda(a, b) = \lambda(O, X)$ iff $ab \equiv OX$. This immediately implies the claim. □

Lemma 23 (Euclidean symmetry group)**.** We consider the symmetries (automorphism group) of the standard Euclidean coordinate structure. Let $\mathbb{E} := (\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$g \in \mathrm{Aut}(\mathbb{E}) \tag{69}$$

if and only if $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection and there exists $\alpha > 0$, $R \in O(2)$, and $\mathbf{d} \in \mathbb{R}^2$ such that, for all $\mathbf{x} \in \mathbb{R}^2$:[24]

$$g(\mathbf{x}) = \alpha\, R\, \mathbf{x} + \mathbf{d}. \tag{70}$$

Definition 29 (Coordinate length function)**.** Let Φ be a Cartesian coordinate system. We define a function

$$\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+ \tag{71}$$

pointwise, by:

$$\Delta_\Phi(p, q) := \Delta_2(\Phi(p), \Phi(q)) \tag{72}$$

Lemma 24. Let Φ be a Cartesian coordinate system. Then, for points p, q, r, s:

$$pq \equiv rs \quad \leftrightarrow \quad \Delta_\Phi(p, q) = \Delta_\Phi(r, s). \tag{73}$$

Proof. From the Representation Theorem for $\mathsf{EG}(2)$ (i.e., Lemma 19, condition (3)), $pq \equiv rs$ iff $\Phi(p)\Phi(q) \equiv_{\mathbb{R}^2} \Phi(r)\Phi(s)$, iff $\Delta_\Phi(p, q) = \Delta_\Phi(r, s)$. □

[24] I do not know the location of a published proof of this, but it's not too hard to work out for oneself. The essential ideas are that a symmetry of the reduct $(\mathbb{R}^2, B_{\mathbb{R}^2})$ must be an affine transformation: $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{d}$, where A is a $GL(2)$ matrix. Then, from the requirement that congruence $\equiv_{\mathbb{R}^2}$ be invariant, one can show that A must be of the form αR, with $\alpha \neq 0$, a multiple of some rotation R in $O(2)$.

Lemma 25. $(\mathbb{P}, \Delta_\Phi)$ is a metric space.

Proof. Using the Definition 29, we can show: $\Delta_\Phi(a,a) = 0$. And $\Delta_\Phi(a,b) = \Delta_\Phi(b,a)$ (symmetry). And $a \neq b \to \Delta_\Phi(a,b) > 0$ (positivity). And, for three points a, b, c, $\Delta_\Phi(a,b) + \Delta_\Phi(b,c) \geq \Delta_\Phi(a,c)$ (Schwarz's (or, the triangle) inequality). $\square$

Lemma 26. Let a coordinate system Φ be based on the 2-frame O, X, Y. Then

$$\Delta_\Phi(O,X) = 1 = \Delta_\Phi(O,Y). \tag{74}$$

Lemma 27. $\lambda(a,b) = \mathbf{u}_\Phi$ iff $\Delta_\Phi(a,b) = 1$.

Proof. By the above lemma, $\lambda(a,b) = \mathbf{u}_\Phi$ iff $ab \equiv OX$. But $ab \equiv OX$ iff $\Delta_\Phi(a,b) = \Delta_\Phi(O,X)$. So, $\lambda(a,b) = \mathbf{u}_\Phi$ iff $\Delta_\Phi(a,b) = \Delta_\Phi(O,X)$. But $\Delta_\Phi(O,X) = 1$. So, $\lambda(a,b) = \mathbf{u}_\Phi$ iff $\Delta_\Phi(a,b) = 1$. $\square$

So, for a coordinate system, an arbitrary length is equal to the unit length of that system just if its coordinate length is equal to 1.

The following is, I feel, a basic result, characterizing our coordinate-based approach:

Lemma 28. For any coordinate system Φ, for any points $a, b, c, d \in \mathbb{P}$:

$$\overbrace{\lambda(a,b)}^{\text{abstract length}} = \overbrace{\lambda(c,d)}^{\text{abstract length}} \leftrightarrow \overbrace{\Delta_\Phi(a,b)}^{\text{coordinate length}} = \overbrace{\Delta_\Phi(c,d)}^{\text{coordinate length}}. \tag{75}$$

As we show below (Lemma 51), this equivalence (for a given coordinate system Φ) guarantees *the existence of a unique bijection* from the lengths to the coordinate lengths wrt Φ. This bijection is then simply the corresponding measurement scale which maps lengths to real numbers.

Lemma 29 (Coordinate transformations)**.** Let Φ, Φ' be Cartesian coordinate systems. Then, there exists $\alpha > 0$, $R \in O(2)$, and $\mathbf{d} \in \mathbb{R}^2$ such that, for all $p \in \mathbb{P}$:

$$\Phi'(p) = \alpha\, R\, \Phi(p) + \mathbf{d}. \tag{76}$$

Proof. Observe that if Φ, Φ' are Cartesian coordinate systems, then $\Phi, \Phi' : (\mathbb{P}, B, \equiv) \to (\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$ are isomorphisms. From this it follows that $\Phi' \circ \Phi^{-1}$ is an automorphism from $(\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$ to $(\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$. Then, from Lemma 23, it follows that we have $\alpha > 0$, $R \in O(2)$ and $\mathbf{d} \in \mathbb{R}^2$ such that, for any $\mathbf{x} \in \mathbb{R}^2$, $(\Phi' \circ \Phi^{-1})(\mathbf{x}) = \alpha\, R\, \mathbf{x} + \mathbf{d}$. Let $\mathbf{x} = \Phi^{-1}(p)$. Then, for any $p \in \mathbb{P}$, $\Phi'(p) = \alpha\, R\, \Phi(p) + \mathbf{d}$, as claimed. $\square$

Lemma 30 (Coordinate length scaling)**.** Let Φ, Φ' be Cartesian coordinate systems. Then, there exists $\alpha > 0$, such that, for all $p, q \in \mathbb{P}$,

$$\Delta_{\Phi'}(p,q) = \alpha\, \Delta_\Phi(p,q) \tag{77}$$

Proof. The following calculation verifies the claim:

$$\begin{aligned}
\Delta_{\Phi'}(p,q) &= \sqrt{((\Phi')_1(p) - (\Phi')_1(q))^2 + ((\Phi')_2(p) - (\Phi')_2(q))_2} && (78)\\
&= \sqrt{(((\alpha\, R\, \Phi(p) + \mathbf{d})_1 - (\alpha\, R\, \Phi(q) + \mathbf{d})_1)^2 + \ldots} && (79)\\
&= |\alpha| \sqrt{\sum_{i,j,k} R_{k,i} R_{k,j}\, (\Phi^i(p) - \Phi^i(q))\, (\Phi^j(p) - \Phi^j(q))} && (80)\\
&= |\alpha| \sqrt{\sum_{i,j} \delta_{i,j}\, (\Phi^i(p) - \Phi^i(q))\, (\Phi^j(p) - \Phi^j(q))} && (81)\\
&= |\alpha|\, \Delta_\Phi(p,q) && (82)
\end{aligned}$$

In this reasoning, we see that the displacement $\mathbf{d}$ "cancels" when we subtract the coordinates; and the scale parameter α "factors" out of the large square rooo. We are left with a matrix product, involving the rotation matrix R. But since the rotation $R \in O(2)$ we have: $RR^T = 1$. I.e., $\sum_{k=1}^2 R_{k,i} R_{k,j} = \delta_{i,j}$.

□

Lemma 31 (Segment ratios)**.** Let $a \neq b$ and let a, b, c be collinear. Let $\varphi_{a,b}$ be the local coordinate system on the line $\ell(a,b)$. Let Φ be a Cartesian coordinate system. Then:

$$|\varphi_{a,b}(c)| = \frac{\Delta_\Phi(a,c)}{\Delta_\Phi(a,b)} \tag{83}$$

Proof. Since $a \neq b$, first we select a coordinate system Φ' based on a 2-frame a, b, Y (i.e., Y is a new point not on the line $\ell(a,b)$, with $aY \equiv ab$ and $aY \perp ab$). In this coordinate system, for any point $c \in \ell(a,b)$, we have $\Phi(c) = (\varphi_{a,b}(c), 0)$. So, $\Phi(a) = (\varphi_{a,b}(a), 0) = (0,0)$ and $\Phi(b) = (\varphi_{a,b}(b), 0) = (1,0)$. So, $\Delta_{\Phi'}(a,b) = \Delta_2(\Phi(a), \Phi(b)) = 1$. And $\Delta_{\Phi'}(a,c) = |\varphi_{a,b}(c)|$. Consequently, $\frac{\Delta_\Phi(a,c)}{\Delta_\Phi(a,b)} = |\varphi_{a,b}(c)|$. Next, let Φ be any coordinate system. Then there is a fixed $\alpha > 0$ such that, for any points $p, q \in \mathbb{P}$: $\Delta_\Phi(p,q) = \alpha \Delta_{\Phi'}(p,q)$. And so, in particular, for any $c \in \ell(a,b)$, $\frac{\Delta_\Phi(a,c)}{\Delta_\Phi(a,b)} = \frac{\Delta_{\Phi'}(a,c)}{\Delta_{\Phi'}(a,b)} = |\varphi_{a,b}(c)|$, as claimed. □

Lemma 32 (Covariance)**.** The following covariance claims hold: for any two Cartesian coordinate systems Φ, Φ' we have, for any points $a, b, c, d, e, f \in \mathbb{P}$:

(1) $\Delta_\Phi(a,b) = \Delta_\Phi(c,d) \leftrightarrow \Delta_{\Phi'}(a,b) = \Delta_{\Phi'}(c,d)$.
(2) $\Delta_\Phi(a,b) = \Delta_\Phi(c,d) + \Delta_\Phi(e,f) \leftrightarrow \Delta_{\Phi'}(a,b) = \Delta_{\Phi'}(c,d) + \Delta_{\Phi'}(e,f)$.

Proof. Let Φ, Φ' be coordinate systems. Hence, by Lemma 30, there is a fixed $\alpha > 0$, for any points $p, q \in \mathbb{P}$, we have: $\Delta_{\Phi'}(p,q) = \alpha\, \Delta_\Phi(p,q)$.

For (1): $\Delta_\Phi(a,b) = \Delta_\Phi(c,d)$ holds iff $\alpha\Delta_\Phi(a,b) = \alpha\Delta_\Phi(c,d)$, which holds iff $\Delta_{\Phi'}(a,b) = \Delta_{\Phi'}(c,d)$ holds.

For (2): $\Delta_\Phi(a,b) = \Delta_\Phi(c,d) + \Delta_\Phi(e,f)$ holds iff $\alpha\Delta_\Phi(a,b) = \alpha\Delta_\Phi(c,d) + \alpha\Delta_\Phi(e,f)$, iff $\Delta_{\Phi'}(a,b) = \Delta_{\Phi'}(c,d) + \Delta_{\Phi'}(e,f)$.

□

Lemma 33 (Addition of coordinate lengths)**.** Let Φ be a Cartesian coordinate system. Then:

$$\mathtt{Bet}(a,b,c) \to (\Delta_\Phi(a,c) = \Delta_\Phi(a,b) + \Delta_\Phi(b,c)) \tag{84}$$

Proof. This holds when $a = b$ because, in that case, $\Delta_\Phi(a,b) = 0$. So, assume $a \neq b$ and consider the line $\ell(a,b)$. We assume also that $c \in \ell(a,b)$. Consider a coordinate system Φ' whose x-axis lies along $\ell(a,b)$ and with benchmarks a, b, Y (Y a point so that $aY \perp ab$). Then, we have the coordinates: $\Phi(c) = (\varphi_{a,b}(c), 0)$, and $\varphi_{a,b}(c) \geq 1$, because $\mathtt{Bet}(a,b,c)$. In particular, $\Phi(a) = (\varphi_{a,b}(a), 0) = (0,0)$ and $\Phi(b) = (\varphi_{a,b}(b), 0) = (1,0)$. A calculation verifies that $\Delta_{\Phi'}(a,b) = 1$, $\Delta_{\Phi'}(a,c) = \varphi_{a,b}(c)$, and $\Delta_{\Phi'}(b,c) = \varphi_{a,b}(c) - 1$. So, $\Delta_{\Phi'}(a,c) = \Delta_{\Phi'}(a,b) + \Delta_{\Phi'}(b,c)$.

But, by Lemma 32, the statement $\Delta_\Phi(a,c) = \Delta_\Phi(a,b) + \Delta_\Phi(b,c)$ is *coordinate invariant*; so, since it holds in the given system Φ', it holds in all. □

The converse of Lemma 33 also holds:

Lemma 34 (Addition of coordinate lengths 2)**.** Let a, b, c be points. Let Φ be a Cartesian coordinate system. Then:

$$\Delta_\Phi(a,c) = \Delta_\Phi(a,b) + \Delta_\Phi(b,c) \to \mathtt{Bet}(a,b,c) \tag{85}$$

I leave the proof of this to the reader.

3.3 Defining $\mathbf{0}$, $\oplus$, $\preceq$ and $\bullet$

To prove Theorem 2, I must first define the following:

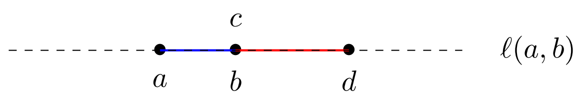
(1) A zero length: $\mathbf{0}$.
(2) A length addition operation: $\oplus$.
(3) A length ordering relation: $\preceq$.
(4) A scalar multiplication of lengths (by reals): $\bullet$.

We have already defined $\mathbf{0}$. The definition of $\oplus$ proceeds in terms of segment concatenation.[25]

Definition 30. Segments ab and cd are *concatenated* on ℓ just if $a, b, c, d \in \ell$ and $b = c$ and $\mathtt{Bet}(a,b,d)$. We write this: $\mathtt{concat}_\ell(a,b,c,d)$.

[25] Our segment concatenation is *linear*: along a line. In his Ellis (1966), Brian Ellis defines a peculiar non-standard form of segment concatenation. ab and cd count as "concatenated" just when $b = c$ and $ab \perp cd$! This is *right-angled concatenation*. Ellis goes on to argue that the resulting notion of "length addition", $\oplus_e$, is equally legitimate. But it is clear that the resulting structure is not an extensive quantity as we've defined it, because it does not respect the vector space structure given by scalar multiplication (where scalar multiplication is defined via the usual theory of proportions, which is our approach, given in a moment). For, assuming $l \neq \mathbf{0}$, we get: $l \oplus_e l \neq 2 \bullet l$.

Figure 7: Concatenation: $\texttt{concat}_\ell(a, b, c, d)$



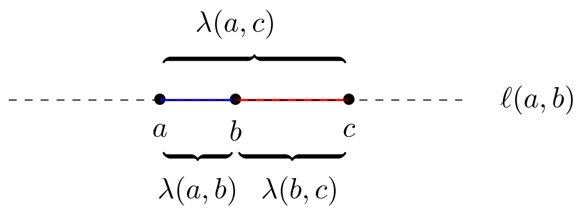
Definition 31 (Definition of $\oplus_\ell$)**.** Fix a line ℓ. We define the operation $\oplus_\ell$ on the line ℓ. Let $a, b, c, d, e, f \in \ell$. Then:

$$\lambda(e, f) = \lambda(a, b) \oplus_\ell \lambda(c, d) := \quad (\exists c', d')(\texttt{concat}_\ell(a, b, c', d') \wedge c'd' \equiv cd \wedge ef \equiv ad') \tag{86}$$

Lemma 35. Fix a line ℓ, with $a, b, c \in \ell$ and $\texttt{Bet}(a, b, c)$. Then:

$$\lambda(a, c) = \lambda(a, b) \oplus_\ell \lambda(b, c) \tag{87}$$

Figure 8: Length addition: $\lambda(a, c) = \lambda(a, b) \oplus_\ell \lambda(b, c)$



Definition 32 (Definition of $\oplus$)**.** We declare that $l_3 = l_1 \oplus l_2$ if and only if, there exists a line ℓ containing points a, b, c such that:

(1) $l_1 = \lambda(a, b)$.
(2) $l_2 = \lambda(b, c)$.
(3) $l_3 = \lambda(a, c)$.
(4) $\lambda(a, c) = \lambda(a, b) \oplus_\ell \lambda(b, c)$

We next define the blunt order $\preceq$:

Definition 33 (Definition of $\preceq$)**.** $l_1 \preceq l_2 := (\exists l_3 \in \mathbb{P})\,(l_2 = l_1 \oplus l_3)$.

The corresponding sharp order $\prec$ is defined accordingly:

Definition 34 (Definition of $\prec$)**.** $l_1 \prec l_2 := l_1 \preceq l_2 \wedge l_2 \neq l_2$.

Lemma 36. The following claims hold for the structure $(\mathbb{L}, \oplus, \preceq)$:

(1) $l \oplus \mathbf{0} = l = \mathbf{0} \oplus l$.
(2) $l_1 \oplus l_2 = l_2 \oplus l_1$.
(3) $l_1 \oplus (l_2 \oplus l_3) = (l_1 \oplus l_2) \oplus l_3$.
(4) $l \preceq l$
(5) $l_1 \preceq l_2 \wedge l_2 \preceq l_1 \to l_1 = l_2$.
(6) $l_1 \preceq l_2 \wedge l_2 \preceq l_3 \to l_1 \preceq l_3$.
(7) $l_1 \preceq l_2 \vee l_2 \preceq l_1$.
(8) $\mathbf{0} \preceq l$
(9) $l_1 \prec l_2 \to (\exists l_3 \in \mathbb{L})\,(l_1 \prec l_3 \prec l_2)$.
(10) $l_1 \preceq l_2 \to l_1 \oplus l_3 \preceq l_2 \oplus l_3$.

I shall not prove these directly, since far easier proofs are available *after* we establish that $\mathbb{L}$ is a positive extensive quantity (Theorem 2). Our approach is a kind of inverse of Otto Hölder's classic article (Hölder (1901)). For Hölder takes a system of axioms for abstract magnitudes, like the properties (1)–(10), as well as some completeness assumptions, and then proves a representation theorem. Instead, we characterize $\mathbb{L}$ geometrically, prove that it is indeed a positive extensive quantity—i.e., isomorphic to the positive cone $(\mathbb{R}^+, +, \leq, \bullet)$—and then we can derive these properties (1)–(10) from that conclusion, by transfer.

Assuming these are so, it follows that the algebraic reduct $(\mathbb{L}, \oplus)$ is a commutative monoid. And the ordered reduct $(\mathbb{L}, \preceq)$ is a dense linear order with a least element $\mathbf{0}$. And together, $(\mathbb{L}, \oplus, \preceq)$ is a densely linearly ordered commutative monoid with a least element.

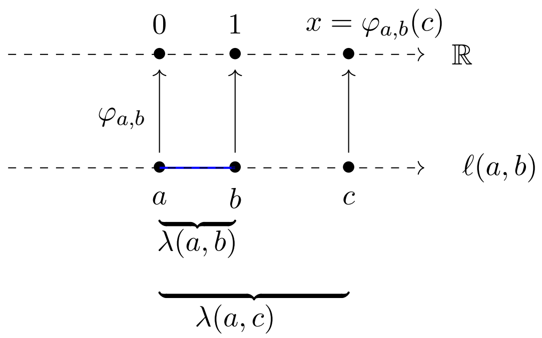
Finally, we must define scalar multiplication $(\bullet)$. There are two ways to do this, but they are equivalent.[26]

Definition 35 (Definition of $\bullet_\ell$)**.** Fix a line ℓ. We now define the scalar multiplication operation $\bullet_\ell$ on the line ℓ. Let $x \in \mathbb{R}^+$, and $a, b, c \in \ell$. Then:

$$\lambda(a,c) = x \bullet_\ell \lambda(a,b) \quad := \quad \begin{cases} \lambda(a,c) = \mathbf{0} \text{ if } a = b \\ x = |\varphi_{a,b}(c)| \text{ if } a \neq b \end{cases} \tag{88}$$

[26] The other way works geometrically with configurations of three points representing given reals, and defines scalar multiplication geometrically, by proportions. My definition is analytic, invoking the local coordinate system $\varphi_{a,b}$ on a line $\ell(a,b)$ with distinct points a, b.

Figure 9: Scalar multiplication: $\lambda(a,c) = x \bullet_\ell \lambda(a,b)$



Definition 36 (Definition of $\bullet$)**.** For $x \in \mathbb{R}^+$, and $l_1, l_2 \in \mathbb{L}$, we declare that $l_2 = x \bullet l_1$ holds if and only if, there exists a line ℓ containing points a, b, c such that:

(1) $l_1 = \lambda(a,b)$.
(2) $l_2 = \lambda(a,c)$.
(3) $\lambda(a,c) = x \bullet_\ell \lambda(a,b)$

Lemma 37. Let a, b, c lie on a line ℓ, with $a \neq b$. Then

$$\lambda(a,c) = x \bullet_\ell \lambda(a,b) \quad \leftrightarrow \quad x = |\varphi_{a,b}(c)|. \tag{89}$$

Proof. This follows from Definition 35. □

Lemma 38. For any $x \in \mathbb{R}^+$, $x \bullet \mathbf{0} = \mathbf{0}$.

Proof. Pick a point, a. Then $\mathbf{0} = \lambda(a,a)$. Let $l = x \bullet \mathbf{0}$. Then, from Definition 36, $l = x \bullet \mathbf{0}$ if and only if there exists a line ℓ containing points a, c such that $l = \lambda(a,c)$ and $\lambda(a,c) = x \bullet_\ell \lambda(a,a)$. But from Definition 35, it follows that $\lambda(a,c) = \mathbf{0}$. So, $l = \mathbf{0}$. □

Definition 37 (Definition of the structure $\mathbb{L}$)**.** Let $\mathbb{L} := (\mathbb{L}, \oplus, \preceq, \bullet)$.

3.4 Defining h_Φ

Lemma 39. For each $l \in \mathbb{L}$, there is a unique $x \in \mathbb{R}^+$ such that:

$$(\exists p, q \in \mathbb{P})\,(l = \lambda(p,q) \wedge x = \Delta_\Phi(p,q)) \tag{90}$$

Proof. Let l be given. The existence axiom $(L2)$ for λ implies that the length l has at least one representative p, q: $l = \lambda(p,q)$. Then $x = \Delta_\Phi(p,q)$. And $x \in \mathbb{R}^+$, because $\Delta_\Phi(p,q) \in \mathbb{R}^+$. For uniqueness, suppose

$$(\exists p, q \in \mathbb{P})\,(l = \lambda(p,q) \wedge x = \Delta_\Phi(p,q)) \tag{91}$$
$$(\exists p', q' \in \mathbb{P})\,(l = \lambda(p',q') \wedge x' = \Delta_\Phi(p',q')) \tag{92}$$

So, we have p, q, p', q' with $l = \lambda(p', q')$ and $l = \lambda(p, q)$ and $x = \Delta_\Phi(p, q)$ and $x' = \Delta_\Phi(p, q)$. Hence, $\lambda(p', q') = \lambda(p, q)$. By the abstraction axiom $(L1)$ for λ, we have: $p'q' \equiv pq$. Then, by the representation theorem for coordinate lengths, we have: $\Delta_\Phi(p', q') = \Delta_\phi(p, q)$. So, $x' = x$. □

Definition 38 (Definition of h_Φ)**.** Assume a coordinate system Φ is given. I define the function:

$$h_\Phi : \mathbb{L} \to \mathbb{R}^+ \tag{93}$$

as follows. For any $l \in \mathbb{L}$, we define:

$$x = h_\Phi(l) \quad := \quad (\exists p, q \in \mathbb{P})\,(l = \lambda(p, q) \wedge x = \Delta_\Phi(p, q)) \tag{94}$$

This correctly defines a function h_Φ by the previous lemma. So, for each length l, the value $h_\Phi(l)$ is independent of the representative points, p, q, chosen such that $l = \lambda(p, q)$.

Lemma 40. For any points $a, b \in \mathbb{P}$:

$$h_\Phi(\lambda(a, b)) = \Delta_\Phi(a, b). \tag{95}$$

Proof. This is immediate from the definition. □

Lemma 41. Let Φ be a coordinate system based on the 2-frame O, X, Y. Then

$$h_\Phi(\mathbf{u}_\Phi) = 1. \tag{96}$$

Proof. By Definition 28, $\mathbf{u}_\Phi = \lambda(O, X)$. So, $h_\Phi(u_\Phi) = h_\Phi(\lambda(O, X)) = \Delta_\Phi(O, X) = 1$. □

We'll next drop the subscript on "h_Φ", and leave it implicit. We will show that the function h just defined above is an isomorphism from $\mathbb{L}$ to the standard extensive quantity, $\mathcal{E}_0$. This will establishes that h is a measurement scale for $\mathbb{L}$. Thus, I must now show that h is bijective, and also that we have the isomorphism conditions: for any $x \in \mathbb{R}^+$, and $l_1, l_2 \in \mathbb{L}$,

$$\begin{aligned} h(l_1 \oplus l_2) &= h(l_1) + h(l_2) & (97) \\ l_2 \preceq l_2 &\leftrightarrow h(l_1) \leq h(l_2) & (98) \\ h(x \bullet l) &= x \cdot h(l) & (99) \end{aligned}$$

3.5 Properties of h_Φ

Lemma 42. *$h(\mathbf{0}) = 0$.*

Proof. Recall that $\mathbf{0} = \lambda(a, a)$. So, by Lemma 40, $h(\mathbf{0}) = h(\lambda(a, a)) = \Delta_\Phi(a, a) = 0$. □

Lemma 43. *$h(l) \geq 0$.*

Proof. By $(L2)$, we may let $l = \lambda(a, b)$, for points $a, b \in \mathbb{P}$. By the definition of h, we have: $h(l) = \Delta_\Phi(a, b)$. For any a, b, $\Delta_\Phi(a, b) \geq 0$, since Δ_Φ is a metric. Hence, $h(l) \geq 0$. □

Lemma 44. *h is a bijection.*

Proof. I first show injectivity of h. Suppose $h(l_1) = h(l_2)$. I claim: $l_1 = l_2$. Let $l_1 = \lambda(p, q)$ and $l_2 = \lambda(r, s)$. Such points exist by the existence axiom for λ. We reason with what I call a "conditional stream", as follows:

$$\begin{aligned}
& h(l_1) = h(l_2) && && (100)\\
\Rightarrow\ & h(\lambda(p, q)) = h(\lambda(r, s)) && && (101)\\
\Rightarrow\ & \Delta_\Phi(p, q) = \Delta_\Phi(r, s) && \text{(by Lemma 40)} && (102)\\
\Rightarrow\ & pq \equiv rs && \text{(Representation theorem, Lemma 19)} && (103)\\
\Rightarrow\ & \lambda(p, q) = \lambda(r, s) && \text{(Abstraction axiom L1 for } \lambda) && (104)\\
\Rightarrow\ & l_1 = l_2 && && (105)
\end{aligned}$$

as claimed.

For surjectivity, suppose $x \in \mathbb{R}^+$. I claim there exists l such that $h(l) = x$.

Consider the function $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+$. Its range is $\mathbb{R}^+$. For recall that each line $\ell(O, X)$ (with O, X distinct parameters) is isomorphic to the reals. So, there is a point $p \in \ell(O, X)$ such that $\Delta_\Phi(O, p) = x$. Consequently, $h(\lambda(O, p)) = x$. Let $l = \lambda(O, p)$: then $h(l) = x$. So, h is surjective. □

A more abstract proof of this is given below, in Lemma 52.

We next show that h respects length addition $\oplus$:

Lemma 45. *For $l_1, l_2 \in \mathbb{L}$: $h(l_1 \oplus l_2) = h(l_1) + h(l_2)$.*

Proof. Let $l_3 = l_1 \oplus l_2$. So, we claim: $h(l_3) = h(l_1) + h(l_2)$. By our definition, there exists a line ℓ containing points a, b, c such that $l_1 = \lambda(a, b)$, $l_2 = \lambda(b, c)$, $l_3 = \lambda(a, c)$ and $\lambda(a, c) = \lambda(a, b) \oplus_\ell \lambda(b, c)$. Now $h(l_1) = \Delta_\Phi(a, b)$, $h(l_2) = \Delta_\Phi(b, c)$, $h(l_3) = \Delta_\Phi(a, c)$. So, we claim: $\Delta_\Phi(a, c) = \Delta_\Phi(a, b) + \Delta_\Phi(b, c)$. But this is simply Lemma 33.

□

Lemma 46. *$h(\lambda(p, q) \oplus \lambda(q, r)) = \Delta_\Phi(p, q) + \Delta_\Phi(q, r)$.*

Proof. By Lemma 45, $h(\lambda(p,q) \oplus \lambda(q,r)) = h(\lambda(p,q)) + h(\lambda(q,r))$. By the definition of h, we have $h(\lambda(p,q)) + h(\lambda(q,r)) = \Delta_\Phi(p,q) + \Delta_\Phi(q,r)$. $\square$

Lemma 47. $\mathtt{Bet}(p,q,r) \leftrightarrow \lambda(p,q) \oplus \lambda(q,r) = \lambda(p,r)$.

Proof. By Lemma 33 and Lemma 34, we have, for any coordinate system Φ,

$$\mathtt{Bet}(p,q,r) \leftrightarrow \Delta_\Phi(p,q) + \Delta_\Phi(q,r) = \Delta_\Phi(p,r) \tag{106}$$

Consequently, we can reason with a simple biconditional stream, as follows:

$$\begin{aligned}
\mathtt{Bet}(p,q,r) \quad &\Leftrightarrow \quad \Delta_\Phi(p,q) + \Delta_\Phi(q,r) = \Delta_\Phi(p,r) && (107)\\
&\Leftrightarrow \quad h(\lambda(p,q) \oplus \lambda(q,r)) = h(\lambda(p,r)) \qquad \text{(by Lemma 46 above)} && (108)\\
&\Leftrightarrow \quad \lambda(p,q) \oplus \lambda(q,r) = \lambda(p,r) \qquad \text{(injectivity of } h) && (109)
\end{aligned}$$

$\square$

Lemma 48. For $l_1, l_2 \in \mathbb{L}$: $l_1 \preceq l_2 \leftrightarrow h(l_1) \leq h(l_2)$.

Proof. By Definition 33, $l_1 \preceq l_2$ iff $\exists l_3\, (l_2 = l_1 \oplus l_3)$. We reason as follows using a biconditional stream:

$$\begin{aligned}
l_1 \preceq l_2 \quad &\Leftrightarrow \quad l_2 = l_1 \oplus l_3 \qquad \text{(for some } l_3) && (110)\\
&\Leftrightarrow \quad h(l_2) = h(l_1 \oplus l_3) \qquad \text{(for some } l_3) && (111)\\
&\Leftrightarrow \quad h(l_2) = h(l_1) + h(l_3) \qquad \text{(for some } l_3) && (112)\\
&\Leftrightarrow \quad h(l_1) \leq h(l_2) && (113)
\end{aligned}$$

where to obtain the final line, we used the fact (Lemma 43) that $h(l_3) \geq 0$.

$\square$

Finally, we must show that h respects scalar multiplication:

Lemma 49. For $x \in \mathbb{R}^+$, $l \in \mathbb{L}$: $h(x \bullet l) = x \cdot h(l)$.

Proof. First, assume $l = \mathbf{0}$. Hence, by Lemma 42, $h(l) = 0$. Also, we have: $x \bullet l = \mathbf{0}$, for any $x \in \mathbb{R}^+$, by Lemma 38. So, $h(x \bullet l) = 0$. So, $h(x \bullet l) = x \cdot h(l)$ as claimed.

So, instead, assume $l \neq \mathbf{0}$. Let $l' = x \bullet l$. So, we claim: $h(l') = x \cdot h(l)$. Since $l' = x \bullet l$, from our definition, we infer, there exists a line ℓ containing points a, b, c such that $l = \lambda(a,b)$, $l' = \lambda(a,c)$ and $\lambda(a,c) = x \bullet_\ell \lambda(b,c)$. Note that $a \neq b$. So $h(l) = \Delta_\Phi(a,b)$ and $h(l') = \Delta_\Phi(a,b)$. But, by Lemma 37, we have: $\lambda(a,c) = x \bullet_\ell \lambda(a,b) \to x = |\varphi_{a,b}(c)|$. So, $x = |\varphi_{a,b}(c)|$. By Lemma 31, we have: $\Delta_\Phi(a,c) = |\varphi_{a,b}(c)|\, \Delta_\Phi(a,b)$. Hence, $h(x \bullet l) = x \cdot h(l)$, as claimed. $\square$

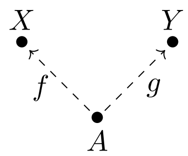
3.6 Hooking Lemma for λ and Δ_Φ

Definition 39. Let $f : A \to X$ and $g : A \to Y$ be functions, with the same domain. We say that f and g are *hooked* just if f and g are surjective, and

$$f(a) = f(b) \leftrightarrow g(a) = g(b) \tag{114}$$

This is a kind of mutual determination condition.

Figure 10: Hooked functions, f, g

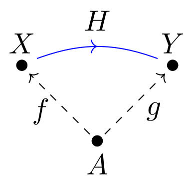


Lemma 50 (Hooking lemma)**.** Suppose that $f : A \to X$ and $g : A \to Y$ are hooked. Then there exists a unique bijection $H : X \to Y$ satisfying, for all $a \in A$:

$$H(f(a)) = g(a) \tag{115}$$

In a picture:

Figure 11: Hooking function, H



Proof. We first define:

$$y = H(x) \quad := \quad (\exists a \in A)\,(x = f(a) \wedge y = g(a)) \tag{116}$$

We must show that this is well-defined.

For existence, let $x \in X$. By surjectivity of f, there exists $a \in A$ with $x = f(a)$. Let $y = g(a)$, and so we have existence.

For uniqueness, let $x \in X$ and let $y_1, y_2 \in Y$. Suppose the defining condition holds in both cases:

$$(\exists a \in A)\,(x = f(a) \wedge y_1 = g(a)) \tag{117}$$
$$(\exists a \in A)\,(x = f(a) \wedge y_2 = g(a)) \tag{118}$$

We claim: $y_1 = y_2$. Skolemize the assumptions, and we have $a_1, a_2 \in A$ st.

$$x = f(a_1) \wedge y_1 = g(a_1) \tag{119}$$
$$x = f(a_2) \wedge y_2 = g(a_2) \tag{120}$$

Thus, $f(a_1) = f(a_2)$. By hooking, $g(a_1) = g(a_2)$. Hence, $y_1 = y_2$.

For injectivity, suppose $H(x_1) = H(x_2)$. We claim: $x_1 = x_2$. Let $y_1 = H(x_1)$ and let $y_2 = H(x_2)$. So, $y_1 = y_2$. We have $a_1 \in A$ such that $x_1 = f(a_1) \wedge y_1 = g(a_1)$; and we have $a_2 \in A$ such that $x_2 = f(a_2) \wedge y_2 = g(a_2)$. Since $y_1 = y_2$, we have: $g(a_1) = g(a_2)$. By hooking, $f(a_1) = f(a_2)$. Hence, $x_1 = x_2$.

For surjectivity, suppose $y \in Y$. By surjectivity of g, we have $a \in A$ such that $y = g(a)$. Let $x = f(a)$. Then there exists $a \in A$ with $x = f(a) \wedge y = g(a)$. So, $y = H(x)$. Hence, H is surjective.

We next show that the claimed condition (115) holds. From the definition,

$$y = H(f(a)) \leftrightarrow (\exists b \in A)\, (f(a) = f(b) \wedge y = g(b)) \tag{121}$$

By "hooking", we have:

$$y = H(f(a)) \leftrightarrow (\exists b \in A)\, (g(a) = g(b) \wedge y = g(b)) \tag{122}$$

But $(\exists b \in A)\, (g(a) = g(b) \wedge y = g(b))$ holds automatically. So, $y = H(f(a))$ and so $g(a) = H(f(a))$, as claimed.

Next, to show that any such bijection is unique, suppose we have another bijection $H' : X \to Y$ satisfying, for all $a \in A$:

$$H'(f(a)) = g(a) \tag{123}$$

Let $x \in X$. By surjectivity of f, we have $a \in A$ such that $x = f(a)$. So, we have $H(f(a)) = g(a)$ and $H'(f(a)) = g(a)$. Hence, $H(x) = H'(x)$. So, $H = H'$.

□

For the next two lemmas, fix a coordinate system Φ.

Lemma 51. The functions $\lambda : \mathbb{P}^2 \to \mathbb{L}$ and $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+$ are hooked.

Proof. Both functions $\lambda : \mathbb{P}^2 \to \mathbb{L}$ and $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+$ are surjections. For Δ_Φ is, because $\Phi : \mathbb{P} \to \mathbb{R}^2$ is a bijection. And λ is, because of the abstraction axiom ($L2$). Moreover, by Lemma 28,

$$\lambda(a, b) = \lambda(c, d) \leftrightarrow \Delta_\Phi(a, b) = \Delta_\Phi(c, d) \tag{124}$$

So, $\lambda : \mathbb{P}^2 \to \mathbb{L}$ and $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+$ are hooked.

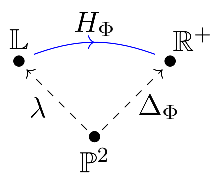
□

Lemma 52. There is a unique bijection $H_\Phi : \mathbb{L} \to \mathbb{R}^+$ satisfying, for all $a, b \in \mathbb{P}$,

$$H_\Phi(\lambda(a, b)) = \Delta_\Phi(a, b) \tag{125}$$

Proof. This follows from Lemma 50 and Lemma 51 . □

Figure 12: H_Φ

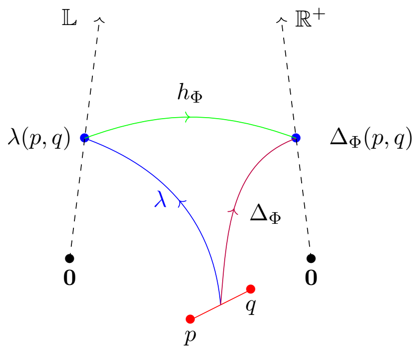


Clearly,

Lemma 53. $H_\Phi = h_\Phi$.

We can picture the relation between the (abstract) *Length* function, λ, the coordinate length function, and the corresponding measurement scale as follows:

Figure 13: *Length* λ, Coordinate Length Δ_Φ and Measurement Scale h_Φ



4 Main Theorems

In the main theorems below, we let Φ be some fixed, but arbitrary Cartesian coordinate chart $\mathbb{P} \to \mathbb{R}^2$. And we let $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+$ be the coordinate distance function on $\mathbb{P}$, determined by Φ. In Definition 38, we defined the function

$$h_\Phi : \mathbb{L} \to \mathbb{R}^+ \tag{126}$$

such that it satisfies, for any points $p, q \in \mathbb{P}$,

$$h_\Phi(\lambda(p,q)) = \Delta_\Phi(p,q) \tag{127}$$

We now are able to obtain our main theorems very quickly. First:

Theorem 1. $h_\Phi : \mathbb{L} \to \mathcal{E}_0^+$ is a measurement scale.

Proof. Let $h_\Phi : \mathbb{L} \to \mathbb{R}^+$ be defined as above. Then (dropping the subscript), as we have shown in §3.5:

(1) $h : \mathbb{L} \to \mathbb{R}^+$ is a bijection.
(2) For any $l_1, l_2 \in \mathbb{L}$: $h(l_1 \oplus l_2) = h(l_1) + h(l_2)$.
(3) For any $l_1, l_2 \in \mathbb{L}$: $l_1 \preceq l_2 \leftrightarrow h(l_1) \leq h(l_2)$.
(4) For any $l \in \mathbb{L}$, any $x \in \mathbb{R}^+$: $h(x \bullet l) = x \cdot h(l)$.

Hence, $h : \mathbb{L} \to \mathcal{E}_0^+$ is an isomorphism, and hence a measurement scale. □

Theorem 2 (Main Theorem)**.** $\mathbb{L}$ is a positive extensive quantity.

Proof. By Theorem 1, $h : \mathbb{L} \to \mathcal{E}_0^+$ is an isomorphism. Therefore, $\mathbb{L}$ is a positive extensive quantity. □

Lemma 54. Let $\mathbf{u} \in \mathbb{L}$ be any unit length (i.e., $\mathbf{0} \prec \mathbf{u}$). Then, for any length $l \in \mathbb{L}$,

$$l = \|l\|_{\mathbf{u}} \bullet \mathbf{u} \tag{128}$$

Proof. By Theorem 2, $\mathbb{L}$ is a positive extensive quantity. So, we apply The Magnitude Lemma 7 above to obtain (128). □

Theorem 3. Let $\mathbf{u} \in \mathbb{L}$ be any unit length. Then, for any points $p, q \in \mathbb{P}$,

$$\lambda(p,q) = \|\lambda(p,q)\|_{\mathbf{u}} \bullet \mathbf{u} \tag{129}$$

Proof. Let p, q be given. Let $l = \lambda(p,q)$. We then obtain Theorem 3 as an immediate corollary of Lemma 54. □

Theorem 4. For any points $p, q \in \mathbb{P}$: $\Delta_\Phi(p,q) = \|\lambda(p,q)\|_{\mathbf{u}_\Phi}$.

Proof. By The Magnitude Lemma 7, we have: $\lambda(p,q) = |\lambda(p,q)|_{\mathbf{u}_\Phi} \bullet \mathbf{u}_\Phi$. Applying h_Φ to both sides, we conclude: $h_\Phi(\lambda(p,q)) = h_\Phi(|\lambda(p,q)|_{\mathbf{u}_\Phi} \bullet \mathbf{u}_\Phi) = |\lambda(p,q)|_{\mathbf{u}_\Phi} h_\Phi(\mathbf{u}_\Phi)$. But $h_\Phi(\mathbf{u}_\Phi) = 1$. Hence, $h_\Phi(\lambda(p,q)) = \|\lambda(p,q)\|_{\mathbf{u}_\Phi}$. And thus: $\Delta_\Phi(p,q) = \|\lambda(p,q)\|_{\mathbf{u}_\Phi}$, as claimed.

□

Theorem 5. For any points $p, q \in \mathbb{P}$: $\lambda(p, q) = \Delta_\Phi(p, q) \cdot \mathbf{u}_\Phi$.

Proof. By The Magnitude Lemma 7, $\lambda(p, q) = \|\lambda(p, q)\|_{\mathbf{u}_\Phi} \cdot \mathbf{u}_\Phi$. By the above, $\Delta_\Phi(p, q) = \|\lambda(p, q)\|_{\mathbf{u}_\Phi}$. This implies: $\lambda(p, q) = \Delta_\Phi(p, q) \cdot u_\Phi$. □

A final corollary of the above is obtained from Lemma 15 and Theorem 2:

Theorem 6. $\text{Aut}(\mathbb{L}) \cong (\mathbb{R}^+ - \{0\}, \times)$.

5 Discussion

For simplicity, I focused on two dimensions. But this can easily be modified to any number of dimensions, including three, of course. Insofar as the betweenness and congruence relations in ordinary three-dimensional physical space are well-approximated by the Euclidean axioms, we can transfer our results to that setting.

Earlier I mentioned being puzzled (when I was a physics undergraduate) by what "1 cm" refers to. Well, we now have established that the *Length* property (better: structure) $\mathbb{L}$ is a continuous positive extensive quantity. So the specific length quantity 1 cm is an example of a *unit length*, an element of the *Length* quantity $\mathbb{L}$. I.e., there are two distinct points in space p, q such that $1\,\text{cm} = \lambda(p, q)$. These representative points are by no means unique, and may be taken to be any two points on a ruler separated by a 1 cm long segment. Likewise, we wondered how measurement reports, like

(1) The length (height) of Nelson's Column = 5159 cm.
(2) The length (height) of Nelson's Column = 2031 inch.

are to be analysed. Let $\mathbf{1}\,\text{cm} \in \mathbb{L}$ be the centimetre unit length and let $\mathbf{1}\,\text{inch} \in \mathbb{L}$ be the inch unit length. Our analysis states:

(3) The length (height) of Nelson's Column = $5159 \cdot \mathbf{1}\,\text{cm}$.
(4) The length (height) of Nelson's Column = $2031 \cdot \mathbf{1}\,\text{inch}$.

Moreover, we have:[27]

(5) $\mathbf{1}\,\text{inch} = 2.54 \cdot \mathbf{1}\,\text{cm}$.

So, $\mathbf{1}\,\text{inch}$ and $2.54 \cdot \mathbf{1}\,\text{cm}$ are thus the precisely the same object (i.e., abstract length). Recall that, in an extensive quantity, we can perform *division* (Lemma 9) of two quantities, obtaining a real number:[28]

$$\frac{\mathbf{1}\,\text{inch}}{\mathbf{1}\,\text{cm}} = 2.54 \tag{130}$$

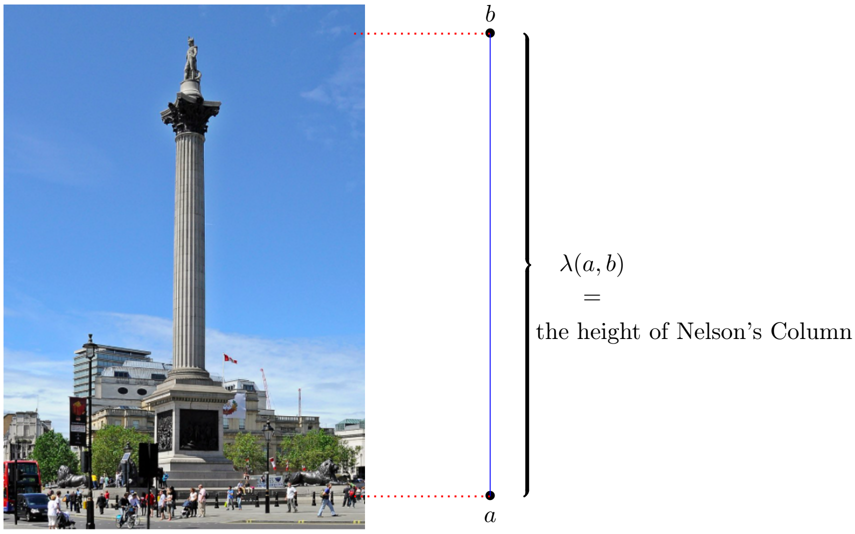
Approximating, obviously, let a, b be points at the very bottom and very top of Nelson's Column (say at a fixed time). Then:

[27] Nowadays, this is not approximate: it is exact, and treated as a definition of $\mathbf{1}\,\text{inch}$.

[28] Cf. "By number we understand not so much a multitude of Unities, as the abstracted Ratio of any Quantity to another Quantity of the same kind, which we take for Unity" (Newton (1707): 2).

(6) $\lambda(a, b) = 5159 \cdot \mathbf{1}\,\text{cm}$.
(7) $\lambda(a, b) = 2031 \cdot \mathbf{1}\,\text{inch}$.

Figure 14: The height of Nelson's Column



Mainly, this paper has been motivated by the aim to clarify the meaning of statements (measurement reports) like (1), (2), (3), (4), (5), (6), (7). I find the conventional "The Representational Theory of Measurement" (RTM) unsatisfactory, because of its excessive operationalism.[29] Although there is an important gist of truth in operationalism (Campbell (1920), Bridgman (1927)), I am sceptical that the implicit reductionism is true for physics. I do not believe, for example, that $\nabla \cdot \mathbf{B} = 0$ is a statement *reducible* to measurement reports, even though it does imply complicated implications between such reports.

Up to various revisions in physical theory, physical quantities are built into the mind-independent Universe, and the relevant properties are completely independent of human assignments and so on. Our approach has not involved "metre rods", or "physical concatenation", or some sort of axiomatization of an "empirical system", as one finds in the measurement literature. Instead, we have treated *Length* as a physical quantity *built into geometry itself*, and we have abstracted individual lengths via the (physical) congruence relation and the abstraction axiom. A metre rod measures length, a mind-independent geometric property of line segments, just as a Hall probe measures the magnetic field.

[29] Classic monographs in the RTM literature are: Krantz et al. (1971), Roberts (1985), Narens (1985), Suppes (2002). The exposition Tal (2015) provides a nice summary of the background philosophical debates and questions. The exposition Eddon (2013) provides a valuable survey of debates about the analysis of quantitative properties.

Hence, we are adopting a *realist* view of the*Length*quantity, and *a fortiori*, of length measurement (Michell (2005), Michell (2021)): "According to this paradigm, real numbers are instantiated in nature in ratios of magnitudes and measurement is the estimation of such ratios" (Michell (2021): 5). Our account developed here is similar, at least in outlook and also in methods, with earlier realist-oriented approaches (Swoyer (1987) and Mundy (1987)) (and also perhaps with the recent monograph Wolff (2020) on the metaphysics of quantities).

We have shown that physical *Length*, on the assumption that Euclidean geometry is true, is a continuous positive extensive quantity—henceforth, a continuous PEQ. The assumption (known to be not quite physically correct) that EG is true can be relaxed. The account can be generalized to the spacetime setting, and yields a completely analogous account of temporal duration. I.e., for spacetime events we shall introduce a duration function τ which maps *events* e_1, e_2 to a time duration, $\tau(e_1, e_2)$. Then temporal durations are governed by an abstract axiom involving the temporal congruence relation:[30]

$$\tau(e_1, e_2) = \tau(e_3, e_4) \leftrightarrow e_1 e_2 \equiv_{\text{tim}} e_3 e_4 \tag{131}$$

So, we have an account of physical length that fits modern physics, as currently understood.

However, perhaps physical *Length* is not, in reality, a continuous PEQ? Perhaps there exists a *minimum non-zero length*, l_p, and all physically exemplified lengths are then multiples, nl_p, for $n \in \mathbb{N}$. In that case, it seems that we could then infer that physical *Length* would be isomorphic to $(\mathbb{N}, +, \preceq)$, which distinguishes itself from $(\mathbb{R}^+, +, \leq)$ by being *rigid*. There are no automorphisms of $(\mathbb{N}, +, \preceq)$ except the identity. I have no idea if this is true, or viable. As I understand it, causal set theory does postulate a minimal length (i.e., directly connected vertices). But this programme has yet not born fruit.[31]

Finally, I wish to briefly mention *Mass*. We have given a geometric basis for talking about abstract lengths (and temporal durations). If this is correct, it establishes that

[30] I also believe it can also be generalized to Riemannian geometry, (M, g). A rough sketch (which might be slightly wrong in technical detail or for pathological metrics) is this. The definition of the *numerical* length of a smooth curve $\gamma : [0,1] \to M$ from a to b is: $L_g(\gamma; a, b) := \int_0^1 dt \sqrt{g(\dot{\gamma}, \dot{\gamma})}$. Then we define $d_g(a, b)$ to be the greatest lower bound of all $L_g(\gamma; a, b)$, over all smooth curves γ from a to b. In stating the abstraction axiom, the Euclidean notion of congruence $ab \equiv cd$, is now replaced by the clause "$d_g(a, b) = d_g(c, d)$": i.e., $\lambda(a, b) = \lambda(c, d) \leftrightarrow d_g(a, b) = d_g(c, d)$. The Hooking Lemma 50 above then guarantees the existence of a unique (bijective) measurement scale h from abstract lengths to numerical values in $\mathbb{R}^+$. Note that metrics g and αg, for a fixed positive real α, give rise to exactly *the same length congruence relation*. This is explained as follows. For given g, the preferred *unit*, $\mathbf{u}_g$ (determined by the metric g) is the length $\lambda(a, b)$ of a geodesic from a to b, whose metric length relative to g, i.e., $d_g(a, b)$ is equal to 1. If the metric tensor g is scaled by a positive real, to $\alpha \cdot g$, say, then while the *numerical length* $d_g(a, b)$ of a geodesic between a and b scales too, the *equality* $d_g(a, b) = d_g(c, d)$ is invariant. I.e., $d_g(a, b) = d_g(c, d) \leftrightarrow d_{\alpha g}(a, b) = d_{\alpha g}(c, d)$. As the numerical length scales *up*, the corresponding unit scales *down* inversely. Consequently, the *physical length* itself, $\lambda(a, b) = \|\lambda\|_{\mathbf{u}_g} \cdot \mathbf{u}_g$, is invariant. This is a form of gauge symmetry.

[31] A detailed exposition of the causal set approach to quantum gravity may be found in Surya (2019). For some further discussion of the issue of the continuum in physics, see John Baez's "Struggles With The Continuum" (Baez (2021)).

Length (and *Time*) is a continuous PEQ, via the assumption of geometrical axioms and the length abstraction principle, with respect to the "*has the same length*" equivalence relation. Is it not also obvious that the physical quantity of *Mass* is also a continuous PEQ? The problem is that I cannot see an analogous geometric abstraction principle for Mass.

There is a standard procedure for mass comparisons amongst a set (Bod, say) of solid macroscopic bodies, like ball bearings or medium-sized lumps of matter, on a weighing scale ("equal-arm balance" in the terminology of Krantz et al. (1971)). And we may define "concatenation" by, say, gluing the ball bearings or putting them into the same pan:

> A comparison relation for mass often developed is using an equal-arm balance. ... The main requirements for such balance are that it should have as little friction as possible and that if a and b balance when a is in one pan and b in the other, then they should also balance when their locations are reversed. Concatenation is simply interpreted as the positioning of both objects together in the pan. (Krantz et al. (1971): 89)

But, note that this will be a partial operation: for Bod will not be closed under $\circ$. Still, using various theorems from RTM, we may then arrive at a numerical representation, $\phi : \text{Bod} \to \mathbb{R}^+$. But this is highly *non-surjective*. Clearly the cardinality $|\text{Bod}|$ is quite small (say, less than 1000). So we do not have, for each real $x \in \mathbb{R}^+$, a ball bearing $b \in \text{Bod}$ such that $\phi(b) = x$. Only under a gigantic idealization can we imagine $|\text{Bod}| = 2^{\aleph_0}$! So, assuming that*Mass*is a continuous PEQ, it seems that individual mass quantities do not, in general, have representatives—at least not these ones, or anything similar. Alternatively, we may attempt to define the measurement of mass using ideas derived from Ernst Mach, essentially that mass *ratios* for a system of N interacting particles can be determined by (measurable) *acceleration ratios* (see Kibble & Berkshire (1996), Ch. 1, §1.3, pp. 9–10).

References

Baez, J., 2021: "Struggles With The Continuum". In M Anel & G Catren, eds., *New Spaces in Physics*, chapter 6, pages 281–326. Cambridge University Press.

Bennett, M., 1995: *Affine and Projective Geometry*. New York: Wiley.

Borsuk, K. & Szmielew, W., 1960: *Foundations of Geometry: Euclidean, Bolyai-Lobachevskian Geometry and Projective Geometry*. Amsterdam: North-Holland Publishing Company. Revised English translation by Erwin Marquit. Reprinted by Dover Books, 2018.

Bridgman, P., 1927: *The Logic of Modern Physics*. The Macmillan Company. Eight printing, 1958.

Burgess, J. & Rosen, G., 1997: *A Subject with No Object*. Oxford: Clarendon Press.

Campbell, N., 1920: *Physics: The Elements*. Cambridge: Cambridge University Press.

Drake, F. & Singh, D., 1996: *Intermediate Set Theory*. Wiley–Blackwell.

Dummit, D. & Foote, R., 2004: *Abstract Algebra*. John Wiley & Sons. Third edition.

Eddon, M., 2013: "Quantitative Properties". *Philosophy Compass* 8(7): 633–645.

Ellis, B., 1966: *Basic Concepts of Measurement*. Cambridge University Press.

Field, H., 1980: *Science Without Numbers*. Princeton: Princeton University Press. Second edition, Oxford University Press (2016).

Frege, G., 1884: *Die Grundlagen der Arithmetic: Eine logisch-mathematische Untersuchung über den Begriff der Zahl*. Breslau: Verlage Wilhelm Koebner. Translation by J.L. Austin as *The Foundations of Arithmetic*. Oxford: Blackwell, 1950.

Halmos, P., 1974: *Naive Set Theory*. New York: Springer-Verlag.

Heck, R., 2011: *Frege's Theorem*. Oxford: Oxford University Press.

Hilbert, D., 1899: *Grundlagen der Geometrie*. Leibzig: Verlag Von B.G. Teubner. Translated (by E.J.Townsend) as *The Foundations of Geometry*. Chicago: Open Court (1950).

Hintikka, J., ed., 1968: *Philosophy of Mathematics*. Oxford: Oxford University Press.

Hölder, O., 1901: "Die Axiome der Quantität und die Lehre vom Mass"". *Berichten der mathematisch-physischen Classe der Königlisch Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikaliche Classe* 53: 1–63. Translations of this work by Joel Michell and Catherine Ernst appear in Hölder (1996), Hölder (1997).

Hölder, O., 1996: "The Axioms of Quantity and the Theory of Measurement". *Journal of Mathematical Psychology* 40(3): 235–252. Part I of Hölder (1901), translated by Joel Michell and Catherine Ernst.

Hölder, O., 1997: "The Axioms of Quantity and the Theory of Measurement". *Journal of Mathematical Psychology* 41(4): 345–356. Part II of Hölder (1901), translated by Joel Michell and Catherine Ernst.

Jech, T., 2002: *Set Theory: The Third Millennium Edition*. New York: Springer. 1st edition published in 1978 and the second edition in 1997. This is the 3rd revised and enlarged edition.

Kibble, T. W. B. & Berkshire, F. H., 1996: *Classical Mechanics*. Addison-Wesley.

Kordos, M., 1969: "On the Syntactic Form of Dimension Axiom for Affine Geometry". *Bull Acad Pol Sci* 17: 833–837.

Kramida, A. E. et al., 2010: "A Critical Compilation of Experimental Data on Spectral Lines and Energy Levels of Hydrogen, Deuterium, and Tritium". *Atomic Data and Nuclear Data Tables* 96 (6): 586–644.

Krantz, D., Luce, R., Suppes, P., & Tversky, A., 1971: *Foundations of Measurement: Volume 1*. Academic Press.

Kunen, K., 1980: *Set Theory: An Introduction to Independence Proofs*. North-Holland.

Machover, M., 1996: *Set Theory, Logic and Their Limitations*. Cambridge: Cambridge University Press.

Mancosu, P., 2016: *Abstraction and Infinity*. Oxford University Press.

Michell, J., 2005: "The Logic of Measurement: A Realist Overview". *Measurement* 38(4): 285–294.

Michell, J., 2021: "Representational Measurement Theory: Is its Number Up?" *Theory & Psychology* 31(1): 3–23.

Mundy, B., 1987: "The metaphysics of quantity". *Philosophical Studies* 51 (1): 29–54.

Narens, L., 1985: *Abstract Measurement Theory*. Cambridge, MA: MIT Press.

Newton, I., 1707: *Arithmetica Universalis*. William Whiston. Translated from the original Latin by Joseph Raphson, in 1720, as "Universal arithmetic: Or, a treatise of arithmetical composition and resolution".

Roberts, F., 1985: *Measurement Theory*. Cambridge: Cambridge University Press.

Rubin, J., 1967: *Set Theory for the Mathematician*. Holden-Day.

Suppes, P., 1960: "A Comparison of the Meaning and Use of Models in Mathematics and the Empirical Sciences". *Synthese* 12: 287–301.

Suppes, P., 2002: *Representation and Invariance of Scientific Structures (*CSLI Lecture Notes: 130*)*. CSLI Publications.

Surya, S., 2019: "The Causal Set Approach to Quantum Gravity". *Living Reviews in Relativity* 22(5).

Swoyer, C., 1987: "The Metaphysics of Measurement". In J Forge, ed., *Measurement, Realism and Objectivity*, pages 235–290. Reidel.

Szczerba, L. & Tarski, A., 1979: "Metamathematical Discussion of Some Affine Geometries". *Fundamenta Mathematicae* 104: 155–192.

Tal, E., 2015: "Measurement in Science". *Stanford Encyclopedia of Phlosophy* https://plato.stanford.edu/entries/measurement-science/.

Tarski, A., 1959: "What is Elementary Geometry?" In L Henkin, P Suppes, & A Tarski, eds., *The Axiomatic Method*. Amsterdam: North Holland. Reprinted in Hintikka (1968).

Tarski, A. & Givant, S., 1999: "Tarski's System of Geometry". *Bull Symbolic Logic* 5: 175–214.

Veblen, O., 1904: "A System of Axioms for Geometry". *Transactions of the American Mathematical Society* 5 (3): 343–384.

Wenzel, M. et al., 2020: "The Isabelle/Isar Reference Manual". The manual is available online.
https://isabelle.in.tum.de/dist/Isabelle2020/doc/isar-ref.pdf.

Wiese, W. L. & Fuhr, J. R., 2009: "Accurate Atomic Transition Probabilities for Hydrogen, Helium, and Lithium". *Journal of Physical and Chemical Reference Data* 38 (3): 565–719.

Wolff, J., 2020: *The Metaphysics of Quantities*. Oxford: Oxford University Press.

Zalta, E., 2021: "Frege's Theorem and Foundations for Arithmetic". *Stanford Encyclo-*

pedia of Philosophy .