Axiomatization of Galilean Spacetime

Jeffrey Ketland*

March 1, 2023

Abstract

In this article, we give a second-order synthetic axiomatization $\mathsf{Gal}(1,3)$ for Galilean spacetime, the background spacetime of Newtonian classical mechanics. The primitive notions of this theory are the 3-place predicate of betweenness $\mathtt{Bet}$, the 2-place predicate of simultaneity $\sim$ and a 4-place congruence predicate, written $\equiv^{\sim}$, restricted to simultaneity hypersurfaces. We define a standard coordinate structure $\mathbb{G}^{(1,3)}$, whose carrier set is $\mathbb{R}^4$, and which carries relations (on $\mathbb{R}^4$) corresponding to $\mathtt{Bet}$, $\sim$ and $\equiv^{\sim}$. This is the standard model of $\mathsf{Gal}(1,3)$. We prove that the symmetry group of $\mathbb{G}^{(1,3)}$ is the (extended) Galilean group (an extension of the usual 10-parameter Galilean group, with two additional parameters for length and time scalings). We prove that each full model of $\mathsf{Gal}(1,3)$ is isomorphic to $\mathbb{G}^{(1,3)}$.

Keywords: Galilean spacetime, Representation theorem, Geometry.

1 Acknowledgements

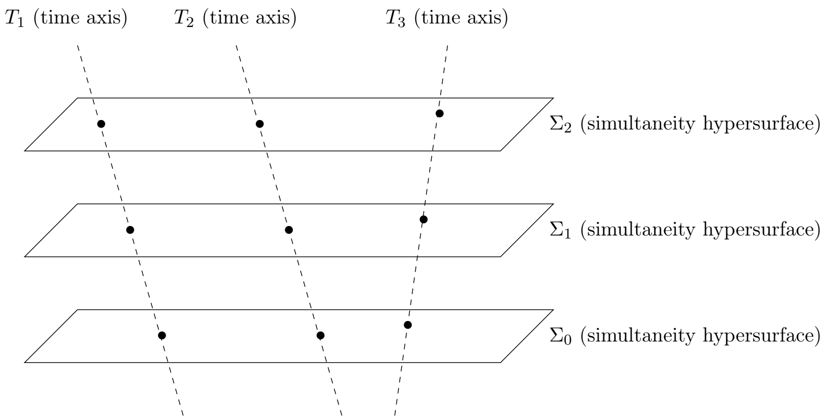
I am grateful to two anonymous referees for helpful comments. I am grateful to Professor Victor Pambuccian and to Professor Robert Goldblatt for helping me clear up some confused thoughts of mine. I am grateful to Professor David Malament for detailed comments on an earlier draft of this article, and for the suggestion of using vectors to simplify things. I am grateful to Joshua Babic and Lorenzo Cocco for some valuable advice. This work was supported by a research grant from The Polish National Science Center (Narodowe Centrum Nauki w Krakowie (NCN), Kraków, Poland), grant number 2020/39/B/HS1/02020.

*Institute of Philosophy, University of Warsaw, 3 Krakowskie Przedmieście, 00-927 Warszawa, POLAND. Email: `jeffreyketland@gmail.com`.

2 Introduction

This article provides a synthetic (and second-order) axiom system, which I call $\mathsf{Gal}(1,3)$, which describes Galilean spacetime and does so categorically.[1] Galilean spacetime is a system $\mathbb{P}$ of points on which are defined three physical geometrical primitives, satisfying certain conditions.[2] Galilean spacetime can be thought of as the background geometry of the system of spacetime events for Newtonian classical mechanics:

Figure 1: Galilean Spacetime



I shall call the carrier set of Galilean spacetime $\mathbb{P}$: this is the domain of "spacetime points" or "events". Going ahead of ourselves a bit, there are three distinguished physical relations on $\mathbb{P}$. A three-place *betweenness relation* B, which gives the whole system an affine "straight-line" structure;[3] a binary *simultaneity relation* $\sim$ which induces a

[1]The parameters "1" and "3" in $\mathsf{Gal}(1,3)$ mean: "1 time and 3 space dimensions". Recall that an axiom system is called *categorical* when it has exactly one model up to isomorphism. Second-order Peano arithmetic, PA_2, is categorical, its unique model being $(\mathbb{N}, 0, S, +, \times)$. The proof (essentially given in Dedekind (1888)) is that if $M \models \mathsf{PA}_2$, we may define using Dedekind's Recursion Theorem a function $\Phi : \mathbb{N} \to \text{dom}(M)$ by $\Phi(0) = 0^M$ and, for all $n \in \mathbb{N}$, $\Phi(n+1) = S^M(\Phi(n))$. The axioms of PA_2 then imply that Φ is a bijection which is an isomorphism from $(\mathbb{N}, 0, S, +, \times)$ to M. In addition to PA_2, the theory ALG of the complete ordered field is also categorical (essentially given in Huntington (1903), using methods developed in Dedekind (1872), Cantor (1897), Hölder (1901)). Various second-order geometrical theories are also categorical. These include the systems denoted $\mathsf{BG}(4)$ and $\mathsf{EG}(3)$ below. Theorems 62 and 63 in Appendix A establish the categoricity (and standard models) of these two systems. The proofs are due to Hilbert (1899), Veblen (1904), Tarski (1959).

[2]I think, informally, of a Galilean spacetime *modally*: a *physically possible world* with certain distinguished, or built-in, geometrical (spatio-temporal) relations. Such metaphysical issues, however, don't matter here, as our whole discussion below is about models of $\mathsf{Gal}(1,3)$.

[3]It is isomorphic to the standard four-dimensional affine space usually called $\mathbb{A}^4$ (see Gallier (2011)), which is gotten from the vector space $\mathbb{R}^4$ by "forgetting its origin". In Gallier's notation, $\mathbb{A}^4$ is $(\mathbb{R}^4, \mathbb{R}^4, +)$, where the first $\mathbb{R}^4$ is the point set, and the second $\mathbb{R}^4$ is the vector space, and $+$ is the action of vectors in $\mathbb{R}^4$ on points in $\mathbb{R}^4$. For the reader whose algebra is rusty, the notion of a *group action* is explained nicely in Dummit & Foote (2004): 41. Or Gallier (2011): 11, or Saunders (2013): 29.

partition of $\mathbb{P}$ into a system of non-intersecting simultaneity hypersurfaces, $\Sigma_0, \Sigma_1, \ldots$, arranged as a "foliation"; and a special four-place *congruence* relation: this is the four-place *sim-congruence* relation, $\equiv^{\sim}$, which induces three-dimensional Euclidean geometry on each hypersurface.[4]

An especially important subset of straight lines are "time axes": a time axis is a *straight line* in the affine geometry which does not lie within a simultaneity hypersurface. Physically, a time axis is the *trajectory of a material point acted on by no forces*—this is Newton's First Law, or the Law of Inertia.[5]

We can bundle the carrier set of Galilean spacetime, and the aforementioned three distinguished physical relations on Galilean spacetime, together: $(\mathbb{P}, B, \sim, \equiv^{\sim})$. Our aim in this paper is to give a *synthetic* axiomatization of this structure $(\mathbb{P}, B, \sim, \equiv^{\sim})$.[6] This means that, in contrast with analytic geometry, the axioms do not quantify over the reals, or introduce a metric function (like a Riemannian metric g_{ab}), or talk about coordinate systems. Instead, the axioms use a number of basic physical predicates on spacetime. And then the existence of special mappings $\Phi : \mathbb{P} \to \mathbb{R}^4$—that is, coordinate systems—becomes a theorem, not an assumption.

Hartry Field (Field (1980)) has carefully studied this approach in order to try and vindicate *nominalism*: this is the claim that there are no mathematical objects at all, and, insofar as numbers, functions, sets, vector spaces, Lie groups and so on are used in physics, and science more generally, they can be dispensed with. It is the claim that physical theories can, in principle, be replaced with theories which are "nominalistic" and the normal use of mathematics is "useful *but false*". It is to Field's enormous credit to have pinned down the two essential uses. These are:

[4] A valuable semi-formal mathematical description of Galilean spacetime, incorporating what has just been said, is given in Arnold (1989) (Ch. 1).

[5] Why do material points move (four-dimensionally) along these "grid lines" in Galilean spacetime? The *physical* answer is that such trajectories *minimize the action*. I.e., $\delta \int dt \, (\dot{q})^2 = 0$.

[6] I have tried to write this paper so that it can be read by those unfamiliar with some of the somewhat arcane details of synthetic geometry. A very useful summary of the main ideas behind the construction of coordinate systems may be found in Burgess & Rosen (1997): 102–111. In my view, a very clear and nice introduction to the topic of affine and projective *incidence* geometry is Bennett (1995), where "geometric addition" and "multiplication" of points on a fixed line are explained clearly, and the core result is proved, that the line, with those operations, is a division ring (if Desargues's Theorem is assumed) and a field (if Pappus's Theorem is assumed). Notable reference works more generally are Coxeter (1969) and Hartshorne (2000). A fairly advanced treatment is Borsuk & Szmielew (1960). Tarski's papers Tarski (1959) and Tarski & Givant (1999) are very accessible. The first of these sketches the representation theorem for first-order Euclidean geometry, and for the second-order Euclidean geometry $\mathsf{EG}(3)$ used below. Tarski focuses on the two-dimensional, first-order ("elementary") case. The book Schwabhäuser et al. (1983) is very detailed (it is in German and there is no English translation). Some recent works have implemented Tarski Euclidean geometry in theorem provers, just as one can implement arithmetic, set theory and type theory in such provers. I have no doubt that this can, in principle, be generalized to our Galilean spacetime geometry and to one or other axiomatization of Minkowski spacetime geometry.

Expressiveness — We can express physical laws by, e.g., "$\nabla \cdot \mathbf{B} = 0$" and so on. So, $\mathbf{B}$ is a *mixed function* which maps each point to some *numbers*. As Feynman put it, "From a mathematical view, there is an electric field vector and a magnetic field vector at every point in space; that is, there are six numbers associated with every point" (Feynman (1970), Vol II, §20-3).

Proof-theoretic — Using mathematics, we can get "quicker proofs" of a non-mathematical conclusion C from a non-mathematical premise P.

As regards the second, in mathematical logic, this is called "speed-up", and it was discovered by Kurt Gödel (Gödel (1936)), as a spin-off from his incompleteness results. Perhaps the most remarkable example of this phenomenon was given in Boolos (1987), a first-order valid inference with a short mathematical proof (it uses second-order comprehension), but whose shortest purely logical derivation, using the rules for the connectives and quantifiers, has vastly more symbols than the number of baryons in the observable universe.[7]

The best survey, and overall evaluation, of a large variety of nominalist approaches for both mathematics and science is Burgess & Rosen (1997).[8] I'm not recommending this as an approach to studying the geometrical assumptions of physical theories, as my own view here is the usual mathematical realist view ("useful *because true*"). Indeed, *Riemannian geometry* is here to stay! Riemannian geometry provides incredible flexibility by assuming the existence of a metric tensor g_{ab} on spacetime.[9] However, for the two special cases of Galilean spacetime and Minkowski spacetime, the *synthetic* approach helps provide a nice example of how the physics (i.e., the basic physical relations: betweenness, congruence, and so on) and mathematics (i.e., real numbers, coordinate systems, vector spaces, and so on) get "entangled".

The basic machinery for the introduction of coordinates is the *Representation Theorem*. Given a synthetic structure satisfying a series of conditions, one proves the existence of an isomorphism to a standard coordinate coordinate structure:[10]

$$\Phi : \text{synthetic structure} \rightarrow \text{coordinate structure} \tag{1}$$

That is, the isomorphism Φ takes each point p in the synthetic structure to its coordinates $\Phi^i(p)$ (usually in $\mathbb{R}^n$) in such a way that a distinguished synthetic relation R holds for $p, q, \dots$ iff a separately defined coordinate relation R' holds for $\Phi(p), \Phi(q), \dots$. (See, for example, (5) below.) Because the synthetic and coordinate structures are *isomorphic*,

[7] See Ketland (2022) for a formalization of the quicker proof in the Isabelle theorem prover.

[8] In that book, Field's approach is called "geometrical nominalism". A technical difficulty that arises for Field's programme in Field (1980), concerning the problem of maintaining *both* a conservativeness condition *and* representation theorems, is briefly described in Remark 14 below.

[9] As Einstein showed, the laws of gravitation amount to certain differential equations constraining g_{ab}, and the energy-momentum tensor T_{ab}. The "low energy limit" of Einstein's field equation is Newton's Law of Gravitation. Two standard textbooks on general relativity are Weinberg (1972) and Wald (1984).

[10] Cf. Terence Tao (Tao (2008)): "More generally, a coordinate system Φ can be viewed as an isomorphism $\Phi : A \rightarrow G$ between a given geometric (or combinatorial) object A in some class (e.g. a circle), and a standard object G in that class (e.g. the standard unit circle)."

the latter is a kind of *map* or *representation* of the former: they share the same *abstract* structure.[11]

However, historically, the analysis of Galilean spacetime did not proceed like this. Modern analysis of Galilean spacetime (sometimes called "neo-Newtonian" spacetime or just "Newtonian spacetime") was developed using the differential geometry methods, developed to study General Relativity: what are now called "*relativistic spacetimes*". This began in the 60s and 70s, with work by Trautman, Penrose, Stein, Ehlers, Earman and others (based on earlier work, such as Cartan's).[13] In Malament (2012) (Ch. 4) David Malament provides details of the differential geometry formulation of this topic. Galilean (or Newtonian) spacetime is defined as a structure of the form

$$\mathcal{A} = (M, \nabla, h^{ab}, t_{ab}), \tag{2}$$

where M is a manifold diffeomorphic to $\mathbb{R}^4$, ∇ is a flat (torsion free) affine connection on M and h^{ab}, t_{ab} are tensor fields on M satisfying compatibility conditions, from which one can construct temporal and spatial metrics, and simultaneity surfaces.[14]

The approach we develop here is entirely *synthetic*. The underlying geometric relations are *betweenness* (written $\texttt{Bet}(p, q, r)$), *simultaneity* (written $p \sim q$) and *sim-congruence* (written $pq \equiv^{\sim} rs$): these are relations on points. Inertial coordinate systems are then proved to exist by a Representation Theorem. An inertial coordinate system Φ is nothing more than an *isomorphism* from the *synthetic* geometrical structure $(\mathbb{P}, B, \sim, \equiv^{\sim})$ of Galilean spacetime (with carrier set $\mathbb{P}$) to a suitable "coordinate structure", built on the carrier set $\mathbb{R}^4$. Below we shall call this standard coordinate structure $\mathbb{G}^{(1,3)}$ (Definition 4). So, we shall obtain, by analogy with (1),

$$\Phi : \overbrace{(\mathbb{P}, B, \sim, \equiv^{\sim})}^{\text{synthetic structure}} \to \overbrace{\mathbb{G}^{(1,3)}}^{\text{coordinate structure}} \tag{3}$$

Euclidean geometry, of course, was also first set out synthetically, in Euclid (1956). However, Euclid's *Elements* does not quite meet modern adequate standards of formal

[11] To be clear, the synthetic and coordinate structures are isomorphic structures *of the same signature*, say σ. This is because it doesn't make mathematical sense to talk of an isomorphism from A to B unless they are both σ-structures.[12] Isomorphisms have to "match up" corresponding relations (operations and constants) in the signature. In logic, automated theorem proving, and so on, even seemingly small changes of the signature of the structures in question can make a large difference. For example, the structure $(\mathbb{N}, 0, S, +)$ is decidable (Presburger (1929)), but $(\mathbb{N}, 0, S, +, \times)$ is undecidable (Gödel (1931), Tarski (1936)). I'm grateful to a referee for mentioning this point, as related ones have arisen in the philosophy of physics.

[13] See Trautman (1966), Stein (1967), Penrose (1968), Earman (1970), Ehlers (1973), Friedman (1983), Earman (1989). One may also find mathematically precise descriptions in Arnold (1989) (Ch. 1) and in Kopczyński & Trautman (1992) (pp. 31–32).

[14] Here I am referring to such things as manifolds, diffeomorphisms, affine connections, tangent spaces, tensor fields and whatnot. An excellent textbook on differential geometry, oriented towards advanced physics students, is Schutz (1980). Also, Malament (2012) and Wald (1984). For useful surveys of some of the surrounding philosophical issues, see Huggett & Hoefer (2015) (absolute vs relational theories of spacetime) and DiSalle (2020) (inertial frames).

rigour. In particular, Moritz Pasch (Pasch (1882)) noted that certain *betweenness* properties of space were merely implicit in Euclid's treatment. Influenced by Pasch and others, the synthetic axiomatization for Euclidean geometry was first made rigorous in Hilbert (1899), which was modified, extended or simplified in a number of ways, one of which is Veblen (1904) (which extracted the purely betweenness part of Hilbert's system: sometimes called the "axioms of order").

Synthetic axiomatization for *Minkowski spacetime geometry* appeared soon after the classic work of Albert Einstein and Hermann Minkowski (i.e., Einstein (1905) and Minkowski (1908)), in Alfred Robb's book Robb (1911). This led to a series of later synthetic developments, including Robb (1936), Ax (1978), Mundy (1986), Goldblatt (1987), Schutz (1997) and, most recently, Cocco & Babic (2021). As is now known, Minkowski spacetime can be axiomatized using a single *binary relation*, usually called λ, with $p\lambda q$ meaning "points p and q can be connected by a light signal"—the light-signal relation.[15] As the reader probably knows, this induces a "light cone structure" on the carrier set of points. So, Minkowski spacetime can be defined as a structure $(\mathbb{P}, \lambda)$ satisfying certain axioms, and one may prove that there is an isomorphism $\Phi : (\mathbb{P}, \lambda) \to (\mathbb{R}^4, \lambda_{\mathbb{R}^4})$.[16] Such an isomorphism is called a "Lorentz coordinate system". Then the automorphism group $\text{Aut}((\mathbb{R}^4, \lambda_{\mathbb{R}^4}))$, of $(\mathbb{R}^4, \lambda_{\mathbb{R}^4})$, is the Poincaré group.[17]

Galilean spacetime, however, is the basic spacetime of classical Newtonian (pre-relativistic) physics. In retrospect, it is a kind of "low energy limit" of Minkowski spacetime (when we let the speed of light approach infinity, and all the light cones get "squashed" into simultaneity surfaces). But, unlike the case with Minkowski spacetime, the synthetic approach did not appear for a long time. As far as I know, the first brief sketch of a synthetic axiom system for Galilean spacetime appeared in Hartry Field's *Science Without Numbers* (Field (1980), Ch. 6), some 80 years after Hilbert's classic monograph, *The Foundations of Geometry*, Hilbert (1899), and close on three hundred years after Newton's *Principia* (Newton (1687)). Shortly after, John Burgess added further work on this, in Burgess (1984), and then again, in Burgess & Rosen (1997). Our work here is a descendant of and stimulated by theirs.[18]

The axiom system $\mathsf{Gal}(1, 3)$ we shall arrive at can be written as follows (see §5):

---

[15] In Goldblatt (1987), a relation of "spacetime orthogonality", $pq \perp rs$, is used, but $\perp$ and λ are interdefinable as Goldblatt shows.

[16] Where the standard coordinate relation $\lambda_{\mathbb{R}^4}$ on $\mathbb{R}^4$ is defined as follows: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$, $\mathbf{x}\lambda_{\mathbb{R}^4}\mathbf{y}$ holds iff $\sum_{i=1}^{3}(x^i - y^i)^2 - (x^4 - y^4)^2 = 0$ (i.e., the Minkowski interval is equal to 0). I have set $c = 1$.

[17] In fact, to be a bit more accurate, I believe it is the "*extended*" Poincaré group, allowing global scaling, $x^\mu \mapsto \alpha x^\mu$ ($\alpha \neq 0$), of coordinates. This is because $(\mathbb{P}, \lambda)$ does not have a special "unit length".

[18] Field states his four axioms very briefly, in a footnote (Field (1980), Ch. 6, p. 54, footnote 33. Field remarks, "Given the Szczerba-Tarski axiom on 'Bet', it is quite trivial to impose requirements on the two new primitives 'Simul' and 'S-Cong' so as to get the desired representation and uniqueness theorems" (p. 54). Although Field takes a slightly different congruence relation as primitive (which he calls `S-cong`), I am reasonably sure that Field's axiom system is definitionally equivalent to the one given here, $\mathsf{Gal}(1, 3)$. I hope to publish the equivalence proof elsewhere. Burgess's sketch of the geometry of Galilean spacetime (Burgess (1984); Burgess & Rosen (1997)) uses our physical primitives and I believe Burgess must have separately established this equivalence.

Gal1	BG(4).
Gal2	$\mathsf{EG}(3)^{\sim}$.
Gal3	$\sim$ is an equivalence relation.
Gal4	$\equiv^{\sim} \subseteq [\sim]^4$.
Gal5	$\equiv^{\sim}$ is translation-invariant.

Here, BG(4) is a group of nine axioms, the subsystem of order axioms for betweenness. See Appendix A. And $\mathsf{EG}(3)^{\sim}$ is a group of eleven axioms, a relativized subsystem of axioms for "sim-congruence" and betweenness, obtained from Tarski's formulation of Euclidean geometry for three dimensions. See Appendix A. The three further axioms Gal3, Gal4, and Gal5 "tie together" these subsystems.[19]

To summarize then how the rest of this paper goes, we shall use the two separate Representation Theorems for BG(4) and EG(3). The first of these (Theorem 62, in Appendix B) asserts the existence of a "global" bijective coordinate system:

$$\Phi : \mathbb{P} \to \mathbb{R}^4 \tag{4}$$

on any (full) model $(\mathbb{P}, B)$ of BG(4), matching any given "*4-frame*" O, X, Y, Z, I, and satisfying the betweenness representation condition, for any points $p, q, r \in \mathbb{P}$:[20]

$$B(p, q, r) \quad \leftrightarrow \quad B_{\mathbb{R}^4}(\Phi(p), \Phi(q), \Phi(r)) \tag{5}$$

where $B_{\mathbb{R}^4}$ is the standard betweenness relation on $\mathbb{R}^4$. The second Representation Theorem (Theorem 63 in Appendix B below) asserts the existence of a global coordinate system ψ on any (full) model $(\mathbb{P}, B, \equiv)$ of *three-dimensional Euclidean geometry* EG(3), matching a given "*Euclidean 3-frame*" O, X, Y, Z and satisfying the representation condition for congruence:

$$pq \equiv rs \quad \leftrightarrow \quad \psi(p)\psi(q) \equiv_{\mathbb{R}^3} \psi(r)\psi(s) \tag{6}$$

where $\equiv_{\mathbb{R}^3}$ is the standard congruence relation on $\mathbb{R}^3$. In our system, the axioms EG(3) are *relativized* to simultaneity hypersurfaces, yielding $\mathsf{EG}(3)^{\sim}$. The relativization implements the requirement that each simultaneity hypersurface is a three-dimensional Euclidean space.

We can then combine these two Representation Theorems, applied to any full model $M \models_2 \mathsf{Gal}(1,3)$, to obtain the Representation Theorem for $\mathsf{Gal}(1,3)$, which is our main theorem, Theorem 55, in §7. That is, assuming $(\mathbb{P}, B, \sim, \equiv^{\sim})$ is a (full) model of $\mathsf{Gal}(1,3)$, the existence of an isomorphism as stated in (3) above:

$$\Phi : \overbrace{(\mathbb{P}, B, \sim, \equiv^{\sim})}^{\text{synthetic structure}} \to \overbrace{\mathbb{G}^{(1,3)}}^{\text{coordinate structure}} \tag{7}$$

[19] $[\sim]^4$ is defined to be: $\{(p, q, r, s) \mid p \sim q \wedge p \sim r \wedge p \sim s\}$. See Definition 12 below.

[20] A 4-frame is an ordered quintuple of points which are not in the same 3-dimensional hypersurface. See Definition 58 below.

The crux of the proof of the main theorem are the Chronology Lemma (Lemma 52) and the Congruence Lemma (Lemma 54).

3 Definitions

Definition 1. The standard Euclidean inner product $\langle ., . \rangle_n$ and norm $\|.\|_n$ on $\mathbb{R}^n$ are defined as follows.[21] For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle_n := \sum_{i=1}^n x^i y^i$ and $\|\mathbf{x}\|_n := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_n}$. The standard Euclidean metrics $\Delta_n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are defined as follows:

$$\Delta_n(\mathbf{x}, \mathbf{y}) \quad := \quad \|\mathbf{x} - \mathbf{y}\|_n. \tag{8}$$

The *standard Euclidean metric space* with carrier set $\mathbb{R}^n$ is:

$$\mathbb{EG}^n_{\text{metric}} \quad := \quad (\mathbb{R}^n, \Delta_n) \tag{9}$$

Definition 2. The following relations are the *standard betweenness relation* $B_{\mathbb{R}^n}$, *standard simultaneity relation* $\sim_{\mathbb{R}^n}$, *standard congruence relation* $\equiv_{\mathbb{R}^n}$, and *standard sim-congruence relation* $\equiv^{\sim}_{\mathbb{R}^n}$ on $\mathbb{R}^n$. For $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u} \in \mathbb{R}^n$:

$$B_{\mathbb{R}^n}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \quad := \quad (\exists \lambda \in [0, 1])(\mathbf{y} - \mathbf{x} = \lambda(\mathbf{z} - \mathbf{x})) \tag{10}$$

$$\mathbf{x} \sim_{\mathbb{R}^n} \mathbf{y} \quad := \quad x^n = y^n \tag{11}$$

$$\mathbf{xy} \equiv_{\mathbb{R}^n} \mathbf{zu} \quad := \quad \Delta_n(\mathbf{x}, \mathbf{y}) = \Delta_n(\mathbf{z}, \mathbf{u}) \tag{12}$$

$$\mathbf{xy} \equiv^{\sim}_{\mathbb{R}^n} \mathbf{zu} \quad := \quad \Delta_n(\mathbf{x}, \mathbf{y}) = \Delta_n(\mathbf{z}, \mathbf{u}) \ \& \ \mathbf{x} \sim_{\mathbb{R}^n} \mathbf{y} \ \& \ \mathbf{x} \sim_{\mathbb{R}^n} \mathbf{z} \ \& \ \mathbf{x} \sim_{\mathbb{R}^n} \mathbf{u} \tag{13}$$

For the one dimensional case, we have two alternative but equivalent definitions. First, $B_{\mathbb{R}}(x, y, z) := (x \leq y \leq z)$; second, $B_{\mathbb{R}}(x, y, z) := |x - y| + |y - z| = |x - z|$.[22]

Definition 3. It will be useful below to define the following special five points in $\mathbb{R}^4$:

$$\mathbf{O} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{X} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{Y} := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{Z} := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{I} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

In other words, the *origin*, and the "unit points" on the four axes. I call the ordered tuple $\mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{I}$ the *standard (4-)frame* in $\mathbb{R}^4$.

Definition 4. The *standard coordinate structures* are:[23]

[21] We use the abbreviation $\mathbf{x} = (x^1, \dots, x^n)$, for n-tuples in $\mathbb{R}^n$. Similarly, for $\mathbf{y}, \mathbf{z}, \dots$. Hopefully it will be clear that these don't mean *powers* of x.

[22] The second of these in fact generalizes to $n > 1$ *if* we have a metric function: $B_{\mathbb{R}^n}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \Delta_n(\mathbf{x}, \mathbf{y}) + \Delta_n(\mathbf{y}, \mathbf{z}) = \Delta_n(\mathbf{x}, \mathbf{z})$.

[23] Regarding the definitions of $\mathbb{BG}^n$, $\mathbb{EG}^n$ and $\mathbb{G}^{(1,n)}$. These still make sense if we replace $\mathbb{R}$ in the definition by a Euclidean ordered field F (an ordered field where all non-negative elements are squares). Cf. Szczerba & Tarski (1979), Definition 1.5, p. 160, who call a space $\mathbb{BG}^n(F)$ a "Cartesian affine space" over F.

$\mathbb{BG}^n$	Betweenness geometry in n dimensions over $\mathbb{R}$	$:= (\mathbb{R}^n, B_{\mathbb{R}^n})$.
$\mathbb{EG}^n$	Euclidean space in n dimensions over $\mathbb{R}$	$:= (\mathbb{R}^n, B_{\mathbb{R}^n}, \equiv_{\mathbb{R}^n})$.
$\mathbb{G}^{(1,n)}$	Galilean spacetime in $n+1$ dimensions over $\mathbb{R}$	$:= (\mathbb{R}^{n+1}, B_{\mathbb{R}^{n+1}}, \sim_{\mathbb{R}^{n+1}}, \equiv^{\sim}_{\mathbb{R}^{n+1}})$.

Our central interest is $\mathbb{G}^{(1,3)}$, the *standard coordinate structure for four-dimensional Galilean spacetime*. The carrier set of $\mathbb{G}^{(1,3)}$ is $\mathbb{R}^4$. Its distinguished relations are betweenness (10), simultaneity (11) and sim-congruence (13), on $\mathbb{R}^4$. Note that $\mathbb{G}^{(1,3)}$ does *not* carry a metric or distance function.

4 Derivation of (Extended) Galilean Transformations

What is the *symmetry group* of the standard coordinate structure $\mathbb{G}^{(1,3)}$ for Galilean spacetime? We will see that its symmetry group is a certain Lie group $\mathcal{G}^e(1,3)$, a 12-dimensional Lie group which extends the usual Galilean group $\mathcal{G}(1,3)$ by two additional parameters, which determine coordinate scalings.

Definition 5. A is an element of the *extended Galilean matrix group* $\mathsf{Mat}^e_{\mathsf{Gal}}(4)$ if and only if A is a 4×4 matrix with real entries, and has the (block matrix) form

$$A = \begin{pmatrix} \alpha_1 R & \vec{v} \\ 0 & \alpha_2 \end{pmatrix} \tag{15}$$

where,

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \tag{16}$$

is in $O(3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $\alpha_1, \alpha_2 \in \mathbb{R} - \{0\}$. The $O(3)$ matrix R is called the *rotation* of A; the 3-vector $\vec{v}$ is called the *(relative) velocity* of A; the constant α_1 is called the *spatial scaling factor* of A and the constant α_2 is the *temporal scaling factor* of A.

Lemma 6. $\mathsf{Mat}^e_{\mathsf{Gal}}(4)$ is a subgroup of $GL(4)$.

Proof. This is a routine verification. The main part is to check that $\mathsf{Mat}^e_{\mathsf{Gal}}(4)$ is closed under matrix multiplication and each element in $\mathsf{Mat}^e_{\mathsf{Gal}}(4)$ has an inverse in $\mathsf{Mat}^e_{\mathsf{Gal}}(4)$. □

Definition 7. Let $h : \mathbb{R}^4 \to \mathbb{R}^4$. We say that h is an *extended Galilean transformation* just if, there exists an extended Galilean matrix A, and a displacement $\mathbf{d} \in \mathbb{R}^4$, such that, for all $\mathbf{x} \in \mathbb{R}^4$,

$$h(\mathbf{x}) = A\mathbf{x} + \mathbf{d}. \tag{17}$$

Lemma 8. The set of extended Galilean transformations forms a group.

Proof. This is a detailed verification of the group properties, analogous to the above. □

Definition 9. $\mathcal{G}^e(1,3) :=$ the group of extended Galilean transformations.

Theorem 10 (Automorphisms of $\mathbb{G}^{(1,3)}$)**.** $\mathrm{Aut}(\mathbb{G}^{(1,3)}) = \mathcal{G}^e(1,3)$.

Proof. I give a sketch of the proof. To show $\mathcal{G}^e(1,3) \subseteq \mathrm{Aut}(\mathbb{G}^{(1,3)})$, we verify that each extended Galilean transformation is a symmetry of $\mathbb{G}^{(1,3)}$. Since $\mathbb{BG}^4$ is a reduct of $\mathbb{G}^{(1,3)}$, and each extended Galilean transformation is affine, it follows that betweenness is invariant. The special form of extended Galilean matrices then ensures that simultaneity and sim-congruence are invariant.

To show that $\mathrm{Aut}(\mathbb{G}^{(1,3)}) \subseteq \mathcal{G}^e(1,3)$ is more involved. Since $\mathbb{BG}^4$ is a reduct of $\mathbb{G}^{(1,3)}$, it follows that any symmetry h of $\mathbb{G}^{(1,3)}$ must be affine, and so there exists a $GL(4)$ matrix A, and displacement $\mathbf{d} \in \mathbb{R}^4$ such that, for any $\mathbf{x} \in \mathbb{R}^4$,

$$h(\mathbf{x}) = A\mathbf{x} + \mathbf{d} \tag{18}$$

To determine the sixteen components A_{ij} of A, one must then examine the conditions that simultaneity and sim-congruence be invariant. By examining certain choices of points, the invariance of simultaneity enforces that A must have the form

$$A = \begin{pmatrix} C & \vec{v} \\ 0 & \alpha_2 \end{pmatrix} \tag{19}$$

where C is 3×3 matrix, and α_2 is a non-zero constant. The invariance of sim-congruence enforces that the upper 3×3 block C must be a multiple $\alpha_1 R$, of an $O(3)$ matrix R by a non-zero real factor α_1:

$$A = \begin{pmatrix} \alpha_1 R & \vec{v} \\ 0 & \alpha_2 \end{pmatrix} \tag{20}$$

But this is an extended Galilean matrix. Consequently, $\mathrm{Aut}(\mathbb{G}^{(1,3)}) \subseteq \mathcal{G}^e(1,3)$.

Together, these results imply that $\mathrm{Aut}(\mathbb{G}^{(1,3)}) = \mathcal{G}^e(1,3)$. □

The constants α_1, α_2 in any *extended* Galilean matrix A determine *scalings* of the spatial, and temporal coordinates, respectively. So, given some A in the extended Galilean matrix group, and any $(\vec{x}, t) \in \mathbb{R}^4$,

$$A(\vec{x}, t) = (\alpha_1 R\vec{x} + \vec{v}t, \alpha_2 t) \tag{21}$$

Let's set the relative rotation R to be $\mathbb{I}$ and set the relative velocity $\vec{v}$ to be zero:

$$A(\vec{x}, t) = (\alpha_1 \vec{x}, \alpha_2 t) \tag{22}$$

Thus, the spatial coordinates are scaled by α_1 and the temporal coordinate is scaled by α_2. Instead, let us set these scalings α_1, α_2 at 1, and consider the image $(\vec{x}', t')$ of the point with coordinates $(\vec{x}, t)$ under an extended Galilean transformation:

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{d} \tag{23}$$
$$t' = t + d_t \tag{24}$$

These are the usual Galilean transformations as given in physics textbooks, in usually simplified form (e.g., Sears et al. (1979): 252; or Longair (1984): 87; or Rindler (1969): 3). The conventional Galilean group $\mathcal{G}(1,3)$ is normally understood to be this 10-parameter Lie group: the ten parameters are these: four parameters for the spatial and temporal translations, **d**; three parameters (i.e., determined by the three Euler angles) for the rotation matrix R; three parameters for the velocity $\vec{v}$.

As we have defined it, the *extended* Galilean group $\mathcal{G}^e(1,3)$ is a 12-parameter Lie group: the two additional parameters, α_1, α_2, permit coordinate scalings. These two extra degrees of freedom are a consequence of our synthetic treatment, and this is completely analogous to Euclidean betweenness and congruence being invariant under coordinate scaling. Indeed, α_1 and α_2 are gauge parameters in the oldest sense of the word.

5 Axiomatization of Galilean Spacetime: $\mathsf{Gal}(1,3)$

To begin, we state the informal physical meanings of our three primitive symbols:[24]

Betweenness predicate: $\mathtt{Bet}$	$\mathtt{Bet}(p,q,r)$ means that q lies on a straight line inclusively between p and r (allowing the cases $q=p$ and $q=r$)
Simultaneity predicate: $\sim$	$p \sim q$ means that the points p,q are simultaneous.
Sim-congruence predicate: $\equiv^{\sim}$	$pq \equiv^{\sim} rs$ means the points p,q,r,s are simultaneous, and the length of the segment pq is equal to the length of the segment rs.

We are now ready to state the (synthetic) axioms for Galilean spacetime.

Definition 11. The theory $\mathsf{Gal}(1,3)$ is a two-sorted theory with sorts $\{\mathtt{point}, \mathtt{pointset}\}$ and variables $\mathsf{Var}_{\mathtt{point}} = \{p_1, p_2, \dots\}$ and $\mathsf{Var}_{\mathtt{pointset}} = \{\mathsf{X}_1, \mathsf{X}_2, \dots\}$. The signatures σ_{Gal} and $\sigma_{\mathsf{Gal},\in}$ are given by: $\sigma_{\mathsf{Gal}} = \{\mathtt{Bet}, \sim, \equiv^{\sim}\}$ and $\sigma_{\mathsf{Gal},\in} = \{\mathtt{Bet}, \sim, \equiv^{\sim}, \in\}$. By $L(\sigma_{\mathsf{Gal}})$, I shall mean the first-order language with restricted signature σ_{Gal} over the single sort $\mathtt{point}$. Its atomic formulas are of the four forms: $p_1 = p_2$, $\mathtt{Bet}(p_1,p_2,p_3)$, $p_1 \sim p_2$, and $p_1p_2 \equiv^{\sim} p_3p_4$, where "$p_i$" are point variables, and remaining formulas are built up using the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and quantifiers $\forall$ and $\exists$, as per the usual recursive definition of "formula of $L(\sigma)$".[25] By $L(\sigma_{\mathsf{Gal},\in})$, I mean the "monadic second-order" language, with signature $\sigma_{\mathsf{Gal},\in}$. Its atomic sentences include those above, along with formulas: $p_i \in \mathsf{X}_j$ and $\mathsf{X}_i = \mathsf{X}_j$. (A parser for this language counts the strings $p_i = \mathsf{X}_j$, $\mathsf{X}_j = p_i$, and $\mathsf{X}_i \in p_j$ and $p_i \in p_j$, as ill-formed.) The remaining formulas are built up using the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and quantifiers $\forall$ and $\exists$, including the new quantifications $\forall \mathsf{X}_i\, \varphi$ and $\exists \mathsf{X}_i\, \varphi$.

[24]Cf. The "interpretive principles" given in Malament (2012), pp. 120–121.

[25]Informally, we liberalize notation for point variables, occasionally using "p", "q", "r", "s", "u", "x", "y", "z", and the like, with natural number subscripts.

In discussing a full model M of, say, $\mathsf{BG}(4)$, I shall generally write "$M \models_2 \mathsf{BG}(4)$", to make it clear that M is a *full* model of $\mathsf{BG}(4)$. In other words, if $M = (\mathbb{P}, \dots)$, then $M \models_2 \forall \mathsf{X}_i\, \varphi(\mathsf{X}_i)$ if and only if, for *every* subset $U \subseteq \mathbb{P}$, $\varphi[U]$ is true in M.

Definition 12. $(p, q, r, s) \in [\sim]^4$ iff $p \sim q, p \sim r, p \sim s$.

Definition 13. The (non-logical) axioms of $\mathsf{Gal}(1, 3)$ are as follows:

Axioms $\mathsf{Gal}(1, 3)$ for Galilean Spacetime

Gal1	$\mathsf{BG}(4)$
Gal2	$\mathsf{EG}(3)^{\sim}$
Gal3	$\sim$ is an equivalence relation
Gal4	$\equiv^{\sim} \subseteq [\sim]^4$
Gal5	$\equiv^{\sim}$ is translation-invariant

$\mathsf{BG}(4)$ is really an axiom group of nine axioms for $\mathtt{Bet}$.[26] These are given in Definition 56 in Appendix A. But, to simplify the description here, one may take their conjunction.[27] $\mathsf{EG}(3)$ is also an axiom group, this time of eleven axioms. These are given in Definition 57 in Appendix A. The axiom $\mathsf{EG}(3)^{\sim}$ listed above requires further explanation.[28]

This construction is sketched, very briefly, in Field (1980), p. 54, footnote 33. First, one replaces $\equiv$ by $\equiv^{\sim}$, in each $\mathsf{EG}(3)$ axiom. Next one *relativizes* each axiom to the formula $p \sim z$ (treating z as a parameter), so that the resulting axiom says that it holds for all points simultaneous with z.[29] Next one prefixes the result with $\forall z$ and then takes the conjunction of the axioms. For example, under relativization, the $\equiv$-Transitivity axiom (E3) and the Pasch axiom (E6) become:

[26] I use the moniker "BG" to mean: "*betweenness geometry*" (n dimensions), for several reasons. First, because there doesn't seem to be a standard name for these geometries. Second, they are sometimes called "affine geometries", but word "affine" has too many meanings, including two different meanings, each having nothing to do with the betweenness relation. These are "*affine plane*" (see, e.g., Bennett (1995)) and "*affine space*" (see, e.g, Gallier (2011)). Sometimes the terminology "ordered geometry" is used (Pambuccian (2011)). But "OG" seems to me ugly. Since the terminology is not entirely uniform, I use "betweenness geometry" and hence $\mathsf{BG}(4)$, etc. I should note that these axiom systems contain Euclid's Parallel Postulate in some form.

[27] The system $\mathsf{BG}(4)$ corresponds precisely to what Burgess called GEOM_4 in Burgess (1984). The system $\mathsf{BG}(4)$ also corresponds to what Szczerba & Tarski called $\mathsf{GA}_4^* +$ Euclid in Szczerba & Tarski (1979) (and Szczerba & Tarski (1965)). The term "GA" is used to mean a system of *absolute* or *neutral geometry* (i.e., without the Parallel Postulate), which is why (Euclid) is added. Note that (Euclid) is formulated entirely using $\mathtt{Bet}$, and congruence does not appear. The subscript denotes the dimension, and the asterisk denotes that the axiom system is second-order: this means the *Continuity Axiom* is second-order, rather than a scheme. A system essentially equivalent to GA_3^* is studied carefully in the monograph Borsuk & Szmielew (1960). The axioms of $\mathsf{BG}(4)$ are the result of simplifying the categorical system of "order axioms" given in Veblen (1904), where the relevant categoricity or representation theorem (i.e., our Theorem 62 in Appendix B) was first given.

[28] $\mathsf{EG}(3)$ itself corresponds to the *second-order* version of the *three-dimensional* version of Tarski's system for synthetic Euclidean geometry in Tarski (1959), and simplified somewhat in Tarski & Givant (1999). As with "BG", I use the moniker "EG" to mean: "*Euclidean geometry*". In my notation, Tarski's 1959 paper is mostly about the first-order theory $\mathsf{EG}_0(2)$, which is $\mathsf{EG}(2)$ "little's brother".

[29] The relativization is more precisely defined as a translation $^\circ$, which acts as the identity on atomic formulas, which commutes with the Boolean logical connectives, and, for quantifiers, maps $\forall p\, \varphi$ to $(\forall p \sim z)\, \varphi^\circ$, maps $\exists p\, \varphi$ to $(\exists p \sim z)\, \varphi^\circ$, maps $\forall \mathsf{X}\, \varphi$ to $(\forall \mathsf{X} \subseteq \Sigma_z)\, \varphi^\circ$, and maps $\exists \mathsf{X}\, \varphi$ to $(\exists \mathsf{X} \subseteq \Sigma_z)\, \varphi^\circ$.

$\equiv^{\sim}$-Transitivity	$\forall z\,[(\forall p,q,r,s,t,u \sim z)\,(pq \equiv^{\sim} rs \wedge pq \equiv^{\sim} tu \to rs \equiv^{\sim} tu)]$.
Pasch	$\forall z\,[(\forall p,q,r,s,u \sim z)\,(\mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(s,u,q) \to (\exists x \sim z)\,(\mathtt{Bet}(r,x,s) \wedge \mathtt{Bet}(p,u,x)))]$.

In addition to the given non-logical axioms, we also have the customary axioms for second-order logic:

Comprehension	$\exists \mathsf{X}_1\, \forall p\,(p \in \mathsf{X}_1 \leftrightarrow \varphi)$	(variable X_1 not free in φ)
Extensionality	$\forall \mathsf{X}_1\, \forall \mathsf{X}_2\,(\forall p\,(p \in \mathsf{X}_1 \leftrightarrow p \in \mathsf{X}_2) \to \mathsf{X}_1 = \mathsf{X}_2)$	

I shall, however, in effect, assume an *ambient set theory*.[30] The reason is that I am not concerned with narrow proof-theoretic matters concerning the whole theory (for example, completeness), but rather with establishing some facts about the *full* models of the theory $\mathsf{Gal}(1,3)$. Since we consider just full models, Comprehension and Extensionality are satisfied more or less by fiat.[31] This is completely analogous to our approach in giving the usual proof, essentially that of Dedekind (1888), of the categoricity of second-order arithmetic PA_2, although, as a matter of fact, the categoricity of PA_2 can be "internalized" as a proof *inside* PA_2 itself (see Simpson & Yokoyama (2013)).

The three Galilean axioms $\mathsf{Gal3}$, $\mathsf{Gal4}$ and $\mathsf{Gal5}$ are the glue that holds together the betweenness axioms $\mathsf{BG}(4)$ and the Euclidean axioms $\mathsf{EG}(3)^{\sim}$. The content of $\mathsf{Gal3}$ and $\mathsf{Gal4}$ seems evident. The final axiom $\mathsf{Gal5}$ is the sole axiom which needs some further explanation.[32] This axiom expresses the *translation invariance* of the $\equiv^{\sim}$ relation, and may be expressed using vector notation as follows:

$$pq \equiv^{\sim} rs \to (p+\mathbf{v})(q+\mathbf{v}) \equiv^{\sim} (r+\mathbf{v})(s+\mathbf{v}) \tag{25}$$

In other words, if the (simultaneous) segments pq and rs have the same length, then the (simultaneous) segments $(p+\mathbf{v})(q+\mathbf{v})$ and $(r+\mathbf{v})(s+\mathbf{v})$ have the same length, for any vector $\mathbf{v}$.[33]

An equivalent axiom can be expressed solely using the primitives $\mathtt{Bet}$, $\sim$ and $\equiv^{\sim}$, and quantifying over points. Roughly, the axiom $\mathsf{Gal5}$ is equivalent to the following rather long-winded claim:

> If p,q,r,s, and p',q',r',s' are points such that the vectors $\mathbf{v}_{p,p'}$, $\mathbf{v}_{q,q'}$, $\mathbf{v}_{r,r'}$, $\mathbf{v}_{s,s'}$ are all equal, and $pq \equiv^{\sim} rs$, then $p'q' \equiv^{\sim} r's'$

[30] See also Borsuk & Szmielew (1960): pp. 7-8, on this topic.

[31] A suitable "ambient set theory", a system of axioms for the existence of sets, where the points will now be *urelements* or *atoms* (i.e., not sets or classes), and where comprehension, separation and replacement schemes can be applied to any urelement predicate (e.g., $\mathtt{Bet}$ and so on) is given in Ketland (2021). The ambient set theory is called $\mathsf{ZFU}_{V(T)}$ in Field (1980): 17.

[32] The axiom $\mathsf{Gal5}$ is so obvious that it occurred to me that it might indeed be provable from the remainder. However, I've not found a proof of this. So, I retain it. It is needed to show that the vector translation of a Galilean 4-frame is also a Galilean 4-frame (Lemma 53 below).

[33] The fact that if the points p,q,r,s are simultaneous, then the points $p+\mathbf{v}$, $q+\mathbf{v}$, $r+\mathbf{v}$ and $s+\mathbf{v}$ are also simultaneous, is given in Lemma 45 below.

Note that the equality clause "$\mathbf{v}_{p,p'} = \mathbf{v}_{q,q'}$" means "$p, q, p', q'$ is a *parallelogram*", and the 4-place predicate "p_1, p_2, p_3, p_4 is a parallelogram" can be defined using Bet. (See Definition 15.)

The second-order theories BG(4) and EG(3), with their point *set* variables, contain the second-order Continuity Axiom (Tarski (1959): 18):

$$[\exists r\,(\forall p \in \mathsf{X}_1)\,(\forall q \in \mathsf{X}_2)\,\mathtt{Bet}(r, p, q)] \rightarrow [\exists s\,(\forall p \in \mathsf{X}_1)\,(\forall q \in \mathsf{X}_2)\,\mathtt{Bet}(p, s, q)] \tag{26}$$

This geometrical continuity axiom, it may be noted, is closely analogous to the "Dedekind Cut Axiom" which may be used as an axiom in the formalization of second-order theory ALG of real numbers:[34]

$$(\forall \mathsf{X}_1 \subseteq \mathbb{R})\,(\forall \mathsf{X}_2 \subseteq \mathbb{R})\,(\mathsf{X}_1 \neq \varnothing \wedge \mathsf{X}_2 \neq \varnothing \wedge \overbrace{(\forall x \in \mathsf{X}_1)\,(\forall y \in \mathsf{X}_2)\,(x \leq y)}^{\mathsf{X}_1 \text{ "precedes" } \mathsf{X}_2}$$
$$\rightarrow \exists s\,\overbrace{(\forall x \in \mathsf{X}_1)\,(\forall y \in \mathsf{X}_2)\,(x \leq s \wedge s \leq y)}^{\text{the point } s \text{ "cuts" } \mathsf{X}_1 \text{ and } X_2} \tag{27}$$

The second-order theories BG(4) and EG(3), foundationally speaking, strong and both interpret ALG. They have *first-order* versions—their "little brothers", so to speak, which I shall call $\mathsf{BG}_0(4)$ and $\mathsf{EG}_0(3)$—obtained by replacing the single Continuity Axiom by infinitely many instances of the Continuity *axiom scheme*: in these instances, there are only point variables.

The little brothers, $\mathsf{BG}_0(4)$ and $\mathsf{EG}_0(3)$, are meta-mathematically somewhat different from their big brothers. In particular, they are, in fact, *complete* (and, since they are recursively axiomatized, *decidable*), as established by a celebrated theorem of Alfred Tarski (Tarski (1948)). But the big brothers are *incomplete*, because they interpret Peano arithmetic (PA), and then Gödel's incompleteness results apply. This observation leads to an important difficulty faced by Field's nominalism:

Remark 14. The *second-order* nature of BG(4)—i.e., its point variables range over points and its set variables range over *sets of points*—is what lies at the root of the technical problem for Hartry Field's nominalist programme (Field (1980)) highlighted, first informally by John Burgess, Saul Kripke and Yiannis Moschovakis, and then, in detail, by Stewart Shapiro in Shapiro (1983), and also mentioned in Burgess (1984) (last section). The required *representation theorem* indeed holds for BG(4), with respect to full models (and from this, the other representation theorems can be built up: just as we

[34] I follow Burgess (1984) in calling this theory ALG. A standard axiomatization of ALG is given in Apostol (1967), p. 18, p. 20, p. 25. An equivalent axiomatization appears in Rudolf Carnap's neglected textbook Carnap (1958): §45, 183–185. See also Tarski (1995): 215, for a similar and equivalent formulation to (27), but Tarski uses the notion "the set X strictly precedes the set Y" (using $<$ instead of $\leq$) and "s separates the sets X and Y" (again using $<$ instead of $\leq$). But these Continuity axioms are equivalent. And both are equivalent to the usual Dedekind cut axiom given in an analysis textbook: "any non-empty bounded subset of $\mathbb{R}$ has a supremum" (e.g, Apostol (1967): 25).

do below). This is Theorem 62 below. But, unfortunately, adding additional *set theory* axioms to $\mathsf{BG}(4)$ is *non-conservative*. This is because $\mathsf{BG}(4)$ interprets Peano arithmetic. And then, by Gödel's incompleteness results (Gödel (1931); Raatikainen (2020)), there is a consistency sentence $\mathsf{Con}(\mathsf{BG}(4))$ in the language of $\mathsf{BG}(4)$ itself such that $\mathsf{BG}(4)$ does not prove $\mathsf{Con}(\mathsf{BG}(4))$. $\mathsf{Con}(\mathsf{BG}(4))$ is indeed true in the standard coordinate structure, since $\mathsf{BG}(4)$ *is* consistent (for it has a model). This sentence becomes *provable* when further set axioms are added. On the other hand, $\mathsf{BG}(4)$ has a little brother, $\mathsf{BG}_0(4)$, which is a *first-order* theory (we replace the Continuity Axiom by infinitely many instances of the Continuity axiom scheme). Then conservativeness holds for $\mathsf{BG}_0(4)$ because it is *complete*! (As we know from the aforementioned celebrated result by Tarski (Tarski (1948).) But now the required representation theorem does *not* hold for the little brother $\mathsf{BG}_0(4)$. Instead, a rather different representation theorem holds, replacing $\mathbb{R}^n$ by F^n, for "some real-closed field F". This is a revision of theoretical physics, for physics works with a *manifold*, a point set equipped with a system of charts, which are maps into $\mathbb{R}^n$. Field's programme required both conservativeness (to vindicate the claimed "instrumentalist nature" of mathematics) and representation (to vindicate the claimed "purely representational" feature of applied mathematics). But the technical snag is that we cannot have *both* conservativeness and the representation theorem.

6 Main Results about $\mathsf{Gal}(1,3)$

6.1 Definitions: Betweenness Geometry

Definition 15. The formula $\mathtt{Bet}(p,q,r) \vee \mathtt{Bet}(q,r,p) \vee \mathtt{Bet}(r,p,q)$ expresses that points p,q,r are *collinear*. Assuming $p \neq q$, we use $\ell(p,q)$ to mean the set of points collinear with p and q: i.e., the line through p,q. It can be proved in $\mathsf{BG}(4)$ that each line is determined by exactly two points. We may express notions of *coplanarity*, *cohyperplanarity*, and so on, through all positive integer dimensions, using formulas that I write as $\mathtt{co}_n(p_1,\ldots,p_{n+2})$.[35] So $\mathtt{co}_1(p,q,r)$ means that p,q,r are collinear. $\mathtt{co}_2(p,q,r,s)$ means that p,q,r,s are coplanar. And so on through higher dimensions. Lines $\ell(p,q)$ and $\ell(r,s)$ are *parallel* if and only if $\mathtt{co}_2(p,q,r,s)$ and either $\ell(p,q) = \ell(r,s)$, or $\ell(p,q)$ and $\ell(r,s)$ do not intersect (i.e., have no point in common). For this, we write: $\ell(p,q) \parallel \ell(r,s)$. Four distinct points p,q,r,s form a *parallelogram* just if $\ell(p,q) \parallel \ell(r,s)$ and $\ell(p,s) \parallel \ell(q,r)$ (see Bennett (1995): 49). The notion of what I call a *4-frame* is given below (Definition 58, in Appendix B): an ordered quintuple O,X,Y,Z,I which do not lie in the same 3-dimensional space.

The theory $\mathsf{BG}(4)$ proves the existence of a 4-frame: this is simply the Lower Dimension Axiom (the axioms are listed in Appendix A). It can be proved in $\mathsf{BG}(4)$ that, given a line ℓ and a point p, there is a unique line ℓ' parallel to ℓ and containing p (this is called Playfair's Axiom, and is an equivalent of Euclid's Parallel Postulate). From Playfair's

[35] The precise definitions of the predicates $\mathtt{co}_n$ are given in Szczerba & Tarski (1979): 190. (Szczerba & Tarski call these predicates $\mathbf{L}_n$.) The definition is recursive: for $n > 1$, each $\mathtt{co}_n$ is defined in terms of the previous ones. These definitions are due to Kordos (1969).

Axiom, it can be proved in $\mathsf{BG}(4)$ that $\parallel$ is an equivalence relation. A number of other theorems from plane and solid geometry can be established, including Desargues's Theorem and Pappus's Theorem. See Bennett (1995) for explanation of these theorems. It can be proved that there is a bijection between any pair of lines. The claims mentioned so far are sufficient (the assumptions required include Desargues's Theorem and Pappus's Theorem) to establish that, given distinct parameters p, q, the line $\ell(p, q)$ is isomorphic to an ordered field.[36] The Continuity Axiom of $\mathsf{BG}(4)$ then ensures that this field is order-complete. From this we conclude that there is (unique) isomorphism $\varphi_{p,q} : \ell(p, q) \to \mathbb{R}$: i.e., $\varphi_{p,q}(p) = 0$ and $\varphi_{p,q}(q) = 1$. See also the proof sketch for Theorem 62 below.

6.2 Definitions: Galilean Geometry

Turning to the system $\mathsf{Gal}(1, 3)$, we need separate definitions of notions pertaining to simultaneity ($\sim$) and sim-congruence ($\equiv^{\sim}$).

Definition 16. A *time axis* T is a line $\ell(p, q)$ where $p \not\sim q$.

Definition 17. A *simultaneity hypersurface* Σ_p is the set $\{q \mid q \sim p\}$ of points simultaneous with p.

Beyond the notion of a 4-frame, we need a few more specialized notions of "frame" for Galilean spacetime.

Definition 18 (sim 4-frame)**.** A *sim 4-frame* is a sequence of five points O, X, Y, Z, I such that O, X, Y, Z are simultaneous and not coplanar, and I is not simultaneous with O. A sim 4-frame is automatically a 4-frame.

Definition 19 (Euclidean sim 3-frame)**.** A *Euclidean sim 3-frame* is a sequence of four points O, X, Y, Z which are simultaneous, are not $\mathtt{co}_2$, and OX, OY, OZ have the same length and are mutually perpendicular. That is, $OX \equiv^{\sim} OY$, $OX \equiv^{\sim} OZ$ and $OY \equiv^{\sim} OZ$; and $OX \perp^{\sim} OY$, $OX \perp^{\sim} OZ$ and $OY \perp^{\sim} OZ$.[37]

Definition 20 (Galilean 4-frame)**.** A *Galilean 4-frame* is a sequence of five points O, X, Y, Z, I which are a sim 4-frame, and such that the four points O, X, Y, Z are a Euclidean sim 3-frame. Note that $O \not\sim I$, and then the line $\ell(O, I)$ is called the *time axis* of the Galilean 4-frame. A Galilean 4-frame is automatically a 4-frame. We shall simply call it a Galilean frame.

---

[36] The required definitions of geometrical addition $+$ and geometrical multiplication $\times$ (which go back to Hilbert (1899)) are given in Bennett (1995). The definition of the order on a fixed line, in terms of `Bet`, is given in Tarski (1959), proof of Theorem 1.

[37] Perpendicularity $OX \perp^{\sim} OY$, for three distinct simultaneous points O, X, Y, is defined just as in Definition 59 in Appendix B, but replacing the ordinary congruence predicate $\equiv$ by the sim-congruence predicate $\equiv^{\sim}$.

6.3 Soundness

It is straightforward to demonstrate that $\mathsf{Gal}(1,3)$ is true in the coordinate structure $\mathbb{G}^{(1,3)}$, by verifying that each axiom of $\mathsf{Gal}(1,3)$ is true in $\mathbb{G}^{(1,3)}$.

Lemma 21 (Soundness Lemma)**.** $\mathbb{G}^{(1,3)} \models_2 \mathsf{Gal}(1,3)$.

6.4 Lemmas

Lemma 22. Given a point p, and a time axis T, there is a unique line $\ell' \parallel T$ st $p \in \ell'$. (This is Playfair's Axiom, a theorem of $\mathsf{BG}(4)$, and an equivalent of Euclid's parallel postulate.)

Lemma 23. Any five simultaneous points are $\mathtt{co}_3$ (i.e., cohyperplanar$_3$).

Proof. This follows from the Upper Dimension Hyperplane Axiom in $\mathsf{EG}(3)^{\sim}$. □

Lemma 24 (Non-Triviality)**.** There are at least two non-simultaneous points.

Proof. By the Upper Dimension axiom in $\mathsf{BG}(4)$, there is a 4-frame of five points, O, X, Y, Z, I which are not co$_3$. By Lemma 23, any five simultaneous points are co$_3$. If $O \sim X \sim Y \sim Z \sim I$, they'd be $\mathtt{co}_3$, a contradiction. So there are at least two non-simultaneous points. □

Lemma 25 (Galilean Frame Lemma)**.** There is a Galilean frame O, X, Y, Z, I.

Proof. By Lemma 24, let O, I be two non-simultaneous points. By $\mathsf{EG}(3)^{\sim}$, Euclidean three-dimensional geometry holds on simultaneity hypersurface Σ_O. So, there exists O, X, Y, Z, a Euclidean sim 3-frame in Σ_O. Since O and I are not simultaneous, O, X, Y, Z, I form a Galilean frame (whose time axis is $\ell(O, I)$). □

Lemma 26. Any simultaneity hypersurface Σ is a three-dimensional affine space.

Proof. If Σ_p is a simultaneity hypersurface, then, by $\mathsf{EG}(3)^{\sim}$, the restriction $(\Sigma_p, B\restriction_{\Sigma_p}, (\equiv^{\sim})\restriction_{\Sigma_p})$ is a Euclidean three-space isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$, by Theorem 63. Since the reduct $(\Sigma_p, B\restriction_{\Sigma_p})$ (i.e., forgetting the congruence relation) of a Euclidean 3-space is an affine 3-space, Σ_p is an affine three-space, and indeed isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3})$. □

6.5 Vector Methods

In this section, we assume that we are considering a full model $M \models_2 \mathsf{BG}(4)$, with $M = (\mathbb{P}, B)$ (i.e., $B \subseteq \mathbb{P}^3$ is the interpretation in M of the predicate $\mathtt{Bet}$). And we assume the material in Appendix D, which introduces the new sorts: *reals* and *vectors*.[38] The

[38] See also Malament (2009) for a nice exposition of these ideas.

vector displacement from p to q is written: $\mathbf{v}_{p,q}$.[39] In particular, recall that, by Theorem 68, the vector space $\mathbb{V}$ of displacements is isomorphic to $\mathbb{R}^4$ (as a vector space).[40]

Since $M \models_2 \mathsf{BG}(4)$, we know, by Theorem 62, that there exists a coordinate system $\Phi : \mathbb{P} \to \mathbb{R}^4$ on M: i.e., an isomorphism $\Phi : (\mathbb{P}, B) \to (\mathbb{R}^4, B_{\mathbb{R}^4})$.

Definition 27. Let O, X, Y, Z, I be a 4-frame in M. Define the four vectors:

$$\mathbf{e}_1 := \mathbf{v}_{O,X} \quad \mathbf{e}_2 := \mathbf{v}_{O,Y} \quad \mathbf{e}_3 := \mathbf{v}_{O,Z} \quad \mathbf{e}_4 := \mathbf{v}_{O,I} \tag{28}$$

Lemma 28. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for $\mathbb{V}$.

This is established inside the detailed proof of Theorem 68 below.

Definition 29. Given a coordinate system Φ on M, we can define the *associated 4-frame*, O, X, Y, Z, I of points in M:

$$O := \Phi^{-1}(\mathbf{O}) \quad X := \Phi^{-1}(\mathbf{X}) \quad Y := \Phi^{-1}(\mathbf{Y}) \quad Z := \Phi^{-1}(\mathbf{Z}) \quad I := \Phi^{-1}(\mathbf{I}) \tag{29}$$

Definition 30. Given a coordinate system Φ, we define four basis vectors:

$$\mathbf{e}_1^{\Phi} := \mathbf{v}_{O,X} \quad \mathbf{e}_2^{\Phi} := \mathbf{v}_{O,Y} \quad \mathbf{e}_3^{\Phi} := \mathbf{v}_{O,Z} \quad \mathbf{e}_4^{\Phi} := \mathbf{v}_{O,I} \tag{30}$$

Lemma 31. $\{\mathbf{e}_1^{\Phi}, \mathbf{e}_2^{\Phi}, \mathbf{e}_3^{\Phi}, \mathbf{e}_4^{\Phi}\}$ is a basis for $\mathbb{V}$.

This is a corollary of Lemma 28.

Given coordinate system Φ, and a point p, the four components of $\Phi(p)$ are written as follows:

$$\Phi(p) = \begin{pmatrix} \Phi^1(p) \\ \Phi^2(p) \\ \Phi^3(p) \\ \Phi^4(p) \end{pmatrix} \tag{31}$$

Lemma 32. For any point p, we have:

$$\mathbf{v}_{O,p} = \sum_{a=1}^{4} \Phi^a(p)\, \mathbf{e}_a^{\Phi} \tag{32}$$

Proof. Consider some of the details of the proof of the Representation Theorem for $\mathsf{BG}(4)$ (see Theorem 62 below). Examining the vector $\mathbf{v}_{O,p}$ from the origin O to p, one can see that:

[39] Some geometry texts (e.g., Coxeter (1969): 213) will write: $\overrightarrow{pq}$. E.g., Chasles's Relation then becomes: $\overrightarrow{pq} + \overrightarrow{qr} = \overrightarrow{pr}$.

[40] I am grateful for a referee for bringing to my attention Saunders (2013), whose discussion of Galilean spacetime uses similar vector methods and the notion of affine space.

$$\mathbf{v}_{O,p} = \mathbf{v}_{O,p_X} + \mathbf{v}_{O,p_Y} + \mathbf{v}_{O,p_Z} + \mathbf{v}_{O,p_I}. \tag{33}$$

where p_X, p_Y, p_Z and p_I are the "ordinates" on the four axes. Note first that $\mathbf{v}_{O,p_X} = \varphi_{O,X}(p_X)\mathbf{v}_{O,X} = \varphi_{O,X}(p_X)\mathbf{e}_1^\Phi$, and similarly for the other three vectors. So:

$$\mathbf{v}_{O,p} = \varphi_{O,X}(p_X)\mathbf{e}_1^\Phi + \varphi_{O,Y}(p_Y)\mathbf{e}_2^\Phi + \varphi_{O,Z}(p_Z)\mathbf{e}_3^\Phi + \varphi_{O,I}(p_I)\mathbf{e}_4^\Phi \tag{34}$$

Note second that $\Phi^1(p)$ is defined to be $\varphi_{O,X}(p_X)$, and $\Phi^2(p)$ is defined to be $\varphi_{O,Y}(p_Y)$, and similarly for Z and I. Hence:

$$\mathbf{v}_{O,p} = \Phi^1(p)\mathbf{e}_1^\Phi + \Phi^2(p)\mathbf{e}_2^\Phi + \Phi^3(p)\mathbf{e}_3^\Phi + \Phi^4(p)\mathbf{e}_4^\Phi \tag{35}$$

□

Lemma 33. $\mathbf{v}_{p,q} = \sum_{a=1}^4 (\Phi^a(q) - \Phi^a(p))\, \mathbf{e}_a^\Phi$.

Proof. This is verified as follows: $\mathbf{v}_{p,q} = \mathbf{v}_{p,O} + \mathbf{v}_{O,q} = (-\mathbf{v}_{O,p}) + \mathbf{v}_{O,q} = \mathbf{v}_{O,q} - \mathbf{v}_{O,p} = \sum_{a=1}^4 \Phi^a(q)\, \mathbf{e}_a^\Phi - \sum_{a=1}^4 \Phi^a(p)\, \mathbf{e}_a^\Phi = \sum_{a=1}^4 (\Phi^a(q) - \Phi^a(p))\, \mathbf{e}_a^\Phi$, where we used Chasles's Relation (i.e., $\mathbf{v}_{p,q} + \mathbf{v}_{q,r} = \mathbf{v}_{p,r}$), and some properties of vectors, and then Lemma 32 to expand $\mathbf{v}_{O,q}$ and $\mathbf{v}_{O,p}$ into their components in the Φ-basis.

□

Note that the vector $\mathbf{v}_{p,q}$ from p to q is entirely coordinate-independent.

Definition 34. We define the *horizontal, or simultaneity, vector subspace* $\mathbb{V}^\sim$ as follows:

$$\mathbb{V}^\sim := \{\mathbf{v}_{p,q} \in \mathbb{V} \mid p \sim q\}. \tag{36}$$

Definition 34 yields:

Lemma 35. $p \sim q$ iff $\mathbf{v}_{p,q} \in \mathbb{V}^\sim$.

From Lemma 26, we obtain:

Lemma 36. $\mathbb{V}^\sim$ is a three-dimensional linear subspace of $\mathbb{V}$.

Definition 37. We define $p + \mathbb{V}^\sim := \{q \in \mathbb{P} \mid \mathbf{v}_{p,q} \in \mathbb{V}^\sim\}$.

Lemma 38. $q \in p + \mathbb{V}^\sim$ if and only if $p \sim q$.

Proof. This is immediate from Definition 37 and Lemma 35. □

Lemma 39. $\Sigma_p = p + \mathbb{V}^\sim$.

Proof. $q \in \Sigma_p$, if and only if $p \sim q$, if and only if (Lemma 38) $q \in p + \mathbb{V}^\sim$. □

Lemma 40. Let a Galilean frame O, X, Y, Z, I be given and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be defined as in Definition 27 above. Then the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{V}^\sim$.

Proof. The proof is that the vectors $\mathbf{v}_{O,X}$, $\mathbf{v}_{O,Y}$ and $\mathbf{v}_{O,Z}$ each lie in $\mathbb{V}^\sim$, and, moreover, given any point $q \in \Sigma_p$, the vector $\mathbf{v}_{O,q}$ is a linear combination of $\mathbf{v}_{O,X}$, $\mathbf{v}_{O,Y}$ and $\mathbf{v}_{O,Z}$. □

Lemma 41. Given a coordinate system Φ, the set $\{\mathbf{e}_1^\Phi, \mathbf{e}_2^\Phi, \mathbf{e}_3^\Phi\}$ is a basis for $\mathbb{V}^\sim$.

This is a corollary of the previous lemma.

Lemma 42. Let a Galilean frame O, X, Y, Z, I be given and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be defined as in Definition 27 above. Let $\mathbf{v} \in \mathbb{V}$ with $\mathbf{v} = \sum_{i=1}^4 v^i \mathbf{e}_i$. Then

$$\mathbf{v} \in \mathbb{V}^\sim \leftrightarrow v^4 = 0 \tag{37}$$

Proof. Let p be any point and consider:

$$p' = p + \mathbf{v} = p + \sum_{i=1}^{3} v^3 \mathbf{e}_i + \sum_{i=1}^{4} v^4 \mathbf{v}_{O,I} \tag{38}$$

So, $\mathbf{v} = \mathbf{v}_{p,p'}$. If $\mathbf{v}_{p,p'} \in \mathbb{V}^\sim$, we infer that: $\mathbf{v}_{p,p'} = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 + \alpha^3 \mathbf{e}_3$ (for some coefficients $\alpha^i \in \mathbb{R}$), by Lemma 36. Equating coefficients, we conclude that $\alpha^i = v^i$ (for $i = 1, 2, 3$) and $v^4 = 0$, as claimed. Conversely, if $v^4 = 0$, we infer: $\mathbf{v}_{p,p'} = \sum_{i=1}^3 v^i \mathbf{e}_i + \sum_{i=1}^4 0.\mathbf{v}_{O,I} = \sum_{i=1}^3 v^i \mathbf{e}_i$. And thus, $\mathbf{v}_{p,p'} \in \mathbb{V}^\sim$. This implies that $\mathbf{v} \in \mathbb{V}^\sim$.

□

Definition 43. Let Σ_p and Σ_q be simultaneity hypersurfaces. We say that Σ_p is *parallel* to Σ_q if and only if either $\Sigma_p = \Sigma_q$ or there is no intersection of Σ_p and Σ_q. This is written: $\Sigma_p \parallel \Sigma_q$.

Lemma 44. All simultaneity hypersurfaces are parallel.

Proof. Let Σ_p and Σ_q be simultaneity hypersurfaces. For a contradiction, suppose $\Sigma_p \not\parallel \Sigma_q$. So, $\Sigma_p \neq \Sigma_q$ and there is an intersection $r \in \Sigma_p \cap \Sigma_q$. So, $r \sim p$ and $r \sim q$. Hence, $p \sim q$. Hence, $\Sigma_p = \Sigma_q$, a contradiction. □

Lemma 45 (Translation Invariance of Simultaneity)**.** If $p \sim q \to (p + \mathbf{v}) \sim (q + \mathbf{v})$.

Proof. Suppose $p \sim q$. So, we have: $\mathbf{v}_{p,q} \in \mathbb{V}^\sim$. Consider $p' = p + \mathbf{v}$ and $q' = q + \mathbf{v}$. Let $\mathbf{w} = \mathbf{v}_{p,q}$. Since $q = p + \mathbf{w}$, we have $q + \mathbf{v} = (p + \mathbf{w}) + \mathbf{v}$, which implies (using some properties of vector addition, and the action) $q' = p' + \mathbf{w}$. Hence, $\mathbf{w} = \mathbf{v}_{p',q'}$. So, $\mathbf{v}_{p',q'} = \mathbf{v}_{p,q}$. Since $\mathbf{v}_{p,q} \in \mathbb{V}^\sim$, we infer: $\mathbf{v}_{p',q'} \in \mathbb{V}^\sim$. From this, it follows that $p' \sim q'$.

□

Lemma 46. Given a simultaneity hypersurface Σ and time axis T, there is a unique intersection point lying in both Σ and T.

Proof. Let hypersurface Σ and time axis T be given. There cannot be two distinct intersections, say q and q', for then we should have $q \sim q'$, contradicting the assumption that T is a time axis. To establish the existence of at least one intersection, let us fix a Galilean frame O, X, Y, Z, I with $O, I \in T$. I.e., $T = \ell(O, I)$. For any point p, we have that there exist unique coefficients v^i and v^4 such that:

$$p = O + \sum_{i=1}^{3} v^i \mathbf{e}_i + v^4 \mathbf{v}_{O,I} \tag{39}$$

Pick any point $p \in \Sigma$ (so $\Sigma = \Sigma_p$). Next define the point q:

$$q \quad := \quad O + v^4 \mathbf{v}_{O,I} \tag{40}$$

Then we infer $\mathbf{v}_{O,q} = v^4 \mathbf{v}_{O,I}$, which implies that $q \in T$. Next consider $\mathbf{v}_{q,p}$:

$$\mathbf{v}_{q,p} = \mathbf{v}_{q,O} + \mathbf{v}_{O,p} = -v^4 \mathbf{v}_{O,I} + \sum_{i=1}^{3} v^i \mathbf{e}_i + v^4 \mathbf{v}_{O,I} = \sum_{i=1}^{3} v^i \mathbf{e}_i \tag{41}$$

Since $\mathbf{v}_{q,p} = \sum_{i=1}^{3} v^i \mathbf{e}_i$ and $\sum_{i=1}^{3} v^i \mathbf{e}_i \in \mathbb{V}^{\sim}$, it follows that $q \sim p$. This implies that $q \in \Sigma_p$, and therefore $q \in \Sigma$. The defined point q is therefore the required intersection of T and Σ. $\square$

Definition 47. Let $\ell = \ell(p, q)$ (with $p \neq q$) be a line and let Σ be a simultaneity hypersurface. We say that ℓ is *parallel* to Σ if and only if either $\ell \subseteq \Sigma$ or there is no intersection $r \in T \cap \Sigma$. This is written: $\ell \parallel \Sigma$.

Lemma 48. No time axis is parallel to a simultaneity hypersurface.

Proof. Let $T = \ell(p, q)$ be a time axis (i.e., $p \not\sim q$). Let Σ be a simultaneity hypersurface. For a contradiction, suppose $T \parallel \Sigma$. So, either $\ell(p, q) \subseteq \Sigma$ or there is no intersection $r \in T \cap \Sigma$. But, by Lemma 46, there is a unique intersection $r \in T \cap \Sigma$. So, we must have: $\ell(p, q) \subseteq \Sigma$. Then, since $p, q \in \ell(p, q)$, we have $p, q \in \Sigma$. Hence, $p \sim q$, a contradiction. Therefore, $T \nparallel \Sigma$. $\square$

Lemma 49. Let lines $\ell(p, q)$ and $\ell(r, s)$ be parallel. Then, for some $\alpha \neq 0$, $\mathbf{v}_{p,q} = \alpha \mathbf{v}_{r,s}$.

Proof. This follows from the detailed construction of $\mathbb{V}$ (based on parallelograms and equipollence) which yields Theorem 68 . $\square$

Lemma 50. Any line parallel to a time axis is a time axis.

Proof. Suppose line $\ell(p, q)$ is parallel to a time axis $T = \ell(O, I)$, with $O \not\sim I$. Then, by Lemma 49, $\mathbf{v}_{p,q} = \alpha \mathbf{v}_{O,I}$, with $\alpha \neq 0$. Since $O \not\sim I$, we have $\mathbf{v}_{O,I} \notin \mathbb{V}^{\sim}$. In general, for any $\alpha \neq 0$, $\mathbf{v} \in \mathbb{V}^{\sim}$ if and only if $\alpha \mathbf{v} \in \mathbb{V}^{\sim}$. So, it follows that $\mathbf{v}_{p,q} \notin \mathbb{V}^{\sim}$. Hence, $p \not\sim q$. Thus, $\ell(p, q)$ is a time axis. $\square$

6.6 Representation

Definition 51. Let $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$ be a σ_{Gal}-structure (i.e., B interprets $\mathtt{Bet}$, and $\sim$ interprets $\sim$ and $\equiv^{\sim}$ interprets $\equiv^{\sim}$). Suppose that $M \models_2 \mathsf{Gal}(1,3)$. Let $\Phi : \mathbb{P} \to \mathbb{R}^4$ be a function. We say:

$B_{\mathbb{R}^4}$ *represents* B wrt Φ	iff	for all $p, q, r \in \mathbb{P}$: $B(p,q,r) \leftrightarrow (\Phi(p), \Phi(q), \Phi(r)) \in B_{\mathbb{R}^4}$.
$\sim_{\mathbb{R}^4}$ *represents* $\sim$ wrt Φ	iff	for all $p, q \in \mathbb{P}$: $p \sim q \leftrightarrow \Phi(p) \sim_{\mathbb{R}^4} \Phi(q)$.
$\equiv^{\sim}_{\mathbb{R}^4}$ *represents* $\equiv^{\sim}$ wrt Φ	iff	for all $p, q, r, s \in \mathbb{P}$: $pq \equiv^{\sim} rs \leftrightarrow \Phi(p)\Phi(q) \equiv^{\sim}_{\mathbb{R}^4} \Phi(r)\Phi(s)$

If Φ is a bijection, and each of the three above representation conditions holds, then Φ is an *isomorphism* from M to $\mathbb{G}^{(1,3)}$.

In order to prove the Representation Theorem for $\mathsf{Gal}(1,3)$ we need to establish three main lemmas. I call these the Chronology Lemma, the Galilean Frame Translation Invariance Lemma, and the Congruence Lemma.

6.7 The Chronology Lemma

Lemma 52 (Chronology Lemma)**.** Let $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$ be a σ_{Gal}-structure, with $M \models_2 \mathsf{Gal}(1,3)$. Let O, X, Y, Z, I be a *sim* 4-frame in M. Since $(\mathbb{P}, B) \models_2 \mathsf{BG}(4)$, let $\Phi : (\mathbb{P}, B) \to (\mathbb{R}^4, B_{\mathbb{R}^4})$ be an isomorphism matching O, X, Y, Z, I. Then $\sim_{\mathbb{R}^4}$ represents $\sim$ wrt Φ.

Proof. Since O, X, Y, Z, I is a *sim* 4-frame, the points O, X, Y, Z are simultaneous, not coplanar, and $O \not\sim I$. Given that Φ matches O, X, Y, Z, I, with O, X, Y, Z simultaneous, the associated basis $\{\mathbf{e}_1^{\Phi}, \mathbf{e}_2^{\Phi}, \mathbf{e}_3^{\Phi}\}$ is a basis for the simultaneity vector space $\mathbb{V}^{\sim}$, by Lemma 41. Since a sim 4-frame is a 4-frame, $\{\mathbf{e}_1^{\Phi}, \mathbf{e}_2^{\Phi}, \mathbf{e}_3^{\Phi}, \mathbf{e}_4^{\Phi}\}$ is a basis for $\mathbb{V}$. Let points p, q be given. We claim:

$$p \sim q \quad \leftrightarrow \quad \Phi^4(p) = \Phi^4(q). \tag{42}$$

From Lemma 35, we have that $p \sim q$ holds if and only if $\mathbf{v}_{p,q} \in \mathbb{V}^{\sim}$. Using Lemma 33, we next expand $\mathbf{v}_{p,q}$ in the basis $\{\mathbf{e}_a^{\Phi}\}$ determined by Φ:

$$\mathbf{v}_{p,q} = \sum_{a=1}^{4} (\Phi^a(q) - \Phi^a(p))\mathbf{e}_a^{\Phi} \tag{43}$$

From Lemma 42, we conclude that $\mathbf{v}_{p,q} \in \mathbb{V}^{\sim}$ iff $(\mathbf{v}_{p,q})^4 = 0$. That is, $p \sim q$ iff $\Phi^4(q) - \Phi^4(p) = 0$. And therefore, $p \sim q$ iff $\Phi^4(q) = \Phi^4(p)$, as claimed. $\square$

6.8 The Galilean Frame Translation Invariance Lemma

Lemma 53 (Galilean Frame Translation Invariance Lemma)**.** Let $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$ be a σ_{Gal}-structure, with $M \models_2 \mathsf{Gal}(1,3)$. Let O, X, Y, Z, I be a Galilean 4-frame in M. Since $(\mathbb{P}, B) \models_2 \mathsf{BG}(4)$, let $\Phi : (\mathbb{P}, B) \to (\mathbb{R}^4, B_{\mathbb{R}^4})$ be an isomorphism matching O, X, Y, Z, I. Let $\mathbf{v} \in \mathbb{V}$. Let $O' = O + \mathbf{v}, X' = X + \mathbf{v}, Y' = Y + \mathbf{v}, Z' = Z + \mathbf{v}, I' = I + \mathbf{v}$. Then O', X', Y', Z', I' is a Galilean 4-frame.

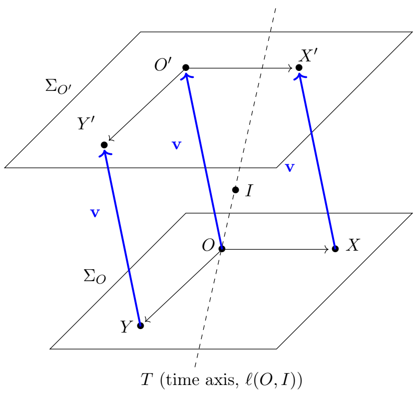
That is, leaving the assumptions as stated, when we apply a translation (given by a vector $\mathbf{v}$) to a Galilean frame, so $O' = O + \mathbf{v}$, etc., the result is also a Galilean frame:

$$O, X, Y, Z, I \text{ is a Galilean 4-frame iff } O', X', Y', Z' \text{ is a Galilean 4-frame} \tag{44}$$

Proof. Without loss of generality, we may suppose that $\mathbf{v}$ does not lie in the simultaneity hypersurface Σ_O. For if it does, the vector will simply translate the frame "horizontally", along within Σ_O and the Euclidean axioms, along with the fact that the temporal benchmark point I also moves "horizontally" too within the hypersurface Σ_I, guarantee that O', X', Y', Z', I' is a Galilean 4-frame

I will sketch how the proof goes. It is best illustrated by Figure 2:

Figure 2: "Transformed Galilean frame" on $\Sigma_{O'}$ (axis $\ell(O,Z)$ and point I' suppressed)



This is the sole part of our analysis appealing to the axiom $\mathsf{Gal}5$, stating the translation invariance of $\equiv^{\sim}$.

The five points O, X, Y, Z, I form a Galilean frame, and thus the four points O, X, Y, Z form a Euclidean sim 3-frame. So, in the lower simultaneity hypersurface, Σ_O, we have a Euclidean sim 3-frame O, X, Y, Z: the three legs OX, OY and OZ are perpendicular and of equal length. (The point Z and the axis $\ell(O, Z)$ are suppressed in Figure 2.)

Consider the hypersurface $\Sigma_{O'}$. By assumption, each of the points O', X', Y', Z' is obtained by adding the *same* displacement vector: $\mathbf{v} = \mathbf{v}_{O,O'}$:

$$O' = O + \mathbf{v} \quad X' = X + \mathbf{v} \tag{45}$$

$$Y' = Y + \mathbf{v} \quad Z' = Z + \mathbf{v} \tag{46}$$

Since O, X, Y, Z are simultaneous, it follows using Lemma 45, that O', X', Y', Z' are simultaneous. So, all four points lie in $\Sigma_{O'}$.

Next we use the Translation Invariance axiom $\mathsf{Gal5}$ of $\mathsf{Gal}(1,3)$: $\equiv^{\sim}$ is translation invariant. Since O, X, Y, Z form a Euclidean sim 3-frame, we may conclude, from the translation invariance of $\equiv^{\sim}$, that O', X', Y', Z' is also a Euclidean sim 3-frame. Since $\mathbf{v}$ does not lie parallel to Σ_O, I' is not simultaneous with O', X', Y', Z'. And so O', X', Y', Z', I' is a Galilean 4-frame. □

6.9 The Congruence Lemma

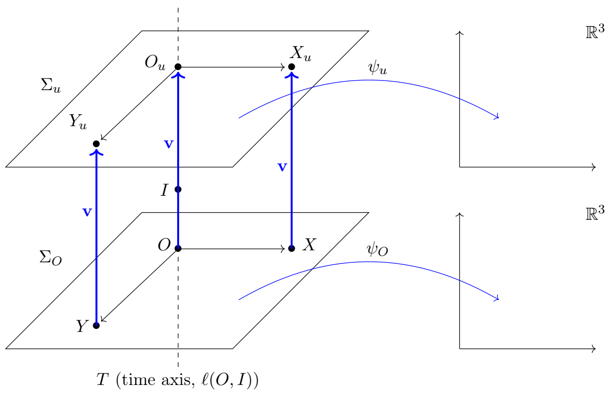
Lemma 54 (Congruence Lemma)**.** Let $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$ be a σ_{Gal}-structure, with $M \models_2 \mathsf{Gal}(1,3)$. Let O, X, Y, Z, I be a Galilean 4-frame in M. By the Chronology Lemma (Lemma 52), there is an isomorphism $\Phi : (\mathbb{P}, B, \sim) \to (\mathbb{R}^4, B_{\mathbb{R}^4}, \sim_{\mathbb{R}^4})$ matching O, X, Y, Z, I. Then $\equiv^{\sim}_{\mathbb{R}^4}$ represents $\equiv^{\sim}$ with respect to Φ.

Proof. We are given a structure $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$, a Galilean frame, O, X, Y, Z, I in M, and an isomorphism $\Phi : (\mathbb{P}, B, \sim) \to (\mathbb{R}^4, B_{\mathbb{R}^4}, \sim_{\mathbb{R}^4})$, matching O, X, Y, Z, I. We shall call Φ the "global isomorphism". We claim that $\equiv^{\sim}_{\mathbb{R}^4}$ represents $\equiv^{\sim}$ with respect to Φ: that is, for simultaneous points p, q, r, s, we have:[41]

$$pq \equiv^{\sim} rs \leftrightarrow \Delta_3(\vec{\Phi}(p), \vec{\Phi}(q)) = \Delta_3(\vec{\Phi}(r), \vec{\Phi}(s)) \tag{47}$$

Consider Figure 3:

Figure 3: "Lifted Euclidean frame" on Σ_u (axis $\ell(O, Z)$ suppressed)



[41]Where $\vec{\Phi}(p)$ is the triple $(\Phi^1(p), \Phi^2(p), \Phi^3(p)) \in \mathbb{R}^3$, and Δ_3 is the metric function on $\mathbb{R}^3$.

By hypothesis, the five points O, X, Y, Z, I form a Galilean frame, and thus the four points O, X, Y, Z form a Euclidean sim 3-frame. For points in the lower simultaneity hypersurface, Σ_O, we have, from the Euclidean axiom group in $\mathsf{Gal}(1,3)$, and the Representation Theorem for Euclidean geometry (Theorem 63), the existence of an isomorphism (i.e., coordinate system on Σ_O),

$$\psi_O : \Sigma_O \to \mathbb{EG}^3. \tag{48}$$

that matches this Euclidean sim 3-frame O, X, Y, Z. So, in the hypersurface Σ_O, a "mini-representation theorem" holds. For any $p, q, r, s \in \Sigma_O$,

$$pq \equiv^{\sim} rs \leftrightarrow \vec{\psi}_O(p)\vec{\psi}_O(q) \equiv_{\mathbb{R}^3} \vec{\psi}_O(r), \vec{\psi}_O(s) \tag{49}$$

Let Φ_O be $\Phi \restriction_{\Sigma_O}$: the restriction of the global isomorphism Φ to the hypersurface Σ_O. We are also given that Φ_O also matches O, X, Y, Z. By the uniqueness of coordinate systems which match the same frame (Lemma 66), we conclude

$$\psi_O = \Phi_O \tag{50}$$

Thus, Φ_O satisfies:

$$pq \equiv^{\sim} rs \leftrightarrow \vec{\Phi}_O(p)\vec{\Phi}_O(q) \equiv_{\mathbb{R}^3} \vec{\Phi}_O(r), \vec{\Phi}_O(s) \tag{51}$$

We now repeat the same argument for an arbitrary simultaneity surface, Σ_u.

Given any point u, we consider the hypersurface Σ_u. By Lemma 46, the time axis $\ell(O, I)$ intersects Σ_u at the corresponding "origin", O_u. By Lemma 22, there are unique lines through X, Y and Z, each parallel to $\ell(O, O_u)$. By Lemma 46 again, these intersect Σ_u at points X_u, Y_u, Z_u. By Lemma 44, the hypersurfaces Σ_O and Σ_u are parallel; this guarantees that each of the points O_u, X_u, Y_u, Z_u is obtained by adding the *same* displacement vector: $\mathbf{v} = \mathbf{v}_{O,O_u}$:

$$O_u = O + \mathbf{v} \quad X_u = X + \mathbf{v} \tag{52}$$
$$Y_u = Y + \mathbf{v} \quad Z_u = Z + \mathbf{v} \tag{53}$$

By the Translation Invariance of Galilean frames, Lemma 53, since O, X, Y, Z, I form a Galilean 4-frame, we may conclude that O_u, X_u, Y_u, Z_u, I_u (where $I_u = I + \mathbf{v}$) also form a Galilean 4-frame. And thus O_u, X_u, Y_u, Z_u form a Euclidean sim 3-frame. By the Representation Theorem for Euclidean geometry, there is an isomorphism ψ_u, which matches O_u, X_u, Y_u, Z_u. By similar reasoning to the case of Σ_O, we define the restriction Φ_u to be $\Phi \restriction_{\Sigma_u}$—i.e., the restriction of the global isomorphism Φ to the hypersurface Σ_u. We can conclude,

$$\psi_u = \Phi_u \tag{54}$$

Thus, Φ_u satisfies the following: for any points $p, q, r, s \in \Sigma_u$,

$$pq \equiv^{\sim} rs \quad \leftrightarrow \quad \vec{\Phi}_u(p)\,\vec{\Phi}_u(q) \equiv_{\mathbb{R}^3} \vec{\Phi}_u(r)\,\vec{\Phi}_u(s) \tag{55}$$

This is equivalent to (47).

□

7 Representation Theorem for $\mathsf{Gal}(1,3)$

Our main theorem is then the following:

Theorem 55 (Representation Theorem for Galilean Spacetime)**.** Let $M = (\mathbb{P}, B, \sim, \equiv^{\sim})$ be a full σ_{Gal}-structure. Then

$$M \models_2 \mathsf{Gal}(1,3) \quad \text{if and only if} \quad \text{there is an isomorphism } \Phi : M \to \mathbb{G}^{(1,3)} \tag{56}$$

Proof. For the right-to-left direction, suppose there is an isomorphism $\Phi : M \to \mathbb{G}^{(1,3)}$. So, $M \cong \mathbb{G}^{(1,3)}$. By the Soundness Lemma 21, $\mathbb{G}^{(1,3)} \models_2 \mathsf{Gal}(1,3)$. Since isomorphic structures satisfy the same sentences, it follows that $M \models_2 \mathsf{Gal}(1,3)$.

For the converse, let $M \models_2 \mathsf{Gal}(1,3)$. From the Galilean Frame Lemma 25, a Galilean frame O, X, Y, Z, I exists. This is a 4-frame. By the Representation Theorem for $\mathsf{BG}(4)$ (Theorem 62), we conclude that there is a global isomorphism:

$$\Phi : \mathbb{P} \to \mathbb{R}^4 \tag{57}$$

such that Φ matches the frame O, X, Y, Z, I and $\Phi : (\mathbb{P}, B) \to (\mathbb{R}^4, B_{\mathbb{R}^4})$ is an isomorphism. So, $B_{\mathbb{R}^4}$ represents the betweenness relation B of M with respect to Φ. Recall that the global isomorphism Φ matches a *Galilean* frame O, X, Y, Z, I. Since O, X, Y, Z, I is a Galilean frame, O, X, Y, Z, I is a *sim* frame. By the Chronology Lemma (Lemma 52), we conclude that the relation $\sim_{\mathbb{R}^4}$ represents the simultaneity relation $\sim$ of M, with respect to Φ. What is more, again, since O, X, Y, Z, I is a Galilean frame, we can appeal to the Congruence Lemma (Lemma 54). From this, we conclude that $\equiv^{\sim}_{\mathbb{R}^4}$ represents the sim-congruence relation $\equiv^{\sim}$ of M with respect to Φ.

Assembling this, $\Phi : M \to \mathbb{G}^{(1,3)}$ is an isomorphism, as claimed. □

Such isomorphisms $\Phi : M \to \mathbb{G}^{(1,3)}$ are *inertial charts* on Galilean spacetime. They correspond, one-to-one, with Galilean frames. As we have seen, the transformation group between these isomorphisms (or, if you wish, between the Galilean frames) is precisely $\mathcal{G}^e(1,3)$—the extended Galilean group.

Appendices

Appendix A Axioms

Definition 56. The non-logical axioms of $\mathsf{BG}(4)$ in $L(\sigma_{\mathsf{Gal},\in})$ are the following nine:[42]

B1.	$\mathtt{Bet}$-Identity	$\mathtt{Bet}(p,q,p) \to p = q$.
B2.	$\mathtt{Bet}$-Transitivity	$\mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(q,r,s) \wedge q \neq r \to \mathtt{Bet}(p,q,s)$.
B3.	$\mathtt{Bet}$-Connectivity	$\mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(p,q,r') \wedge p \neq q \to (\mathtt{Bet}(p,r,r') \vee \mathtt{Bet}(p,r',r))$.
B4.	$\mathtt{Bet}$-Extension	$\exists p\, (\mathtt{Bet}(p,q,r) \wedge p \neq q)$.
B5.	Pasch	$\mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(s,u,q) \to \exists x\, (\mathtt{Bet}(r,x,s) \wedge \mathtt{Bet}(p,u,x))$.
B6.	Euclid	$\mathtt{Bet}(a,d,t) \wedge \mathtt{Bet}(b,d,c) \wedge a \neq d \to \exists x\, \exists y\, (\mathtt{Bet}(a,b,x) \wedge \mathtt{Bet}(a,c,y) \wedge \mathtt{Bet}(x,t,y))$
B7.	Lower Dimension	There exist five points which are not $\mathtt{co}_3$.
B8.	Upper Dimension	Any six points are $\mathtt{co}_4$.
B9.	Continuity Axiom	$[\exists r\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathtt{Bet}(r,p,q)] \to \exists s\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathtt{Bet}(p,s,q)$

See Szczerba & Tarski (1979), pp. 159–160, for the first-order two-dimensional theory GA_2 (for "neutral" or "absolute geometry"), which lacks the Euclid Parallel axiom (which is called (E) in Szczerba & Tarski (1979) and is called (Euclid) above). Their system includes Desargues's Theorem. But for us, this axiom is no longer required, as it provable from the remaining axioms in dimensions above two (Szczerba & Tarski (1979): 190). The above axiom system is the second-order, four-dimensional theory, and containing (E), i.e., (Euclid). The relevant representation theorem follows from Theorem 5.12 of Szczerba & Tarski (1979) (see also Example 6.1). The same theorem is stated, somewhat indirectly, on pp. 196–197 of Borsuk & Szmielew (1960). The representation theorem itself goes back to Veblen (1904).

Definition 57. The non-logical axioms of $\mathsf{EG}(3)$ in $L(\sigma_{\mathtt{Bet},\equiv,\in})$ are the following eleven:

E1.	$\mathtt{Bet}$-Identity	$\mathtt{Bet}(p,q,p) \to p = q$.
E2.	$\equiv$-Identity	$pq \equiv rr \to p = q$.
E3.	$\equiv$-Transitivity	$pq \equiv rs \wedge pq \equiv tu \to rs \equiv tu$.
E4.	$\equiv$-Reflexivity	$pq \equiv qp$.
E5.	$\equiv$-Extension	$\exists r\, (\mathtt{Bet}(p,q,r) \wedge qr \equiv su)$.
E6.	Pasch	$\mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(s,u,r) \to \exists x\, (\mathtt{Bet}(q,x,s) \wedge \mathtt{Bet}(u,x,p))$.
E7.	Euclid	$\mathtt{Bet}(a,d,t) \wedge \mathtt{Bet}(b,d,c) \wedge a \neq d \to \exists x\, \exists y\, (\mathtt{Bet}(a,b,x) \wedge \mathtt{Bet}(a,c,y) \wedge \mathtt{Bet}(x,t,y))$
E8.	5-Segment	$(p \neq q \wedge \mathtt{Bet}(p,q,r) \wedge \mathtt{Bet}(p',q',r') \wedge pq \equiv p'q' \wedge qr \equiv q'r' \wedge ps \equiv p's' \wedge qs \equiv q's') \to rs \equiv r's'$.
E9.	Lower Dimension	There exist four points which are not $\mathtt{co}_2$.
E10.	Upper Dimension	Any five points are $\mathtt{co}_3$.
E11.	Continuity Axiom	$[\exists r\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathtt{Bet}(r,p,q)] \to \exists s\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathtt{Bet}(p,s,q)$

The original source of this axiomatization is Tarski (1959) and Tarski & Givant (1999). See Tarski (1959), pp. 19–20, for a formulation of the first-order two-dimensional theory, with twelve axioms and one axiom scheme (for continuity); and Tarski & Givant (1999) for

[42] These axioms are given originally in Szczerba & Tarski (1965) and Szczerba & Tarski (1979). See also Goldblatt (1987): 165 for the corresponding first-order theory, which we have called $\mathsf{BG}_0(4)$. Goldblatt calls this "the first-order theory of ordered affine fourfolds over real-closed fields".

a simplification down to ten axioms and one axiom scheme (for continuity). The above axiom system is the second-order, four-dimensional theory (i.e., the single Continuity Axiom is the second-order one).

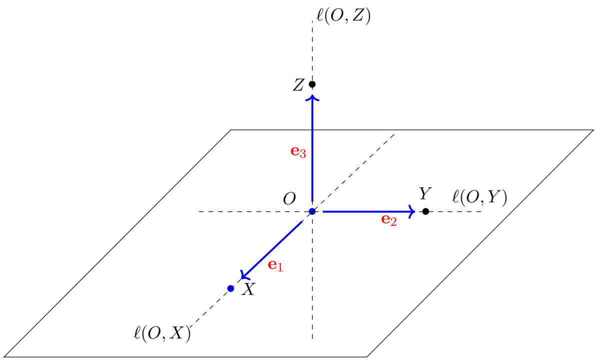
Appendix B Representation Theorems

Definition 58 (4-frame)**.** For betweenness geometry, a *4-frame* is an ordered tuple of five points O, X, Y, Z, I which are not $\mathtt{co}_3$.[43]

Definition 59 (Perpendicularity)**.** In Euclidean geometry, perpendicularity $OX \perp OY$, for three distinct points O, X, Y, is defined as follows: $OX \perp OY$ holds iff $XY \equiv (-X)Y$, where $(-X)$ is the unique point p on $\ell(O, X)$ such that $p \neq X$, and $Op \equiv OX$.

Definition 60 (Euclidean 3-frame)**.** For Euclidean geometry, a Euclidean 3-frame is an ordered quadruple O, X, Y, Z of points which are not $\mathtt{co}_2$ (i.e., not coplanar), and such that the segments OX, OY, OZ are mutually perpendicular and of equal length.

Figure 4: Euclidean 3-Frame



Definition 61 (Matching)**.** Suppose that $M = (\mathbb{P}, B)$ is a $\sigma_{\mathtt{Bet}}$-structure with $M \models_2 \mathsf{BG}(4)$ and suppose that O, X, Y, Z, I is a 4-frame in M. Suppose that $\Phi : \mathbb{P} \to \mathbb{R}^4$ is a function. We say that Φ *matches* O, X, Y, Z, I just if:[44]

$$\Phi(O) = \mathbf{O} \quad \Phi(X) = \mathbf{X} \quad \Phi(Y) = \mathbf{Y} \quad \Phi(Z) = \mathbf{Z} \quad \Phi(I) = \mathbf{I} \tag{58}$$

[43]Burgess refers to such systems of points as "benchmarks": Burgess & Rosen (1997): 107. For example, in the two-dimensional case, one may imagine marking three non-collinear points O, X, Y on a bench. This will be a "2-frame", and will determine a two-dimensional coordinate system, with O at the origin, and $\ell(O, X)$ the "x-axis" and $\ell(O, Y)$ the "y-axis".

[44]A similar definition, *mutatis mutandis*, can be applied to $\mathsf{BG}(n)$ in general, and to $\mathsf{EG}(n)$ in general.

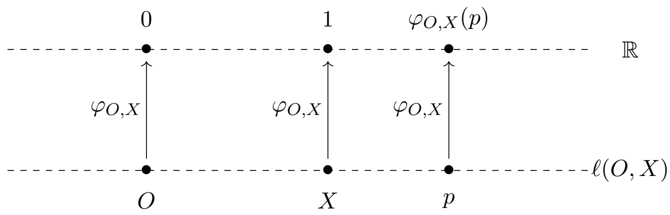
The following two theorems are due, primarily, to Hilbert (1899), Veblen (1904), Tarski (1959):[45]

Theorem 62 (Representation Theorem for $\mathsf{BG}(4)$)**.** Let $M = (\mathbb{P}, B)$ be a $\sigma_{\mathtt{Bet}}$-structure. Assume that $M \models_2 \mathsf{BG}(4)$. Suppose O, X, Y, Z, I is a 4-frame in M. Then there exists a bijection $\Phi : \mathbb{P} \to \mathbb{R}^4$ such that:

(a) Φ matches O, X, Y, Z, I.
(b) For all $p, q, r \in \mathbb{P}$: $(p, q, r) \in B \leftrightarrow B_{\mathbb{R}^4}(\Phi(p), \Phi(q), \Phi(r))$.

Proof. I give a brief sketch. Given a 4-frame O, X, Y, Z, I in M, we first define four lines $\ell(O, X), \ell(O, Y), \ell(O, Z)$ and $\ell(O, I)$: these are the "x-axis", "y-axis", "z-axis" and "t-axis" of the 4-frame. One can define (as Hilbert does in Hilbert (1899)) geometrical operations $+$, $\times$, and a linear order $\leq$, on each axis (relative to the two fixed parameters which determined that axis). These definitions are explained very clearly in Bennett (1995): for $+$ at p. 48 and for $\times$ at p. 62. Also see Goldblatt (1987): 23–27. The definition of $\leq$ is given in Tarski (1959), in the proof of Theorem 1. Then, using the betweenness axioms, one shows that, on each axis, $\ell(O, X), \ell(O, Y), \ell(O, Z)$ and $\ell(O, I)$, these definitions specify an ordered field. For details (ignoring the order aspect), see Bennett (1995): 48–72, especially Theorem 1, p. 72. What is more, the Continuity Axiom guarantees that this ordered field is a *complete ordered field*. Up to isomorphism, there is exactly one complete ordered field, and this is also rigid. Consequently, there is a (unique) isomorphism $\varphi_{O,X} : \ell(O, X) \to \mathbb{R}$ (and similarly on each axis):

Figure 5: Isomorphism from $\ell(O, X)$ to $\mathbb{R}$



Given any point p, one then constructs four "ordinates" p_X, p_Y, p_Z, p_I on the four axes $\ell(O, X), \ell(O, Y), \ell(O, Z), \ell(O, I)$, by certain parallel lines to these axes. Then one defines the coordinate system Φ as follows. Given any point $p \in \mathbb{P}$, define:

$$\Phi(p) := \begin{pmatrix} \varphi_{O,X}(p_X) \\ \varphi_{O,Y}(p_Y) \\ \varphi_{O,Z}(p_Z) \\ \varphi_{O,I}(p_I) \end{pmatrix} \tag{59}$$

[45] See also Borsuk & Szmielew (1960), Szczerba & Tarski (1965), Szczerba & Tarski (1979)

It is clear that Φ matches O, X, Y, Z, I. Finally, one shows that Φ is a bijection and that it satisfies the required isomorphism condition. Namely, for $p, q, r \in \mathbb{P}$: $B(p, q, r)$ iff $B_{\mathbb{R}^4}(\Phi(p), \Phi(q), \Phi(r))$. □

Theorem 63 (Representation Theorem for $\mathsf{EG}(3)$)**.** Let $M = (\mathbb{P}, B, \equiv)$ be a $\sigma_{\mathtt{Bet},\equiv}$-structure. Assume that $M \models_2 \mathsf{EG}(3)$. Suppose O, X, Y, Z is a Euclidean 3-frame in M. Then there exists a bijection $\Phi : \mathbb{P} \to \mathbb{R}^3$ such that:

(a) Φ matches O, X, Y, Z.
(b) For all $p, q, r \in \mathbb{P}$: $(p, q, r) \in B \leftrightarrow B_{\mathbb{R}^3}(\Phi(p), \Phi(q), \Phi(r))$.
(c) For all $p, q, r, s \in \mathbb{P}$: $pq \equiv rs \leftrightarrow \Phi(p)\Phi(q) \equiv_{\mathbb{R}^3} \Phi(r)\Phi(s)$.

Roughly, this corresponds to Theorem 1 of Tarski (1959) and a sketch of the proof is given there. The differences are that Tarski considers the two-dimensional first-order theory, whose axioms are what we've called $\mathsf{EG}_0(2)$, with the first-order continuity axiom scheme. The Representation Theorem in Tarski (1959) asserts that, given a model $M \models \mathsf{EG}_0(2)$ and a Euclidean frame, there is a real-closed field F such that the conditions (a), (b), (c) hold, with $\mathbb{R}$ replaced by that field, and "3" replaced by "2". When we strengthen to the second-order Continuity axiom, it follows that this field is in fact $\mathbb{R}$.

Appendix C Automorphisms and Coordinate Systems

Theorem 64. The automorphism (symmetry) groups of the structures defined in Definition 1 and Definition 4 are characterized as follows. Let $h : \mathbb{R}^n \to \mathbb{R}^n$.

Aut Group	Condition
$h \in \text{Aut}(\mathbb{BG}^n)$	$(\exists A \in GL(n))\,(\exists \mathbf{d} \in \mathbb{R}^n)\,(\forall \mathbf{x} \in \mathbb{R}^n)\,[h(\mathbf{x}) = A\mathbf{x} + \mathbf{d}]$
$h \in \text{Aut}(\mathbb{EG}^n)$	$(\exists R \in O(n))\,(\exists \lambda \in \mathbb{R} - \{0\})(\exists \mathbf{d} \in \mathbb{R}^n)\,(\forall \mathbf{x} \in \mathbb{R}^n)\,[h(\mathbf{x}) = \lambda R\mathbf{x} + \mathbf{d}]$
$h \in \text{Aut}(\mathbb{EG}^n_{\text{metric}})$	$(\exists R \in O(n))\,(\exists \mathbf{d} \in \mathbb{R}^n)\,(\forall \mathbf{x} \in \mathbb{R}^n)\,[h(\mathbf{x}) = R\mathbf{x} + \mathbf{d}]$

Proof. I give a brief summary. For the first, the proof relies on the requirement that straight lines get mapped to straight lines, and parallel lines get mapped to parallel lines. The outcome is that any such mapping h must be an affine transformation generated by a $GL(n)$ matrix A and a translation $\mathbf{d}$. So, the automorphism group is what is usually called $\text{Aff}(n)$, the *affine group* in n dimensions. For the third, the symmetry group is the isometry group of the metric space $\mathbb{EG}^n_{\text{metric}}$—thus, what's usually called the *Euclidean group* $E(n)$: rotations, inversions, reflections and translations (reflections and inversions are $O(n)$ matrices with determinant -1). For the second, which is less familiar, the symmetries include rotations, inversions, reflections and translations again, but also include *scalings* too:

$$\mathbf{x} \mapsto \lambda \mathbf{x} \tag{60}$$

The latter are sometimes called *similitudes* or *dilations* (the non-zero factor λ represents this scaling). Although the metric distance between two points is not invariant, nonetheless *metric equalities* are invariant. Imagine a rubber sheet, pinned at some central point, say, O, and imagine "stretching" it uniformly and radially from O, by some factor. The *distance* between two points on the sheet is not invariant under the stretching: $\Delta(\mathbf{x}, \mathbf{y}) \mapsto |\lambda|\Delta(\mathbf{x}, \mathbf{y})$; but equality between distances of points (i.e., congruence) is invariant.

□

Lemma 65 (Coordinate Transformations)**.** Given two coordinate systems $\Phi, \Psi : \mathbb{P} \to \mathbb{R}^4$, on a full model $M = (\mathbb{P}, B)$ of $\mathsf{BG}(4)$, they are related as follows: there is a $GL(4)$ matrix A and a translation $\mathbf{d} \in \mathbb{R}^4$ such that, for any point $p \in \mathbb{P}$, we have:

$$\Psi(p) = A\Phi(p) + \mathbf{d} \tag{61}$$

This follows from two facts. First, if $\Phi, \Psi : M \to (\mathbb{R}^4, B_{\mathbb{R}^4})$ are isomorphisms, then $\Psi \circ \Phi^{-1} \in \text{Aut}((\mathbb{R}^4, B_{\mathbb{R}^4}))$. Second, we have $\text{Aut}((\mathbb{R}^4, B_{\mathbb{R}^4})) = \text{Aff}(4)$. (This is the result given in Theorem 64 for the automorphisms of the standard coordinate structure $(\mathbb{R}^4, B_{\mathbb{R}^4})$ for $\mathsf{BG}(4)$.)

Lemma 66. Given a 4-frame O, X, Y, Z, I and two coordinate systems, Φ, Ψ on a model M of $\mathsf{BG}(4)$, both of which match the frame O, X, Y, Z, I, we have:

$$\Psi = \Phi \tag{62}$$

The proof applies the coordinate transformation equation (61) to the five points, O, X, Y, Z, I, which gives five specific instances. The first of these implies that $\mathbf{d} = \mathbf{0}$. The remaining four imply that the $GL(4)$ matrix A is the identity matrix. Similar reasoning applies in any dimension, and also to the Euclidean case.

Appendix D Reals and Vectors

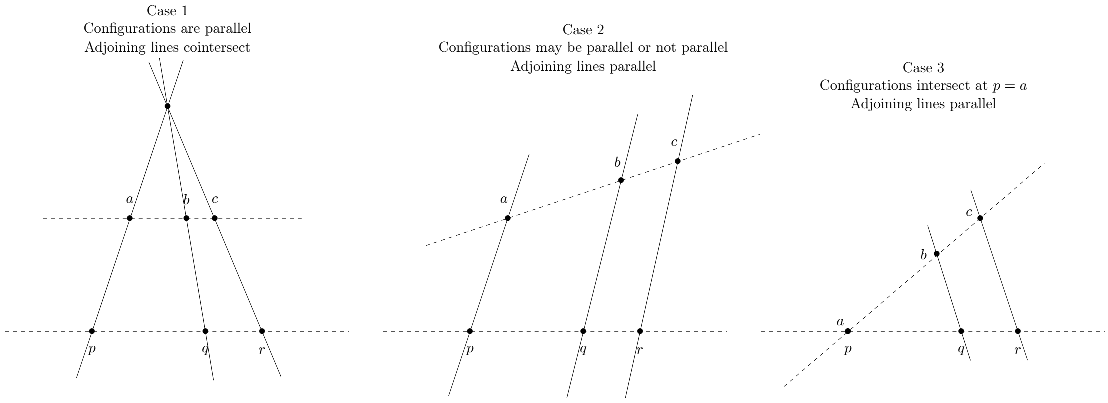
Given a model $(\mathbb{P}, B) \models_2 \mathsf{BG}(4)$, we know, by Theorem 62, that it is isomorphic to the standard coordinate structure $(\mathbb{R}^4, B_{\mathbb{R}^4})$.

Using abstraction (or, equivalently, a quotient construction), we can extend $(\mathbb{P}, B)$ with a new sort (or "universe", or carrier set) $\Re$ (of ratios), and operations $0, 1, +, \times, \leq$, to a two-sorted structure $(\mathbb{P}, \Re; B; 0, 1, +, \times, \leq)$ where the reduct $(\Re; 0, 1, +, \times, \leq)$ is isomorphic to $\mathbb{R}$ (as an ordered field).[46] Call a triple p, q, r of points a *configuration* just if $p \neq q$ and p, q, r are collinear. This abstraction proceeds by the equivalence relation, on configurations (p, q, r), of *proportionateness*. In geometrical terms, there are three basic cases of proportionateness:[47]

[46] I included this Appendix, in part, because I had difficulty locating the material elsewhere. One important textbook Bennett (1995), where the definitions of addition $+$ and multiplication $\times$ on a line $\ell(p, q)$, and the proof that these induce a division ring (given Desargues' Theorem) or a field (given Pappus's Theorem), are explained very clearly, is out of print. Also because I need, in the main part of the article, to refer to a couple of the summary theorems at the end of this Appendix.

[47] Burgess & Rosen (1997): 110 list two basic cases, our Case 1 and Case 3. In a sense, Case 3 is a limiting case of Case 2, by "sliding" the configuration abc, parallel to the three parallel lines, until a now coincides with p

Figure 6: Proportionate Configurations



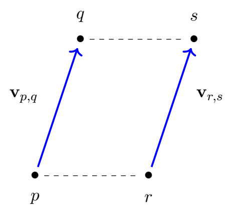
A real, or ratio, is then an equivalence class $[(p, q, r)]$ with respect to proportionateness, and $\Re$ is the set of these equivalence classes. One may define a zero 0 as $[(p, q, p)]$ and a unit 1 as $[(p, q, q)]$. One defines field operations $+$, $\times$, $\leq$ in terms of the corresponding operations on a fixed line (see Bennett (1995)). One readily checks that the result is that $\Re$, with these operations, is a complete ordered field (and can then be identified with $\mathbb{R}$). Although we described this model-theoretically, this construction can be "internalized" within $\mathsf{BG}(4)$ by adding suitable abstraction axioms (a "definition by abstraction") for a new sort, with variables ξ_i, and a 3-place function symbol $\xi(p, q, r)$, and then explicitly defining 0, 1, $+$, $\times$ and $\leq$ on these new objects, and then proving that the resulting abstracta, i.e., the $\xi(p, q, r)$ for any configuration p, q, r, satisfy the second-order axioms for a complete ordered field.[48]

We may further extend, with a new universe $\mathbb{V}$ (of displacements, or vectors), and operations $\mathbf{0}, +, \boldsymbol{\cdot}$, to a three-sorted structure $(\mathbb{P}, \Re, \mathbb{V}; B, 0, 1, +, \times, \leq; \mathbf{0}, +, \boldsymbol{\cdot})$, where the reduct $(\mathbb{V}, \Re; 0, 1, +, \times; \mathbf{0}, +, \boldsymbol{\cdot})$ is isomorphic to $\mathbb{R}^4$ (as a vector space).[49] This abstraction proceeds by the equivalence relation, on ordered pairs (p, q), of *equipollence*: (p, q) is equipollent to (r, s) just if p, q, s, r is a parallelogram:

[48]The details are given in Burgess (1984). What we've called "configurations", Burgess calls "suitable configurations". For the simple case of "extension by abstraction", with a formula $\varphi(x, y)$ which can be shown to be equivalence relation in the basic theory T, an extension of T by abstraction is obtained by abstraction axioms (i): $\xi(x) = \xi(y)$ iff $\varphi(x, y)$; and, (ii): $\forall \xi \, \exists x \, (\xi = \xi(x))$, where ξ is a new variable sort, and $\xi(x)$ is a function symbol (which Burgess writes as "$[x]$"). See Burgess (1984): 381. Burgess shows (Theorem 1.3) that this (indeed any) "extension by abstraction" is a *conservative extension* of the original theory T, and may be *interpreted* into the original theory. For the geometrical case, the abstraction axioms are (i): $\xi(p, q, r) = \xi(p', q', r')$ iff the configurations p, q, r, and p', q', r' are proportionate; and, (ii): $\forall \xi \, \exists p, q, r \, (p \neq q \wedge \mathsf{co}_1(p, q, r) \wedge \xi = \xi(p, q, r))$. See Burgess (1984), 387, axioms (1) and (2).

[49]The two pluses (+) here have been overloaded: the first is the *field* addition, and the second is the *vector* addition.

Figure 7: Equipollence



A displacement, or vector, is then an equivalence class $[(p, q)]$ with respect to equipollence, and $\mathbb{V}$ is the set of these equivalence classes. An equivalence class $[(p, q)]$ is written $\mathbf{v}_{p,q}$. One may define the zero vector $\mathbf{0}$ as $\mathbf{v}_{p,p}$. One defines vector addition $+$ so that $\mathbf{v}_{p,q} + \mathbf{v}_{q,r} = \mathbf{v}_{p,r}$ holds (usually called *Chasles's Relation*). One may define the scalar multiplication $\bullet$ so that, when $p \neq q$, $\alpha \bullet \mathbf{v}_{p,q} = \mathbf{v}_{p,r}$ just if $\alpha = [(p, q, r)]$; and, otherwise, $\alpha \bullet \mathbf{0} = \mathbf{0}$. One checks that the vector space axioms are true, and that $\mathbb{V}$ is 4-dimensional.

Finally, by an explicit definition of an action $+ : \mathbb{P} \times \mathbb{V} \to \mathbb{P}$, we can further extend to $(\mathbb{P}, \Re, \mathbb{V}; B; 0, 1, +, \times, \leq; \mathbf{0}, +, \bullet; +)$ such that $(\mathbb{P}, \mathbb{V}, +)$ is isomorphic to the affine space $\mathbb{A}^4$.[50] The definition of the action $(p, \mathbf{v}) \mapsto p + \mathbf{v}$ is: $q = p + \mathbf{v}$ iff $\mathbf{v} = \mathbf{v}_{p,q}$. One may then show that $+$ is free and transitive action of $\mathbb{V}$ on $\mathbb{P}$. The affine space obtained in this way (basically, from the *vector space* $\mathbb{R}^4$, by "forgetting the origin") is called $\mathbb{A}^4$.

The discussion and constructions above may be summarized in the following three theorems (I follow the usual practice of conflating the name of a structure with the name of its carrier set):

Theorem 67. $\Re$ is isomorphic to the complete ordered field $\mathbb{R}$.

Theorem 68. $\mathbb{V}$ is isomorphic to the vector space $\mathbb{R}^4$.

Theorem 69. $(\mathbb{P}, \mathbb{V}, +)$ is isomorphic to the affine space $\mathbb{A}^4$.

References

Apostol, T., 1967: *Calculus Volume 1: One Variable Calculus, With an Introduction to Linear Algebra*. Wiley & Sons Ltd.

Arnold, V., 1989: *Mathematical Methods of Classical Mechanics*. New York: Springer. Second edition. This is the English translation by K. Vogtmann and A. Weinstein of the second Russian edition. (First Russian edition 1974.).

Ax, J., 1978: "The Elementary Foundations of Spacetime". *Foundations of Physics* 8: 507—546.

[50]The three pluses $(+)$ here are overloaded: the first is the field addition; the second is the vector addition; the third is the action, $(p, \mathbf{v}) \mapsto p + \mathbf{v}$.

Bennett, M., 1995: *Affine and Projective Geometry.* New York: Wiley.

Boolos, G., 1987: "A Curious Inference". *Journal of Philosophical Logic* 16: 1–12.

Borsuk, K. & Szmielew, W., 1960: *Foundations of Geometry: Euclidean, Bolyai-Lobachevskian Geometry and Projective Geometry.* Amsterdam: North-Holland Publishing Company. Revised English translation by Erwin Marquit. Reprinted by Dover Books, 2018.

Burgess, J., 1984: "Synthetic Mechanics". *Journal of Philosophical Logic* 13: 379–95.

Burgess, J. & Rosen, G., 1997: *A Subject with No Object.* Oxford: Clarendon Press.

Cantor, G., 1897: "Beiträge zur Begründung der transfiniten Mengenlehre". *Mathematische Annalen* 49: 207–246.

Carnap, R., 1954: *Einführung in die symbolische Logik.* Julius Springer.

Carnap, R., 1958: *Introduction to Symbolic Logic and its Applications.* Dover. English translation of Carnap (1954).

Cocco, L. & Babic, J., 2021: "A System of Axioms for Minkowski Spacetime". *Journal of Philosophical Logic* 50: 149–185.

Cohen, I. B., Whitman, A., & Budenz, J., 1999: *Isaac Newton: The Principia, Mathematical Principles of Natural Philosophy (preceded by "A Guide to Newton's Principia").* University of California Press. This is the second major translation of the third edition of Newton's *Principia* (Newton (1687)), following Andrew Motte's translation of 1729.

Coxeter, H., 1969: *Introduction of Geometry.* New York: Wiley. Second edition.

Dedekind, R., 1872: *Stetigkeit und irrationale Zahlen.* Braunschweig: Friedrich Vieweg. Translation *Continuity and irrational numbers*, by W.W. Beman in Dedekind 1901, *Essays on the Theory of Numbers*, Open Court, Chicago. Paperback edition. New York: Dover, 1963.

Dedekind, R., 1888: *Was sind und was sollen Die Zahlen?* Brunswick: Vieweg. Translation *The Nature and Meaning of Numbers*, by W.W. Beman in Dedekind 1901, *Essays on the Theory of Numbers*, Open Court, Chicago. Paperback edition. New York: Dover, 1963.

DiSalle, R., 2020: "Space and Time: Inertial Frames". *Stanford Encyclopedia of Philosophy* (online) First published Sat Mar 30, 2002; substantive revision Wed Apr 15, 2020. `https://plato.stanford.edu/entries/spacetime-iframes/`.

Dummit, D. & Foote, R., 2004: *Abstract Algebra.* John Wiley & Sons. Third edition.

Earman, J., 1970: “Who’s Afraid of Absolute Space?” *Australasian Journal of Philosophy* 48: 287–319.

Earman, J., 1989: *World Enough and Spacetime: Absolute and Relational Theories of Motion*. Cambridge, MA.: MIT Press.

Ehlers, J., 1973: “The Nature and Structure of Spacetime”. In J Mehra, ed., *The Physicist’s Conception of Nature in the Twentieth Century*. Dordrecht: Reidel.

Einstein, A., 1905: “Zur Elektrodynamik bewegter Körper”. *Annalen Der Physik* 17(10): 891–921. Translation “On the Electrodynamics of Moving Bodies” appeared in Sommerfeld (1923).

Euclid, 1956: *The Thirteen Books of Euclid’s Elements*. New York: Dover. Originally written around 300 BCE. Translated by T.L Heath.

Feynman, R., 1970: *Feynman Lectures on Physics: Vols I–III*. Reading, Ma.: Addison-Wesley. Edited by R.B. Leighton & M. Sands. Page references to 2006 edition.

Field, H., 1980: *Science Without Numbers*. Princeton: Princeton University Press. Second edition, Oxford University Press (2016).

Friedman, M., 1983: *Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science*. Princeton:: Princeton University Press.

Gallier, J., 2011: *Geometric Methods and Applications: For Computer Science and Engineering*. New York: Springer. Second edition.

Gödel, K., 1931: “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I”. *Monatshefte für Mathematik und Physik* 38: 173–198. English translation “On Formally Undecidable Propositions in Principia Mathematica and Other Systems I” appeared as a Dover Book, 1951. This translation “On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems” also appears in van Heijenoort (ed.) 1967.

Gödel, K., 1936: “Über die Länge von Beweisen”. *Ergebinisse Eines Mathematischen Kolloquiums* 7: 23–24. Reprinted with English translation, Vol 1 of Gödel’s collected works.

Goldblatt, R., 1987: *Orthogonality and Spacetime Geometry*. Springer.

Hart, W., ed., 1996: *The Philosophy of Mathematics*. Oxford: Oxford University Press.

Hartshorne, R., 2000: *Geometry: Euclid and Beyond*. New York: Springer.

Hilbert, D., 1899: *Grundlagen der Geometrie*. Leibzig: Verlag Von B.G. Teubner. Translated (by E.J.Townsend) as *The Foundations of Geometry*. Chicago: Open Court (1950).

Hintikka, J., ed., 1968: *Philosophy of Mathematics*. Oxford: Oxford University Press.

Hölder, O., 1901: "Die Axiome der Quantität und die Lehre vom Mass"". *Berichten der mathematisch-physischen Classe der Königlisch Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikaliche Classe* 53: 1–63. Translations of this work by Joel Michell and Catherine Ernst appear in Hölder (1996), Hölder (1997).

Hölder, O., 1996: "The Axioms of Quantity and the Theory of Measurement". *Journal of Mathematical Psychology* 40(3): 235–252. Part I of Hölder (1901), translated by Joel Michell and Catherine Ernst.

Hölder, O., 1997: "The Axioms of Quantity and the Theory of Measurement". *Journal of Mathematical Psychology* 41(4): 345–356. Part II of Hölder (1901), translated by Joel Michell and Catherine Ernst.

Huggett, N. & Hoefer, C., 2015: "Absolute and Relational Theories of Space and Motion". *Stanford Encyclopedia of Philosophy* (online) First published Fri Aug 11, 2006; substantive revision Thu Jan 22, 2015. https://plato.stanford.edu/entries/spacetime-theories/.

Huntington, E. V., 1903: "Complete Sets of Postulates for the Theory of Real Quantities". *Transactions of the American Mathematical Society* 4 (3): 358–370.

Ketland, J., 2021: "Foundations of Applied Mathematics I". *Synthese* 199,: 4151—4193.

Ketland, J., 2022: "A Formalization of Boolos's "Curious Inference" in Isabelle/HOL". *Archive for Formal Proofs* .

Kopczyński, W. & Trautman, A., 1992: *Spacetime and Gravitation*. Wiley.

Kordos, M., 1969: "On the Syntactic Form of Dimension Axiom for Affine Geometry". *Bull Acad Pol Sci* 17: 833–837.

Longair, M., 1984: *Theoretical Concepts in Physics*. Cambridge University Press.

Malament, D., 2009: "Notes on Geometry and Spacetime". Unpublished lecture notes.

Malament, D., 2012: *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago: University of Chicago Press.

Minkowski, H., 1908: "Raum und Zeit". *Physikalische Zeitschrift* 10: 75—88. Repriinted in translation, as "Space and Time", in Sommerfeld (1923).

Mundy, B., 1986: "Optical Axiomatization of Minkowski Space-Time Geometry". *Philosophy of Science* 53(1): 1–30.

Newton, I., 1687: *Principia Mathematica Philosophia Naturalis*. London: Benjamin Motte. The first edition was published in Latin in 1687. The second and third editions were 1713 and 1726. An English translation of the third edition by Andrew Motte appeared in 1729. A new translation (Cohen et al. (1999)) by I.B. Cohen et al appeared in 1999.

Pambuccian, V., 2011: "The Axiomatics of Ordered Geometry I. Ordered Incidence Spaces". *Expositiones Mathematicae* 29: 24–66.

Pasch, M., 1882: *Vorlesungen über neuere Geometrie*. Leipzig: Teubner.

Penrose, R., 1968: "Structure of Spacetime". In CM DeWitt & JA Wheeler, eds., *Battelle Rencontres*. New York: Benjamin.

Presburger, M., 1929: "Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt". *Comptes Rendus du I congrès de Mathématiciens des Pays Slaves, Warszawa* pages 92–101. English translation in Stansifer (1984).

Raatikainen, P., 2020: "Gödel's Incompleteness Theorems". *Stanford Encyclopedia of Philosophy* (online) .

Rindler, W., 1969: *Essential Relativity*. New York: Springer-Verlag. Revised second edition, 1979.

Robb, A., 1911: *Optical Geometry of Motion, a New View of the Theory of Relativity*. Cambridge: Heffer & Sons.

Robb, A., 1936: *The Geometry of Time and Space*. Cambridge University Press.

Saunders, S., 2013: "Rethinking Newton's Principia". *Philosophy of Science* 80(1): 22–48.

Schutz, B., 1980: *Geometrical Methods of Mathematical Physics*. Cambridge: Cambridge University Press.

Schutz, J., 1997: *Independent Axioms for Minkowski Space-Time*. Routledge.

Schwabhäuser, W., Szmielew, W., & Tarski, A., 1983: *Metamathematische Methoden in der Geometrie*. Berlin: Springer-Verlag (Hochschultext).

Sears, F., Zemasky, M., & Young, H., 1979: *University Physics*. Addison-Wesley. Fifth edition.

Shapiro, S., 1983: "Conservativeness and Incompleteness". *Journal of Philosophy* 80: 521–31. Reprinted in Hart (1996).

Simpson, S. G. & Yokoyama, K., 2013: "Reverse Mathematics and Peano categoricity". *Annals of Pure and Applied Logic* 164(3): 284–293.

Sommerfeld, A., ed., 1923: *The Principle of Relativity. A Collection of Original Papers on the Special and General Theory of Relativity By H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl with Notes By A. Sommerfeld*. London: Methuen & Co. Reprinted as *The Principle of Relativity* (Dover Books on Physics), 1952.

Stansifer, R., 1984: "Presburger's Article on Integer Arithmetic: Remarks and Translation". Technical Report TR84-639, Cornell University, Computer Science Department. http://techreports.library.cornell.edu:8081/Dienst/UI/1.0/Display/cul.cs/TR84-639.

Stein, H., 1967: "Newtonian Space-Time". *Texas Quarterly* 10: 174–200.

Szczerba, L. & Tarski, A., 1965: "Metamathematical Properties of Some Affine Geometries". In Y Bar-Hillel, ed., *Logic, Methodology, and Philosophy of Science*, page 166. Amsterdam: North Holland.

Szczerba, L. & Tarski, A., 1979: "Metamathematical Discussion of Some Affine Geometries". *Fundamenta Mathematicae* 104: 155–192.

Tao, T., 2008: "What is a Gauge?" Online. https://terrytao.wordpress.com/2008/09/27/what-is-a-gauge/.

Tarski, A., 1936: "Der Wahrheitsbegriff in den formalisierten Sprachen". *Philosophia Studia* 1: 261–400. English translation by J.H. Woodger "The Concept of Truth in Formalized Languages", in A. Tarski 1956: *Logic, Semantics and Metamathematics*. Oxford: Clarendon Press. The 2nd edition, ed. by J Corcoran, appeared in 1983 (Indianapolis).

Tarski, A., 1948: "A Decision Method for Elementary Algebra and Geometry". Technical Report R-109, RAND Corporation.

Tarski, A., 1959: "What is Elementary Geometry?" In L Henkin, P Suppes, & A Tarski, eds., *The Axiomatic Method*. Amsterdam: North Holland. Reprinted in Hintikka (1968).

Tarski, A., 1995: *Introduction to Logic and the Methodology of the Deductive Science*. New York: Dover Publications Inc. Unabridged republication of the 1946 second, revised edition of the work originally published by Oxford University Press, New York, in 1941.

Tarski, A. & Givant, S., 1999: "Tarski's System of Geometry". *Bull Symbolic Logic* 5: 175–214.

Trautman, A., 1966: "Comparison of Newtonian and Relativistic Theories of Space-Time". In B Hoffmann, ed., *Perspectives in Geometry and Relativity*. Bloomington. Essays in Honor of V. Hlavaty.

Veblen, O., 1904: "A System of Axioms for Geometry". *Transactions of the American Mathematical Society* 5 (3): 343–384.

Wald, R., 1984: *General Relativity*. Chicago: University of Chicago Press.

Weinberg, S., 1972: *Gravitation and Cosmology*. New York: Wiley.