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Authorised translation

"ARS INVENIENDI" TODAY

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> "I don't pay much attention to specific discoveries. What I most desire is to perfect the Art of Invention, and to provide methods rather than solutions to problems, for one single method comprises an infinity of solutions".
> (Leibniz, letter to Duke Ernst-August, *GPS*, VII, 25.)

In his obituary of Abraham Robinson[1], Simon Kochen gave a portrait of Kurt Gödel (1906-1978) that will be of interest to Leibniz scholars. Gödel, who, it should be borne in mind, was particularly interested in Leibniz's philosophy, regarded Abraham Robinson's work as the best realisation of the Leibnizian ideal of logic serving as an *ars inveniendi* for mathematics. In this paper, I would like to examine which aspect of *ars inveniendi* is done justice by this claim, and which characteristic of contemporary logic it highlights. Given that we are dealing with Abraham Robinson's work, I will only discuss model theory, although there are other domains where logical results may facilitate mathematical discoveries.

Leibniz's project of a "universal calculus"[2] has been widely discussed in the light of the findings of modern logic. By contrast, the idea, which is essential to Leibniz's mathematics and philosophy, that the establishment of calculus presupposes a "concept analysis" has been hardly considered from the point of view of modern logic. Yet this was what Gödel, for whom conceptual analysis is a precious component of scientific research and logic a useful discipline for mathematicians[3], had in mind.

For Leibniz, analysis is essential. First of all, it provides the elements of calculus, represented by primitive symbols and corresponding to basic notions – or at least the notions that are basic for us. Secondly, it is inseparable from the generalisation of the very idea of calculus, which refers to an *operatio per Characteres*[4], of which numerical or algebraic calculus is but a specific example. Thirdly,

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[1] On Abraham Robinson's work in mathematical logic, *The bulletin of the London mathematical society*, vol. 8, part 3, November 1976, 312-315

[2] Cf. in particular, Hans Hermes, Ideen von Leibniz zur Grundlagenforschung: Die ars inveniendi und die ars judicandi, *Studia Leibnitiana, Supplementa,* III (1969), 92-102. And more recently M. Sanchez-Mazas, La caractéristique numérique de Leibniz comme méthode de décision, *Studia Leibnitiana, Supplementa*, XXI (1980), 168-182. I have also previously discussed the coincidence of the arithmetisation of logic at the start of this century with Leibniz's "universal calculus" project (La logique comme *ars inveniendi,* in A. Robinet (ed.), *Doctrines et concepts. Cinquante ans de philosophie de langue française.* Paris, Vrin, 1988, 321).

[3] Cf. Hao Wang's testimony in *Reflections on Kurt Gödel*, MIT Press, Cambridge (Mass.), London, 1987, 167.

[4] Letter to Tschirnhaus, May 1678, *GMS,* IV, 462 (for bibliographic abbreviations, see the end of the paper, p.11)

it is often coupled with a purely "characteristical" analysis whose object is the calculus itself, or rather the *form* of the calculus.

As regards the first aspect – but all the more so the second and third ones –, conceptual analysis is a key element of *ars inveniendi.* This is what I would like to show in this paper. I will then explain how *model theory* has nowadays placed logical analysis at the service of mathematical discovery, thus serving as a *method* rather than a *foundation.*

I. "ARS INVENIENDI", "ARS COMBINATORIA", "CHARACTERISTICA LOGICA"

1. It is generally acknowledged that the main key to understanding *ars inveniendi* is found in *De Arte Combinatoria,* whose subtitle, "Logica inventionis semina," shows the close proximity between the notions of combinatorics and invention. Leibniz moreover specifies that invention is the main application of combinatorics[5], also known as [combinatorial] "synthesis" and sometimes as *ars formularia*[6].

Combinatorics is certainly anchored in arithmetic (*sedes doctrinae istius Arithmetica*), as it involves finding the *number* of all possible combinations of elements under certain conditions. But its applicability is not restricted to this field, as the elements in question are not numbers but *characters* whose interpretation ranges from algebra to geometry, through logic, music, and cryptography. Hence the basic role of characteristics in combinatorics and the fact that many of Leibniz's texts use these terms as mutually interchangeable. For example, in his famous letter to Tschirnhaus of May 1678, Leibniz describes combinatorics as "a science of forms or of the similar and dissimilar" (*scientia de formis seu de simili et dissimili*), which is not too different from "a general science of characteristics" whose characters can "signify" or "represent"[7] algebraic signs, musical notes, and logical concepts[8]. Characteristics, also known as "symbolics"[9] or, in the French phrase used by Leibniz, *spécieuse universelle*[10], is absolutely essential to the progress of *ars inveniendi*[11]. It is even consubstantial to it[12]. Characters "beckon the mind, spur it on" and drive it to "conceive of universal notions"[13]. They are our "instruments" for inventing.

2. However, *ars inveniendi* is presented (following the famous debates on the two parts of *dialectica,* invention and method) as one part, or the essential part[14], of the "Art of thinking" or logic[15]. Leibniz wrote to Princess Sophie that *ars inveniendi* is the "true logic"[16]. It also appears in its broader

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5 *GMS,* V, 39.

6 *C,* 37.

7 In the rest of the paper, the words in quotation marks are generally Leibniz's original terms.

8 *GMS*,IV, 459-460; V, 241. On characteristics cf. *C*,98-99, 326-327. On combinatorics, cf. *GPS*, V, 7 ; VII, 10, 297-298; GMS, VII, 159 ; C, 531-533.

9 *GMS*, IV, 465 and *C*, 511.

10 *GMS*, VII, 159; *GPS*, VII, 297-298; *C*, 336. In most cases, *spécieuse*, characteristics, and combinatorics tend to blend into each other.

11 "Progressus Artis inventoriae rationalis pro magna parte pendet a perfectione artis characteristicae" (*GPS*, VII, 98). Likewise *GMS*, V, 307 and *LH,* XXXV 1, 27, Bl. 3-10, in the *Vorausedition*, zur Reihe VI, vol. 6 (1987), 1369.

12 Cf. Historia et commendatio linguae charactericae universalis quae simul sit ars inveniendi et judicandi, *Œuvres philosophiques*, ed. R. Eric Raspe, Amsterdam and Leipzig, 1765, 533-540.

13 De linea ex lineis..., *GMS*, V, 269.

14 "…illa logicae pars praestantissima…" (*GMS,* VI, 206).

15 "Unter der Logik oder Denkkunst verstehe ich die Kunst den Verstand zu gebrauchen, also nicht allein was fürgestellt zu beurtheilen, sondem auch was verborgen zu erfinden". (*GPS*,VII, 517).

16 *GPS,* IV, 292. Also *GPS,* VII, 172.

meaning as the "general science" which, together with memory and mastery of the passions[17], etc., encompasses *ars judicandi* (extensively developed by Aristotle's work on syllogism) and *ars inveniendi* (which was still in its early days). No doubt because it was still so far from being perfected, *ars inveniendi* was of great importance for Leibniz. The fact that it is not so completely different from *ars demonstrandi*[18] is no matter – it sits nonetheless at the top of the scale of the values of knowledge. Whereas truths "that are confusedly and imperfectly known" can be handled with the "method of certainty" [*ars demonstrandi*], those that are not known at all require the art of invention. And "it is naturally far easier to demonstrate inventions than to disclose their origin, thus making the art of invention itself progress"[19]. Nor does Leibniz hesitate to identify the art of invention with general science[20], or at least include it among its *initia*[21]. Being thus different from proof through calculus, the art of invention is a "palpable thread that guides research" through combinatorics and analysis and makes it possible to establish, on an exact or temporary basis, sciences or parts of sciences, such that anyone can find *ex datis* and *non casu sed ratione.* The art of invention does not merely entail finding the outcome sought, but foreseeing it[22]. Or, as Leibniz says elsewhere[23], "the invention of demonstrations [relies] on a *certain Method*". The palpable thread is a "rule to move from one thought to another"[24].

3. But we already have three terms – characteristics, combinatorics, general science – to explain a single one: *ars inveniendi.* Despite being tempted to do so, we cannot strictly identify the latter with any one of those three terms. And refusing to so identify it entails, in particular, not reducing any relation between two or several terms to a relation of inclusion. For example, *ars inveniendi* is an application of *ars combinatoria:* in turn, *ars combinatoria* is an aspect, a *species*[25] or a "method"[26], of the former. These two definitions are compatible without necessarily entailing the identity of both arts. Looking closely, they are two ways of expressing the same thing – namely, that combinatorics is a general method of discovery, but not the only one.

There remains the problem of the relation of combinatorics and characteristics to logic. If the logic is the art of thinking universally, it comprises all that is thinkable, and, in particular, combinatorics[27], potentially through the art of invention[28]. But if the *art* of thinking is to be a general

[17] "(Scientia generalis) tractate ergo debet turn de modo bene cogitandi, hoc est inveniendi, judicandi, affectus regendi, retinendi ac reminiscendi, turn vero de totius Encyclopaediae Elementis, et summi Boni investigatione, cujus causa omnis meditatio suscipitur; est enim nihil aliud sapientia quam scientia felicitatis » (*GPS*,VII,3).Cf. also *C*, 228-229, 511.

[18] *GPS*,VII, 183. But Leibniz also writes that there is "a significant difference" between demonstrating and inventing (*GMS*, II, 223).

[19] "Sane facilius multo est inventionum dare demonstrationem, quam originem, quae auget ipsam inveniendi artem"; (letter to Ch. Wolff, *GMS*,V, 384). Many other texts show that the art of invention is not reduced to the art of demonstrating, e.g., *GMS,* II, 276: "I am not upset that M. de la Hire is willing to take the trouble, which I would not wish to take in any way, to reduce through demonstrations, as the ancients did, what we can easily discover through our Methods. But it would be even better if he made use of new means capable of advancing the art of invention…"

[20] *GPS*, VII, 168-169, 173, 180, 183; *C*, 219, 228-229.

[21] Initia et specimina Scientia Generalis, *GPS*, VII, 57.

[22] *C*, 161, or *LH*, IV 7 A Bl. 4, Vorausedition, VI, vol. 4 (1985), 706: "Est autem inventio vel conjecturalis vel secum demonstrationem ferens. Et quae demonstrative est, non tamen satis perfecta est, nisi ante aggressionem solutionis demonstrative praevideri possit, hac method necessario ad exitum ventum iri, alioqui inventio ex parte casui debetur."

[23] *C*, 153. Cf. also *GMS*, V, 258 : "...ac methodos potius quam specialia... aestimavi."

[24] "Methodus inveniendi consistit in quodam cogitandi filo id est regula transeundi de cogitatione in cogitanionem" (*LH*, XXXV, 1, 27, Bl. 3-10, *Vorausedition*, VI, vol. 6 (1987), 1364).

[25] *GPS*, VII, 57.

[26] *C*, 557.

[27] *C*, 511. This is one of the aspects of Leibniz's "panlogicism" highlighted by Couturat when he claims that "Leibniz's metaphysics is solely based on the principles of his logic, and follows from it in its entirety". A symmetrical

science, then, like any rigorous and fruitful science, it must be symbolic and use the synthetic method. Whatever their type, relations between combinatorics and logic go in both directions. And they are all the more complex or multiple in that each of the two disciplines appears both under the aspect of a science and under that of an art. For logic, this is very clear from my short discussion. For combinatorics, even though the aspect of an art generally prevails in Leibniz's writings, the aspect of a science is not absent[29]. In *Plus ultra sive initia et specimina Scientiae Generalis,* it seems that the *art* of combinatorics is identified with synthesis, whereas "special" combinatorics is defined as the "science of forms or of *qualities*" in general, or the science of the similar, as opposed to the "special" analysis or "science of *quantities* in general" ("special" combinatorics and "special" analysis together constitute general *mathesis*). As for the art of invention, it most often appears as a method, as in *Plus ultra,* or it comes just after the "general calculus", with which it thus cannot be identified, just before Synthesis and Analysis. But in the same way as Synthesis and Analysis give rise to sciences – mathematical combinatorics and analysis – invention also generates a "topic" that includes Algebra, but also dialectics, rhetorics, and the science of divination[30].

We can thus see two things. Firstly: the art of invention invokes analysis as often as synthesis[31], so that the subdivision of logic into *ars judicandi* and *ars inveniendi* is superimposed by a subdivision into analysis and synthesis. Thus, we must consider the specific contribution of analysis to the art of invention. Secondly: mathematics offers us the best "samples" of the art of invention, of analysis and synthesis. It is on these grounds that symbolism can be more easily established, and we are able to reason in such a profitable way on the basis of characters, "notes", rather than things[32]. Mathematics also works as a sort of laboratory for general science. This, in turn, can make mathematicians more willing to generalise their methods and systematise their results. But general science functions as an integral component, not an external aid. The best evidence of this is that the new calculus, invented through the generalisation of the ordinary rules of algebra to the infinitesimals, carries its own justification within itself. It does not seek its justification in logic any more than it owes it to metaphysics. How can we then talk of Leibniz's "logicism", if this term designates the reduction of mathematics to logic as its foundation?

II. The art of invention and analysis

1. The reason why the art of invention proceeds through analysis as well as through synthesis is clear, which I mentioned at the start: the establishment of the elements of characteristics, which is so necessary for the exercise of the art of combinatorics and the art of invention, relies on analysis[33]. Analysis is involved again, in an essential manner, in the search for *general* and *systematic* solutions, or in the examination of procedures that are *already* formal, which sets in motion further formalisation.

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tendency, which cannot be exclusive of the previous one, is to turn the Monadology into the key to the entire system. This "panlogicism" is very quickly identified with logicism strictly speaking, by which mathematical notions or proofs are ultimately reduced to logical notions or proofs.

[28] *GPS,* VII, 57.b

[29] Cf. "Plus ultra sive initia et specimina Scientiae Generalis...", *GMS*, VII, 49-50. In an unpublished fragment, Leibniz writes: "Aliud est ars Combinatoria, aliud Scientia combinationum", *LH*, XXXV, 8, 30, Bl. 79, *Vorausedition*, VI, vol. 6, 1372.

[30] *C,* 37, 219.

[31] *GPS,* VI, 292 et seq.; 477; VII, 57; *GMS,* VII, 17; *C,* 162, 165, 350-351, 558, 560, 563; *LH,* XXXV, 1, 27, Bl. 3-10, *Vorausedition,* VI, vol. 6, 1356-1371.

[32] *C,* 155, 176.

[33] *C,* 159.

As for synthesis, "part of the secret of analysis consists of characteristics, which is the art of properly employing the notes that we use".[34] And it should not be thought that analysis is less certain than demonstration[35] – it only has a broader range and "opens the way" for many results.

Analysis is long and difficult. If it is finite, it constitutes a demonstration, or "the invention of a common measure"[36]. But it can be infinite. While synthesis consists in combining data to invent more complex ones, analysis consists in decomposing the complex into the simpler, until the primitive elements – or at least those elements that are primitive for us – are reached[37]. It is a sort of "anatomy" of things[38], accounting for everything or for as much as possible[39]. It goes from the conditional to its conditions, from the effects to the causes. It has no need for foreign suppositions[40], such as the demands of geometers[41]. Analysis becomes all the more "pure" the more it focuses on the problem posed and solves it in its own terms with no external aids[42]. Thus, "analysis must be pushed to the limit", transforming suppositions into theorems, and finding the "origin", that is, the principle, of inventions[43]. In this way we will find the most general methods and the source of many other inventions. Something can be certainly proven without discovering its origin, through a synthetic method[44]. But "seeing" its origin gives its full power to the art of combinations[45].

2. Analysis is indeed rarely employed on its own; most of the time, it is mixed with synthesis[46], and serves, for example, to solve the questions that synthesis makes it possible to formulate[47]. And solving means "finding the key to something that is hidden"[48], making the implicit explicit, finding the form, the "canon", the general formula through which one will systematically treat all specific cases, and will thus be relieved of calculations[49]. Algebra, the analysis of finite quantities, is the sample *par excellence* of universal characteristics, of the *ars inveniendi* and of the "method of universality". It familiarises us with the indeterminacy of signs through the very fact that they are "character-based", and encourages us to conceive, through generalisation, of ambiguous signs that simultaneously represent various operations; or, through analogy, of signs that represent something other than numbers: points, relations, quantities, or qualities of logical propositions, etc. In this way it teaches us

[34] *GM,* V, 240.

[35] Leibniz ironizes about those who believe analysis to be defective from the point of view of certainty (*LH,* XXXV, 1, 27, Bl. 3-10, *Vorausedition,* VI, Fasc. 6, 1368).

[36] *C,* 1. The notion of a common measure refers, of course, to the theory of proportions treated by Euclid in Book V of the *Elements*.

[37] *C,* 220-221, cf. rules 2 and 5 stated in On Wisdom, *GPS,* VII, 83, or *LH,* IV, 7 C, Bl. 160-161, *Vorausedition,* VI, fasc. 1 (1982), 193: "…catalogus *notionum primitivarum, seu earum* quas nullis definitionibus clariores reddere possumus."

[38] *C,* 167.

[39] On Wisdom, 5th rule.

[40] Cf. the 3rd rule in On Wisdom, *GPS,* VII, 83; *C,* 165.

[41] *C*, 181. Leibniz remarks that the existence of suppositions or axioms is "the main reason why it has not yet been possible to turn the synthesis of Geometers into Analysis".

[42] *LH,* IV, 7 A, Bl. 4, *Vorausedition,* VI, fasc. 4 (1985), 706 or *LH,* XXXV, 1, 27, Bl. 3-10, *ibid.,* fasc. 6 (1987), 1367-1368. Leibniz calls pure analysis "anagogics" by contrast to "zetetics", or turning a problem that is already a mix of analysis and synthesis into an equation.

[43] "Origin" is opposed to history, but it also differs from demonstration (*GMS,* II, 284; V, 384). It is one thing to show truths, another one to simultaneously reveal their origin (*LH,* IV, 7, Bl. 4, *Vorausedition,* VI, fasc. 4 (1985), 707. For Leibniz, the origin of inventions should always be sought (*LH,* XXXV, 1, 27, Bl. 3-10, *ibid.,* fasc. 6, 1367).

[44] *LH,* XXXV, 1, 27, Bl. 3-10, *ibid.,* 1368.

[45] *GMS,* V, 89.

[46] *C,* 165; *GMS,* VII, 206-207.

[47] *C,* 167.

[48] *C,* 563; *LH,* XXXV, 1, 26, Bl. 3-4, *Vorausedition,* VI, fasc. 3 (1984), 624:… "et hoc propriedicitur *analysis,* est enim velut invention clavis in aliquo Cryptographemate."

[49] Letter to Placcius, 8 September 1690 (Dutens, VI, I, 49), cited, with other similar texts, in Couturat, *La logique de Leibniz,* Paris, PUF (1901), 479, n. 3. Cf. also *GMS,* VII, 189.

the art of generality and of analogy. It exercises us in the practice of this "constant formality" which we should follow in all our reasonings[50]. Even though it is neither the entirety of mathematics[51] and it is not necessarily best way to invent[52], algebra plays a remarkable inductive role. As Leibniz writes, algebra need not be introduced everywhere, but rather we should establish, by analogy, "universal formulas" to generalise and formalise reasonings, so as to "mathematically reason on matters… that are entirely removed from mathematics"[53]. And so the analysis of algebraic methods leads to the idea of a symbolic activity whose fruitfulness exceeds the calculus of finite quantities, and even the sphere of the calculable. Even though the idea of universal calculus represents "the ultimate perfection of the art of invention", it does not exhaust all possibilities[54]. We can invent through other means than calculus, through a simple "view of the mind"[55], or even through divination or through "various attempts"[56].

3. If calculus is only the goal of *ars inveniendi,* characteristics is truly its source. It is characteristics that constitutes the trait common to both analysis and synthesis, to *ars demonstrandi* and *ars inveniendi,* to logic and mathematics. Hence its fundamental role, which far exceeds that of an assistant to our thoughts. It does not only significantly support the analysis of notions and the establishment of "universal formulas". It is in itself, and independently from a possible calculus, a demonstrative or inventive approach. All truths in a formal language (a characteristics) are provable through calculus or, more generally, through "the sole manipulation of characters in a fixed form"[57]. Mathematical calculus is nothing but the application of this "characteristic" formality to specific characters, numbers or letters. Moreover, taking it as the object of analysis in itself provides often significant "openings".

If we consider the rules of algebraic calculus rather than its elements, finite quantities, then we are standing on the threshold of infinite analysis. It has been often stressed, echoing Leibniz, that the invention of infinitesimal calculus came from the idea of universal characteristics. Yet more needs to be said. The combinatory spirit, which pays attention to forms, to order, to "characteristic" similarities, gives rise to a formal analysis of formality. "Characteristic" analysis thus generates new things. It is a well-known fact, for example, that after conceiving "fictional" or symbolic numbers to represent the coefficients of equations in a system of several equations, Leibniz came to prefigure our current determinants.[58] Another well-known example, which is highly instructive for us, is that given by Couturat in Appendix III to this book *La logique de Leibniz.* This is the symbolic analogy between finding the power of a binomial and finding the differential of the product of two factors[59]. If, in the second of the following equations:

[50] Letter to Princess Elisabeth (1678), works of G.W. Leibniz edited by L. Prenant, Aubier Montagné (1972), 131.
[51] Algebra cum Mathesi universali non videtur confundenda, *GMS,* VII, 205.
[52] Geometry does not intrinsically include considerations of size, equality, or proportion. Hence the idea of *Analysis situs* to directly study – and not through the detour of numbers, determinate (arithmetic) or indeterminate (algebra) – the relations of position and special configurations (*C,* 152, 563-568). But algebra and infinite analysis are to the art of invention as species is to genus (*GMS,* VII, 206).
[53] *GMS,* II, 229.
[54] *GPS,* VII, 169.
[55] *GMS,* II, 246.
[56] *C,* 262.
[57] *LH,* IV, 7 C, Bl. 160-161, *Vorausedition,* VI, fasc. 1 (1982), 195.
[58] *GMS*, II, 229, 239-240, 269; *GMS*, V, 348-349 and the work of E. Knobloch, of which we have an idea from the article published in the Studia Leibnitiana, Supplémenta XXII (1982), 96-118.
[59] This analogy, to which Leibniz often returns, is the subject of a specific essay, Symbolismus memorabilis Calculi Algebraici et Infinitesimalis…, *GMS,* V, 377-381.

$(x+y)^1 = x^1y^0 + x^0y^1$

$(dxy) = d^1x\, d^0y + d^0x\, d^1y$

x and *y* are considered to be indices of the letter *d* and the higher-order differentials are considered to be powers of the first differential, d^1x or *dx*, then the identity of both operations can be established. In this way, raising to a power and differentiation, which are distinct operations in themselves, can be written in the same way. Algebraic calculus and differential calculus have parallel languages and thus parallel structures. Here is the germ of the model theory approach: establishing a meta-theory of mathematical theories through the analysis of their characteristics, that is, of their language.

The bipolarity, essential to all of Leibniz's thinking, between the method and its specifications, the general and its specialisations, the form and its samples[60], the *sign* and its *interpretations*, should be borne in mind. It has been recently recognised[61] as an ancestor of the pair constituted by a formal system and interpretation (or an abstract theory and a specific model). And this has given rise to a brilliant thesis on how Leibniz's system (metaphysics) and his mathematical models complement each other[62]. But we must still see why Gödel suggested such a strong affinity between the methods of model theory and Leibniz's *ars inveniendi.*

III. – Characteristics and model theory

1. There is no doubt a certain paradox in bringing model theory closer to the Leibnizian notion of characteristics, a general language or science, inasmuch as Leibniz's goal was to establish a *universal* characteristics or language. For it was by resolutely turning its back on this idea of a universal language that model theory could start to be developed in the 1930s. In doing this, it learnt from Richard's paradox (1905) and Gödel's incompleteness theorem (1931). These show that every exact language is susceptible of an arithmetic treatment of its means of expression. The primitive symbols and the well-formed expressions in the language generate a countable formalism. But mathematics exceeds this formalism. Hence the requirements of an exact language and those of a universal language cannot be reconciled, as Leibniz, and later Frege (1848-1925) and Russell (1872-1970), believed. Logic can no longer seek to establish universal rules for all sciences, not even for all of mathematics.

Nonetheless, the institution of a universal characteristics and a universal calculus represents for Leibniz a "perfection" that cannot be achieved in a single stroke. What is essential for us is to build as much as possible without requiring the means to complete the analysis or definitively guarantee its foundations[63]. *In fact,* Leibniz continually introduces new mathematical notations, shows formal analogies, establishes connections. The imperative of progress prevails over any interest in questions of status. Independently from whether infinitesimals exist actually or potentially, they can be regarded

[60] This term reappears constantly in Leibniz's writings to designate not only a fragment of a science, but also a specific realisation, a specific model of a formal scheme. Cf. the letter to Nicaise (5 June 1692), *GPS,* II, 535, as well as *GMS,* IV, 465; V, 141-171; VII, 206.

[61] Cf. for example, N. Rescher's study, Leibniz's interpretation of his logical calculi, *The Journal of symbolic logic,* 19 1954, 1-13.

[62] M. Serres, *Le système de Leibniz et ses modéle mathématiques,* Paris, PUF (1968).

[63] *GPS,* VII, 165.

as "ideal notions" or "fictions" that serve to abbreviate discourse and facilitate discovery[64]. The ideal of universality, which gave rise to a logicism *avant la lettre,* is largely counterbalanced by Leibniz's formalism, which was recognised early on[65]. This mathematical formalism, required by the need "to advance our knowledge", is not a principled position, but the almost natural attitude of a mathematician seeking to make discoveries. Formality flushes out error and increases certainty. But above all, it opens up perspectives, makes visible what was hidden, spurs us on to "conceive universal notions". We are not far from the idea – supported in Hilbert's school – that abstract axiomatic is an instrument of mathematical research.

2. It is remarkable to find, as an explicit reflection, a similar attitude in Abraham Robinson (1918-1974). For him, only a formalist point of view makes it possible to accept symbolic entities or abstract theories - that is, those theories whose interpretation is not direct, does not have a finite model but extrapolates from the finite to the infinite. But the history of mathematics proves that mathematicians have practically never baulked at abstractions ("fictions" in Leibniz's sense, "ideal elements" in Hilbert's sense). And logic itself (model theory, proof theory, generalised recursion) has never ceased to use infinite procedures or ceased to refer to infinite sets. This is but a "natural" and "fruitful extension" of the mathematical formalism promoted by set theory and by the setting up of abstract structures. Thus, the formalist position accounts for a *fact.* It is commanded *a posteriori* by a practice that goes back two thousand years, not *a priori* by the idea of restraining all mathematical truths within the pre-established straitjacket of the provable, which was once the obsession of Hilbert's early *Beweistheorie.*

Strictly speaking, moreover, there is no ontological difference between "ideal" elements and other elements. The generation of idealities is a constant and essential form of mathematical proliferation. We must acknowledge the full extent of the role played by the formalist disposition in invention – it is the lion's part! But any formal arborescence, however rich and complex it may be, is rooted in a controllable node, interpreted or dominated by finite procedures. Mathematical idealities grow in the interval of these swings between the finite and the infinite, not within a to-and-fro between "real" and "fiction". For example, an infinitesimal in a non-standard model is "no more and no less real" – and, we should add, no more or less fictional - than a standard irrational[66]. Abraham Robinson departs from Leibniz and from all those who in any way associate mathematical formalism and metaphysical realism[67]. Formalism stands by itself, on the basis of its results. There is no need for realism as a foil or as an aid. Infinite processes should not be admitted *despite* their lack of "reality" (Leibniz) or *because* of their "reality" (the adepts of the actual infinite in mathematics). *In fact,* mathematicians allow them because they are a fruitful extension of finite processes, and because rejecting them means cutting invention's wings. Leibniz's idea of "useful fictions" can be retained provided that it is retained by itself, dissociated from Leibniz's prejudice when he said that the infinitesimals "are *nothing but* fictions".

3. It is only to be expected that a formalist should pay the greatest attention to language, to symbolism, to notations, to the forms of expression not only out of rigorousness, but also to facilitate discovery, if not more so. But model theory has turned this attention into a *method.* It systematically

[64] *GMS,* IV, 92-93, 98, 110.
[65] Husserl, *Logische Untersuchungen,* I, § 60 (1900); French translation, Paris, PUF, 1959, 238-241.
[66] *Non Standard Analysis,* Amsterdam, North-Holland (1966), 281-282.
[67] From a formalist's point of view, *Dialectica,* 23, 1969, 45-49.

applies the logical analysis of mathematical language to the discovery of logical or *mathematical* procedures of a general nature that cannot be accessed by any other means. By trying to turn logic into "an effective instrument for mathematical research", Robinson seeks no less than to make it fully play the role of an *ars inveniendi.*

We cannot infer from Robinson's writings that he read Leibniz's works, other than certain texts regarding the justification of infinitesimal calculus. However, they are driven by an explicit reflection on the art of analogy and generality which is firstly anchored in the study of algebraic structures. The logical analysis of these leads to subordinating their multiplicity to general principles that turn the *mathematical analogies* between certain structures into formal *logical identity*. This is possible because the essence of model theory is that it deals with "sets of theorems or different sets of axiom systems simultaneously, whereas ordinary mathematics is content with deducing specific theorems for specific structures, or else for all structures that satisfy a specific axiom system"[68]. For example, he or she considers the theory of algebraically closed fields[69] and/or the theory of real closed fields[70], that is, simultaneously, the set of the models of the former theory and/or the set of models of the latter. The simultaneous examination of multiplicities, their comparison, as Leibniz would have said[71], discovers "metamathematical theorems of algebra"[72], that is, general algebraic theorems discovered through logical methods.

Thus, for the set of real closed fields, any elementary statement (formulated in first-order logic) that is true in the Archimedean field of the real numbers is equally true in any real closed field, be it Archimedean or not. Because it is possible to move from a specific field to any other model of a certain class, it can be said that there is a "transfer principle" at play.

The mathematical usefulness of transfer principles has become well-known, even independently from model theory work. Lefschetz's principle, formulated in 1953, states the conditions in which a theorem verified in the field of complex numbers can be generalised to any algebraically closed field[73]. But by then Alfred Tarski (1901-1983) had already proven the logical counterpart of a transfer principle for the class of closed real fields, namely the completeness of the elementary theory of real closed fields[74]. And Robinson drew the consequence of the fact that any complete theory gives rise to a transfer principle[75]. The systematic search for transfer principles is included in the programme of model theory[76]. The challenge is clear: being able to generalise an elementary theorem, even if its known proof is not elementary. There are, for example, theorems in the field of the real numbers that have been proven through topological methods specific to this particular

[68] L'application de la logique formelle aux mathématiques. *Actes du 2e Colloque international de logique mathématique,* Paris, Gauthier-Villars and Louvain, Nauwelaerts, 1952, 51-64.

[69] It should be borne in mind that an algebraically closed field is a field such that any polynomial with coefficients within this field has all its roots in this field. A familiar prototype: the field of the complex numbers.

[70] A field is real if (-1) cannot be written in it as a sum of squares. A field is real and closed if all its positive elements are squares and if every polynomial of an odd degree with coefficients in this field has at least one root in this field. A familiar prototype: the ordered field of real numbers.

[71] For example, *C,* 561-562.

[72] *On the metamathematics of algebra,* Amsterdam, North-Holland, 1951, 9.

[73] S. Lefschetz, *Algebraic Geometry*, Princeton, 1953.

[74] *The completeness of elementary algebra and geometry* (1939), Paris, Institut Blaise Pascal, 1967 repr. in Tarski, *Collected Papers*, S.R. Givant and R.N. McKenzie eds, Birkhäuser, 1986, 289-346; *A decision method for elementary algebra and geometry*, University of California Press, Berkeley and Los Angeles, 1948, second revised ed. 1951. Repr. in *Collected Papers*, III, 297-368.

[75] Indeed, a theory is complete if all its models verify the same statements.

[76] *On the application of symbolic logic to algebra,* Proc. Intern. cong. math. Cambridge (Mass.), 1950, 686-694.

field. Topological methods are not elementary. And yet being able to state the theorems in question in an elementary language, or associate them with elementary statements that are logically equivalent to them, is enough to conclude that they are valid in any real closed field (for example, in the field of the real algebraic numbers, which is strictly included in the field of real numbers). We can see how the logical analysis of language joins forces with the study of specific mathematical models to advance mathematics itself. Model theory is but "a natural development" of modern mathematics "through the means of formal logic"[77].

4. The analogy between raising the power of a binomial and finding the differential of the product of two factors led Leibniz to his theorem of symbolic calculus. The analogy between different algebraic structures led Robinson to metastructural concepts that can multiply the mathematical analogies in which they originate. This is the case of the algebraic closure and the real closure of a field[78], whose common scheme is established by Robinson under the term of the model completion of a theory[79]. The notion of "differential field", or a field in which certain pairs of elements satisfy a binary relation R*xy*, interpreted as "*y* is the derivative of *x*", following the usual rules in the calculus of derivatives, was already known. Robinson invented the notion of differential closure, the model completion of the theory of the differential fields, equating the notions of algebraic and real closure. Thus with the help of logic, we have created a mathematical structure that is evidently useful for a certain type of problems. But the notion of model completion also enables us to make explicit and increase the connections between the two known classes of algebraically closed fields and real closed fields. Robinson manages, indeed, to isolate the logical structure common to different theorems that determine the existence of a common solution to several algebraic equations. These theorems[80] provide algorithmic procedures for the effective calculus of this solution. These procedures can be formalised as conjunctions of logically simple statements inasmuch as they do not involve any quantifier. Specifically, these statements are decidable, that is, whether they are true or false can be easily determined (through the use of the truth tables for propositional connectors). Hence, we can see that the mathematical notion of algorithm or an effective method is, in fact, a "sample" of what the logician calls more generally a "decision procedure". A simple numerical calculus is already a "decision procedure".

These few examples, selected here because they can easily appear independently from the technical sophistication that Robinson's various papers necessarily involve, show well the logical conceptualisation performed by model theory through the analysis of the language useful for characterizing concepts forged by classical or structural mathematics. Structural mathematics has sufficiently persuaded us of the generative power of structure concepts. Model theory shows us the generative power of meta-structural concepts. And we can see how conceptual analysis, as practiced in

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[77] Cf. the introduction to the article cited *supra*, no. 67.

[78] These are respectively the smallest algebraically closed extension of a field and the smallest closed algebraic and real closed extension of a real field.

[79] Some problems of definability in the lower predicate calculus, *Fundamenta mathematicae,* 44, 1957, 309-329.

[80] These are, for algebraically closed fields, the classic theorems of the theory of algebraic elimination or Hilbert's *Nullstellensatz* (every polynomial f, null for all the values of the common roots of a series of polynomials has a power f^k belonging to the ideal generated by the elements of this series), and for real closed fields Sturm's theorem (Sturm's theorem expresses the number of distinct real roots of a polynomial $p(x)$ located in an interval $[a,b]$; applied to the interval of all the real numbers, it gives the total number of real roots of $p(x)$. Cf. Tarski, *The completeness of elementary algebra and geometry*).

model theory, improves the Art of Invention by providing methods rather than just solutions. It has shown us the "origin" of a set of results which is just the beginning of a prosperous future.

Bibliographic abbreviations

GMS *Leibnizens matematische Schriften*, ed. C. I. Gerhardt, Berlin and Halle, 1849-1863,7 vol., reprinted in Hildesheim, Georg Olsm Verlagsbuchhandlung, 1962.

GPS *Die philosophischen Schriften von G.W.Leibniz*, ed. C.I. Gerhardt, Berlin,1875-1890, 7 vol., reprinted in Hildesheim, Georg Olms Verlagsbuchhandlung, 1960-1961.

C *Opuscules et fragments inédits de Leibniz*, ed. L. Couturat, Paris, PUF, 1903, reprinted in Hildesheim, Georg Olms Verlagsbuchhandlung, 1961.

LH *Leibniz Handschriften*

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