Duality and Categorical Equivalence: A Look at Gauge/Gravity

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Abstract

It is well known among physicists that many distinct physical theories are equivalent, in that the state space of one can be formally mapped to the other (and vice versa). Yet this introduces a number of problems: what are the formal and conceptual criteria for theoretical equivalence? In other words, when do two distinct yet interchangeable mathematical structures represent the same physical system? Notice that difference in structure does not mean inequivalence of theories. Each involves difference in structure. To complicate things further, theories can be dual to another theory. Duality is a special yet notably hard-to-define relationship. Although similar to theoretical equivalence, it remains unclear whether dual theories are another case of equivalence or stand as a unique type of theory relation. Indeed, "the complete physical meaning of the duality symmetry is still not clear, but a lot of work has been dedicated in recent years to understand the implications of this type of symmetry" (Alvarez- Gaume et al, "Duality in Quantum Field Theory and String Theory," CERN). Perhaps the most extensive treatment of duality in philosophy of physics literature comes from De Haro and Butterfield, who argue that a) dual theories are not equivalent and b) formally, a duality is an isomorphism between models of the same theory. In this paper I identify important problems with this view—specifically, problems of bijective mappings between theory-models. To revise the account given by De Haro and Butterfield, I propose a category-theoretic account of duality in physics that qualifies as a specified form of so-called "categorical equivalence" for scientific theories. By introducing category theory as a formal framework, I show how these problems are avoided, while also demonstrating the extensive "reach" of this proposal by applying it to gauge/gravity.

1 Introduction

In this paper I present some problems facing the account of duality in physics given by De Haro and Butterfield (2019), where a) dual theories are not equivalent and b) formally, a duality is an isomorphism between models of the same theory. We will provide more details below for this account, but for now it suffices to state that after the problems with this view are introduced, I provide reasons not necessarily for rejecting it; rather, I argue that a more robust formalization of their second claim—specifically, a category theoretic framing—is sufficient to avoid its current problems. However, this will come at the expense of their first claim, namely, that duality and equivalence are distinct relations between theory models. Once the problems with their current account are introduced—illustrated through general relativity as an example—and our formal revision is proposed, the resulting solution will show that duality is a subtype of categorical equivalence. We outline the basics of category theory in the beginning of section 3 before identifying duality as a subclass of categorical equivalence. To give further reason for adopting this revised account of duality, we note its considerable breadth, ranging from electromagnetic duality to gauge/gravity as examples, and suggest this indicates that the revision is both more robust and more successful in its explanatory utility. Far from simply fixing this account of duality, our proposal has the upside of offering a new area of exploration in theoretical equivalence. As a subset of categorical equivalence, dual theories identify a specific type of equivalent intertheoretic relationship.

2 Revising the Schema

De Haro and Butterfield have presented a "Schema" for understanding how two physical theories can be dual to each other. The mathematical structure of a bare theory and its respective models is crucial for their view. All relevant information within a theory's model —say, a model that satisfies the field equations in general relativity—is encapsulated by satisfying a bare theory's triple: $(\mathbf{S}, \mathbf{Q}, \mathbf{D})$ meaning a set of states, a set of quantities, and dynamics. For instance, we might have a Hilbert space, an algebra of operators (observables), and the Schrödinger equation. Models of a theory are obtained by way of representation, in the strict algebraic sense, meaning the bare theory structure can be homomorphically mapped (to a model).[1] A duality—mapping a model triple to a different model triple—is such that both the Hilbert space and algebraic relations are isomorphic to another state-space and algebra, with the dynamics of the two models being equivariant. A bare theory can have multiple realizations, meaning it can allow multiple homomorphic maps to different model triples. If two of these model triples are isomorphic to one another, then the models are dual.

Although this account of duality offers a lot, it faces a few problems. In the proceeding sections, I identify these problems, and in revising the Schema in a way that avoids them, I offer an improved version of the Schema. There are a few reasons for working within the parameters of De Haro and Butterfield's Schema instead of rejecting it wholesale. Their account covers a lot of important ground in the case of bosonization, where a detailed and formal explication of this duality is desirable. Additionally, the problems I shall note concern the Schema's formal framework, the revision of which can salvage the otherwise promising content.

2.1 Relative Isomorphisms

In general relativity, one considers relativistic spacetimes as models for the theory. A relativistic spacetime is a pair (M, g), M being a smooth manifold of four dimensions and g

[1] Homomorphisms are specific to algebraic structures, but we can classify this as a more specific account of models realizing a theory's structure. Indeed, what matters is that a model is a "mathematical instantiation" of a theory's structure, much like models in first-order logic.

being the Lorentzian metric. These two objects in the pair ensure that our spacetime is a Lorentzian manifold. According to the Schema, isomorphic model tuples (in this case, a pair) constitute a duality. However, this is surely an inadequate account of duality in light of the hallmark principle of relativistic spacetimes: general covariance.

Consider two manifolds M and N and a C^∞ map $\psi : M \to N$. We'd like to know if there is a way to compare the structures of each. For N (and M respectively) we can construct a tangent space T_pN at point $p \in N$ by parametrizing a curve $\gamma(\lambda)$ through p, such that a smooth map $\gamma : I \to N$ obtains for an open interval $I \in \mathbb{R}$, with $\lambda \in I$ and $\gamma(\lambda_p) = p \in N$. We then denote a map $f \in C^\infty \mid f : N \to \mathbb{R}$, by which equidistant contour measures can be made for point p, allowing for the following differential operation: $\frac{df}{d\lambda} = \frac{d}{d\lambda}(f \circ \gamma)$, the directional derivative at a point along a curve. Considering the same point p, the set of all directional derivatives for each curve through p, $(\gamma(\lambda), \chi(\alpha), \sigma(\beta),...)$, forms a tangent space $\{\frac{d}{d\lambda_p}, \frac{d}{d\alpha_p}, \frac{d}{d\beta_p}, ...\}$. For a coordinate adapted basis for the tangent space, we need a chart to map our tangent vector. We begin with $\phi(p) : N \to \mathbb{R}^n$, so that $f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma : \lambda_p \to \alpha_p$. Let $x^\mu(\lambda)$ be $(\phi \circ \gamma)^\mu$ and $f(x^\mu)$ be $(f \circ \phi^{-1})$, so that

$$\frac{df}{d\lambda} = \frac{d}{d\lambda}(f \circ \phi^{-1} \circ \phi \circ \gamma) = \frac{\partial(f \circ \phi^{-1})}{\partial(\phi \circ \gamma)^\mu} \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \tag{1}$$

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} \tag{2}$$

Thus, the directional tangent vector is $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$. Next, in order to compare manifolds M and N, we can pull the function $f : N \to \mathbb{R}$ back to M (thus, the "pullback") by composing ψ and f: $\psi_* f = (f \circ \psi)$. To compare the derivatives of smooth functions in each manifold (tangent spaces), we can then take a directional tangent vector V in M and find its action on a function in N, which we call the "pushforward":

$$(\psi^* V)^\mu \partial_\mu(f) = V^\nu \partial_\nu(f \circ \psi) \tag{3}$$

$$= V^\nu \frac{\partial y^\mu}{\partial x^\nu} \partial_\mu f \tag{4}$$

for basis vectors ∂_μ in N—introduced in (2) above—and ∂_ν in M. One can generalize this to covectors and tensors, but the details are not important here.[2]. If $\psi : M \to N$ is one-to-one, onto, and has a C^∞ inverse, then the map is a diffeomorphism (a C^∞ isomorphism). This means (a) that M and N have "identical manifold structure," (b) if ψ maps tensors, the diffeomorphism is a symmetry transformation for tensors, and (c) if the metric tensor $g_{\mu\nu}$ is mapped, ψ is an isometry.[3] With this we can state that if $\psi : M \to N$ $\psi : g_{\mu\nu} \to g_{\mu'\nu'}$ is invertible, the two relativistic spacetime models (M, g) and (N, g') are isomorphic. We can also think of this diffeomorphism as an "external symmetry" on spacetime models, "an external transformation which preserves the solutions of the theory."[4]

[2]For more, see Wald (1984) appendix C

[3]Ibid.

[4]Dewar, 2019, § 2

Isomorphic spacetime models are a problem for the Schema as it stands, for if a duality means isomorphic model tuples (pairs in this case), then it allows too much. Any diffeomorphic spacetime model satisfies this condition; however, diffeomorphic invariance (or "general covariance") is just a feature of general relativity, by which tensorial structures can be compared through symmetry transformations. A wide range of models are permitted, whose manifold structure can be identical and whose field-theoretic solutions are invariant under diffeomorphism. *But this is not a duality.* It allows far too many cases of isomorphic models, such that nothing accounts for common duality examples (AdS/CFT, bosonization, etc) being distinct in their symmetric models. As De Haro and Butterfield have noted, dualities in physics are universally "surprising," meaning that they are isomorphic in a way that differs from symmetries in ordinary theory structure. Thus, we can either formalize this distinction or reject the notion that there is a formal dichotomy between duality and symmetric models of a theory. To reiterate: in order to salvage the Schema, we need to revise it in such a way that models of a theory are isomorphic in a particular way.

Mathematical structures admit a variety of definitions of isomorphism that are relative to those structures. To paraphrase Shapiro (1997), underdetermination of isomorphism means that we can always ask, "isomorphic in terms of what?" The Schema, as it stands, faces two problems: a) it cannot specify the isomorphic relation between models, and b) it exclusively focuses on isomorphic models, yet a duality in physics also involves an isomorphism of the relations between models. Defining formal structure in terms of both models and their morphisms is not peculiar to physical theories; it is a feature of mathematical structures in general, where

> the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of (Lawvere, 1966)

For example, in the AdS/CFT correspondence we map AdS isometries to conformal symmetries on the CFT side. Similarly, in electromagnetic duality we map Minkowski space isometries to Minkowski space isometries. We are mapping both models and their morphisms to corresponding models and morphisms.

Fortunately, category theory provides a formal framework in which we can explicate how a theory's models are isomorphic to models in a corresponding dual theory. More generally, category theory exposes the symmetric properties of mathematical structures. As opposed to foundations of mathematics in ZFC, category theory does not formalize on the basis of set-membership; rather, it defines a mathematical object in terms of its possible morphisms.[5] We are neither endorsing category theory as a foundation for mathematics, nor are we presenting the revised Schema as a form of mathematical structuralism (see Awodey, 1996). We are, however, associating the problems faced in the Schema (a project in foundations of physics) with more general problems faced in foundations of mathematics.

Category theory may not be required to address underdetermined isomorphic–or, bijective–mappings if we are dealing with first-order theories. In (possibly many-sorted) first-order logic, two models of a set of axioms can only be compared relative to the signature of those axioms (see Halvorson, 2019, 173 for an example). Model theory provides robust definitions

[5] Ernst (2017)

for other related mappings, such as elementary embedding and extension. But physical theories rarely admit first-order axiomatization, meaning we need a formal framework that can generalize these results to higher-order structure (Barrett and Halvorson, 2015, Thm 5.1).

3 Duality as Restrictive Categorical Equivalence

Here, we introduce the basics of category theory:

Definition 3.1. A *category* $\mathbf{C}$ consists of

- a collection of objects: A, B, C, ...
- a collection of arrows (also called morphisms): f, g, h, ...
- for each arrow f objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ called the *domain* and *codomain* of f. If $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$, we also write $f : A \to B$,
- given $f : A \to B$ and $g : B \to C$, so that $\mathrm{dom}(g) = \mathrm{cod}(f)$, there is an arrow $g \circ f : A \to C$,
- an arrow $\mathbf{1}_A : A \to A$ for every object A of $\mathbf{C}$,

such that

(Associative law) for every $f : A \to B$, $g : B \to C$ and $h : C \to C$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

(Unit laws) for every $f : A \to B$ we have

$$f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f.$$

For instance the category **Set** has sets as objects and functions between them as morphisms. If we take a set X with a collection τ of open subsets in X that satisfies particular axioms, then we have new category **Top** of topological spaces (objects), whose morphisms are continuous functions (homeomorphisms). Such functions are important because if we have a well-defined metric between elements of the set X, then we can define locally Euclidean spaces in this topological space, thus defining topological manifold. We can perform operations between categories through functors, which are structure-preserving maps between categories, mapping objects to objects and arrows (morphisms) to arrows.

Definition 3.2. A *functor* $F : \mathbf{C} \to \mathbf{D}$ maps objects in $\mathbf{C}$ to objects in $\mathbf{D}$ and arrows in $\mathbf{C}$ to arrows in $\mathbf{D}$ such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$,
- $F(g \circ f) = F(g) \circ F(f)$,
- $F(1_A) = 1_{F(A)}$

Consider a functor F : **Top** $\rightarrow$ **Set**. We call this functor "forgetful" because it forgets the topology we added to the category of sets. Two categories are equivalent if a functor between them is full, faithful, and essentially surjective. A functor $F : C \rightarrow D$ is full if for all objects c_1, c_2 in C and morphisms $g : Fc_1 \rightarrow Fc_2$ in D there exists a morphism $f : c_1 \rightarrow c_2$ in C such that $Ff = g$. The functor is faithful if $Ff = Fg$ implies that $f = g$ for all morphisms $f : c_1 \rightarrow c_2$ and $g : c_1 \rightarrow c_2$ in C. Finally, the functor is essentially surjective if for every object d in D there exists an object c in C such that $Fc \cong d$. Compare this with the functor F : **Top** $\rightarrow$ **Set**, where topological structure is not preserved (thus, it is not "full"). So, to have categorical equivalence, it is required that a "functor F forgets nothing" (Weatherall 2015, 6).

Categorical equivalence offers a way to formalize the sense in which different formulations of the mathematical structures are equivalent. Just as the theories of abelian groups or topological spaces might admit different yet equivalent axiomatic formulations (e.g. Kuratowski closure axioms), so can two categories admit a generalized equivalence relation between them. Thus, we can relate different types of mathematical structures, such as topological spaces and Boolean algebras (Stone's representation theorem). For physical theories, we will consider the objects of the category to be the models of a theory. For instance, general relativity has smooth four dimensional manifolds with a Lorentzian metric as its models. We can call two theories "categorically equivalent" if their category of models are equivalent. We can give a nice example of categorical equivalence in the wild by considering Lagrangian and Hamiltonian mechanics. To understand the models for each, we look to the statespace structure of each theory. For Lagrangian mechanics, position and velocity of particles are encoded in a tangent bundle, whereas Hamiltonian mechanics represents position and momentum of particles via a cotangent bundle. A model for each pairs the respective bundle with the governing smooth scalar function for measuring the energy of a system. Although the mathematical models for each theory are structurally different compared with the other theory (tangent vs cotangent bundles), the above criteria for categorical equivalence can be satisfied in this case (See Barrett, 2015 for the proof).

Notice how both models and the relationships between models need to be appropriately mapped for the categories of models to be equivalent. This feature was introduced earlier as a necessity for a duality between physical theories. Indeed, we can motivate the association of duality with categorical equivalence by noting that the framework we have introduced does offer a way to formalize the type of symmetric theory structure found in a duality. We can now specify the ways in which dual theories are isomorphic, as "categorical equivalence captures a sense in which theories have 'isomorphic semantic structure'...if T_1 and T_2 are categorically equivalent, then the relationships that models of T_1 bear to one another are 'isomorphic' to the relationships that models of T_2 bear to one another."[6] De Haro and Butterfield rightly identify something peculiar about dual models that we can reformulate in our new framework, namely, that a duality relates models of the *same* theory. We have already seen why this cannot obtain as an isomorphism relative to model structure; however, we can state this as an isomorphism in terms of categorical structure. By this I mean that the following two claims are compatible:

- Under a duality mapping, categories of models are isomorphic

[6]Barrett and Halvorson, 2015

- A duality mapping is a functor from one category to itself

To unpack this, consider categorically equivalent theories in physics, such as Lagrangian and Hamiltonian mechanics. The categories of models, though isomorphic, are distinct categories–that is, we are relating two separate theories by virtue of positing different mathematical structures (tangent vs cotangent bundles). However, a duality relates models of the same theory, meaning it is an *automorphism* of one and the same category of models. The two claims above are compatible if we take duality to be a restrictive case of categorical equivalence, restrictive insofar as it requires an endofunctor acting on a category of models–an endofunctor being a functor from a category to itself.

S–duality in electromagnetism provides an apt illustration. Consider Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0 \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c}(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}) \tag{6}$$

where an electric field **E** and a magnetic field **B** are solutions. An identical solution to this occurs when **B** replaces **E** and **-E** replaces **B**. As a category of models, we can say that our objects are Faraday tensors $F_{\mu\nu}$ with Minkowski space isometries as arrows (morphisms). The tensor is a 2-form:

$$\frac{1}{2}F_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \tag{7}$$

$$= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy + E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt \tag{8}$$

We require this to be a solution to Maxwell's equations, invariant under isometries. Call this category of models **M**. S–duality can be represented as a Hodge star operator acting as an endofunctor from $H : \mathbf{M} \to \mathbf{M}$ such that

$$H(F_{\mu\nu}) : F_{\mu\nu} \to \star F_{\mu\nu} \tag{9}$$

$$H(f : F_{\mu\nu} \to F_{\mu\nu}) = H(f) : \star F_{\mu\nu} \to \star F_{\mu\nu} \tag{10}$$

Weatherall (2019) calls this an "autoequivalence," since the functor is an operator that returns objects and arrows back to the same category. As opposed to general cases of categorical equivalence, a duality entails that you end where you begin. This actually helps the Schema's "realization" story because other forms of categorical equivalence can be seen as translating between different categories of models, whereas a duality is simply an auto-transformation from one category of models to itself. Thus, under such a full, faithful, and essentially surjective endofunctor operation, these isomorphic categories of models are the separate "realizations" of the same bare theory. We have simply clarified a feature of the Schema in a way that avoids underdetermined isomorphisms. Our revised Schema can be stated as follows:

- *A duality is an automorphism of a category of models*

With this in place, we can now turn to our main example, AdS/CFT. We show in the next section that this revised schema can account for this case of duality. Specifically, we show that the models of one theory and their morphisms are isomorphic to the models and morphisms of a theory on the asymptotic boundary. Crucially, these will be the same category of models, meaning we will have an equivalence of categories under automorphism. For this to work, we need to show that an endofunctor on one category of models can produce the respective dual models and morphisms. If we can accomplish this, then we will have obtained a restrictive class of categorically equivalent theories, which we call "duality."

4 AdS/CFT

To demonstrate the utility of the revised Schema, we now ask whether it helps elucidate other cases of duality. We use gauge/gravity as an example, specifically the non-trivial correlation between:

(a) Type IIB string theory on compactified $AdS \times S^5$

(b) $\mathcal{N} = 4$ Supersymmetric Yang-Mills gauge theory on the AdS boundary.

The first is our bulk theory–that is, our theory for a spacetime region–and the second is our theory for that region's boundary. Work on AdS/CFT ranges from establishing the correlation and the dictionary between bulk and boundary terms to the renormalization problem.[7] The dictionary is established insofar as "the expectation value of the boundary stress-energy tensor is determined by functionally differentiating the on-shell gravitational action with respect to the boundary metric," and renormalization solves the divergent on-shell gravitational action. [8] In other words all physical information in the bulk spacetime is encoded on a quantum field theory on the region's boundary. It is worth mentioning this because we will have to leave out, at least in this paper, a good portion of the dictionary material. Indeed, once an AdS/CFT prescription for correlating boundary and bulk fields is derived, the majority of the work afterwards is identifying appropriate boundary conditions and deriving the boundary–bulk propagator, each involving the asymptotic limit of the bulk field. We will not have the space to address those details in full, and although they play a determining role regarding the success of the revised Schema, such details are mostly irrelevant to the claim that the revised Schema accounts for this duality. What matters is that we can account for the matching symmetries, statespaces, and observables between the bulk and boundary.

[7] See Andrade et al 2009: "The holographic calculation of field theory correlators dual to an action I has three different faces. The dictionary problem establishes the relationship between bulk fields and dual operators. This is where most of the AdS/CFT physics resides. The renormalization and Dirichlet problems, namely, to find B to make the bulk action finite and well-defined for the given boundary conditions. Finally, the fluctuation problem involves imposing boundary conditions in the deep interior and find non-local relations among the boundary data."

[8] de Haro et al, 2000

Since our models will mainly be of D-branes, we will briefly review the string theoretic background. A relativistic string has tension T

$$T = \frac{1}{2\pi\alpha'} \tag{11}$$

We parameterize the worldsheet Σ (the two dimensional path of a string) in terms of its target space, which is the spacetime $\mathcal{M}$ in which it is embedded. Thus, the function $X^\mu(\tau, \sigma)$ embeds the worldsheet in terms of the proper time τ and spatial interval σ of the string. With this, we can express the Nambu-Goto action for the string,

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}} \tag{12}$$

where $\eta_{\mu\nu}$ is the target spacetime metric. Strings may be open or closed. D-branes are hyperplanes at the ends of open strings (even if the endpoints are located on the same D-brane). We can use the Nambu–Goto action to write the Dirac-Born-Infeld action for D-branes:

$$S_{DBI} = -T_{D_p} \int d^{p+1}x \sqrt{-\det(g_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu})} \tag{13}$$

where T_{D_p} is the D-brane tension $\frac{1}{(2\pi)^p g_s l_s^{p+1}}$, $g_{\mu\nu}$ is the induced metric, and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Fermionic terms may be added. If there is a stack of N coincident D-branes, the gauge group for A_μ goes from *U(1)* to *SU(N)*. For a 3+1 worldvolume we have D3-branes and need a metric solution for a stack of N coincident branes. Doing this requires viewing D3-branes as sources for closed strings (gravitons). Consider the metric

$$ds^2 = f(r)(-dt^2 + dx^2) + h(r)(dr^2 + r^2 d\Omega_5^2) \tag{14}$$

and its SUGRA solution

$$f(r) = \frac{1}{h(r)} = H^{-1/2}(r), \quad H(r) = 1 + \frac{R^4}{4}, \tag{15}$$

$$R^4 = N\frac{4}{\pi^2} G_N T_3 = N 4\pi g_s \alpha', \quad G_N = g\pi^6 g_s^2 \alpha'^2 \tag{16}$$

The (bulk) near horizon limit $r \to 0$ gives us the $AdS \times S^5$ metric

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \tag{17}$$

where $d\Omega_5^2$ are the compactified S^5 components (expanded into Kaluza-Klein modes below). The (boundary) asymptotic limit $r \to \infty$ brings $H(r) \to 1$, giving us flat Minkowski space. We can also view D3-branes from the original open string perspective, and if we take the asymptotic, low-energy limit (to the boundary), the DBI action above reduces to

$$S = -\frac{1}{2\pi g_s} \mathrm{tr} \int d^4x (\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{6} \partial^\mu \phi^i \partial_\mu \phi^i - \pi g_s \sum_{i,j=1}^{6} [\phi^i, \phi^j]^2) \tag{18}$$

where ϕ^i are scalar fields in the adjoint representation of *U(N)*. This is the action for $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) gauge theory, provided the $\mathcal{N}=4$ SYM coupling is identified with $2\pi g_s$. The gauge multiplet for this theory is $(A_\mu, \lambda^a_\alpha, X^i)$, the gauge field, left Weyl fermions, and six scalar fields (same as ϕ^i above) respectively. With constants C_i^{ab} which relate to Clifford Dirac matrices, the following Lagrangian is invariant under Poincaré supersymmetry,[9]

$$\mathcal{L} = \mathrm{tr}\{-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X^i \tag{19}$$

$$+\sum_{a,b,i} gC_i^{ab}\lambda_a[X^i,\lambda_b] + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a[X^i,\bar{\lambda}^b] + \frac{g^2}{2}\sum_{i,j}[X^i,X^j]^2\} \tag{20}$$

Importantly, with scale invariance the symmetry group becomes $SU(2,2 \mid 4)$, which matches the isometry group of the bulk $AdS \times S^5$ theory, meaning that the symmetry groups of each theory are equivalent.

Although the symmetry groups (the relationships between models) are isomorphic, we still would like to obtain the objects on which the symmetry groups will act. To do this we consider CFT operators and AdS fields. Since we have compatified S^5, a canonical supergravity field can be reduced (Kaluza-Klein reduction), such that it is encoded on a tower of fields on AdS. We then parameterize the tower in terms of spherical harmonics and coordinates of S^5: $Y_l(\Omega)$ and Ω,

$$\phi(x,\Omega) = \sum_l \phi_l(x)Y_l(\Omega) \tag{21}$$

the Laplacian eigenvalues of which indicate the fields are invariant under $SU(2,2 \mid 4)$ representations, with scaling dimension Δ. As established, the asymptotic limit of the bulk theory is $\mathcal{N}=4$ SYM, whose field theoretic operators are derived through the symmetrized trace of the scalar fields above X^i,

$$\mathcal{O}_\Delta(x) = \mathrm{str}(X^{(i_1}(x)...X^{i_\Delta)}(x)) \tag{22}$$

of the same conformal dimension for the bulk field Δ, meaning the operators also transform under representations of $SU(2,2 \mid 4)$. Finally, to demonstrate the duality, consider canonical supergravity fields in the bulk $\phi(x,z)$ that satisfy the action,

$$S = \frac{1}{8\pi G}\int^{d+1} z\sqrt{\bar{g}}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3}b\phi^3...) \tag{23}$$

We can derive the bulk–boundary propagator from,

$$\frac{\delta S}{\delta\phi} = (-\Box + m^2)\phi + b\phi^2 + ... = 0 \tag{24}$$

[9] D'Hoker and Freedman, 2002

with boundary conditions

$$\phi(z_0, \vec{z}) \to z^{d-\Delta}\phi(\vec{z}) \tag{25}$$

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + 4m^2} \tag{26}$$

Once an iterative bulk–boundary propagator is established, the relationship between two sides of the duality are:

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = (-)^{n-1}\frac{\delta}{\delta\phi(x_1)}...\frac{\delta}{\delta\phi(x_n)}\delta[\phi]\Big|_{\bar{\phi}=0} \tag{27}$$

where $\mathcal{O}(x)$ are correlation functions for field operators in the gauge theory. This is the basis of the duality, which the revised Schema shall both account for and elucidate, insofar as we will get a better sense of how the duality functions and formally mirrors other cases of dual or equivalent theories.

4.1 The Duality and the Categorical Schema

Consider a category of models **Bulk** whose objects are tuples $(M, g_{\mu\nu}, X^i)$ where M is a maximally symmetric Lorentzian manifold with AdS metric $g_{\mu\nu}$ (in Euclidean signature) and X^i are canonical scalar fields defined on M, whose trace $Tr[X^i]$ corresponds to physical observables.[10] Fields in the bulk will look like (21), and their Fourier modes ω serve as states in some algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, such that

- ω is a G–invariant state
- G being the supergroup $SU(2,2 \mid 4)$

Thus, the arrows between the objects in **Bulk** are invertible transformations that represent actions α of the superconformal group $SU(2,2 \mid 4)$.

Suppose we take apply the endofunctor boundary operator on our category of models $D : M \to \partial M, \partial M \to \partial M$. Our spacetime $(M, g_{\mu\nu})$ is isomorphic to the boundary spacetime. We can confirm the correspondence presented in (27) as follows. Observables on the boundary also correspond to a trace on fields, but we require primary field operators, such that together with any descendant operators, we have the SYM gauge multiplet $(A_\mu, \lambda^a_\alpha, X^i)$. The trace of our primary operator will be the gauge invariant observable, meaning they transform under the adjoint gauge group. Only the scalar fields X^i in the multiplet can satisfy this, insofar

[10]We require a Euclidean signature because we have performed a Wick rotation. The (Poincaré) metric takes the upper half of hyperbolic space $H_{d+1} = \{(z_0, \vec{z}),\ z_o \in \mathbf{R}^+, \vec{z} \in \mathbf{R}^d\}$ and will read $ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2)$ We should also note that the specificity of the relevant mathematical structures is important for the particular case of holographic duality in which we are interested. However, it is desirable to have a generalized statement of holographic duality, in which bulk and boundary terms are related. We mention this because an expanded and less specific account should accommodate algebraic holography (Rehren, 2000; Anderson, 2004), CS–WZW correspondence (Witten, 1989; Gawedzki, 1999), and twisted holography (Costello and Paquette, 2020; Costello and Gaiotto, 2021). Accommodation is necessary but not sufficient for a general account; this categorical Schema must also be continuous with the formal results of these three examples.

as the fields define the gauge invariant operator from (22). We classify the fields in terms of an algebra of operators $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ and the states ω in the algebra. Finally, the whole symmetry group is the superconformal group $SU(2, 2 \mid 4)$. Therefore, our model tuples and their arrows are isomorphic under the boundary endofunctor $D : \mathbf{Bulk} \to \mathbf{Bulk}*$

5 Conclusion

In this paper we have introduced problems faced by De Haro and Butterfield's Schema for duality. Using relativistic spacetime models in general relativity, we showed that the criterion of isomorphic models for dual theories allows too much, considering that any diffeomorphic spacetime would then qualify. We proposed the framework of category theory as a solution, by which the Schema's criterion is qualified with the stipulation that models *and* their morphisms must be isomorphic in terms of categorical structure. This is accomplished through an automorphism of a category of models through an endofunctor mapping, meaning duality is a special case of categorical equivalence. To see this proposal at work, we looked at gauge/gravity (specifically, a version of AdS/CFT), with the result being that the revised, categorical Schema accounts for the duality and avoids the problems in the original Schema. We can anticipate further developments in the following areas. As mentioned, the revised Schema should be able to accommodate duality examples with more rigorous formulations, well-known in mathematical physics. Next, the revised Schema should also be continuous with said examples, such that our account can serve as a rigorous account of dual theories in general. Although rather technical, this paper intersects with contemporary structuralist debates in philosophy of science. Rickles (2017) employs a structuralist approach to duality, with a defensive claim–dual theories are not problematic for structural realists–and a positive claim–structuralism can best account for dualities). With respect to the current paper, we cannot say much beyond agnosticism, but we can expect the category theoretic underpinnings of the revised Schema to have a non-trivial bearing on structualist approaches to duality.

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