Does the Curvature Structure of Spacetime Determine Its Topology?*

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Abstract

We explore the title question. After some topological preliminaries, we define a scalar "curvature isomorphism" between spacetimes. We introduce a hierarchy of curvature conditions and show that at a certain level, a curvature isomorphism must be a homeomorphism. The highest level of the hierarchy is satisfied by a spacetime if every smooth scalar function on its manifold is an invariant scalar curvature function. We show that such "maximally structured" spacetimes exist and that a curvature isomorphism between them must be a diffeomorphism. We highlight a number of connections between our project and the one in which the topology of spacetime is determined from a causal relation between spacetime points (Malament 1977). We emphasize that analogous results are obtained here by considering only invariant properties of spacetime points – no relations needed.

1 Introduction

It is well known that the causal structure of spacetime, if sufficiently rich, determines the topology of spacetime (Malament 1977). For each spacetime (M, g), a casual relation $\ll$ on M is determined by the metric g. A bijection $\varphi : M \to M'$ between spacetimes (M, g) and (M', g') is a "causal isomorphism" if, for all $p, q \in M$, we have $p \ll q$ if and only if $\varphi(p) \ll \varphi(q)$. One then considers: under which conditions is it the case that a causal isomorphism between spacetimes must be a homeomorphism? A hierarchy of causal conditions plays a central role in exploring the question. At a certain level of the hierarchy (past and future distinguishability) one finds that a causal isomorphisms must be a homeomorphism – indeed a diffeomorphism. This is false at a slightly lower level (past or future distinguishability).

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In the natural way, one can consider an analogous question with respect to (scalar) curvature structure: if the curvature structure of spacetime is sufficiently rich, does it determine the topology of spacetime? After some topological preliminaries, we define a "curvature isomorphism" between spacetimes. We introduce a hierarchy of curvature conditions and show that at a certain level, a curvature isomorphism must be a homeomorphism. The highest level of the hierarchy is satisfied by a spacetime if every smooth scalar function on its manifold is an invariant scalar curvature function. We show that such "maximally structured" spacetimes exists and that a curvature isomorphism between them must be a diffeomorphism. We close with a brief remark concerning the significance of the work.

2 Topological Preliminaries

In order to construct a hierarchy of curvature conditions, we need to review some basic topology (Willard 1970 pp. 52-57). In what follows, let Y and I be sets. For each $i \in I$, let X_i be a topological space and let $f_i : Y \to X_i$ be an associated function. The *weak topology* on Y induced by the family of functions $\{f_i\}$ is the smallest topology on Y for which f_i is continuous for all $i \in I$. The characteristic property of the weak topology is the following (Willard 1970, Theorem 8.10).

Lemma 1. *Let the set Y have the weak topology induced by the family $\{f_i\}$ of functions $f_i : Y \to X_i$ and let Z be a topological space. A function $f : Z \to Y$ is continuous if and only if $f_i \circ f$ is continuous for each $i \in I$.*

Let $X = \prod X_i$ be the *direct product* of the sets X_i which is defined as the set of functions $x : I \to \bigcup X_i$ such that $x(i) \in X_i$ for each $i \in I$. For each $x \in X$ the value of x at $i \in I$ will be denoted x_i rather than $x(i)$. For each $i \in I$, we have a *projection map* $\pi_i : X \to X_i$ defined by $\pi_i(x) = x_i$. Now let X have the weak topology induced by the family of projection maps $\{\pi_i\}$. This is the *product topology* on X. The *evaluation map* $e : Y \to X$ induced by the family $\{f_i\}$ of functions $f_i : Y \to X_i$ is defined by $e(y) = \{f_i(y)\}$ for all $y \in Y$. We see that $\pi_i \circ e = f_i$ for all $i \in I$. Let us say the family $\{f_i\}$ of functions $f_i : Y \to X_i$ *separates points* on the set Y if, for any distinct $y_1, y_2 \in Y$, we have $f_i(y_1) \neq f_i(y_2)$ for some $i \in I$. For any topological spaces W and Z, we say that the map $h : W \to Z$ is an *embedding* if the range restricted map $h : W \to h[W]$ is a homeomorphism where $h[W]$ has the subspace topology induced from Z. We have now come to a foundational result (Willard 1970, Theorem 8.12).

Proposition 1. *The evaluation map $e : Y \to X$ induced by the family $\{f_i\}$ of functions $f_i : Y \to X_i$ is an embedding if and only if: (i) Y has* *the weak topology induced by the family $\{f_i\}$ and (ii) the family $\{f_i\}$ separates points on Y.*

Given a topological space Y and a family $\{f_i\}$ of continuous functions $f_i : Y \to X_i$, a useful condition allows one to determine if the topology on Y is the weak topology on Y induced by family $\{f_i\}$. Let us say that the family $\{f_i\}$ of functions $f_i : Y \to X_i$ *separates points from closed sets* on the topological space Y if, for any closed set $C \subset Y$ and any $y \notin C$, we have $f_i(y) \notin \overline{f_i[C]}$ for some $i \in I$. One can show that if a family $\{f_i\}$ of continuous functions $f_i : Y \to X_i$ separates points from closed sets on Y, then the topology on Y is the weak topology on Y induced by the family $\{f_i\}$. One can also show that if the topological space Y is such that $\{y\}$ is closed for each $y \in Y$ (i.e. if Y is a T_1 space), then whenever a family $\{f_i\}$ of functions $f_i : Y \to X_i$ separates points from closed sets on Y, then the family $\{f_i\}$ also separates points on Y. We have the following (Willard 1970, Theorem 8.16).

Corollary 1. *If Y is a T_1 space, then the evaluation map $e : Y \to X$ induced by the family $\{f_i\}$ of continuous functions $f_i : Y \to X_i$ is an embedding if the family $\{f_i\}$ separates points from closed sets on Y.*

3 Curvature Isomorphisms

For any manifold M, let $\mathscr{S}(M)$ be the collection of smooth real-valued functions on M. Associated with each spacetime (M, g) is a collection $\mathscr{C}(M, g) \subseteq \mathscr{S}(M)$ of smooth invariant curvature functions $f : M \to \mathbb{R}$ constructed from the metric g and its associated Riemann curvature tensor (Hawking and Ellis 1973, p. 35). There are a number of senses of "constructed" that one might consider here. Under any sense, the Ricci scalar curvature function $R : M \to \mathbb{R}$ counts as a member of $\mathscr{C}(M, g)$. Following the literature, one might also permit the use of basic algebraic operations so that, for example, a function like $108R^{-5/2}$ counts as a member of $\mathscr{C}(M, g)$ when well-defined (Page and Shoom 2015). In what follows, we shall even permit the use of arbitrary smooth functions so that, for example, the function $\alpha(R)$ also counts as a member of $\mathscr{C}(M, g)$ if $\alpha : \mathbb{R} \to \mathbb{R}$ is smooth. The important thing to keep in mind is that even under this liberal understanding of metric construction, all functions in $\mathscr{C}(M, g)$ are invariant under isometries in the following sense. Consider a pair of spacetimes (M, g) and (M', g') and an isometry $\psi : M \to M'$. Let $f \in \mathscr{C}(M, g)$ and $f' \in \mathscr{C}(M', g')$ be such that f is constructed from the metric g and its associated Riemann curvature tensor in the same way that f' is constructed from the metric g' and its associated Riemann curvature tensor. Because ψ is an isometry, we have $\psi^*(g') = g$ where ψ^* is the pull back associated with ψ. So $\psi^*(f') = f$ and thus $f' \circ \psi = f$.

There are a number of senses in which the collection $\mathscr{C}(M,g)$ can be used to determine the local geometry of spacetime (Sternberg 1964; Ehlers 1981). But in general, the collection $\mathscr{C}(M,g)$ does not determine the topology of spacetime; one finds, for example, that "local invariants cannot distinguish a plane and a flat torus" (MacCallum 2015, p.6). Let us make this claim precise.

Definition 1. Let (M,g) and (M',g') be spacetimes and let $\mathscr{C}(M,g)$ and $\mathscr{C}(M',g')$ be indexed by a set I to generate the respective families of functions $\{f_i\}$ and $\{f'_i\}$ such that f_i and f'_i correspond to the same metric construction for each $i \in I$. We say a bijection $\varphi : M \to M'$ is a (scalar) *curvature isomorphism* if $\{f_i\} = \{f'_i \circ \varphi\}$.

A curvature isomorphism between spacetimes captures a sense in which the (scalar) curvature structures are preserved. One can verify that for all spacetimes (M,g) and (M',g'), if $\varphi : M \to M'$ is a curvature isomorphism, then so is φ^{-1}. We can now show that, in general, a curvature isomorphism need not be a homeomorphism.

Example 1. Consider Minkowski spacetime (M,g) and a flat spacetime (M',g') where M' is a torus. Let $\mathscr{C}(M,g)$ and $\mathscr{C}(M',g')$ be indexed by a set I to generate the respective families of functions $\{f_i\}$ and $\{f'_i\}$ such that f_i and f'_i correspond to the same metric construction for each $i \in I$. Consider any bijection $\varphi : M \to M'$. Because both spacetimes are flat, we know that for any points $p \in M$ and $p' \in M'$, there are open neighborhoods O and O' of the points p and p' respectively, and an isometry $\psi : O \to O'$ such that $\psi(p) = p'$. It follows that for any points $p \in M$ and $p' \in M'$, we have $f_i(p) = f'_i(p')$ for each $i \in I$. So $f_i = f'_i \circ \varphi$ for each $i \in I$. So $\{f_i\} = \{f'_i \circ \varphi\}$ and thus φ is a curvature isomorphism. But of course, M and M' have different topologies by construction so φ cannot be a homeomorphism.

4 A Hierarchy of Curvature Conditions

One wonders under which conditions is it the case that a curvature isomorphism between spacetimes must be a homeomorphism. In order to answer the question, we now construct a hierarchy of curvature conditions on spacetimes. As we do so, we will directly import some of the central topological conditions considered above. We will also rely heavily on Lemma 1 and Proposition 1. Let (M,g) be a spacetime and let $\mathscr{C}(M,g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Recall that the family $\{f_i\}$ separates points on M if, for any distinct $p, q \in M$, we have $f_i(p) \neq f_i(q)$ for some $i \in I$. Consider the following.

Definition 2. Let (M,g) be a spacetime and let $\mathscr{C}(M,g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. We say the spacetime (M, g) *separates points* if the family $\{f_i\}$ separates points on M.

This condition is analogous to the past and future distinguishing condition on the causal structure of spacetime (Hawking and Ellis 1973, p. 192). Just as that condition prohibits distinct points from having the same causal structure, the separating points condition prohibits distinct points from having the same (scalar) curvature structure. As we shall see, the condition of separating points is the weakest of the curvature conditions considered here. Even so, it sits atop a hierarchy of spacetime "asymmetry" conditions (Manchak and Barrett 2023). Because the strongest of these asymmetry conditions will be needed later on, let us briefly review them here.

We say a spacetime (M, g) is *giraffe* if the identity map is the the only isometry $\varphi : M \to M$. We say a spacetime (M, g) is *locally giraffe* if, for any open connected set $O \subseteq M$, the spacetime (O, g) is giraffe. Finally, we say that a spacetime (M, g) is *Heraclitus* if, for any distinct points $p, q \in M$ and any open neighborhoods O_p and O_q of these points respectively, there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. One can show that (i) any spacetime that separates points must be a Heraclitus spacetime, (ii) any Heraclitus spacetime must be locally giraffe, and (iii) any locally giraffe spacetime must be giraffe. There are examples showing that a giraffe spacetime need not be locally giraffe and that a locally giraffe spacetime need not be Heraclitus. But it is unknown if a Heraclitus spacetimes can fail to separate points. Let us start now to keep track of open questions.

Question 1. Can a Heraclitus spacetime fail to separate points?

We now come to the next curvature condition in the hierarchy. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Recall that the weak topology on M induced from the family $\{f_i\}$ is the smallest topology for which f_i is continuous for each $i \in I$. We can think of the weak topology on M as a type of "curvature topology" induced from $\mathscr{C}(M, g)$. Consider the following.

Definition 3. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. We define the *curvature topology* on M to be the weak topology on M induced by $\{f_i\}$.

For any spacetime (M, g), one can verify that the curvature topology on M is a subset of the manifold topology on M. This follows from the fact that the manifold topology on M counts each member of $\mathscr{C}(M, g)$ as continuous while the curvature topology on M is defined as the smallest topology on M with this property.

Proposition 2. *For any spacetime, its curvature topology is a subset of its manifold topology.*

Proof. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Let τ_m and τ_c be the manifold and curvature topologies on M respectively. So τ_c is the smallest topology for which f_i is continuous for each $i \in I$. Suppose there is an $O \in \tau_c$ such that $O \notin \tau_m$. We show a contradiction. Since the intersection of any pair of topologies is a topology, we know $\tau = \tau_m \cap \tau_c$ is a topology on M. Since $O \in \tau_c$ but $O \notin \tau_m$, we know that τ is a proper subset of τ_c. Consider the function $f_i : M \to \mathbb{R}$ for any $i \in I$ and let $U \subseteq \mathbb{R}$ be any open set. Since both τ_m and τ_c count f_i as continuous, we have $f_i^{-1}[U] \in \tau_m$ and $f_i^{-1}[U] \in \tau_c$. So $f_i^{-1}[U] \in \tau$. So τ makes the function f_i continuous for any $i \in I$. Since τ is a proper subset of τ_c and τ makes the function f_i continuous for each $i \in I$, we see that τ_c is not the smallest topology with this property: a contradiction. $\square$

In general, the manifold and curvature topologies do not coincide. Indeed, the curvature structure can be so impoverished in spacetimes that the associated curvature topology is trivial. Consider the following example.

Example 2. Consider Minkowski spacetime (M, g) and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Consider the function $f_i : M \to \mathbb{R}$ for any $i \in I$. We know that for any points $p, q \in M$, there is an isometry $\psi : M \to M$ such that $\psi(p) = q$. It follows that for any points $p, q \in M$, we have $f_i(p) = f_i(q)$. So the function f_i is constant; there is some $a \in \mathbb{R}$ such that $f_i(p) = a$ for all $p \in M$. Now consider the trivial topology $\{M, \varnothing\}$ on M and let $U \subseteq \mathbb{R}$ be any open set. If $a \in U$, then $f_i^{-1}[U] = M$ which is open in the trivial topology. If $a \notin U$, then $f_i^{-1}[U] = \varnothing$ which is also open in the trivial topology. Either way, $f_i^{-1}[U]$ is open in the trivial topology and thus f_i is continuous relative to the trivial topology. Since f_i was arbitrarily chosen, we know that f_i is continuous relative to the trivial topology for any $i \in I$. Since the weak topology on M induced from $\{f_i\}$ is the smallest topology on M with this property, the weak topology on M must be the trivial topology on M. So the curvature topology on Minkowski spacetime is the trivial topology.

Here, we find that the situation is completely analogous with one concerning the causal structure. For each spacetime (M, g), one can construct a "causal topology" on M, i.e. the Alexandrov topology (Hawking and Ellis 1973, p. 196). One can show that the Alexandrov topology is a subset of the manifold topology but that, in general, the two topologies do not coincide. Indeed, we find that the causal structure can be so impoverished in some spacetimes (e.g. Gödel spacetime) that the resulting Alexandrov topology is trivial. Given the situation, a natural causal condition to consider is one in which a spacetime is required to have its manifold and Alexandrov topologies coincide. This condition is equivalent to the so called strong causality condition and implies a number of conditions at lower levels of the causal hierarchy including the past and future distinguishing condition. Something like this is true for the curvature case as well: if a spacetime is such that its manifold and curvature topologies coincide, then it must separate points.

Proposition 3. *If a spacetime is such that its manifold and curvature topologies coincide, then it separates points.*

Proof. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Suppose that (M, g) does not separate points. So the family $\{f_i\}$ does not separate points on M. So there are distinct points $p, q \in M$ such that $f_i(p) = f_i(q)$ for all $i \in I$. Consider the function $f : M \to M$ defined such that $f(p) = q$, $f(q) = p$, and $f(r) = r$ for all $r \in M - \{p, q\}$. By construction, f is not continuous relative to the manifold topology on M. Since $f_i(p) = f_i(q)$ for all $i \in I$, one can verify that $f_i \circ f = f_i$ for all $i \in I$. We know that, for all $i \in I$, the function f_i is continuous relative to the the weak topology on M induced from the family $\{f_i\}$. So for all $i \in I$ the function $f_i \circ f$ is continuous relative to the weak topology on M induced from the family $\{f_i\}$. By Lemma 1, the function f must also be continuous relative to the weak topology on M induced from the family $\{f_i\}$. Since the function f is continuous relative to the weak topology on M but not continuous relative the manifold topology on M, we know that these topologies do not coincide. So the manifold and curvature topologies do not coincide. □

One wonders if the other direction also holds.

Question 2. Can a spacetime separate points if its manifold and curvature topologies differ?

We now come to the next curvature condition in the hierarchy. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Recall that the family $\{f_i\}$ separates points from closed sets on M if, for any closed set $C \subset M$ and any $p \notin C$, we have $f_i(p) \notin \overline{f_i[C]}$ for some $i \in I$. Consider the following.

Definition 4. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. We say the spacetime (M, g) *strongly separates points* if the family $\{f_i\}$ separates points from closed sets on M.

We have the following.

Proposition 4. *If a spacetime strongly separates points, then its manifold and curvature topologies coincide.*

Proof. Let (M, g) be a spacetime and let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Suppose (M, g) strongly separates points. So the family $\{f_i\}$ separates points from closed sets on M. So the manifold topology on M is the weak topology on M induced from the family $\{f_i\}$ (recall the discussion just after Proposition 1). So the manifold and curvature topologies coincide. $\square$

One wonders if the condition of strongly separating points is equivalent to the condition of having the manifold and curvature topologies coincide. If so, it would be yet another parallel with the causal structure case in which strong causality is equivalent to the condition that the manifold and Alexandrov topologies coincide. We have the following.

Question 3. Can a spacetime fail to strongly separate points if its manifold and curvature topologies coincide?

We now come to the highest level of the curvature hierarchy. This condition requires that every smooth scalar function on a spacetime is an invariant scalar curvature function.

Definition 5. We say the spacetime (M, g) is *maximally structured* if $\mathscr{C}(M, g)$ is the collection $\mathscr{S}(M)$ of smooth functions on M.

One can show the following.

Proposition 5. *If a spacetime is maximally structured it also strongly separate points.*

Proof. Let (M, g) be a maximally structured spacetime. So $\mathscr{C}(M, g) = \mathscr{S}(M)$. Let $\mathscr{C}(M, g) = \mathscr{S}(M)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Let $C \subset M$ be any closed set and let $p \notin C$. A foundational result states: for any manifold M, any point $p \in M$, and any neighborhood O of p, there is a smooth bump function $f : M \to \mathbb{R}$ such that $f(p) = 1$ and $f(q) = 0$ for all $q \in M - O$ (Lu 2011, p. 144). It follows that there is a smooth function $f : M \to \mathbb{R}$ such that $f(p) = 1$ and $f(q) = 0$ for all $q \in C$. So $\overline{f[C]} = \overline{\{0\}} = \{0\}$ and thus $f(p) \notin \overline{f[C]}$. Since f is smooth and $\mathscr{C}(M, g) = \mathscr{S}(M)$, we know $f = f_i$ for some $i \in I$. So $f_i(p) \notin \overline{f_i[C]}$ for some $i \in I$. So the family $\{f_i\}$ separates points from closed sets on M. So (M, g) strongly separates points. $\square$

Naturally, we have the following.

Question 4. Can a spacetime fail to be maximally structured if it strongly separate points?

Let us now put all the results together to form a hierarchy of conditions. Consider the diagram below. The highest four levels are curvature conditions, the lowest three are asymmetry conditions. Each of the six arrows is an implication relation. The lowest two arrows do not run in the other direction. It is unknown if the highest four arrows run in the other direction. Could it be that the Heraclitus condition is equivalent to the maximally structured condition?

Maximally Structured

$\Downarrow$

Strongly Separates Points

$\Downarrow$

Manifold Topology=Curvature Topology

$\Downarrow$

Separates Points

$\Downarrow$

Heraclitus

$\Downarrow$

Locally Giraffe

$\Downarrow$

Giraffe

5 Topology from Curvature Structure

Let us now show that a curvature isomorphism must be a homeomorphism under sufficiently strong curvature conditions. If the manifold topology and curvature topology coincide, the desired result is straightforward (see Proposition 6 below). Again we have an analogy with the causal structure situation. If the manifold topology coincides with the Alexandrov topology, it is straightforward to show that a causal isomorphism must be a homeomorphism (Hawking and Ellis 1973, p. 197).

Proposition 6. *If a pair of spacetimes is such that each one has its manifold and curvature topologies coincide, then a curvature isomorphism between them is a homeomorphism.*

Proof. Let (M, g) and (M', g') be spacetimes and such that the manifold and curvature topologies coincide on M and similarly for M'. Let

$\mathscr{C}(M,g)$ and $\mathscr{C}(M',g')$ be indexed by a set I to generate the respective families of functions $\{f_i\}$ and $\{f'_i\}$ such that f_i and f'_i correspond the same metric construction for each $i \in I$. Suppose $\varphi : M \to M'$ is a curvature isomorphism. So $\{f_i\} = \{f'_i \circ \varphi\}$.

For each $i \in I$, let X_i be a copy of $\mathbb{R}$ and let $X = \prod X_i$ be the direct product with the product topology. Let $e : M \to X$ and $e' : M' \to X$ be the evaluation maps induced from the families $\{f_i\} = \{f'_i \circ \varphi\}$ and $\{f'_i\}$ respectively. Since (M,g) is such that its manifold and curvature topologies coincide, from Proposition 3 we know that (M,g) separates points. So the family $\{f_i\} = \{f'_i \circ \varphi\}$ separates points on M. Since (M,g) is such that its manifold and curvature topologies coincide, the manifold topology on M is the weak topology on M induced from the family $\{f_i\} = \{f'_i \circ \varphi\}$. Since M has the weak topology induced from the family $\{f_i\} = \{f'_i \circ \varphi\}$ and since $\{f_i\} = \{f'_i \circ \varphi\}$ separates points on M, it follows from Proposition 1 that the evaluation maps e and e' are embeddings. Let $e[M]$ and $e'[M']$ be given the subspace topology induced from X. Since e and e' are embeddings, we can restrict the ranges in the natural way to construct the homeomorphisms $h : M \to e[M]$ and $h' : M' \to e'[M']$.

We now show that $h' \circ \varphi = h$. Consider any point $p \in M$. Since the evaluation map e was induced from the family $\{f_i\} = \{f'_i \circ \varphi\}$, we see that $h(p) = e(p) = \{f_i(p)\} = \{(f'_i \circ \varphi)(p)\} = \{f'_i(\varphi(p))\}$. Now consider $h' \circ \varphi$. Since the evaluation map e' was induced from the family $\{f'_i\}$, we see that $(h' \circ \varphi)(p) = h'(\varphi(p)) = e'(\varphi(p)) = \{f'_i(\varphi(p))\}$. So $h' \circ \varphi = h$. It follows that $\varphi = h'^{-1} \circ h$ which is homeomorphism since h and h'^{-1} are both homeomorphisms. $\square$

Of course, one wonders if the result will go through if the curvature condition is weakened. Is a curvature isomorphism between two spacetimes that separate points necessarily a homeomorphism? This question is analogous to the question: is a causal isomorphism between past and future distinguishing spacetimes necessarily a homeomorphism? The answer to the second question eventually turned out to be yes though this was not straightforward (Malament 1977). We have the following.

Question 5. Is there a curvature isomorphism between spacetimes that separate points that is not a homeomorphism?

Naturally one also looks for the strongest condition under which a curvature isomorphism fails to be a homeomorphism. One can easily find a counterexample among locally giraffe spacetimes. Consider the following.

Proposition 7. *A curvature isomorphism between locally giraffe spacetimes need not be a homeomorphism.*

Proof. We know there exist locally giraffe spacetimes that are not Heraclitus (Manchak and Barrett 2023). Let (M, g) be any such spacetime. Let $\mathscr{C}(M, g)$ be indexed by a set I to generate the family of functions $\{f_i\}$. Since the spacetime is not Heraclitus, there are distinct points $p, q \in M$, respective open neighborhoods O_p and O_q of these points, and an isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. Since $\psi(p) = q$, we know that $f_i(p) = f_i(q)$ for all $i \in I$. Let $\varphi : M \to M$ be the bijection defined such that $\varphi(p) = q$, $\varphi(q) = p$, and $\varphi(r) = r$ for all $r \in M - \{p, q\}$. By construction, φ is not continuous and therefore not a homeomorphism. But since $f_i(p) = f_i(q)$ for all $i \in I$, we have $f_i \circ \varphi = f_i$ for all $i \in I$. So $\{f_i \circ \varphi\} = \{f_i\}$ and thus φ is a curvature isomorphism. □

Can one find a counterexample among Heraclitus spacetimes? We have the following.

Question 6. Is there a curvature isomorphism between Heraclitus spacetimes which fails to be a homeomorphism?

6 Maximal Curvature Structure

It turns out that maximally structured spacetime have sufficiently rich curvature structure that, not only is a curvature isomorphism between such spacetimes a homeomorphism, it must also be a diffeomorphism. We have the following.

Proposition 8. *A curvature isomorphism between maximally structured spacetimes is a diffeomorphism.*

Proof. Let (M, g) and (M', g') be maximally structured spacetimes. Let $\mathscr{C}(M, g)$ and $\mathscr{C}(M', g')$ be indexed by a set I to generate the respective families of functions $\{f_i\}$ and $\{f'_i\}$ such that f_i and f'_i correspond the same metric construction for each $i \in I$. Since (M, g) and (M', g) are maximally structured, we have $\mathscr{C}(M, g) = \mathscr{S}(M)$ and $\mathscr{C}(M', g') = \mathscr{S}(M')$. Suppose $\varphi : M \to M'$ is a curvature isomorphism. So $\{f_i\} = \{f'_i \circ \varphi\}$. Let $f' : M' \to \mathbb{R}$ be any smooth function on M'. Since $\mathscr{C}(M', g') = \mathscr{S}(M')$, we know $f' = f'_i$ for some $i \in I$. Since $\{f_i\} = \{f'_i \circ \varphi\}$ it follows that the function $f_i = f'_i \circ \varphi = f' \circ \varphi$ is a member of $\mathscr{C}(M, g) = \mathscr{S}(M)$. So $f' \circ \varphi$ is smooth and thus φ is smooth. An analogous argument shows that φ^{-1} is smooth. So φ is a diffeomorphism. □

Of course, one wonders if the result goes through if the curvature condition is weakened. Is a curvature isomorphism between two spacetimes that strongly separate points necessarily a diffeomorphism?

Question 7. Is there a curvature isomorphism between spacetimes that strongly separate points that fails to be a diffeomorphism?

We know there exist spacetimes that separate points (Manchak and Barrett 2023). But we have yet to show the existence of spacetimes that satisfy the three stronger conditions on the curvature hierarchy. It turns out that even maximally structured spacetimes exist. We have the following.

Proposition 9. *A maximally structured spacetime exists.*

Example 3. Let the manifold M be the set of points $(t, x) \in \mathbb{R}^2$ such that (i) $t^2 - x^2 > 0$, (ii) $t > 0$, and (iii) $x > 0$. Let $\Omega : M \to \mathbb{R}$ be the smooth positive function defined by $\Omega(t, x) = 1/(t^2 + x^2)$. Let g_{ab} be the metric on M given by $\Omega^2[-\nabla_a t \nabla_b t + \nabla_a x \nabla_b x]$. Let $R : M \to \mathbb{R}$ be the Ricci scalar curvature function and let $Q : M \to \mathbb{R}$ be the scalar curvature function defined by $g^{ab}(\nabla_a R)\nabla_b R$. One can show that $R(t, x) = 8(t^2 - x^2)$ and $Q(t, x) = -32R(t, x)\Omega(t, x)^{-2} = -256(t^2 - x^2)(t^2 + x^2)^2$ (Manchak and Barrett 2023). Because of condition (i), we find that $Q < 0 < R$ on M.

Let $C_1 : M \to \mathbb{R}$ and $C_2 : M \to \mathbb{R}$ be the scalar curvature functions defined by $C_1 = R/8$ and $C_2 = (-Q/32R)^{1/2}$. One can verify that because $Q < 0 < R$ on M and $t^2 + x^2 > 0$, the functions C_1 and C_2 are well-defined and smooth on M. Indeed, we find that $C_1(t, x) = t^2 - x^2$ and $C_2(t, x) = t^2 + x^2$. Now let $T : M \to \mathbb{R}$ and $X : M \to \mathbb{R}$ be the scalar curvature functions defined by $T = ((C_1 + C_2)/2)^{1/2}$ and $X = ((C_2 - C_1)/2)^{1/2}$. One can verify that since $C_1 + C_2 = 2t^2 > 0$ and $C_2 - C_1 = 2x^2 > 0$ and because of conditions (ii) and (iii) above, the functions T and X are well-defined and smooth on M. Indeed, we find that $T(t, x) = t$ and $X(t, x) = x$.

Now let $f : M \to \mathbb{R}$ be any smooth function. Of course, f is just a rule which maps certain ordered pairs of real numbers to real numbers. Let us use this rule f to construct a smooth invariant scalar curvature function $F : M \to \mathbb{R}$. We do this by setting $F(p) = f(T(p), X(p))$ for all points $p \in M$. Since $T(t, x) = t$ and $X(t, x) = x$, we find that $F(t, x) = f(T(t, x), X(t, x)) = f(t, x)$ which shows that $F = f$. Since f is well-defined and smooth, so is F. Since F is a scalar invariant function, we know $F \in \mathscr{C}(M, g)$ and thus $f \in \mathscr{C}(M, g)$. Since f was an arbitrarily chosen smooth function on M, we have $\mathscr{C}(M, g) = \mathscr{S}(M)$. So (M, g) is maximally structured.

7 Remark

We have shown a sense in which the (scalar) curvature structure of spacetime determines its topology. Along the way, we have drawn attention to a number of ways in which the situation with respect to curvature structure in analogous to the one with respect to causal structure. Here, we which to highlight a sense in which they come apart. The causal structure of spacetime is encoded in a particular relation between spacetime points. The result of Malament (1977) shows that, under certain conditions, one can use this relation to uniquely determine the topology of spacetime. Now consider the scalar curvature structure of spacetime. It differs significantly from the causal structure of spacetime in the sense that it is not encoded in a relation (or in a collection of relations) between spacetime points. Instead, each scalar curvature function gives rise to an invariant property of each spacetime point. The scalar curvature structure of spacetime is encoded in the collection of all such properties of all spacetime points. Here we have shown that, under certain conditions, one can use these properties of spacetime points to uniquely determine the topology of spacetime – no relations needed.

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