

The local validity of special relativity: the standard story articulated and defended

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Abstract

Morally, the ‘standard story’ that special relativity is locally valid in general relativity is surely correct. However, rendering the standard story precise and unambiguous has proved to be a vexed issue. In this article, we articulate and defend the standard story from an effective field theory (EFT)-inspired perspective.

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1 Introduction

Conventional wisdom has it that general relativistic physics is effectively special relativistic for ‘sufficiently small neighbourhoods.’¹ This position is reflected in leading textbook presentations of the matter (often in relation to principle(s) of equivalence)—consider, for example, this passage from Steven Weinberg’s seminal textbook on gravitation:

The Principle of Equivalence rests on the equality of gravitational and inertial mass, demonstrated by Galileo, Huygens, Newton, Bessel, and Eötvös [...] Einstein reflected that, as a consequence, no external static homogeneous gravitational field could be detected in a freely falling elevator, for the observers, their test bodies, and the elevator itself would respond to the field with the same acceleration [... But e]ven the observers in Einstein’s freely falling elevator would in principle be able to detect the earth’s field, because objects in the elevator would be falling radially toward the center of the earth [...] Although inertial forces do not exactly cancel gravitational forces for freely falling systems in an inhomogeneous or time-dependent gravitational field, we can still expect an approximate cancellation if we restrict our attention to such a small region of space and time that the field changes very little over the region. Therefore we formulate the equivalence principle as the statement that *at every space-time point in an arbitrary gravitational field it is possible to choose a “locally inertial coordinate system” such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.* There is a little vagueness here about what we mean by “the same form as in unaccelerated Cartesian coordinate systems,” so to avoid any possible ambiguity we can specify that by this we mean the form given to the laws of nature by special relativity [...] There is also a question of how small is “sufficiently small.” Roughly speaking, we mean that the region must be small enough so that the gravitational field is sensibly constant throughout it, but we cannot be more precise until we learn how to represent the gravitational field mathematically [Weinberg, 1972, p. 68, emphasis in original]

So, according to Weinberg, within ‘a sufficiently small region’, laws in general relativity (GR) take the same form as in special relativity (SR), i.e., as in the inertial frames of SR.² (It follows that Weinberg is also saying that the physical effects

¹We include scare quotes here because, of course, ‘sufficiently small neighbourhoods’ is a term of art which, ultimately, will require unpacking.

²[Hartle, 2003, p. 140], for instance, just as well renders the idea of the local validity of SR in GR as being cashed out in terms of Riemannian normal coordinates (more on which later):

A concrete expression of this physical idea [i.e., of the equivalence principle] is the requirement that, given a metric $g_{\alpha\beta}$ in one system of coordinates, at each point P of spacetime it is possible to introduce new coordinates x'^{α} such that

$$g'_{\alpha\beta}(x'_p) = \eta_{\alpha\beta}$$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric of flat spacetime and $x'_p{}^{\alpha}$ are the coordinates locating the point P . This requirement is one of the assumptions of general relativity.

modelled by the mathematical concept of ‘spacetime curvature’ can be neglected in the right regime. We will return to that later on.) In fact, this is arguably exactly what Einstein himself had in mind—or so he writes in a letter to Painlevé:³

According to the special theory of relativity the coordinates x, y, z, t are directly measurable via clocks at rest with respect to the coordinate system. Thus, the invariant ds , which is defined via the equation $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, likewise corresponds to a measurement result. The general theory of relativity rests entirely on the premise that each infinitesimal line element of the spacetime manifold physically behaves like the four-dimensional manifold of the special theory of relativity. Thus, there are infinitesimal coordinate systems (inertial systems) with the help of which the ds are to be defined exactly like in the special theory of relativity. The general theory of relativity stands or falls with this interpretation of ds . It depends on the latter just as much as Gauss’ infinitesimal geometry of surfaces depends on the premise that an infinitesimal surface element behaves metrically like a flat surface element [...] .
[Einstein, 1921]

Let us call what is expressed in the Weinberg and Einstein quotes above the *standard story* about the effective emergence of SR within GR. Although Weinberg’s above presentation of the standard story leaves room for explication—as he readily admits—we think that it makes eminent sense. All that remains is to carry out these explications, and articulate clearly the way in which the standard story is baked into a sensible interpretation of the geometric formalism of GR.⁴

Our view of the standard story should be uncontroversial for the physics community. Indeed, most contemporary physicists hold that the local validity of SR within GR is expressed by means of an interdependent cluster of *mathematical* concepts, one of which is the existence of normal coordinate systems, to which the above quotes allude. Nonetheless, there are some philosophers who have claimed to find the standard presentation inscrutable and/or confused—see in particular recent works of Fletcher and Weatherall [2022a,b]. While we disagree with this charge, we do believe that there remains valuable conceptual work to be done in (a) clarifying how a network of mathematical concepts is recruited in a particular modelling context in order to account for the local validity of SR within GR, and (b) highlighting the richness and subtlety of this mode of representation, as well as the way in which it interacts with the concept of ‘approximate symmetry’.

The bulk of our efforts in this paper are positive, rather than critical or polemical. In §2, we explain how the standard story regarding the local validity of SR in GR is cashed out in terms of the geometric formalism of GR. In this section, the formal characterization of SR is geometric (in terms of Minkowski spacetime); however, as is well-known, we can also characterize SR in terms of the Poincaré

³The following translation from [Lehmkuhl, 2021, p. 135].

⁴By ‘geometric formalism’ we are not taking a stance in the dynamical/geometric debate (on which see e.g. [Brown and Read, 2021]). In the following, ‘geometry’ is to be thought of in terms of mathematics—it is *prima facie* not any less ‘dynamical’ to talk in terms of a geometrical *formalism*.

symmetries of Minkowski spacetime—thus, one can also pose the effective emergence of SR from GR in terms of the effective emergence of Poincaré symmetry—or, in terms of approximate Poincaré symmetry.⁵ We thus take up the matter of a *symmetry*-based understanding of the ‘effective’ emergence of SR in §3. Since the description of SR with which we are concerned is an ‘effective’ one (i.e., one that is approximately valid in a certain regime, relative to the background theory of GR), one must here confront a difficult question: how should one articulate the relevant notion of ‘approximation’ in ‘approximate symmetry’? In this section, we argue that Harte [2008] has provided a comprehensive and clear solution. We then show that this symmetry-based understanding dovetails nicely with the account of the standard story which we gave in §2. Finally, and as an application of our positive work, we turn our attention to some challenges to the standard story that have been raised by philosophers of physics. In §4, we assess Fletcher and Weatherall’s challenges [2022a, 2022b] as well as Gomes’ responses [2022], in light of our clarifications of the standard story.

2 The local validity of special relativity from normal coordinates

In this section, we first present the standard story regarding the local validity of SR in GR in terms of normal coordinates (§2.1). We then explain how this sense in which GR is locally special relativistic is best understood from an effective field theory (EFT)-like perspective (§2.2).

2.1 Normal coordinates

In the context of classical spacetime theories formulated in terms of Lorentzian manifolds, normal coordinates are coordinate systems which are defined on a normal convex neighbourhood⁶ relative to an embedded submanifold S with vanishing intrinsic curvature such that, on S , they are formally indistinguishable from a Lorentzian frame in Minkowski spacetime (i.e., a frame in which the metric field diagonalises).⁷ For example, Riemann normal coordinates (RNCs) are defined on

⁵There is a precedent for discussing these issues in the recent philosophy literature: see [Read et al., 2018, Fletcher, 2020].

⁶The normal convex neighbourhood $\mathcal{N}(p)$ relative to a point p is the set of points that are uniquely connected to p by geodesics [Poisson et al., 2011, p. 42]. The normal convex neighbourhood relative to *set* of points S is the union of normal convex neighbourhoods $\mathcal{N}(p)$ for each point $p \in S$, i.e. $\mathcal{N}(S) = \cup_{p \in S} \mathcal{N}(p)$.

⁷Strictly speaking, these are ‘orthonormal’ (as opposed to just ‘normal’) coordinates—i.e., the coordinates in which the metric diagonalises (orthonormality) and in which the connection coef-

a normal convex neighbourhood relative to a sub-neighbourhood which is a *point*; Fermi normal coordinates (FNCs) are defined on a normal convex neighbourhood relative to a sub-neighbourhood which is a *line*, and so forth. (We will be more precise about the definitions of these different forms of normal coordinates in what follows.)

In this subsection, we would like to emphasize that RNCs are most helpfully thought of in terms of *invariant* geodesic deviation structure. To this end, we first introduce the ideas of (i) a (manifold-)coordinate-independent tetrad lab frame (§2.1.1), and that of (ii) Synge’s world function, which codifies geodesic deviation structure (§2.1.2). Having done so, we will be in a position to show that (iii) RNCs are nothing but projections of the (coordinate-independent) world function onto the (coordinate-independent) tetrad ‘legs’ (i.e., the four vector fields which comprise the tetrad) (§2.1.3). We end by commenting on some other normal coordinate systems (§2.1.4).

2.1.1 A coordinate-independent lab frame

Consider the point x' —called a ‘base point’ in what follows—with normal convex neighbourhood $\mathcal{N}(x')$ and a certain set of coordinates $\{z^\mu\}$ defined on that normal neighbourhood.⁸ Relative to the base point x' , define an orthonormal ‘lab’ frame e_I^μ such that⁹

$$g_{\mu\nu}e_I^\mu e_J^\nu(x') = \eta_{IJ}(x'), \quad (1)$$

where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Here and in what follows, we follow the convention whereby the coordinate values of geometric objects at the base point are denoted by primed indices—so, (1) can equivalently be written as

$$g_{\mu'\nu'}e_I^{\mu'} e_J^{\nu'} = \eta_{IJ}. \quad (2)$$

The so-called dual tetrads are defined as $e^I_{\mu'} := \eta^{IJ}g_{\mu'\nu'}e^{\nu'}_J$; thereby, $g_{\mu'\nu'} = \eta_{IJ}e^I_{\mu'}e^J_{\nu'}$.

As the lab frame is not specific to a special set of coordinates $\{z^\mu\}$, we can talk of a (manifold-)coordinate-independent tetrad $e_I^{a'}$ where a' is an abstract index.

ficients of the associated compatible derivative operator vanish (normality). In the course of this article we will generally mean ‘orthonormal normal’ by ‘normal’ coordinates. See [Knox, 2013] for philosophical discussion of this distinction, and [Iliev, 2006] for more technical background.

⁸Our presentation here largely follows that of Poisson et al. [2011].

⁹In this article, we use Greek indices μ, ν, \dots for coordinate indices on the manifold, capital Latin letters I, J, \dots for coordinate indices on the tangent space, and lowercase Latin indices a, b, \dots for abstract indices on the manifold. Note also that we follow the convention whereby the base point with respect to which a given normal coordinate system is defined is denoted with a prime.

The existence of such a frame is special insofar as it is adapted to the canonical tangent space of GR.¹⁰

At this stage, as we are apparently dealing with a lab frame defined at a ‘point’, it is worth contemplating what is meant by a ‘point’ in GR to begin with. Notably, a manifold point $p \in M$ does not represent an extensionless event—rather, it can be understood as a mathematical notion which we bring to bear upon some particular modelling context. The context which will concern us in this article is that in which the characteristic length scale of some object (a black hole, a lump of matter, etc.) is sufficiently small relative to some other relevant background length scale that the object is well-modelled by a (mathematical) point. Thus understood, the point is rather analogous to point particles.

2.1.2 The world function

For any $x \in \mathcal{N}(x')$ there is a unique geodesic linking x to x' ; denote the geodesic by $z^\mu(\lambda)$ with λ the affine parameter ranging from λ_0 to λ_1 such that $z(\lambda_0) = x'$ and $z(\lambda_1) = x$. Now, the world function relative to the base point x' and its neighbouring point x is defined as

$$\sigma(x, x') = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} g_{\mu\nu}(z(\lambda)) t^\mu t^\nu d\lambda, \quad (3)$$

with $t^\mu := \frac{dz^\mu}{d\lambda}$ tangent to the geodesic. Note that the world function evaluates to $1/2\varepsilon(\lambda_1 - \lambda_0)^2$, where $\varepsilon := g_{\mu\nu}t^\mu t^\nu = \text{const}$ across the geodesic, with $\varepsilon = \pm 1$ in the spacelike/timelike case, and 0 for the null case.¹¹ Again, just like the tetrad frame, the world function is a coordinate-independent expression relative to the manifold.

In particular, $\sigma^{a'}(x, x') := \nabla^{a'}\sigma(x, x')$ gives the tangent vector of the unique geodesic between x' and x at x' ;¹²

the associated function $\sigma_{x'}^{a'} : \mathcal{N}(x') \subset M \rightarrow T_{x'}M, x \mapsto \sigma^{a'}(x, x')$ is the inverse to the familiar exponential function (defined here as $U \subset T_{x'}M \rightarrow \mathcal{N}(x') \subset M, v \mapsto \gamma_v(1)$ where v is a tangent vector at x' with parameterization λ , and $\gamma_v(1)$ the unique point after parameter distance 1).

The world function can be said to encode geodesic structure in various ways. Most importantly:

¹⁰It has been questioned whether the right tangent space for specific GR spacetimes is always the four-dimensional tangent space isomorphic to \mathbb{R}^n [Chamseddine and Mukhanov, 2010]. However, these considerations in fact go beyond GR; one generalised theory allowing for other tangent spaces is Einstein-Cartan theory (see, for instance, [Trautman, 2006]).

¹¹Further properties of the world function are described by Poisson et al. [2011].

¹²Moreover, in preparation for the following, define $\sigma^a(x, x') := \nabla^a\sigma(x, x')$; as well as $\sigma_{ab} := \nabla_{ab}\sigma(x, x')$, $\sigma_{a'b} = \nabla_{a'b}\sigma(x, x')$, $\sigma_{ab} = \nabla_{ab}\sigma(x, x')$, and $\sigma_{a'b'} = \nabla_{a'b'}\sigma(x, x')$ where all expressions are invariant under swapping of indices then (see [Poisson et al., 2011, §3.2]).

- The world function $\sigma(x, x')$ is tantamount to a Lagrangian. In combination with Hamilton's principle, it encodes the geodesic equations [Ballmann, 2015]. Alternatively, the world function can be seen to encode information regarding geodesic flow: upon Legendre transformation, the resulting Hamiltonian is the generator of geodesic flow [Cardin and Marigonda, 2004].
- The derivative of the world function at base point, x' seen as a function $\sigma_{x'}^{a'}(x)$ of x , encodes geodesic deviation structure, in the sense that it is associated to Jacobi vector fields.¹³

2.1.3 Riemann normal coordinates

Relative to the fixed base point x' and the tetrad $e_I^{a'}$, the world function can be used to assign coordinates to a neighbouring point x' of form

$$\hat{x}^I = e_I^{a'} \sigma^{a'}(x, x'). \quad (5)$$

Notably, the RNCs are a down-projection of the (coordinate-independent) bitensorial object $\sigma^{a'}(x, x')$ by means of the duals to the tetrad vectors $e_I^{a'}$ at the base point x' . In particular, one can easily switch back and forth between the coordinate-dependent and coordinate-independent viewpoint relative to the base point x' . This demonstrates clearly that RNCs have directly underlying coordinate-independent geometric structure: this coordinate-independent geometric structure is specified in terms of the derivative of the world function at base point x' (see §2.1.2).

One often sees RNCs constructed at the base point x' via (i) the exponential map $\exp_{x'} : T_{x'}M \rightarrow \mathcal{N}(x') \subset M$, $v \mapsto \gamma_v(1)$ where γ_v is the unique geodesic for which $\gamma_v(0) = x'$ and for which $\gamma'_v(0) = v$ —with uniqueness guaranteed from requiring $\mathcal{N}(x')$ to be a normal convex neighbourhood—, and (ii) the canonical

¹³Consider a 1-parameter family of geodesics as given by $s : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow T_{x'}M$, $(t, u) \mapsto tX + utY$ (where t is the 'time' parameter, u is the 'deviation' parameter and $(-\varepsilon, \varepsilon)$ denotes the range of the deviation parameter) composed with the exponential map $\exp_{x'} : U \subset T_{x'}M \rightarrow M$ so that $H = \exp_{x'} \circ s : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow M$, $(t, u) \mapsto \exp(tX + utY)$ and $H(t, u = \text{const})$ is a geodesic for any sufficiently small $|u|$. Notably, the variation vector $V(t) := \left. \frac{\partial}{\partial u} H \right|_{u=0}$ relative to the geodesic $H(t, 0)$ then fulfils the Jacobi equation

$$\frac{d^2}{dt^2} V(t) = R \left(\frac{dH(t, 0)}{dt}, V(t) \right) \frac{dH(t, 0)}{dt} \quad (4)$$

with Ricci tensor R and the initial conditions $V(0) = 0$ and $\left. \frac{dV}{dt} \right|_{t=0} = Y$. Now, as the differential of the exponential map $d(\exp_{x'})(tX) : T_{tX}(T_{x'}M) \cong T_{x'}M \rightarrow T_{\exp_{x'}(tX)}M$, $tY \mapsto V(t)$ is associated directly to a Jacobi field. (See, for instance, [Wilkins, 2005] for further details.) and thus geodesic deviation structure, so is then the differential of its inverse, $d\sigma_{x'}^{a'}(x) = d\sigma_{x'}^{a'}(x(t, u))$ (after all, the differential of the inverse of a bijective function f is just the inverse of the differential df).

isometric isomorphism E between $(T_x M, g_x)$ and (\mathbb{R}^n, η) , so that $\phi = E \circ \exp_x^{-1} : M \supset \mathcal{N}(x') \rightarrow \mathbb{R}^n$ assigns to a point $x \in \mathcal{N}(x')$ the normal coordinates $(\hat{x}^1, \dots, \hat{x}^n)$. But this construction is in fact just the one above: the inverse to \exp is generated by the world function as $\sigma_x^{a'} = \exp^{-1} : x \mapsto \sigma^{a'}(x)$ (as explicated in §2.1.2); E is given by (e^1, \dots, e^n) with $e^I : T_x M \rightarrow \mathbb{R}, v^{a'} \mapsto e_{a'}^I v^{a'}$. Thus, $\phi(x) = (e_{a'}^1 \sigma^{a'}, \dots, e_{a'}^n \sigma^{a'}) = (\hat{x}^1, \dots, \hat{x}^n)$.

Within GR, of paramount importance is that RNCs provide an asymptotic expansion of the metric:

$$g_{IJ} = \eta_{IJ} - \frac{1}{3} R_{ILJK} \hat{x}^L \hat{x}^K + O(x^3) \quad (6)$$

where $R_{ILJK} := R_{a'b'c'd'} e_I^{a'} e_b^{b'} e_J^{c'} e_K^{d'}$. Considering that $[R_{ILJK}] = \text{length}^{-2}$ and $[\hat{x}^I] = \text{length}$, one obtains upon restoring units that

$$g_{IJ} = \eta_{IJ} - \delta^2 \frac{1}{3} R_{ILJK} \hat{x}^L \hat{x}^K + O(x^3) \quad (7)$$

with $\delta := \frac{l_{\text{probing}}}{l_{\text{curvature}}}$, i.e. the ratio of characteristic lengths for probing and curvature.¹⁴

It is worth stressing that expanding ‘around a point’ is not to be understood literally. Based upon what we said about the representational nature of a point in §2.1.1, we should really think of the expansion point as standing for some object of completely negligible extent in the modelling context; the effective physics in the modelling context is to be described in the normal coordinates around it with the relevant metric for instance described by (7). The individual points in the neighbourhood are not to be seen as basic, dimensionless entities but ultimately as (albeit not the only¹⁵) reminders of the resolutive restriction in the modelling context (cf. how the moon may be modelled as a point particle when describing its orbit around the sun at a certain effective scale, with the effective physics being described through an elliptic curve).

2.1.4 Other normal coordinates

There are two different ways in which to generalise the notion of RNCs: first, one can construct arbitrarily more adapted normal coordinates at x' , i.e. for which

¹⁴ This is not to say that this is the only well-behaved asymptotic expansion possible: a liberated sense of Riemann normal coordinates around maximally symmetric spacetimes more generally (and not just Minkowski spacetime) is given by Hari and Kothawala [2020] in their equation (25). As the authors themselves point out, this expansion is not as purely adapted to the inertial structure as the Riemann normal coordinates (the rest frame is ‘non-inertial’).

¹⁵ Clearly, the expansion (7) itself provides clear restrictions on the validity of the effective physical description.

not only $\partial g = 0$ ('1-adapted at x' ') and $\partial^2 g = 0$ ('2-adapted at x' ') so that for all $m \leq n$ $\partial^m g = 0$.¹⁶ Second, one can extend the restrictions from a point x' to a larger sub-neighbourhood. Fermi normal coordinates, for instance, extend the strict conditions of the RNCs for a point to a timelike curve. This gives the Fermi normal coordinates a natural interpretation as the frame of an observer, and that of an inertial observer specifically if the curve is also a geodesic.

For all these different adaptation and construction schemes beyond RNCs, we note that they—just as RNCs themselves—are related to the geodesic structure through second-order contact. The remarkable aspect of RNCs is that they are, however, the immediate 'down-projection' of the world function and thus of the geodesic deviation structure (see again §2.1.3); they are therefore the coordinates that are immediately connected to the geodesic deviation structure.

2.2 An EFT-like aspect

Above, we sketched how the inter-related formal tools of geodesic deviation and normal coordinates are used to spell out the effective emergence of SR within GR in a particular regime (i.e., they provide an asymptotic expansion of the metric). In this subsection, we explain how this way of modeling the 'effective physics of SR' is just an application of general effective field theory (EFT)-like ideas within the differential geometric formalism of GR. Having done so, we will thereby be in a position to understand why such an expansion around Minkowski spacetime is singled out over that around other backgrounds.

Let us preface our explanation by emphasizing that, by 'EFT-like', we do not mean the literal application of the *Lagrangian* EFT formalism (familiar from high energy and condensed matter physics) to the theory of GR (though that can be done—see [Donoghue, 1995]). Rather, by an EFT-like account of the emergence of SR, we mean an account whose conditions, motivations, rough structural features, and explanatory strategy bear a strong analogy to what we typically think of as EFT models of physics.

In order to motivate EFT-like accounts, let us begin by recalling that physics always concerns the description of subsystems that are in some sense isolated from external influences; furthermore, we are not in the business of describing perfectly precise quantities (e.g. real-number valued positions) of these subsystems but only such quantities up to a precision that makes sense relative to the physics of the subsystem and the environment (including the physics of the measuring apparatus). The stability or isolation of these subsystems implies that there is a separation of scales—the scale(s) of the subsystem from the scale(s) of other physical phenom-

¹⁶The adaptation scheme for this is nicely explained by Bryant [2015].

ena from which the subsystem is decoupled.

All this is just part and parcel of the everyday practice of physics. EFT-like models of physics try to go further by explicitly representing this separation of scales as part of the model, and using it to define a small expansion parameter δ (often called the ‘power counting parameter’) such that quantities can be (formally) calculated to some order n in the expansion parameter δ , where each higher-order term represents a correction to the previous term and thus a more precise description. This is of course not to say that physical (as opposed to formally-defined) quantities have a precision that can be improved arbitrarily, but rather that explicitly parametrizing the precision by the order of δ gives us a powerful form of idealization and thereby a means of describing quantities up to some physically meaningful, finite, precision.

In order to illustrate these features, consider a very simple textbook example: the multipole expansion in electrostatics. In this scenario, there are two scales in play: (i) the typical spacing a between the charges, which plays the role of the short-distance scale; and (ii) the distance R between the localized charge distribution and some distant observer O . If the scales are widely separated, *viz.*, $R \gg a$, then $\delta := a/R$ plays the role of the small expansion parameter. The electrostatic potential $V(r)$ can then be expanded in terms of δ by means of the familiar multipole expansion

$$V(r) = \frac{1}{r} \sum_{l,m} c_{lm} \delta^l Y_{lm}(\Omega), \quad (8)$$

where c_{lm} are dimensionless coefficients and the $Y_{lm}(\Omega)$ are spherical harmonics. The powerful form of idealization referred to above arises when we formally take the limit in which the scales are ‘infinitely’ separated, *i.e.* $\delta \rightarrow 0$, in which case $V(r)$ is just given by the term for a point charge, or monopole. In more contemporary EFT-parlance, the monopole plays the role of an ‘attractive fixed point’ for the space of models parametrized by (8), since all these models flow to the monopole as $\delta \rightarrow 0$.

Similarly, in the practice of GR, we also come across EFT-like thinking in terms of an expansion parameter $\delta = \frac{l_{\text{probing}}}{l_{\text{curvature}}}$ all the time—for example:

Approximate Killing vectors in curved spacetime: Jacobson [1995] achieves his famous reformulation of the field equations as thermodynamic equations of state by assuming an approximate Killing observer as well as an approximate Rindler observer. The feasibility of the assumption of such observers is controlled by δ . That this control is not merely superficial cosmetics of rigour becomes clear in a follow-up work by Jacobson [2012], in which he ponders on the validity of such an assumption, allowing him to argue that

only GR but not higher-order corrections can be taken to have a meaningful thermodynamic reinterpretation in the spirit of the 1995 article.¹⁷

Geometrical optics-limit in curved spacetime The high-frequency limit for electromagnetic waves (or waves more generally) in curved spacetime involves the idealisation of letting δ go to zero [Misner et al., 1973, §22.5].¹⁸

We should like to emphasize that yet another instance of such EFT-like reasoning is the asymptotic expansion of the metric around Minkowski in GR with δ as defined in the previous section through (7). Here, it is the Minkowski metric term η_{IJ} which plays the role of an attractive fixed point, in the sense that the models parametrized by (7) flow towards this term as $\delta \rightarrow 0$ (furthermore, it is of course true that Minkowski spacetime would be self-similar under such a flow). We note that the value of δ , and therefore the size of the neighbourhood both relative to the probing scale and relative to the background curvature scale, genuinely controls the approximation; by contrast, this is not a property possessed by the analysis of Fletcher and Weatherall [2022a], which simply works via a fiducial Minkowski metric, the physical significance of which is not specified (more on this in §4.1).

So, the geodesic structure of GR bakes in an EFT-like attractor fixed point in the theory. This also clarifies the sense in which Minkowski spacetime is special: of course, some other non-Minkowskian spacetime can approximate any spacetime in some regime (as Fletcher and Weatherall [2022a] recognise). However, for the asymptotic expansion in terms of ‘pure’ geodesic deviation structure in the sense of (7),¹⁹ only the approximation to Minkowski spacetime becomes ever more accurate for ever smaller δ .

3 The local validity of special relativity from approximate Killing fields

RNCs and the associated coordinate-independent geodesic deviation structure capture completely what is meant by effective special relativistic geometry and/or behaviour. But just as one often characterises special relativistic geometry and/or

¹⁷What is common to many instances of working with approximate Killing symmetries in the literature is that they rarely specify an exact notion, therefore leaving it open to the (unacquainted) reader to assess whether there is really a precise and preferred(!) sense in which the notion can be fleshed out. Such a sense is provided by the account of Harte [2008], which we will present in §3.1.

¹⁸See for instance [Linnemann and Read, 2021] for discussion of the geometrical optical limit in this spirit.

¹⁹As opposed to the the expansions by Hari and Kothawala [2020] mentioned in footnote 14.

behaviour in terms of symmetry structure (the Killing vectors of Minkowski spacetime), one might also want to account for the effective special relativistic geometry and/or behaviour in terms of approximate Killing vectors.²⁰

For instance, in their seminal book, Hawking and Ellis [1973] make use of an approximate Killing vector setup in RNCs in order to argue for the existence of locally approximately conserved currents in curved spacetime even in the absence of global Killing symmetries:

If the metric is not flat there will not, in general, be any Killing vectors and so the [...] integral conservation laws will not hold. However, in a suitable neighbourhood of a point q one may introduce normal coordinates $\{x^\alpha\}$. Then at q the components g_{ab} of the metric are $e_a\delta_{ab}$ (no summation), and the components Γ_{bc}^a of the connection are zero. One may take a neighbourhood \mathcal{D} of q in which the g_{ab} and Γ_{bc}^a differ from their values at q by an arbitrarily small amount; then the [covariant derivatives of flat spacetime Killing vectors] $L_{\alpha}^{(a;b)}$ and $M_{\alpha\beta}^{(a;b)}$ will not exactly vanish in \mathcal{D} , but will in this neighbourhood differ from zero by arbitrarily small amount. Thus

$$\int_{\partial\mathcal{D}} P_{\alpha}^b d\sigma_b \quad \text{and} \quad \int_{\partial\mathcal{D}} P_{\alpha\beta}^b d\sigma_b$$

[where $P^a = T^{ab}K_b$, and K_b stands for the flat spacetime Killing vector] will still be zero in the first approximation; that is to say, one still has approximate conservation energy, momentum and angular momentum in a small region of space-time. [Hawking and Ellis, 1973, pp. 62-63]

In order to demonstrate that the approximate symmetry account from the RNC perspective has a clear coordinate-independent counterpart, slightly more work needs to be put in than was necessary for showing that the RNC perspective corresponds to coordinate-independent geometric structure. Fortunately, this work has recently been undertaken by Harte [2008], whose results we present and contextualise over the course of this section. In §3.1, we introduce Harte’s constructions; in §3.2, we demonstrate how to think about local approximate symmetries using this machinery; in §3.3, we once again highlight the EFT-like aspect of these considerations.

3.1 Harte’s construction

In this subsection, we will see via the work of Harte [2008] how an approximate Killing field is immediately associated to geodesic structure and the constraint that at a base point x' one has Killing symmetry:²¹

$$\mathcal{L}_{\Psi_K^{a'}} g_{b'c'} = 0. \tag{9}$$

²⁰That the effective geometry of Minkowski spacetime is associated directly to an effective dynamics as in Minkowski spacetime renders superfluous the attempt by Fletcher and Weatherall [2022b] to look for senses of local special relativistic nature of GR that are decidedly ‘dynamical’.

²¹We will see in the course of the section why such a constraint can be imposed at any point.

In fact, following Harte [2008], we first introduce the more general notion of an approximate ‘affine colineation field’ ψ^d that is immediately associated to geodesic structure and an affine colineation constraint at the base point x' :²²

$$\nabla_{a'} \mathcal{L}_{\psi^d} g_{b'c'} = 0. \quad (10)$$

Notably, an approximate Killing field is automatically an approximate affine colineation. As we will learn in a second step, these approximate affine colineation fields (and thus in particular approximate Killing fields) are geodesic deviation fields themselves (also known as Jacobi fields).

The basic strategy of Harte is to use the exponential map and its inverse to induce uniquely the values for ψ^d at $x \neq x'$ from deformations of $\psi^{d'}$ on the tangent space of x' . Importantly, there are two contributions to a deviation field of this kind: first, linear deformations of vectors X in the tangent space of x' correspond to—via the exponential map relative to x' —deformations of x' to a point x by a path segment δx on the manifold. The path segment δx in turn corresponds to—via the inverse of the exponential map relative to point x —a (deviation) vector at that point x . This is the first contribution to ψ^d . And second, changes of the base point x' itself that do not affect x can be taken to induce a transformation at the point x —and thus also a second contribution to ψ^d —if the change of x' is re-interpreted as an active transformation which leaves x' —after all the center of reference—the same but changes x .²³ Bringing both forms of deviations together, the full set of transformations is

$$\psi^a = H^a_{a'} \left(\sigma^{a'}_{b'} A^{b'} - \sigma_{b'} B^{b'a'} \right). \quad (11)$$

with $\psi^{a'}(x', x') = A^{a'}$ and $\nabla_{b'} \psi_{a'} = B_{b'a'}$. (In Minkowski spacetime specifically, $\psi^\alpha = A^\alpha + (x - x')^\beta B_\beta^\alpha$.)

As mentioned at the beginning of this section, *approximate* affine colineation fields ψ^a are equivalently characterised as Jacobi fields; these have initial conditions $\psi^{a'}(x', x') = A^{a'}$ and $B^{a'b'} = \nabla^{a'} \psi^{b'}(\gamma, \gamma)$, i.e. ψ^d fulfils the Jacobi equation (also known as the geodesic deviation equation)

$$\sigma^b \sigma^c (\nabla_b \nabla_c \psi_a - R_{abc}{}^d \psi_d) = 0. \quad (12)$$

Here, the corresponding Jacobi propagators $H^a_{a'} \sigma^{a'}_{b'}$ and $H^a_{a'} \sigma_{b'}$ are exactly the deformations as derived in the construction via the exponential map construction. (Exact affine colineations fulfil $\nabla_a \mathcal{L} g_{ab} = 0$ —or rather the equivalent statement $\nabla_b \nabla_c \psi_a - R_{abc}{}^d \psi_d = 0$ —and are thus a *special* subset of Jacobi fields.)

²²The existence of an affine colineation field at a point can be read as the statement that the Levi-Civita connection is conserved at that point. Clearly, any Killing field at point is also an affine colineation at that point.

²³We provide the details for the two forms of deviations in Appendix A.

In sum, then, we obtain reassurance that the notion of approximate affine colineation *qua* Jacobi field is a generalisation of exact affine colineations in terms of geodesic deviation and geodesic deviation structure alone. Note that the special case of picking A and B with B antisymmetric as initial values for the geodesic deviation equation gives a geodesic deviation structure that *only* generalises Killing symmetries (such that $\mathcal{L}_\psi g_{a'b'} = 0$), called *approximate* Killing fields.

3.2 Normal coordinates and approximate Killing symmetries

In this section, we investigate the extent to which normal coordinates stand in a special relation to approximate Killing vector fields. *Prima facie*, one would expect the flow of tetrads along the integral curves of the approximate Killing vector fields to be in continuity with the flow of tetrads along the integral curves of Killing vector fields.

Consider first the behaviour of tetrads under exact Killing symmetries ξ . Chinea [1988] derives that $\mathcal{L}_\xi e^I = \chi_\xi^I{}_J e^J$ with χ_ξ an element of the Lie algebra $\mathfrak{so}(3, 1)$. What is the situation for approximate Killing symmetries? If Killing symmetries rotate/boost/translate tetrads, then approximate Killing symmetries should rotate/boost/translate tetrads at least to zeroth order. In fact, and in analogy to Harte's approximate of the Lie derivative of the metric away from x' [Harte, 2008, eq. 32], one obtains for the Lie derivative on the tetrad

$$\begin{aligned} \mathcal{L}_\psi(e_a^I) &= \sigma_a^{a'} \mathcal{L}_\psi e_{a'}^I + \sigma_a^{a'} X^{c'} [\nabla_c \mathcal{L}_\psi e_a^I]_{a,c \rightarrow a',c'} \\ &\quad + \frac{1}{2} \sigma_a^{a'} X^{d'} X^{c'} [\nabla_d \nabla_c \mathcal{L}_\psi e_a^I]_{a,c,d \rightarrow a',c',d'} + \dots, \end{aligned} \quad (13)$$

where an index limit of form $i \rightarrow i'$ denotes a point limit $x \rightarrow x'$ relative that tensor index. The zeroth order term is simply evaluated to $\chi_\psi^I{}_J e_{a'}^J$ given that ψ is exactly Killing at x' . For further simplification, one uses that ψ is an affine colineation at x' , i.e., connection and Lie derivative commute. The first order term then becomes $[\nabla_c \mathcal{L}_\psi e_a^I]_{a,c \rightarrow a',c'} = \mathcal{L}_\psi \nabla_{c'} e_{a'}^I$. Upon restoring units as in §2.1, we obtain:

$$\begin{aligned} \mathcal{L}_\psi(e_a^I) &= \sigma_a^{a'} \mathcal{L}_\psi e_{a'}^I + \delta \cdot \sigma_a^{a'} X^{c'} \mathcal{L}_\psi \nabla_{c'} e_{a'}^I \\ &\quad + \frac{\delta^2}{2} \sigma_a^{a'} X^{d'} X^{c'} [\nabla_d \nabla_c \mathcal{L}_\psi e_a^I]_{a,c,d \rightarrow a',c',d'} + \dots, \end{aligned} \quad (14)$$

where $\delta = \frac{l_{\text{probing}}}{l_{\text{curvature}}}$ as before.²⁴

²⁴The Lie derivative acts on the tetrad conceived as a set of four vectors on spacetime—not conceived of as a set of four vectors in the internal lab frame. With respect to the latter, i.e., the lab

3.3 The EFT-like aspect again

We can illustrate the EFT-like aspect worked out before also upon the reconceptualisation of special relativistic physics in terms of its characteristic symmetry group—the Poincaré group—and its corresponding Killing symmetries: just as the asymptotic expansion in RNCs renders only the Minkowski metric self-similar (see §2.2), the asymptotic expansion of the Lie derivative on tetrads renders only the exact Killing symmetries as having a self-similar effect.

Now, do all approximate Killing vectors of spacetime effectively ‘align’ with Harte’s approximate symmetries upon some limit? In four dimensions they clearly do: four dimensional spacetime only allows for ten different approximate Killing vectors. These possibilities are exhausted by the full (approximate) symmetry groups dS_4 , AdS_4 , and Poincaré. Upon taking a group contraction tantamount to zooming out—letting the background curvature scale $R \rightarrow \infty$ while keeping the probing scale fixed, so $\delta \rightarrow 0$)—the only possibility remaining is the full symmetry group of Minkowski spacetime.²⁵ Notably, the point generalises for all spacetime dimensions $n \geq 2$. Ultimately, this means that with respect to group contraction, any set of approximate Killing vectors is an approximation of the Poincaré Killing vectors—explicitly, the reasoning proceeds as follows:

1. Any full set of approximate Killing vectors with control parameter δ is an approximation of a full set of exact Killing vectors (with exactness at the base point).
2. In $d = 4$, there are just three sets of exact Killing vectors: AdS, dS and Poincaré Killing vectors.
3. dS and AdS symmetry groups are approximations with control parameter δ to the Poincaré symmetry group (group contraction with respect to $\delta \rightarrow 0$ results in the Poincaré symmetry group).
4. Therefore, any full set of approximate Killing vectors with control parameter δ is ultimately an approximation of the Poincaré Killing vectors (with exactness at the base point).

frame index I alone, it is clear that Lorentz transformations relate one tetrad frame to another. (That transformations between lab frames are exactly the Lorentz transformations can be seen by considering that (1) the transformation from tetrad e_a^μ to \tilde{e}_a^μ is given by $\tilde{e}_a^\mu \tilde{e}_\mu^b$ —a matrix which fulfils ‘ $\Lambda\Lambda = \eta$ ’ and thus counts as Lorentz transformation—; and (2) that a Lorentz transformation acting on a tetrad index I results in another tetrad, i.e. is a transformation between tetrads.)

²⁵See Ayala and Haase [2002] and [Enayati et al., 2022, section 10] for accessible accounts.

4 Responses to some other views

Read et al. [2018] articulate a version of what we have called above the standard story regarding the effective emergence of SR within GR—namely that, in GR, (i) connection coefficients vanish at a point in normal coordinates but generically not in the neighbourhood of that point; (ii) the curvature tensor, if non-zero, will vanish neither at a point nor in the neighbourhood of that point; (iii) SR is valid in GR insofar as, in a neighbourhood of a point, those connection coefficients and curvature terms are negligible—where, of course, what counts as ‘negligible’ will depend upon the experimental apparatuses available, as well as on the pragmatic concerns of practitioners. It is this line of thinking which we hope to have bolstered in this article, in part by making recourse to the EFT paradigm.

Although our exposition has been largely positive, we can see now that it affords the resources with which to respond to critics of the standard story such as Fletcher and Weatherall [2022a,b], who have complained (*inter alia*) that the renderings of Read et al. [2018] are insufficiently mathematically precise.²⁶ To the contrary, in the foregoing sections of this article we have provided a precise account of what should be meant by ‘local special relativistic’ in GR which is compatible with the claims of Read et al. [2018]. For maximal clarity, we now reply explicitly to the major charges levied against the standard story by Fletcher and Weatherall [2022a,b] (§4.1); we also make some comments on the recent response to those authors offered by Gomes [2022] (§4.2).

4.1 On Fletcher and Weatherall

Fletcher and Weatherall [2022a] focus in their first article on geometric aspects of the recovery of SR, and discuss two interpretations of what the ‘local validity of SR’ could mean which are of particular importance:²⁷

Tangent space interpretation: “The tangent space at a point of spacetime is, or is somehow equivalent to, Minkowski spacetime.” [Fletcher and Weatherall, 2022a, p. 7]

and

²⁶See also Fletcher [2020] and Weatherall [2020].

²⁷What Fletcher and Weatherall [2022a] dub the ‘Literal interpretation’ (every spacetime is locally isometric to Minkowski spacetime) and ‘Coordinate-chart interpretation, first pass’ (according to which there are coordinates in which curvature vanishes) are straightforwardly false explications of the local validity of SR (as Fletcher and Weatherall [2022a] go on to point out themselves). Note in this context that Fletcher and Weatherall [2022a] attribute the coordinate-chart interpretation (first pass) to Pauli by identifying ‘gravitational fields’ in his writings with ‘curvature’—but what was meant in the historical context by ‘gravitational fields’ were the *Christoffel symbols*, which indeed do vanish in normal coordinates on the point (Riemann) or on the curve (Fermi) [Lehmkuhl, 2021].

Normal coordinate interpretation:²⁸ “At any point of any relativistic spacetime (or along certain curves), local coordinates may be chosen so that, at that point (or along that curve), (a) the components of the metric agree with the Minkowski metric in standard coordinates and (b) all Christoffel symbols vanish.” [Fletcher and Weatherall, 2022a, p. 10]

Our line defended in this article has been that both the tangent space interpretation and the normal coordinate interpretation are integral parts of cashing out what one means by the local validity of SR in GR:²⁹ (i) the tangent space at a point of spacetime is equivalent to Minkowski spacetime in the sense of acting as the ‘fixed point’ around which a preferred small parameter expansion takes place, and (ii) the normal coordinate interpretation fleshes out concretely that preferred small parameter expansion. By appreciating furthermore that (iii) the expansion is to be understood from an effective field theory point of view, one indeed obtains all-together a precise and preferred sense in which GR is locally special relativistic.

In any case, Fletcher and Weatherall’s most charitable take on the standard story is as either claiming that (A) the existence of normal coordinates expresses certain facts about local geometry, or that (B) there is an interpretive claim about idealized measurement apparatuses of natural motions that somehow gets bundled up with normal coordinates. But they maintain quite emphatically that these claims are unsatisfactory for making sense of the local validity of SR.³⁰ With regard to (A), Fletcher and Weatherall fear that it is unclear how the ‘form’ of such coordinates can tell us something special about the metric and covariant derivative given that there are also always coordinates in which the Christoffel symbols do not vanish.³¹

²⁸ ‘Coordinate Chart Interpretation, second pass’, in the words of Fletcher and Weatherall [2022a].

²⁹ Again it is worth stressing that unlike Fletcher and Weatherall [2022a,b] we do not think it is *de facto* justified to keep separate the notions of local special relativistic *geometry* and local special relativistic dynamical *behaviour*: the effective understanding of local special relativistic geometry (as explicated in terms of geodesic deviation structure and approximate Killing symmetries) is understood directly with respect to how matter moves, both kinematically as well as at the level of the equations of motion.

³⁰ We agree with Fletcher and Weatherall [2022a] here to the extent that if one de-contextualises the understanding of RNCs and normal coordinates more generally, surely they are not sufficient for a convincing sense of special relativistic spacetime. But this is less a flaw of the standard story but of their reconstruction.

³¹ They also find it unclear whether the claim is really meant to be about the existence of normal coordinates, or about the constructibility of a specific subclass of normal coordinates. We have addressed that issue in §2.1.4 by making clear that while there are many forms of coordinates that are ‘normal’, the really important aspect of normal coordinates is the fact that they are adapted to geodesic deviation structure. So the question is about the *existence* of normal coordinates (in virtue of the fact that they are proxies for geodesic deviation structure); the way in which we *construct* such coordinates does not *per se* matter. Rather (to repeat), what matters is that there exist normal coordinate systems such as RNCs which are more naturally adapted to geodesic deviation structure than others by virtue of their construction.

What they fail to appreciate, however, is that RNCs correspond directly to the coordinate-invariant geometric structure of geodesic deviation (see §2.1.2)—other coordinate systems in general lack such an intimate connection to geometric structure.³² With regard to (B), their point is that any such claim goes “far beyond any facts about curvature, local or otherwise” [Fletcher and Weatherall, 2022a, p. 1], implying that much more would have to be said. But this going ‘far beyond’ is simply to interpret the formalism as it is used in theorising—and that is provided by the EFT understanding, or so we have argued above. Rather conversely, a sense of the local validity of SR cannot simply be expected to be just a local geometric fact: SR is a physical theory which must (partly) be understandable in terms of the local behaviour of apparatuses of some kind. Concretely, the connection between apparatus behaviour and normal coordinates is provided by the fact that the validity of normal coordinates is controlled by the scale size relevant for probing with apparatuses etc. *vis-à-vis* the background curvature scale.

The positive work of Fletcher and Weatherall [2022a] intends to make manifest whatever genuine content exists in the notion of the ‘local validity of SR’ by providing a clear notion/interpretation of ‘locally flat’. For this, they introduce a metric distance function on the set of tensors under recourse to a fiducial Riemannian metric on the spacetime manifold, making the set of tensors thereby pointwise a metric space.³³ This leads to (i) a metric space continuity notion of approximation of form. The latter in turn induces (ii) a notion of approximate symmetry. (Note the clear difference to (i’) an asymptotic series notion of approximation, which is what normal coordinates are about, and (ii’) the notion of approximate Killing symmetry *à la* Harte [2008].) (i) and (ii) are used to launch a ‘non-specialness’ argument concerning their corresponding notion of ‘effective/approximate’ Minkowski spacetime: on this notion of approximate symmetry, a Lorentzian metric is not merely locally approximately the Minkowski metric (as follows from their Theorem 1) but really approximately any other metric (see their Theorem 5). However, the sense of approximation which Fletcher and Weatherall [2022a] use with their distance function is unphysical, and thus irrelevant for an explication of ‘local relativistic nature’ of GR. At the very least, Fletcher and Weatherall [2022a] say nothing regarding the physical significance of this distance function. Their notion of approximation does

³²In *specific* spacetimes (such as spacetimes with rich isometry groups), there can of course be other coordinate systems which are very well-adapted to geometric structure.

³³Concretely, for a neighbourhood $U \subset M$ with compact closure, choose any Riemannian metric h_{ab} . Relative to h_{ab} , define a norm on covariant tensors $f_{a_1 \dots a_n}$ at a point by $|f|_h = |h^{a_1 b_1} \dots h^{a_n b_n} f_{a_1 \dots a_n} f_{b_1 \dots b_n}|^{1/2}$; then define distance between $f := f_{a_1 \dots a_n}$ and $f' := f'_{a_1 \dots a_n}$ as

$$d_U(f, f'; h, k) = \max_{j \in \{0, \dots, k\}} \sup_U |\nabla^j |f - f'|_k| \quad (15)$$

where ∇ is the Levi-Civita derivative with respect to h_{ab} .

not incorporate any meaningful physical scale: there is no handle on the validity of their approximation, as one would normally expect in any physical approximation whatsoever.

The other dividends which Fletcher and Weatherall [2022a] ascribe to their method are those of (a) making clear that many coordinates correspond to the same flat metric, and (b) showing that there can be many flat metrics (i.e., metrics with flat associated Levi-Civita derivative operators) in a neighbourhood. But note with respect to (a) that this insight is actually obvious from the tetrad construction of normal coordinates (§2.1). And on the subject of point (b), recall that Theorem 1 of Fletcher and Weatherall [2022a] assumes (but does not prove) the well-known result associated to the Cartan-Ambrose-Hicks theorem (see e.g. [Cheeger and Ebin, 1975] for a formal statement and proof) that on a simply connected and complete sub-neighbourhood a flat metric is always isometric to the Minkowski metric.³⁴In this sense, even they admit that there is uniqueness after all.

To close, it's also worth pointing out a peculiarity in Fletcher and Weatherall's discussion of (b). To illustrate that there are infinitely many flat metrics in the neighbourhood of a point, Fletcher and Weatherall consider one flat derivative operator $\bar{\nabla}$ (assumed to be metric compatible) in a neighbourhood, and another derivative operator $\bar{\nabla}'$ related to the first by way of a non-vanishing difference tensor (see [Fletcher and Weatherall, 2022a, fn. 17]). But one can show using equations (2) and (16) of [Stein, 2015] that the difference tensor of two isometric compatible connections must vanish! So the second derivative operator considered by Fletcher and Weatherall [2022a] must, in fact, not be compatible with any metric (at least as long as we are restricting to a simply connected and complete neighbourhood where according to the Cartan-Ambrose-Hicks theorem every metric is isometric to Minkowski). This is confirmed by checking that the trace of its Riemann tensor on the first two indices does not vanish, which is a necessary condition for a connection to be metric. (On this front, note also that $\bar{\nabla}'$ is only flat on a given timelike geodesic defined by Fletcher and Weatherall, and so seems insufficient to illustrate their point.)

4.2 On Gomes

In direct response to Fletcher and Weatherall [2022a,b], Gomes [2022] separates three senses of the local validity of SR in GR:

³⁴In fact, the precise corollary of the Cartan-Ambrose-Hicks theorem apparently used is stated in [Cheeger and Ebin, 1975, p. 41] as follows:

Let M^n and \bar{M}^n be complete simply connected manifold with constant curvature K . Then M^n and \bar{M}^n are isometric. In fact, given any $p \in M^n$, $\bar{p} \in \bar{M}^n$ and an isometry $I : M_p^n \rightarrow \bar{M}_{\bar{p}}^n$, there exists an isometry $\Phi : M^n \rightarrow \bar{M}^n$ such that $\Phi(p) = \bar{p}$ and $d\Phi_p = I$.

1. At any $p \in M$, the tangent space is Minkowskian.
2. At any $p \in M$, RNCs exist.
3. In the relevant circumstances, geodesic deviation is negligible.

Here, (1) is clearly the tangent space interpretation discussed above; (2) is clearly a version of the normal coordinate interpretation discussed above. Regarding (3), Gomes is surely correct to highlight this as an omission in the discussion offered by Fletcher and Weatherall. However, to stop here (as Gomes does) is to leave the story incomplete, for—as we have made clear in detail in the foregoing sections of this article—(1)-(3) are not independent notions of the local validity of SR in GR, but are in fact all related intimately to one another: (1) underwrites (2), which itself is a way of expressing (3).

5 Conclusions

In this article, we have argued that there is a clear, natural, and ultimately uncontroversial sense in which GR can be said to be locally special relativistic; all that is required is to think of GR in terms of the EFT framework, rather than as a mere set of Lorentzian geometries.

Our position not only is consistent with what leading practitioners of GR have written on this topic (see e.g. [Weinberg, 1972, Hawking and Ellis, 1973, Poisson and Will, 2014]), but also takes seriously what philosophers of science have learned from numerous case studies of the empirical interpretation of a physical theory: said interpretation depends ultimately on bringing formalism into contact with relevant measurement scales, which are *de facto* classical; for this to happen, objects and/or functional roles played by objects in that theory must be associated with (or at least accounted for in) objects and/or functional roles played by objects in a classical measurement theory, or with objects and/or functional roles played by objects in some intermediate theory.³⁵ Mere association, however, is not enough: one must also account for the regime of validity of the approximation. For example, the role played by a coherent state in a quantum theory should be linked effectively to the role played by a configuration in a classical theory—but only for the regime in which the state actually is coherent.

The selfsame thinking underlies the account of the local validity of SR in GR which we have presented in this article: the local spacetime structure is to be linked effectively to a special relativistic—and ultimately a non-relativistic—structure;

³⁵Recently, similar points have been made by Grimmer [2022] in the context of measurements in QFT.

only if some expansion parameter is used can we thereby get an estimate of how good the measurement is according to GR. From this point of view—and given that GR is empirically relevant—it is not surprising that there exists an asymptotic series of the metric with the Minkowski metric as ‘base point’, or that a general relativistic spacetime has approximately and locally the symmetries of Minkowski spacetime.

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A Harte’s construction of the deviation field

Importantly, there are two contributions to a deviation field ψ^a of the kind presented by Harte [2008]. First: linear deformations of vectors X in the tangent space of x' correspond, via the exponential map relative to x' , to deformations of x' to a point x by a path segment δx on the manifold. The path segment δx in turn corresponds, via the inverse of the exponential map relative to point x , to a (deviation) vector at that point x . This is the first contribution to ψ^a . At a mere technical level then, the goal is to state the deformation at x , denoted by the tangent vector ψ^a , in terms of a tangent vector at x' . To achieve this goal, consider the linear transformation

$$X^{a'} \rightarrow X^{a'}(x, x') + \epsilon \Psi^{a'} \quad (16)$$

on the tangent space $T_{x'}M$ with $\Psi^{a'} := B_{b'}^{a'} X^{b'}$. These transformations are associated directly with a transformation $x \rightarrow x + \delta x$ on the manifold M via the exponential map $x = \exp_{x'} X$. The inverse of the exponential map is given by

$$X_{a'} = -\sigma_{a'}(x, x') \quad (17)$$

in terms of the world function. From Taylor expanding the inverse, we learn that $X^{a'}(x + \delta x, x') = -\sigma_{a'}(x + \delta x, x') = -\sigma_{a'}(x, x') - (\epsilon \psi^b) \sigma_{ba'}(x, x')$. By comparison with (16), it follows immediately that

$$B_{b'}^{a'} X^{b'}(x, x') = \psi^b \sigma_{ba'}(x, x'). \quad (18)$$

Upon defining $H^b_{a'} = \left[-\sigma^{a'}_b(x, x') \right]^{-1}$, we finally get

$$\psi^b = -H^b_{a'} B^{b'a'} \sigma_{b'}, \quad (19)$$

where we replaced X'_b by $-\sigma'_b$ (in virtue of (17)). Note that $B_{b'a'} = \nabla_{b'} \psi_{a'}(x', x')$: $\nabla_{b'} \psi_{a'}(x', x') = \nabla_{b'} \left(-H^{a'}_{c'} \sigma_{a'} B^{a'c'} \right) = B_{b'a'}$ where the last step follows from the facts that $\sigma'_d(x', x') = 0$ and $\nabla_{b'} \sigma'_d = \sigma_{b'd'}$.

For this, regard $X^{a'}(x, x' + \delta x')$ as the result of a ‘passive transformation’ of the description of x relative to x' (characterised by the fact that the reference point x' but not x changes); from Taylor expanding, we learn that $X^{a'}(x, x' + \delta x') = -\sigma_{a'}(x, x') - \sigma_{a'b'} A^{b'}$. One then looks for the corresponding ‘active transformation’ by equating $X^{a'}(x, x' + \delta x')$ to $X^{a'}(x + \delta x, x')$ (characterised by the fact that x but not x' changes). We already know that a transformation relative to a base point x' on a neighbouring x generally has the form $X^{a'}(x + \delta x, x') = -\sigma_{a'}(x + \delta x, x') = -\sigma_{a'}(x, x') - \psi^b \sigma_{ba'}(x, x')$. Using the posited equality between a ‘passive transformation’—the base point *qua* reference system is moved but not the point to be described—and an ‘active transformation’—the point to which we refer via the base point is moved but not the base point—, ψ^a of the latter conception can be determined to be $\psi^a = H^a_{a'} \sigma^{a'}_{b'} A^{b'}$. Note that clearly $A^{a'} = \psi^{a'}(x', x')$ (the ‘prefactor’ to A becomes an identity with the coincidence at x' ; $B_{b'a'}(x', x') = 0$ for translations in any case).

Bringing these results together, the full set of transformations is

$$\psi^a = H^a_{a'} \left(\sigma^{a'}_{b'} A^{b'} - \sigma_{b'} B^{b'a'} \right), \quad (20)$$

with $\psi^{a'}(x', x') = A^{a'}$ and $\nabla_{b'} \psi_{a'} = B_{b'a'}$.

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