Between a Stone and a Hausdorff Space

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Abstract

We consider the duality between General Relativity and the theory of Einstein algebras, in the extended setting where one permits non-Hausdorff manifolds. We show that the duality breaks down, and then go on to discuss a sense in which general relativity, formulated using non-Hausdorff manifolds, exhibits excess structure when compared to Einstein algebras. We discuss how these results bear on a class of algebraically-motivated deflationist views about spacetime ontology. We conclude with a conjecture concerning non-Hausdorff spacetimes with no bifurcating curves.

1 Introduction

In standard textbook treatments of general relativity (GR),[1] possible universes, or regions thereof, are represented by geometric structures: viz. Lorentzian 4-manifolds, or *relativistic spacetimes*, which are smooth four-dimensional manifolds endowed with a smooth metric. But as first observed by Geroch (1972), a different formalism is also available. On this alternative, one would drop all mention of a manifold and instead represent possible (regions of) universes with a certain kind of algebraic structure. These structures, which Geroch called *Einstein algebras*, consist of a commutative ring satisfying certain conditions and endowed with further structure. Indeed, he argued, the entire theory of GR could be developed using just the structure of Einstein algebras.

When Geroch first introduced Einstein algebras, he suggested they might be a step towards a quantum theory of gravity. The idea was that within GR, it is implicitly assumed that events can be treated as "point-like", in the sense that they can be localized in space and time with arbitrary precision. But in a quantum theory, one should not expect arbitrarily precise determinables, including locations.

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[1]Such as Wald (1984). See fn. 6 for further references.

Thus, Geroch reasoned, point-like events, which play an apparently fundamental role in classical GR, were unlikely to persist in a quantum theory of gravity. The Einstein algebra framework, meanwhile, removes, or at least suppresses, the "manifold of point-like events", thereby removing one of the hurdles to constructing a quantum theory.

But philosophers of physics soon became interested in Einstein algebras for a different reason. Not long after Geroch's paper first appeared, John Earman (1977) suggested that Einstein algebras might cast new light on the classic substantivalist/relationalist debate.[2] Roughly speaking, the idea was that standard geometric formulations of spacetime theories are "substantivalist", in the sense that one first posits a structure representing space and time – that is, the manifold – and then one introduces matter by defining fields on that manifold.[3] This approach looks like one according to which spacetime is ontologically independent of, and prior to, matter within spacetime. Einstein algebras, meanwhile, cut out the first step. One can think of the elements of an Einstein algebras as possible global configurations of a (scalar) matter field, and so on this approach, one begins by positing the possible configurations of some form of matter, and then proceeds to develop the theory from there, without ever needing to introduce a manifold. Einstein algebras, in other words, look like a natural formalism for "relationalism".[4] It seems that Earman had hoped that Einstein algebras may lead to a formulation of GR that has less structure than the standard geometric formulation, and this algebraic approach would be a way to excise the excess structure in the geometry.

More recently, Rosenstock et al. (2015) took up Earman's suggestion, but with a twist. They show, using a generalization of the Stone duality theorem, that there is a sense in which the Einstein algebra and the standard geometric formalism of GR are *equivalent*.[5] Roughly, they show that two Einstein algebras are structurally equivalent (i.e. isomorphic) if and only if their corresponding relativistic spacetimes are structurally equivalent (i.e. isometric). This allows us to reconstruct a unique manifold given an algebra of possible matter distributions (in the form of smooth scalar

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[2]For an overview of the recent debate, see Pooley (2013).

[3]Field (1984) famously defends this perspective; see also Friedman (1983).

[4]Earman goes on to argue that Einstein algebras are not fully relationalist, or at least not fully Leibnizian, since that they do not satisfy the principal of sufficient reason (PSR): God has no grounds for choosing between the many distinct-but-isomorphic Einstein algebras (Earman, 1977, 1986). One's views on the PSR aside, this worry seems chimerical to us, on the grounds that the apparent multiplicity of Einstein algebras derives entirely from how we construct mathematical objects in set theory, and not from a multiplicity of the structures themselves (Bradley and Weatherall, 2022; Halvorson and Manchak, 2022).

[5]Their arguments are heavily indebted to prior work by Nestruev (2002). Rynasiewicz (1992) also responded to Earman's work on Einstein algebras by invoking general Stone-type dualities between functions and space.

fields), and vice versa. They go on to suggest that this result supports a deflationary reading of the substantivalist/relationalist debate, at least as understood by Earman and others in the late 20th century. Because of the possibility of unique reconstruction, a manifold can be viewed as (nothing but) a representation of possible configurations of matter, and the points of a manifold are just loci of field (dis)agreement, characterizing the ways in which possible matter configurations might be distinct. Thus, the "substantivalist" formalism of relativistic spacetimes does not posit "more structure" than the "relationalist" formalism of Einstein algebras. The two formalisms are, in this sense, equivalent.

Here we present several results that, while not necessarily undermining this picture, introduce further texture. The principal observation is that the duality between Einstein algebras and relativistic spacetimes on which Rosenstock et al. (2015) base their arguments depends on the topology of the manifolds under consideration. In particular, if one expands the class of relativistic spacetimes to includes ones that violate the Hausdorff condition, the duality no longer holds, because now structurally non-equivalent manifolds can have structurally equivalent algebras of smooth scalar fields. In other words, the Algebra to Geometry direction of the duality fails—we can no longer reconstruct a unique manifold given an algebra of smooth scalar fields. As we go on to show, this breakdown is asymmetric: equivalent manifolds always have equivalent algebras of smooth scalar fields, and thus the Geometry to Algebra direction of the duality continues to hold in this generalized context. We prove several propositions that further probe the character of the breakdown of duality.

We take these results to be of interest for several reasons. Perhaps the most important is that they cast the substantivalist/relationalist debate in GR in new light. As we discuss below, our main results, in Section 3, suggest a strong sense in which the points of (some) non-Hausdorff manifolds must be interpreted as having an ontological status independent of the possible configurations of matter. This suggests that non-Hausdorff manifolds require a "substantivalist" interpretation. Moreover, as we go on to show, in Section 4, there is a precise sense in which not-necessarily-Hausdorff spacetimes have strictly more structure than their algebraic counterparts. In this sense, something like Earman's original vision is realized: in the not-necessarily-Hausdorff context, the geometric formulation of the theory appears to prefer (or require) a substantivalist

interpretation; but one can excise this extra substantivalist structure by moving to an algebraic approach. But despite vindicating Earman's intuitions, the results, and the version of substantivalism at issue here, looks very different from characterizations of substantivalism in the contemporary literature, for reasons we discuss in Section 6.

Moreover, recently, several philosophers of physics have argued that it is fruitful, for some purposes, to drop the Hausdorff condition (Luc and Placek, 2020; Luc, 2020; Manchak, 2020). We take the results here to show one potential cost to adopting this proposal: it blocks the deflationary interpretation of spacetime events (and manifolds) suggested by Rosenstock et al. (2015).

Finally, these results show a sense in which expanding a modal space – that is, the space of possible models allowed by a theory – has consequences for the sensible interpretation of models representing individual possibilities. In other words, if one adopts a principle of uniformity of interpretation, then once we consider non-Hausdorff manifolds, the physical significance of points in Hausdorff models has to change as well.

The remainder of the paper will proceed as follows. We begin in Section 2 with some preliminary remarks to fix ideas and notation, leading to a statement of the duality result that we take as our starting point. We then proceed in Section 3 to state and prove the principal results of the paper, showing that the duality breaks down asymmetrically (Propositions 1 and 2). In Section 4, we revisit the deflationary account in light of these results. This discussion motivates two further results: first, there is a sense in which dropping the non-Hausdorff condition genuinely introduces distinct algebraic possibilities (Proposition 3); and second, that there is a precise sense in which non-Hausdorff manifolds (and, by extension, spacetimes) have more structure than the algebras of smooth functions on them (Proposition 4). Finally, in Section 5, we propose a conjecture that, if true, would lead to a novel hierarchy of "non-Hausdorfness" for manifolds. We conclude in Section 6 by elaborating on some of the themes mentioned above. Proofs of propositions appear in an Appendix.

2 Preliminaries

In standard approaches to GR, one begins with a smooth manifold M, points of which represent events—localized occurrences—in space and time.[6] This manifold is assumed to be equipped with a smooth pseudo-Riemannian metric g_{ab} of Lorentzian signature, encoding spatiotemporal relations between events, including the effects of gravitation. The metric is the principal dynamical field in the theory, and standard treatments generally focus on properties of the metric and, especially, solutions to Einstein's equation, which relates the metric to the distribution of energy-momentum in space and time. For our purposes, however, the metric will play no role. We focus just on the manifold.

The most general definition of a (smooth) manifold is that it is a set M endowed with an *atlas*. An atlas, here, is a collection of $n-$charts (U, φ), where U is a subset of M and φ is an injective map from U to $\mathbb{R}^n$, for fixed n, satisfying certain compatibility and maximality conditions. This atlas may then be seen to induce both a topology and a "smoothness structure" on M, in the sense that it determines which maps from M to other manifolds are smooth.[7]

In standard presentations, however, more is often required. In particular, the atlas on M is assumed to be such that the induced topology has several additional topological properties.[8] Of particular interest for the present paper is that M is standardly assumed to satisfy the *Hausdorff* condition.

Definition 1. *A topological space X is* Hausdorff *if for any points x, y, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Other conditions are also standard. For instance, It is also common to assume that M is *paracompact*.

Definition 2. *A topological space X is* paracompact *if any open cover admits a locally finite refinement.*[9]

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[6] For background on the mathematical structure of GR using conventions similar to ours, see Wald (1984) or Malament (2012); for a treatment directed at mathematicians, see, for instance, O'Neill (1983). For more on smooth manifolds, also directed at mathematicians, see Lee (2013).

[7] See Malament (2012, §1.1) for a compact discussion. Throughout, we work specifically with Malament's definition and freely refer to his notions of $n-$chart, compatibility, maximilaity, etc. in the proofs of our propositions and (occasionally) the main text.

[8] These assumptions are so common that many physicists and mathematical physicists work them into the definition of manifold and then proceed to ignore them. For instance, Wald (1984) and Hawking and Ellis (1973) both assume the Hausdorff and paracompactness conditions; and Malament (2012) assumes the Hausdorff condition and, later, the Countable Cover Condition. Wald's Appendix A provides a nice discussion of some of these conditions; see also Manchak (2020, Appendix). Note that on some presentations, one begins with a topological space satisfying certain conditions and then places compatibility conditions on the atlas. The discussion below goes through *mutatis mutandis* on that approach.

[9] With less jargon: X is paracompact if, for any collection of open sets $\{O_\alpha\}_{\alpha \in A}$ satisfying $\cup_{\alpha \in A} O_\alpha = M$, there

For connected manifolds, in the presence of the Hausdorff condition, paracompactness is equivalent to another condition that is also sometimes assumed (instead), known as *second countability*.[10]

Definition 3. *A topological space X is* second countable *if its topology admits a countable base.*[11]

The condition we relax in Section 3 is the Hausdorff condition. This means that we will consider smooth manifolds in the general sense described above, without also assuming that that the charts are such that the induced manifold topology is Hausdorff. We will call such manifolds "not-necessarily-Hausdorff". It might be instructive to think about the Hausdorff condition by thinking about when it fails. If it fails, we would have points on the manifold that open sets (induced by the atlas) cannot separate. A classic example of a non-Hausdorff space can be constructed from two lines with their standard manifold structures. We identify every point on these two real lines except for the origin. What we are left with is a real line with two origins. These two origins cannot be separated by any two charts.

By default, we assume that not-necessarily-Hausdorff manifolds do satisfy paracompactness and second countability. We state this assumption only to ensure that the claims below are unambiguous and precise. Most of the propositions we state and prove below involve existence claims. The examples we give to prove them generally do satisfy the paracompactness and second-countability conditions. In this sense we do not relax those conditions. Note, however, that Prop. 2, which is a universal claim, does not require *any* of the topological conditions we have stated here. It holds for not-necessarily-Hausdorff manifolds, as stated; but also for not-necessarily-paracompact and not-necessarily-second-countable manifolds.

We now turn to the algebraic side of things. As we noted in the introduction, our motivation for pursuing the questions discussed here is to explore interpretational issues raised by the Einstein algebra reformulation of GR. On that approach, the subject matter of GR is represented by algebras

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exists a collection of open sets $\{V_\beta\}_{\beta \in B}$ such that (a) for every $\beta \in B$, there exists $\alpha \in A$ for which $V_\beta \subseteq O_\alpha$, (b) $\cup_{\beta \in B} V_\beta = M$, and (c) for every point $p \in M$, there exists an open set W containing p that has non-empty intersection with only finitely many elements of $\{V_\beta\}_{\beta \in B}$.

[10]For a clean proof of the equivalence claim, see Tanaka (2014). Note that in some sources, e.g. Malament (2012), one finds the following "countable cover" condition imposed on the atlas associated with a manifold: there exists a countable collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\cup_i U_i = M$. This condition is not a topological condition as stated, but if it holds, then the manifold topology induced by the atlas is second countable. This result holds independently of other topological conditions, including the Hausdorff condition; it follows from the facts that $\mathbb{R}^n$ is second countable and the manifold topology makes chart maps continuous.

[11]That is, if there exists a countable collection of open sets $\{O_i\}_{i \in I}$ such that for any open set U, there exists $J \subseteq I$ for which $\cup_{i \in J} O_i = U$.

satisfying certain further conditions and endowed with further structure that plays the role of a metric and which satisfies dynamical equations.[12] But just as we focused only on manifolds in our discussion above, for present purposes the full mathematics and physics of Einstein algebras will play no role. Only the algebras themselves will be relevant.

To motivate the algebraic approach, consider a smooth manifold M, and let $C^\infty(M)$ denote the collection of smooth maps from M to $\mathbb{R}$.[13] This collection turns out to inherit some nice algebraic properties from the structure of M. In particular, $C^\infty(M)$ forms an Abelian group under pointwise addition, with identity given by the constant function 0; it forms a commutative (associative and unital) ring over that group under pointwise multiplication, with multiplicative identity given by the constant function 1; and it forms a real algebra over that ring via pointwise scalar multiplication by constant functions. Many structures one can define on a manifold M can be re-expressed as structures on $C^\infty(M)$. For instance, a smooth vector field is an $\mathbb{R}$-linear function from $C^\infty(M)$ to itself satisfying the Leibniz rule: $\xi(fg) = f\xi(g) + g\xi(f)$ for all $f, g \in C^\infty(M)$. And so on.

This construction provides the basic idea behind the Einstein algebra approach. One begins with an algebra, which can be represented by $C^\infty(M)$ for some manifold M. This algebra is taken to represent possible configurations of (scalar) matter or properties of matter. Other types of matter fields and possible metrics are represented as tensor fields, which are defined in terms of their action on algebra elements. One can further define structures such as stress-energy tensors, derivative operators, and curvature tensors on the algebra, and state Einstein's equation. In this way, one can drop all reference to the manifold at all.

How are these algebras related to relativistic spacetimes? Consider, first, the case of smooth, Hausdorff, paracompact manifolds. In that case, the relationship between manifolds M and the algebras $C^\infty(M)$ associated to them is very strong. We have the following fundamental result.

Theorem 1 (Generalized Stone). *Given two (smooth, Hausdorff, paracompact) manifolds M and M', M and M' are diffeomorphic if and only if $C^\infty(M)$ and $C^\infty(M')$ are isomorphic.*

Proof. See Rosenstock et al. (2015). ☐

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[12] For more about Einstein algebras and their physical interpretation, see Geroch (1972), Heller (1992), or Rosenstock et al. (2015). Further details about the mathematical structures described here can be found in Nestruev (2002).

[13] For present purposes, the topological considerations above play no role, so we do not assume them.

This Theorem says that if two algebras associated with smooth, connected, Hausdorff, paracompact manifolds are isomorphic, then the manifolds they are associated to are diffeomorphic; and, conversely, if the algebras are *not* isomorphic then the manifolds are not diffeomorphic. This provides a sense in which the entire structure of a smooth manifold is captured by its algebra of smooth functions.

In fact, even more is true. As Nestruev (2002) show and Rosenstock et al. (2015) discuss, one can identify necessary and sufficient conditions on an algebra A for it to be isomorphic to $C^\infty(M)$ for some smooth, connected, Hausdorff, paracompact manifold M. Moreover, given such an algebra one can reconstruct a smooth, connected, Hausdorff, paracompact manifold, unique up to diffeomorphism, that gives rise to that algebra by the construction above. These results show that one can describe the entire subject matter of GR beginning with purely algebraic structures, satisfying certain conditions and endowed with further structure, in such a way that the standard geometric presentation arises as a certain kind of representation of the algebraic structures as algebras of smooth functions on a particular M.

Furthermore, we have a certain precise sense in which GR, formulated in the standard way using Lorentzian manifolds, is theoretically equivalent to a theory that represents spatiotemporal structure using Einstein algebras.[14] As Rosenstock et al. (2015) show, one can establish that the relationship between smooth manifolds and their associated algebras is (contravariantly) functorial in the sense that the relationships between manifolds and algebras lifts to a natural map from diffeomorphisms to algebra isomorphisms, and vice versa; and that the functors determined in the way form a categorical equivalence between a category (actually, groupoid) of smooth, connected, Hausdorff, paracompact manifolds and a certain category of algebras. This equivalence can be extended to spacetimes (i.e., manifolds with metrics) and Einstein algebras, in such a way that it preserves the empirical content of both theories on a natural understanding of what that means.

We now proceed to ask: how much of this picture extends to the case where one considers not-necessarily-Hausdorff manifolds? Is it the case that if not-necessarily Hausdorff manifolds give rise to isomorphic algebras of smooth functions, then they are diffeomorphic? As we will see in the next section, the answer to this question is "no".

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[14]For a recent overview of the theoretical literature, including this particular notion of equivalence, see Weatherall (2019*a,b*). Weatherall (2021) describes some of the limitations of this notion of equivalence.

3 Duality Spoiled

As indicated above, the main results of this section may be summarized as follows. If we allow for not-necessarily-Hausdorff manifolds, then there exist non-diffeomorphic manifolds with isomorphic algebras of smooth maps. In other words, if we consider the duality results discussed above in this new context, we find that the Algebra to Geometry direction breaks down, in the sense that an algebra of smooth functions does not uniquely determine a smooth manifold, even up to diffeomorphism. However, as we show below, the Geometry to Algebra direction still holds, even in the absence of the Hausdorff condition: if two not-necessarily-Hausdorff manifolds are diffeomorphic, then their algebras of smooth functions are isomorphic.

To establish the first result, we will begin with a construction by which one takes an arbitrary Hausdorff manifold, M, and generates a new (non-Hausdorff) manifold, N, by "adding a point". We will then show that M and N are not diffeomorphic. Finally, we show that $C^\infty(M)$ and $C^\infty(N)$ are nonetheless isomorphic. The result that there exist non-diffeomorphic not-necessarily-Hausdorff manifolds with isomorphic algebras of smooth functions follows immediately.

We begin with our basic construction. Let $(M, \mathcal{C})$ be a smooth, not-necessarily-Hausdorff $n-$manifold and let p be an arbitrary point of M. Let N be the set $M \cup \{p'\}$, where $p' \notin M$. First, observe that every $n-$chart in $\mathcal{C}$ is automatically an $n-$chart on N. Now, for each chart $(U, \varphi) \in \mathcal{C}$ containing p in its domain, let U' be the set $U' = \{p'\} \cup U \setminus \{p\}$ and let $\varphi' : U' \to \mathbb{R}^n$ be the (injective) function defined by $\varphi' : q \mapsto \varphi(q)$ if $q \neq p'$ and $q \mapsto \varphi(p)$ if $q = p'$. Then (U', φ') is also an $n-$chart on N; call it the "mirror chart" determined by (U, φ). Finally, let $\mathcal{C}'$ consist of the collection of all $n-$charts on N compatible with both $\mathcal{C}$ and all mirror charts on N. We have the following lemma.

Lemma 1. $(N, \mathcal{C}')$ *as just defined is a smooth, non-Hausdorff manifold. Moreover, it is second countable if* $(M, \mathcal{C})$ *is.*

We will refer to this manifold $(N, \mathcal{C}')$ as "M with an 'additional' p". Observe, now, that if $(M, \mathcal{C})$ is Hausdorff, the manifold that results by adding any "additional" p is not diffeomorphic to the manifold with which we began. To prove this, we first introduce a definition and another lemma.

Definition 4. *Let M be a non-Hausdorff manifold. Call points $p_1, p_2 \in M$ witness points if, given any*

open sets O_1 and O_2 containing p_1 and p_2, respectively, $O_1 \cap O_2$ is non-empty.

Observe that for any manifold M with an "additional" p, the points p and the additional point p' are witness points. We now have the following two results.

Lemma 2. *If p_1 and p_2 are witness points of some smooth, non-Hausdorff manifold M, then for any smooth scalar field $f : M \to \mathbb{R}$, $f(p_1) = f(p_2)$.*

Lemma 3. *Let $(M, \mathcal{C})$ be a smooth, Hausdorff $n-$manifold, let $p \in M$ be some point, and let $(N, \mathcal{C}')$ be M with an "additional" p. Then $(M, \mathcal{C})$ and $(N, \mathcal{C}')$ are not diffeomorphic.*

What we have established thus far is that given any smooth Hausdorff manifold, we can construct a non-diffeomorphic, non-Hausdorff manifold from it by adding an "additional" point to it. Note that in Lemma 3, we assume M to be Hausdorff. We might wonder: does the Lemma hold if M is non-Hausdorff? What we know is that the Lemma fails if M is non-Hausdorff and non-paracompact (think a Hausdorff manifold with p duplicated countably infinite many times. This manifold and it with p duplicated one more time are diffeomorphic). What would happen if M is non-Hausdorff and paracompact? This leads to the following open question, but Proposition 1 holds regardless.

Conjecture 1. *Let $(M, \mathcal{C})$ be a smooth, paracompact, not-necessarily-Hausdorff $n-$manifold, let $p \in M$ be some point, and let $(N, \mathcal{C}')$ be M with an "additional" p. Then $(M, \mathcal{C})$ and $(N, \mathcal{C}')$ are not diffeomorphic.*

The final step in the main result is to establish that any smooth manifold M has an algebra of smooth functions that is isomorphic to that of M with an "additional" point.

Lemma 4. *Let $(M, \mathcal{C})$ be a smooth, not-necessarily-Hausdorff manifold and let $(N, \mathcal{C}')$ be M with some "additional" point p. Then $C^\infty(M) \cong C^\infty(N)$.*

The main result follows immediately from the foregoing lemmas. We have established

Proposition 1. *There exist non-diffeomorphic smooth not-necessarily-Hausdorff manifolds with isomorphic algebras of smooth functions.*

Thus we have a many-to-one relationship, even up to diffeomorphism, between smooth not-necessarily-Hausdorff manifolds and algebras of smooth functions. It follows, as claimed, that

the Algebra to Geometry direction of the duality fails in this context, as it is hopeless to reconstruct a unique not-necessarily-Hausdorff manifold from an algebra of smooth functions.

Now consider the other direction, from Geometry to Algebra. It turns out that this direction of the generalized Stone Theorem does hold, even for not-necessarily Hausdorff manifolds.

Proposition 2. *If not-necessarily-Hausdorff manifolds M and N are diffeomorphic, then $C^\infty(M) \cong C^\infty(N)$.*

Observe that Prop. 2 would still hold even if we relaxed second countability or paracompactness, by an essentially identical proof. It remains an open question, to our knowledge, as to whether the analogues of Prop. 1 would hold if different topological conditions were relaxed.

4   Deflation Revisited, and Two More Results

The results just presented, taken together, have consequences for the interpretation of GR. If we adopt the Hausdorff condition, then the deflationary interpretation of spacetime points described in Section 1 is a viable option for interpreting the points of a manifold representing our actual world: manifolds are (just) ways of encoding the possible configurations of matter; points reflect the ways in which matter configurations can agree or differ. But if we expand to not-necessarily Hausdorff manifolds, this deflationary interpretation is no longer available—at least not for all models. We apparently cannot take spacetime points to be (nothing but) loci for differentiating possible matter configurations.

Instead, it seems that one must adopt an interpretation according to which at least some points (specifically, pairs of witness points) in non-Hausdorff spacetimes represent locations in space and time whose existence cannot be probed, even in principle, via exploring the possible distributions of matter. The differences between those manifolds are not reflected in differences between (any) of the matter configurations. Whatever the differences are, they must be characterized in some other way, such as a brute ontological difference between those points.

As we suggested above, this situation has a distinctly "substantivalist" character. (Some) points, on this interpretation, take on an ontological status independent of, and not exhausted by, spatio-temporal relations between matter properties. It is hard to see how a relationalist or deflationist can

make sense of these points. Indeed, one might argue that if we had empirical evidence that non-Hausdorff manifolds were necessary for physics, it would strike a serious blow to the relationalist and deflationist positions.

How might the algebraically-minded relationalist reply? One promising possibility would be to turn the arguments above on their head. One might insist that the physical significance of the spacetime manifold is exhausted by the possible configurations of matter that could be defined on it. From this perspective, there is no physical difference between universes represented by manifolds with isomorphic algebras of smooth functions. A smooth manifold M and that manifold with an "additional" p (for instance) represent the same facts concerning the possible ways in which matter properties can be distributed. Since matter fields cannot "see" the extra point, it has no physical significance. It is not really there.

An algebraically-minded relationalist may take this line of thought even further. They might hope, for instance, that expanding the class of possibility to non-Hausdorff manifolds does not add distinct algebraic possibilities. More precisely, one might hope that every non-Hausdorff manifold would be an element of an equivalence class of manifolds with the same algebra of smooth functions; and that each of those equivalence classes would contain one (and, by Theorem 1, only one, up to diffeomorphism) Hausdorff representative. If this were true, then expanding the space of models to include non-Hausdorff manifolds would not change the space of physically possible worlds, because the physical content of the additional models would be exhausted by that of the Hausdorff spacetimes. Non-Hausdorff manifolds would turn out to be superfluous.

This hope is moot, however, because of the following result.

Proposition 3. *There exists a non-Hausdorff manifold M whose algebra of smooth functions $C^\infty(M)$ is not isomorphic to that of any Hausdorff manifold.*

The example we give is a one-dimensional manifold with two "branches." This manifold has an associated algebra that is non-isomorphic to that of any Hausdorff manifold. In what follows, we will call non-Hausdorff manifolds whose algebras of smooth functions do not admit a representation as functions on any Hausdorff manifold *detectably non-Hausdorff*.

There are several observations to make about this proposition. First, it follows that one cannot collapse all non-Hausdorff manifolds into Hausdorff manifolds that are physically equivalent in the

sense of admitting the same algebra of smooth functions. Adding non-Hausdorff manifolds to the mix really does increase the space of possibilities, even if the physical significance of a manifold is entirely contained in its associated algebraic structure.

Second, this result has an immediate corollary, concerning the character of the algebras associated with detectably non-Hausdorff manifolds. Recall that we said above that Nestruev (2002) provides necessary and sufficient conditions for a commutative algebra to have a representation as the algebra of smooth functions on a (Hausdorff) manifold. The details require machinery that would be beyond the scope of this paper, but briefly, there are three conditions: an algebra admits a representation as the algebra of smooth functions on a (Hausdorff) manifold if and only if it is *complete* (Nestruev, 2002, §3.28), *geometric* (Nestruev, 2002, §3.7), and *smooth* (Nestruev, 2002, §4.1). Thus we have

Corollary 1. *There exists non-Hausdorff manifolds M for which $C^\infty(M)$ is not complete, geometric, and smooth.*[15]

We do not at present have necessary and sufficient conditions for an algebra to have a representation of smooth functions on a not-necessarily-Hausdorff manifold. We do not claim that this is impossible. But doing so successfully would be a prerequisite to presenting an autonomous generalization of the theory of Einstein Algebras to include (all) non-Hausdorff manifolds.

Let us return, now, to the motivation for stating Prop. 3. The question at issue was whether *all* non-Hausdorff manifolds can be associated with some Hausdorff manifold by first passing to the associated algebra of smooth functions and then identifying the (unique) Hausdorff manifold on which that algebra could be represented. That idea failed. But it is nonetheless true that *some* non-diffeomorphic non-Hausdorff manifolds can be associated with Hausdorff manifolds in this way.

This suggests that once we allow for non-Hausdorff manifolds, something like Earman's original idea (Earman, 1977, 1989) about the relationship between geometric and algebraic approaches gets traction after all: it would seem that not-necessarily-Hausdorff manifolds have *more structure* than algebras of smooth functions, since there are non-diffeomorphic not-necessarily-Hausdorff manifolds

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[15]The assertion stated here is the corollary to Prop. 3. But inspection of the conditions shows that it is the smoothness condition that fails, where smoothness is the requirement that there exists a finite or countable open covering $\{U_k\}$ of the dual space of an algebra $\mathcal{A}$ such that all of the restrictions of $\mathcal{A}$ to the elements of the covering are isomorphic to $C^\infty(\mathbb{R}^n)$ for some fixed n. This condition will fail for open sets containing witness points in some non-Hausdorff manifolds.
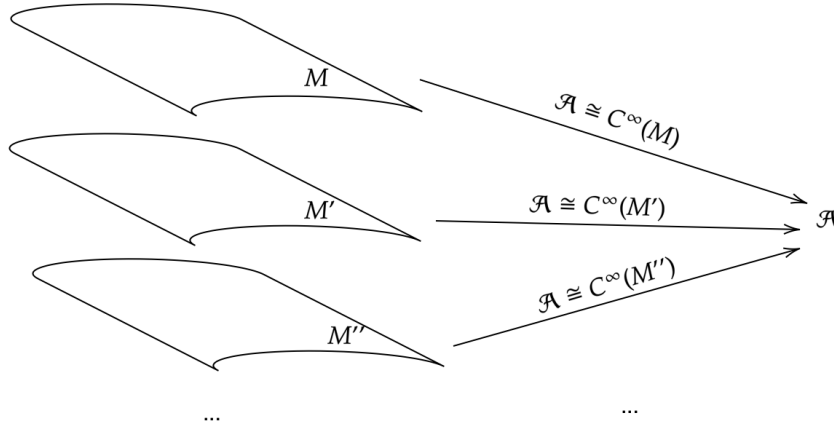
Figure 1

with the same algebras associated with them. The sense of "excess structure" here would be that of Weatherall (2016*b*): there would be two formulations of a theory with the same empirical content, but where there exist distinct, non-equivalent models of one corresponding to a single model of the other. Moving to the algebraic approach would be a way to eliminate that "excess structure," in the following sense: isomorphic algebras would correspond to equivalence classes of (non-diffeomorphic) manifolds, where those equivalence classes would include non-diffeomorphic manifolds (Figure 1).

These ideas can be made precise using the criteria of structural comparison developed by Baez et al. (2006) and exported to structural comparisons of physical theories by Weatherall (2016*b*,*a*).[16] To do so, we first introduce two categories and then consider a functor between them.[17] Our structural analysis will concern the formal properties of this functor.[18] According to the standard of comparison under consideration, the objects of some category **A** of mathematical gadgets has more *structure* than those of another, **B**, relative to a choice of functor $F : \mathbf{A} \rightarrow \mathbf{B}$, if F is not *full*, i.e., its induced action on hom sets is not surjective; more *stuff* if F is not *faithful*, i.e., its induced action on hom sets is not injective; and more *properties* if F is not essentially surjective, i.e., surjective up to isomorphism in **B**. Conversely, if F is full, faithful, and essentially surjective, then **A** and **B** are equivalent (or dual) categories, reflecting that their objects have the same structure.

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[16]See Barrett (2022) for a detailed justification of these criteria in the case of first order theories.

[17]For readers interested in further background reading on category theory, see Leinster (2004).

[18]Strictly speaking, the functor we consider will be contravariant, which means it "reverses" arrows. Thus, we will be using an extended version of the criterion for structural comparison that has previously been considered in the philosophy of physics literature. However, just as Rosenstock et al. (2015) argued that a categorical duality has much the same significance for questions of theoretical equivalence as a categorical equivalence does, we suggest that the upshot of the analysis does not change for the fact that the functor is contravariant.

Weatherall (2016*b*) proposes extending this idea to physical theories by considering functors F between categories of models of physical theories, where F is required to preserve the empirical consequences of the models. One formulation of a theory has more structure than another if there is a functor from the models of the first to the models of the second that preserves empirical consequences, but fails to be full. Intuitively speaking, this means that there are models that are equivalent in the second category, but their corresponding models are not equivalent in the first. The key example Weatherall considers is formulations of electromagnetism on Minkowski spacetime using vector potentials and formulations using Faraday tensors, where the former is shown to have more structure than the latter.

We will follow Rosenstock et al. (2015) closely here—except that we will not worry about metrics or other structure defined on manifolds, or their analogs on algebras. We will consider just categories of not-necessarily-Hausdorff manifolds and algebras of smooth functions on such manifolds, defined as follows.

First, we consider the category **nnHMan**, which has as objects not-necessarily-Hausdorff manifolds (of any finite dimension); and has as arrows diffeomorphisms.[19] Now consider the category **nnHAlg**, whose objects are the algebras of smooth functions on the objects of **nnHMan** and whose arrows are algebra isomorphisms.[20] Finally, observe that this construction gives rise to a natural contravariant functor, F :**nnHMan**→**nnHAlg**, directly analogous to the one considered by Rosenstock et al. (2015), that takes each object M of **nnHMan** to $C^\infty(M)$ and takes each arrow $\alpha : M \to N$ to the algebra isomorphism $\tilde{\alpha} : N \to M$ as defined in the proof of Prop. 2 in the Appendix. One can easily confirm that F is a functor, since it preserve identity maps and composition. Again, by analogy to arguments in Rosenstock et al. (2015), this functor would induce a functor that preserves the empirical consequences of the theories were we to extend our analysis to consider spacetimes and Einstein algebras. So its properties are probative for present purposes.

We now have the following proposition.

Proposition 4. *The functor F :**nnHMan**→**nnHAlg** as defined above is neither full nor faithful.*

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[19] One could of course consider a richer category, with more than just isomorphisms, but since Rosenstock et al. (2015) consider only isometries and we are inheriting our choice of functor from them, we will stick with their choice. See Barrett (2018) for a discussion of this choice.

[20] Note that as discussed we do not have an independent characterization of the objects of **nnHAlg**, but that will not matter for the present discussion.

This proposition shows that the functor F forgets structure, because it is not full. Since we expect F to determine a functor from not-necessarily-Hausdorff spacetimes to their associated Einstein algebras that preserves empirical content (assumed to be exhausted by the algebraic structure), it would follow that non-Hausdorff spacetimes have not just more structure, but also *excess* structure. There is another formalization that captures the same empirical content with less structure.

But the proposition shows more than this. F also forgets *stuff* because it is not faithful.[21] The intuition is that a non-trivial diffeomorphism that permutes a pair of witness points but leaves everything else fixed would map to the identity arrow by the functor. So witness points contribute extra "stuff" to non-Hausdorff manifolds, leading to additional symmetries. Thus, we can think of non-Hausdorff manifolds as exhibiting *both* excess structure and excess ontological commitments. Moving to algebras excises both.

5   A Question and a Conjecture about Bifurcating Curves

In this section, we turn to a question that arises in light of Prop. 3, and whose significance is informed by Prop 4. To motivate this question, first observe that one consequence of Prop. 3 is that there seem to be two classes of non-Hausdorff manifolds: ones, like M with an "additional" p, whose algebra of smooth functions is isomorphic to that of a Hausdorff manifold; and ones, like the example in the proof of Prop. 3, whose algebra is not.

There is another property shared by some, but not all, non-Hausdorff manifolds that has sometimes been used to classify them. This is the property of admitting *bifurcating curves*, as first introduced by Hajicek (1971).

Definition 5 (Bifurcating Curves, Hajicek (1971)). *A manifold M has a bifurcating curve iff there exist (smooth) curves $\gamma_i : [0,1] \to M$ for $i = 1, 2$ and some $t \in (0,1]$ such that $\gamma_1(s) = \gamma_2(s)$ for all $s < t$ and yet $\gamma_i(s) \neq \gamma(s)$ for all $s \geq t$.*

A manifold has a bifurcating curve iff there exists two curves that agree up to a point, and disagree for all subsequent points including this point. Whether a non-Hausdorff manifold contains

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[21]See Nguyen et al. (2020) and Bradley and Weatherall (2022) for a recent discussion about the significance of forgetting stuff.

bifurcating curves or not can be seen as a test for its physical reasonableness. This is because bifurcating (timelike) curves reflect a kind of indeterminism: a particle traversing such a curve, upon reaching the bifurcation point, could equally well continue in either of two (or more) directions, with nothing in the structure of spacetime distinguishing those possible continuations from one another. The thought, then, is that insofar as it is this sort of branching that makes (some) non-Hausdorff manifolds seem "unphysical", then those non-Hausdorff spaces that are free of bifurcating curves may be physically reasonable after all (c.f. Manchak, 2020).

All of the examples of non-Hausdorff manifolds considered thus far in the present paper do admit bifurcating curves.[22] Since we have seen examples of both detectably non-Hausdorff manifolds and non-detectably-non-Hausdorff manifolds, it follows that admitting bifurcating curves can be neither necessary nor sufficient for detectability. But what is not clear is how detectability relates to the *absence* of bifurcating curves. That is: are there non-Hausdorff manifolds with no bifurcating curves that are detectably non-Hausdorff? Are all such manifolds detectably non-Hausdorff?

We propose the following conjecture.

Conjecture 2. *There exist detectably non-Hausdorff manifolds with no bifurcating curves.*

We suggest that the classic example of Misner spacetime with two extensions (Hawking and Ellis, 1973, pp. 173-4) is a likely candidate for a non-Hausdorff manifold that is free of bifurcating curves, but whose algebra of smooth functions has no Hausdorff representation. However, we have not been able to show this. We also think it is more likely than not that *all* non-Hausdorff manifolds without bifurcating curves are detectably non-Hausdorff.

Suppose this conjecture holds. We would then have a (at least) tripartite division of non-Hausdorff manifolds. On one extreme, there would be non-detectable non-Hausdorff manifolds. These would exhibit the most mild non-Hausdorff flavors. (This is so even though some of them have bifurcating curves!) One might reasonably argue either that considering such manifolds is harmless; or one might argue that they are otiose because they introduce additional structure that makes no difference to the empirical structure of the theory. But in both cases, the non-Hausdorff character is highly localized, and arguably eliminable. Indeed, the algebraic

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[22]For the manifold with an "additional" p, consider any curve whose endpoint is p and observe that there is another curve agreeing with it everywhere except that its endpoint is p'. That the example in the proof of Prop. 3 exhibits such curves, meanwhile, is essential to the proof.

relationalist might even argue that expanding the space of models of GR to include non-detectable non-Hausdorff manifolds would not truly change the space of physical possibilities contemplated by the theory at all, insofar as the physical significance of a manifold is exhausted by the field configurations it supports.

On the other extreme, we would have detectably non-Hausdorff manifolds with bifurcating curves. Admitting these manifolds to the space of physical possibility would carry significant costs, such as failures of determinism for particles following the curves that bifurcate. Moreover, these pathological possibilities would be manifest in the algebra of smooth fields on the manifold.

But we would also have another class of non-Hausdorff spacetimes, ones that do not admit such bifurcating curves, but which nonetheless have genuinely novel non-Hausdorff behavior, as reflected in their associated algebras of smooth functions. We do not have a view regarding whether these should count as "physically reasonable"; indeed, we are not even confident if they exist. But it would certainly be of interest to settle this issue and explore the properties of such manifolds, if they do exist.

On the other hand, if the conjecture above is false, and it turns out that admitting bifurcating curves is necessary (but not sufficient) for detectability, then non-Hausdorff manifolds would have considerably *less* interest: either they have the same physical significance as some (unique) Hausdorff manifold; or else they admit unphysical features that should only be accepted under strong empirical pressure.

Table 1 summarizes the situation.

	$\mathbb{R}$	Misner Spacetime with Two Extensions	$\mathbb{R} \cup \{p\}$	Branching $\mathbb{R}$
Hausdorff?	Yes	No	No	No
Free of Bifurcating Curves?	Yes	Yes	No	No
Shares Algebra of Smooth Functions with a Hausdorff Manifold?	Yes	?	Yes	No

Table 1: A tripartite hierarchy of not-necessarily-Hausdorff manifolds.

6 Conclusion

This paper has considered whether the duality between smooth manifolds and algebras of smooth functions thereon – and, by extension, the duality between relativistic spacetimes and Einstein algebras – generalizes to the not-necessarily-Hausdorff case. We found that for not-necessarily-Hausdorff manifolds, it is not the case that if two manifolds have isomorphic algebras of smooth functions, then the manifolds must be diffeomorphic. However, diffeomorphic not-necessarily-Hausdorff manifolds always have isomorphic algebras of smooth functions.

We then showed that there exist non-Hausdorff manifolds whose algebras of smooth functions are not isomorphic to that of any Hausdorff manifold. It follows that the algebras associated with non-Hausdorff manifolds are not necessarily complete, geometric, and smooth in the sense of Nestruev (2002); thus, expanding the collection of relativistic spacetimes to include ones on not-necessarily-Hausdorff manifolds would lead to a corresponding expansion of the collection of Einstein algebras. Some non-Hausdorff manifolds exhibit genuinely novel behavior, as viewed from their algebras of smooth functions.

But not all of them do. Once we consider not-necessarily-Hausdorff manifolds, each smooth algebra of functions on such manifolds is associated with a collection of non-diffeomorphic manifolds. Thus, one can see the passage from not-necessarily-Hausdorff manifolds to algebras of smooth functions as taking equivalence classes of non-diffeomorphic manifolds. This idea is reflected in the fact that the natural functor, analogous to the one analyzed by Rosenstock et al. (2015), from a certain category of not-necessarily-Hausdorff manifolds to a corresponding category of smooth algebras fails to be full (and faithful). Thus, there is a sense in which the algebraic approach posits less structure than the geometric approach in the not-necessarily-Hausdorff context. If the empirical content of not-necessarily-Hausdorff GR is exhausted by the algebras of smooth functions on spacetimes, as apparently is the case for standard GR, then not-necessarily-Hausdorff relativistic spacetimes have excess structure.

Finally, we posed a question about the relationship between our results and the existence of bifurcating curves; and we posed a conjecture that, if true, would establish a hierarchy of degrees of non-Hausdorff behavior.

We will conclude with some brief reflections on the philosophical significance of the arguments

here. Once again, perhaps the most important takeaway is that once we move to not-necessarily-Hausdorff manifolds, something like Earman's original picture of the relationship between manifold-based "substantivalist" models and algebra-based "relationalist" models of GR is realized after all, in two ways. First, the manifold approach now seems to require an interpretation on which spacetime points are ontologically prior to, and independent of, configurations of matter, in the strong sense that there can be points that differ from one another, i.e., are distinct individuals, but be such that those differences cannot be reflected in any difference in field valuations. Matter configurations cannot detect the fact that these points are distinct. It is hard to make sense of spacetime points in this context other than by adopting a version of substantivalism. Second, conversely, algebras in this context have *less* structure than manifolds, and so one can excise structure by moving from a geometric formulation to an algebraic one. Thus, we recover a sense in which the "relationalist" algebraic approach is more structurally parsimonious than the "substantivalist" geometric approach.

It is worth reflecting on how different the substantivalism-relationalism debate looks when viewed from the present perspective, as compared to how that debate has unfolded in the context of GR since the work of Friedman (1983), Field (1984), Earman (1986), Earman and Norton (1987), and other giants of the last quarter of the 20th century. To see the point most starkly, consider that it is tempting to call the version of substantivalism currently under consideration "manifold substantivalism". But that expression would not be appropriate given the literature, since "manifold substantivalism" as understood by Earman and Norton (1987) denies what is called "Leibniz equivalence," which claims that isometric spacetimes represent the same physical possibilities. The form of substantivalism under consideration here, by contrast, can fully embrace "Leibniz equivalence". Since Earman and Norton (1987, 522) famously argue that one must either deny Leibniz equivalence or deny substantivalism, the form of substantivalism under consideration here would not count for them as a substantivalist theory at all! This is so despite the fact that it endorses the claim that spacetime points are individuals whose existence is independent of possible matter configurations.

These considerations suggest a different focus for the debate between the substantivalist and relationalist, concerning whether it makes sense to posit additional "places" in space and time, even if those places cannot be registered, even in principle, in the space of possible matter configurations.

The substantivalist, here, would say "yes"; the relationalist "no".

Another takeaway is that we now have yet another example of how, by expanding the possibility space of GR, we can no longer take for granted previous results we relied on (c.f. Manchak (2021)). The duality between relativistic spacetimes and Einstein's algebra is only one of the many results that can fail in not-necessarily-Hausdorff spaces. Perhaps most interestingly, the present case seems to be an example in (non-quantum) physics where the results that fail matter for plausible interpretations of the theory. Indeed, expanding the space of possible models changes the options for interpreting structures of our actual world. Suppose we adopt the *prima facie* plausible principle that mathematical structures should be interpreted uniformly across the models of a theory. This principle of uniform interpretation would suggest that if non-Hausdorff spacetimes are possible, then insofar as they do not support deflationism, we have to change our strategies for interpreting spacetime points in all models of the theory—including ones representing our actual world. Such action-at-a-distance in modal space is common in quantum theories, where there is a close relationship between the determinable properties and the space of (all) possible states. But it is more difficult to think of similar examples in classical physics.

These considerations, taken together, suggest a novel argument for relationalism—or, at least, for algebraicism. In the not-necessarily-Hausdorff case, by adopting the principle of uniform interpretation, the geometric approach suggests a non-deflationary, substantivalist interpretation of spacetime points, with excess structure as compared to the algebraic approach. Thus parsimony considerations would advise adopting an alebgraic-relationalist view. While one is not forced to that position by the same considerations in standard (Hausdorff) GR, one might argue that parsimony considerations should take into account not just the specific theory under considerations, but also other "closeby" theories. For instance: one might argue that to properly understand Newtonian gravitation, one should consider not only the theory as classically conceived, but also in relation to other theories, such as GR or teleparallel gravity. The fact that for "small" variations on standard GR, the duality between GR and Einstein algebras breaks down, then, might be taken as evidence that *in general* algebraic-relationalist theories are more parsimonious than substantivalist-geometric ones, and thus one should in general prefer the algebraic variants, even in the special cases, such as Hausdorff GR, where they coincide.

Of course, one might reject the arguments just made, perhaps by rejecting the principle of uniform interpretation that we have advanced. For instance, Ruetsche (2011) argues against the viability of what she calls "pristine interpretations" for some physical theories, where a pristine interpretation is one that applies uniformly across all possible applications. But from that perspective, the results above are no less interesting, since they offer a novel example of a case where one might prefer an "adulterated interpertation" of a (non-quantum) theory.

Acknowledgments

Appendix: Proofs of Propositions

In what follows, we adopt conventions and defintions from (Malament, 2012, §1.1).

Lemma 1 $(N, \mathcal{C}')$ as just defined [in Section 3] is a smooth, non-Hausdorff manifold. Moreover, it is second countable if $(M, \mathcal{C})$ is.

Proof. Recall that we defined $\mathcal{C}'$ as the collection of all $n-$charts on N compatible with both $\mathcal{C}$ and all mirror charts on N. We first show that this set is non-empty, by showing that the elements of the collections consisting of $\mathcal{C}$ and the collection of mirror charts are all pairwise compatible. Since $\mathcal{C}$ is an atlas, its elements are pairwise compatible by construction. So we have two other cases to establish: (1) that each mirror chart is compatible with each chart in $\mathcal{C}$; and (2) that mirror charts are pairwise compatible. We consider these in turn.

1. Let (U_1, φ_1) be a mirror chart and let (U_2, φ_2) be in $\mathcal{C}$. Then $p \notin U_1$ and there exists a n-chart, (U, φ) in $\mathcal{C}$, for which (U_1, φ_1) mirrors. We know that (U, φ) and (U_2, φ_2) are compatible because $(M, \mathcal{C})$ is a n-manifold. Suppose first that $p \in U_2$. Then $\varphi_1[U_1 \cap U_2] = \varphi(U \cap U_2) \setminus \{\varphi(p)\}$ and $\varphi_2[U_1 \cap U_2] = \varphi_2(U \cap U_2) \setminus \{\varphi_2(p)\}$ are both open; $\varphi_1 \circ \varphi_2^{-1} : \varphi_2[U_1 \cap U_2] \to \mathbb{R}^2$ is the same map as $\varphi \circ \varphi_2^{-1} : \varphi_2[(U \cap U_2) \setminus \{p\}] \to \mathbb{R}^2$, which is smooth, and for similar reasons $\varphi_2 \circ \varphi_1^{-1} : \varphi_1[U_1 \cap U_2] \to \mathbb{R}^2$ is smooth. So (U_1, φ_1) and (U_2, φ_2) are compatible. No suppose $p \notin U_2$. Then $U_1 \cap U_2 = U \cap U_2$, $\varphi_1(U_1 \cap U_2) = \varphi(U \cap U_2)$, and $\varphi_2(U_1 \cap U_2) = \varphi_2(U \cap U_2)$. Because (U, φ) and (U_2, φ_2) are compatible, (U_1, φ_1) and (U_2, φ_2) are compatible.

2. Now suppose (U_1, φ_1) and (U_2, φ_2) are both mirror charts. In this case, $p' \in U_1 \cap U_2$; and there exist n-charts $(\tilde{U}_1, \tilde{\varphi}_1), (\tilde{U}_2, \tilde{\varphi}_2) \in \mathcal{C}$ that mirror (U_1, φ_1) and (U_2, φ_2), respectively. Now $\varphi_1[U_1 \cap U_2] = \tilde{\varphi}_1[\tilde{U}_1 \cap \tilde{U}_2]$ and $\varphi_2[U_1 \cap U_2] = \tilde{\varphi}_2[\tilde{U}_1 \cap \tilde{U}_2]$ are open, and $\varphi_1 \circ \varphi_2^{-1} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1} = \tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1}$ are smooth. So (U_1, φ_1) and (U_2, φ_2) are compatible.

By construction, $\mathcal{C}'$ is also maximal. Thus, $(N, \mathcal{C}')$ is a smooth, not-necessarily-Hausdorff manifold.

It remains to show that $(N, \mathcal{C}')$ is in fact non-Hausdorff. To do so, we first show that $\mathcal{C}'$ consists *only* of n-charts in $\mathcal{C}$ and their mirror n-charts; there are no additional $n-$charts compatible with this set.

Suppose, to the contrary, that there is some n-chart (O, ψ) that is compatible with every n-chart in $\mathcal{C}$ and every mirror n-chart, but is not itself in $\mathcal{C}$ or a mirror $n-$chart to one in $\mathcal{C}$. It follows that $O \nsubseteq M$, since $(M, \mathcal{C})$ is a manifold, and by the maximality of $\mathcal{C}$, if $O \subseteq M$ and (O, ψ) is compatible with every chart in $\mathcal{C}$, it would be in $\mathcal{C}$ already. Thus $p' \in O$. Moreover, $p \notin O$. To see this, suppose to the contrary that $p \in O$. Then, since ψ is injective, $\psi(p) \neq \psi(p')$. It would follow that there exists $\delta > 0$ such that $p \notin B_\delta(\psi(p'))$. Now choose any n-chart $(U, \varphi) \in \mathcal{C}$ with $p \in U$ and its mirror n-chart (U', φ'). By assumption, (O, ψ) must be compatible with both (U, φ) and (U', φ'); but this is impossible, since $\varphi \circ \psi^{-1}$ is discontinuous at p if $\varphi' \circ \psi^{-1}$ is smooth. So $p \in O$, as claimed. Finally, observe that since (O, ψ) is compatible with all n-charts in $\mathcal{C}$ and their mirror charts, $(\tilde{O}, \tilde{\psi})$, defined by $\tilde{O} = (O \setminus \{p'\}) \cup \{p\}$, $\tilde{\psi}(q) = \psi(q)$ for $q \neq p$ and $\tilde{\psi}(p) = \psi(p')$, is also compatible with all such n-charts, by the arguments presented above. In particular, $(\tilde{O}, \tilde{\psi})$ is compatible with all n-charts in $\mathcal{C}$, so (O, ψ) would be already contained in $\mathcal{C}$. Thus (O, ψ) is a mirror $n-$chart to a chart in $\mathcal{C}$.

Now observe that, since $\mathcal{C}'$ consists only of elements of $\mathcal{C}$ and their mirror n-charts, no pair of n-charts in $\mathcal{C}'$ can separate p from p'. This is because by construction, every chart containing p' in its domain has non-vanishing intersection with every open set containing p. It follows that there could be no open sets in the manifold topology that separate p from p'. So $(N, \mathcal{C}')$ is non-Hausdorff.

Finally, we show that $(N, \mathcal{C}')$ is second-countable if $(M, \mathcal{C})$ is. To see this, suppose that $\{(O_n\}_{n \in \mathbb{N}}$ is a countable base for $(M, \mathcal{C})$. Now let $I \subseteq \mathbb{N}$ be the collection of indices such that $p \in O_i$ for each $i \in I$. We then define a new collection of open sets, $\{\tilde{O}_i\}_{i \in I}$, where for each i, $\tilde{O}_i = (O_i \setminus \{p\}) \cup \{p'\}$. Then $\{(O_n\}_{n \in \mathbb{N}} \cup \{\tilde{O}_i\}_{i \in I}$ is a countable base for $(N, \mathcal{C}')$. $\square$

Lemma 2 If p_1 and p_2 are witness points of some smooth, non-Hausdorff manifold M, then for any smooth scalar field $f : M \to \mathbb{R}$, $f(p_1) = f(p_2)$.

Proof. Suppose, on the contrary, that for two witness points $p_1, p_2 \in M$, there exists a smooth scalar field $f : M \to \mathbb{R}$, $f(p_1) \neq f(p_2)$. Then, there exists $\epsilon > 0$ such that $|f(p_1) - f(p_2)| > 2\epsilon$.

Let (U_1, φ_1) and (U_2, φ_2) be two n-charts of M with $p_1 \in U_1$ and $p_2 \in U_2$. Since f is smooth, $f \circ \varphi_1^{-1}$ and $f \circ \varphi_2^{-1}$ are both smooth. It follows that there exists open sets $O_1 \in \varphi_1[U_1]$ and $O_2 \in \varphi_2[U_2]$ with $\varphi_1(p_1) \in O_1$ and $\varphi_2(p_2) \in O_2$ such that for all $q_1 \in O_1$ and $q_2 \in O_2$, $|f \circ \varphi_1^{-1}(q_1) - f(p_1)| < \epsilon$ and $|f \circ \varphi_2^{-1}(q_2) - f(p_2)| < \epsilon$. Now, since O_1 and O_2 are open, $(\varphi_1^{-1}[O_1], \varphi_1)$ and $(\varphi_2^{-1}[O_2], \varphi_2)$ are both n-charts. It follows that $\varphi_1^{-1}[O_1]$ and $\varphi_2^{-1}[O_2]$ are both open, with $p_1 \in \varphi_1^{-1}[O_1]$ and $p_2 \in \varphi_2^{-1}[O_2]$. Now since p_1, p_2 are witness points, $\varphi_1^{-1}[O_1] \cap \varphi_2^{-1}[O_2]$ is non-empty. Let $q \in \varphi_1^{-1}[O_1] \cap \varphi_2^{-1}[O_2]$. We have both $|f(q) - f(p_1)| < \epsilon$ and $|f(q) - f(p_2)| < \epsilon$. But $|f(p_1) - f(p_2)| > 2\epsilon$, which is a contradiction. $\square$

Lemma 3 Let $(M, \mathcal{C})$ be a smooth, Hausdorff $n-$manifold, let $p \in M$ be some point, and let $(N, \mathcal{C}')$ be M with an "additional" p. Then $(M, \mathcal{C})$ and $(N, \mathcal{C}')$ are not diffeomorphic.

Proof. Suppose, for contradiction, that there exists a diffeomorphism $\alpha : N \to M$. Then because α is injective, $\alpha(p) \neq \alpha(p')$. Since M is Hausdorff, there exist $n-$charts (U_1, φ_1) and (U_2, φ_2) in $\mathcal{C}$ such that $\alpha(p) \in U_1$ and $\alpha(p') \in U_2$ and $U_1 \cap U_2 = \emptyset$. Now, define the following neighborhood of $\varphi_1(\alpha(p))$ contained in $\varphi_1[U_1]$: $B_\epsilon(\varphi_1(\alpha(p))) : \{x : \|x - \varphi_1(\alpha(p))\| < \epsilon\}$ for some appropriate $\epsilon > 0$. Then select a smooth "bump function" $g : \mathbb{R}^n \to \mathbb{R}$ (where n is the dimension of N) centered at $\varphi_1(\alpha(p))$ such that $g(\varphi_1(\alpha(p)) = 1$ and $g = 0$ outside of $B_\epsilon(\varphi_1(\alpha(p_1)))$. Now define $f : M \to \mathbb{R}$ such that $f = g \circ \varphi_1$ for all points in U_1 and $f = 0$ outside of U_1. f is smooth because g is smooth and the n-charts are compatible. But then $f \circ \alpha : N \to \mathbb{R}$ cannot be smooth. This is because according to Lemma 2 all smooth functions from $N \to \mathbb{R}$ assign the same value to a pair of witness points, but $1 = f \circ \alpha(p) \neq f \circ \alpha(p') = 0$. So α is not smooth, and a fortiori, not a diffeomorphism. $\square$

Lemma 4 Let $(M, \mathcal{C})$ be a smooth, not-necessarily-Hausdorff manifold and let $(N, \mathcal{C}')$ be M with some "additional" point p. Then $C^\infty(M) \cong C^\infty(N)$.

Proof. We establish the isomorphism explicitly. For every smooth map f in $C^\infty(M)$, define a map $f' : N \to \mathbb{R}$ by setting $f'(q) = f(q)$ for $q \in M$ and $f'(p') = f(p)$. We first show that S, the collection of such f' for all $f \in C^\infty(M)$, is an algebra. Because all operations are defined pointwise, the only point to check is p'; but that the algebraic properties hold at p' is immediately inherited from the fact that they hold at p. Furthermore, elements of S are smooth maps from N to $\mathbb{R}$. To see this, pick an arbitrary n-chart (U, φ) in $\mathcal{C}'$. If $p' \notin U$, then $f' \circ \varphi^{-1} = f \circ \varphi^{-1}$ is smooth. If $p' \in U$, then $f' \circ \varphi^{-1} = f \circ \tilde{\varphi}^{-1}$ is smooth, where $(\tilde{U}, \tilde{\varphi})$ is the n-chart in $\mathcal{C}$ of which (U, φ) is its mirror n-chart. Finally, we show that $S = C^\infty(N)$. To see this, suppose, on the contrary, that there exists a smooth map $h : N \to \mathbb{R}$ such that $h \notin S$. This means either that $h(p') \neq h(p)$ or else $h_{N \setminus \{p'\}}$ is not a smooth map on $(M, \mathcal{C})$. But the first cannot hold due to Lemma 2; and the second cannot hold because any smooth map on a manifold, when restricted to an open subset, determines a smooth map on that subset with its inherited manifold structure. Therefore, $S = C^\infty(N)$.

Finally, it is easy to verify that the map $g : f \mapsto f'$ for all $f \in C^\infty(M)$ is an isomorphism between $C^\infty(M)$ and $C^\infty(N)$. $\square$

Proposition 2 If not-necessarily-Hausdorff manifolds M and N are diffeomorphic, then $C^\infty(M) \cong C^\infty(N)$.

Proof. Suppose that M and N are diffeomorphic. Then there exists a diffeomorphism $\alpha : M \to N$. From this we know that α and its inverse α^{-1} are both smooth.

Now define $\tilde{\alpha} : C^\infty(N) \to C^\infty(M)$ as follows. For any smooth map $\pi : N \to \mathbb{R}$, $\tilde{\alpha}(\pi) = \pi \circ \alpha$. The map $\tilde{\alpha}$ is well defined because, first, for all π in $C^\infty(N)$, $\pi \circ \alpha$ is an element of $C^\infty(M)$ because α is also smooth. Second, if $\pi_1 = \pi_2$, then $\pi_1 \circ \alpha = \pi_2 \circ \alpha$. This is because α is a function.

Furthermore, $\tilde{\alpha}$ admits an inverse $\tilde{\alpha}^{-1} : C^\infty(M) \to C^\infty(N)$ defined as follows. For any smooth map $\chi : M \to \mathbb{R}$, $\tilde{\alpha}^{-1}(\chi) = \chi \circ \alpha^{-1}$. $\tilde{\alpha}g^{-1}$ is well-defined for analogous reasons. $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ are inverses because for any smooth map $\pi \in C^\infty(N)$, $\tilde{\alpha}^{-1}(\tilde{\alpha}(\pi)) = \pi \circ \alpha \circ \alpha^{-1} = \pi$.

It now suffices to show that $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ are homomorphisms. We first show that $\tilde{\alpha}$ preserves addition. For any $p \in M$ and $\pi_1, \pi_2 \in C^\infty(N)$, $g(\pi_1 + \pi_2)(p) = (\pi_1 + \pi_2) \circ \alpha(p) = \pi_1 \circ \alpha(p) + \pi_2 \circ \alpha(p) = \tilde{\alpha}(\pi_1)(p) + \tilde{\alpha}(\pi_2)(p)$. The second equality holds because $\alpha(p)$ is a point in N and we invoke the additive structure in $C^\infty(N)$. We then show that $\tilde{\alpha}$ preserves scalar multiplication. For any $p \in M, a \in \mathbb{R}$, and $\pi \in C^\infty(N)$, $\tilde{\alpha}(a\pi)(p) = (a\pi) \circ \alpha(p) = a(\pi \circ \alpha(p)) = a\tilde{\alpha}(\pi)(p)$. Again, the second equality holds because $\alpha(p)$ is a point in N and we invoke the scalar multiplication structure in $C^\infty(N)$. Finally, we show that $\tilde{\alpha}$ preserves multiplication. For any $p \in M$ and $\pi_1, \pi_2 \in C^\infty(N)$, $\tilde{\alpha}(\pi_1 \cdot \pi_2)(p) = (\pi_1 \cdot \pi_2) \circ \alpha(p) = (\pi_1 \circ \alpha(p)) \cdot (\pi_2 \circ \alpha(p)) = \tilde{\alpha}(\pi_1)(p) \cdot \tilde{\alpha}(\pi_2)(p)$. Like before, the second equality holds because $\alpha(p)$ is a point in N and we invoke the multiplication structure in $C^\infty(N)$.

Identical arguments show that $\tilde{\alpha}^{-1}$ is a homomorphism. $\square$

Proposition 3 There exists a non-Hausdorff manifold M whose algebra of smooth functions $C^\infty(M)$ is not isomorphic to that of any Hausdorff manifold.

Proof. Note that it suffices to identify a (non-Hausdorff, connected) $n-$manifold M and a smooth function φ on M such that no function with the algebraic properties of φ lie in $C^\infty(N)$ for any connected Hausdorff $n-$manifold N. (We can restrict attention to connected manifolds because no disconnected manifold has an algebra of smooth functions isomorphic to that of a connected manifold. Likewise, manifolds of different dimension have non-isomorphic algebras.) Consider, as M, the one-dimensional non-Hausdorff "branching manifold", i.e., the manifold constructed by taking two copies of $\mathbb{R}$ and identifying all points $x < 0$, but not identifying points $x \geq 0$. Now consider a smooth function φ on M with the following two properties: first, $d_a \varphi$ is nowhere vanishing; and second, φ has two zeros not at M's witness points. (Such a function exists: choose any function φ' on $\mathbb{R}$, with standard coordinates, that is everywhere increasing in the $+x$ direction, and which has a zero at some point $x > 0$; and let φ be the corresponding function on M. Then φ will vanish twice, once on each fork, and it will have nowhere vanishing first derivative relative to any non-vanish vector field.) Such a function would lie in two distinct maximal ideals of $C^\infty(M)$ even though its derivative is nowhere vanishing and thus it is strictly monotonically increasing in one direction. But clearly no function with those (algebraic) properties can exist on any connected Hausdorff manifold (i.e., up to diffeomorphism, the line or circle), by the mean value theorem. $\square$

Proposition 4 The functor $F :$ **nnHMan**$\to$**nnHAlg** as defined above is neither full nor faithful.

Proof. To show that the functor fails to be full, it is sufficient to consider any arbitrary Hausdorff manifold M, any point $p \in M$, and the manifold N, which is M with an "additional" p. By Lemmas 3 and 4, M and N are not diffeomorphic, and so there is no arrow $f : M \to N$. But $F(M) = F(N)$ and so $1_{F(M)} \in \hom(F(M), F(N))$. Thus the action of F from $\hom(M, N)$ to $\hom(F(N), F(M))$ fails to be surjective. So F is not full.

To show that the functor is not faithful, meanwhile, consider the manifold Q, which we defined as $\mathbb{R}^4$ with an "additional" p; and consider the action of F on the diffeomorphism $\beta : Q \to Q$ that takes p to the additional point p'. That this is a diffeomorphism can be see by considering that it is bijective, and that both it and its inverse, when composed with any smooth function $f : Q \to \mathbb{R}$, is smooth, since by Lemma 2, $f \circ \beta = f \circ \beta^{-1} = f$. But now observe that, by the same fact, $F(\beta) = F(1_Q) = 1_{F(Q)}$, since for every $f \in F(Q)$, $\tilde{\beta}(f) = f$. Thus the action of F from $\hom(Q, Q)$ to $\hom(F(Q), F(Q))$ fails to be injective. So F is not faithful. $\qquad\square$

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