Gauge- and Galilei-Invariant Geometric Phases*

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Abstract

Neither geometric phases nor differences in geometric phases are generally invariant under time-dependent unitary transformations (unlike differences in total phases), in particular under local gauge transformations and Galilei transformations. (This was pointed out originally by Aharonov and Anandan, and in the case of Galilei transformations has recently been shown explicitly by Sjöqvist, Brown and Carlsen.) In this paper, I introduce a phase, related to the standard geometric phase, for which phase differences are both gauge- and Galilei-invariant, and, indeed, invariant under transformations to linearly accelerated coordinate systems. I discuss in what sense this phase can also be viewed as geometric, what its relation is to earlier proposals for making geometric phases invariant under gauge or Galilei transformations, and what is its classical analogue. I finally apply this invariant phase to Berry's derivation of the Aharonov–Bohm effect.

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1 Introduction

The field of geometric phases has developed out of Berry's (1984) seminal paper, in which he identified a geometric component of the phase in adiabatic cyclic evolutions. This concept was further developed and generalised in subsequent important papers by B. Simon (1983), Aharonov and Anandan (1987), Samuel and Bhandari (1988), Aitchison and Wanelik (1992), Mukunda and R. Simon (1993), and others. I shall be considering in particular the geometric phase as treated by Aharonov and Anandan (AA) for the case of arbitrary cyclic evolutions and its generalisation by Aitchison and Wanelik (AW) to the non-cyclic case.

As was pointed out by Aharonov and Anandan (1987), the geometric phase is neither *locally gauge-invariant*, nor (as has been shown explicitly by Sjöqvist, Brown and Carlsen (1997)) is it *Galilei-invariant*. In this paper, I shall show how to construct a phase that is closely related to the geometric phase of AA or AW, and such that phase differences (as also done by García de Polavieja (1997)) become both locally gauge- and Galilei-invariant, as well as invariant under transformations to linearly accelerated frames.

The paper is structured as follows. After recalling the basic definitions and properties of geometric phases in Section 2, I shall sketch the proofs of non-invariance under local gauge and Galilei transformations (Section 3). In Section 4 I review previous treatments of this non-invariance, in particular by Anandan and Aharonov (1988) and by García de Polavieja (1997). I introduce an extended geometric phase in Section 5 such that, as shown in Section 6, phase differences are indeed invariant under local gauge and Galilei transformations and under transformations to linearly accelerated coordinate systems. Section 7 discusses the geometric interpretation of this phase, and Section 8 its classical analogue. Finally, Section 9 applies the extended geometric phase to Berry's (1984) calculation of the Aharonov–Bohm (AB) effect, making his identification of the AB phase with a geometric-type phase invariant under the above transformations.

2 Geometric Phases

Aitchison and Wanelik (1992, Appendix; see also Mukunda and Simon (1993)) define the geometric phase, which we shall denote by $\gamma[\lambda]$, of a generally non-cyclic (i.e. open) path $\lambda: \ s \in [0, S] \mapsto \pi(|\psi(s)\rangle)$ in projective Hilbert space (ray space) $\mathcal{P}(\mathcal{H})$ as follows (here $\pi(|\psi(s)\rangle)$ denotes the projection of the vector $|\psi(s)\rangle$ onto the ray containing $|\psi(s)\rangle$, and s is a curve parameter, that is, the mapping λ is locally one-to-one):[1]

$$e^{i\gamma[\lambda]} := \left(\frac{\langle\psi(0)|\psi(S)\rangle}{\langle\psi(S)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^S \left\langle \frac{\psi(s)}{\|\psi(s)\|} \middle| \frac{d}{ds} \middle| \frac{\psi(s)}{\|\psi(s)\|} \right\rangle ds\right\}. \tag{1}$$

If one considers only normalised vectors, $\|\psi(s)\| = 1$, definition (1) reduces to

$$e^{i\gamma[\lambda]} := \left(\frac{\langle\psi(0)|\psi(S)\rangle}{\langle\psi(S)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^S \langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds\right\}, \tag{2}$$

which for the case of a cyclic path ($\lambda(0) = \lambda(S)$) is equivalent to AA's definition. In this last case, if $s \mapsto |\psi(s)\rangle$ is a *single-valued* representation in the Hilbert space of the cyclic path λ, then $\gamma[\lambda]$ can be further written as

$$\gamma[\lambda] = i \oint_\lambda \langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds = i \int_0^S \langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds \tag{3}$$

(whenever we use the notation $\oint_\lambda$, we shall assume that the representation chosen for the path λ is single-valued).

The geometric phase (2) can be expressed as the difference between the *total* phase φ, given by

$$e^{i\varphi} := \left(\frac{\langle\psi(0)|\psi(S)\rangle}{\langle\psi(S)|\psi(0)\rangle}\right)^{1/2}, \tag{4}$$

and what we shall call the *locally accumulated* phase δ, defined by

$$\delta := -i \int_0^S \langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds. \tag{5}$$

[1] This is the so-called *real* geometric phase. AW treat extensively also the *complex* geometric phase, which we do not discuss here.

The phase (2) has two properties that make it a function of the unparametrised path λ in ray space. First of all, it is a *projective* quantity, in the sense that it is independent of the particular (differentiable) curve $s \mapsto |\psi(s)\rangle$ in *Hilbert* space that projects to λ. Another such curve can be written as

$$|\tilde{\psi}(s)\rangle = e^{if(s)}|\psi(s)\rangle \tag{6}$$

for some real function $f(s)$. In fact we have (in obvious notation):

$$\begin{aligned}
e^{i\gamma[\lambda,f]} &= \left(\frac{\langle\psi(0)|e^{-if(0)}e^{if(S)}|\psi(S)\rangle}{\langle\psi(S)|e^{-if(S)}e^{if(0)}|\psi(0)\rangle}\right)^{1/2} \\
&\quad \cdot \exp\left\{-\int_0^S \langle\psi(s)|e^{-if(s)}\frac{d}{ds}e^{if(s)}|\psi(s)\rangle ds\right\} = \\
&= e^{i(f(S)-f(0))}\left(\frac{\langle\psi(0)|\psi(S)\rangle}{\langle\psi(S)|\psi(0)\rangle}\right)^{1/2} \\
&\quad \cdot \exp\left\{-i\int_0^S \frac{d}{ds}f(s)ds - \int_0^S \langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds\right\} = \\
&= e^{i\gamma[\lambda]},
\end{aligned} \tag{7}$$

and $\gamma[\lambda]$ is indeed only a function of the path λ in projective Hilbert space. Incidentally, one sees from (7) that, taken separately, neither φ nor δ are projective quantities.

Secondly, $\gamma[\lambda]$ is *reparametrisation invariant*, in the sense that under any strictly monotonical transformation $s \mapsto \sigma$, the expression (2) is invariant, as can be easily seen by noticing that $\frac{d}{d\sigma} = \frac{d}{ds} \cdot \frac{ds}{d\sigma}$ and $d\sigma = ds \cdot \frac{d\sigma}{ds}$.

We shall now recall what is perhaps the most natural geometric interpretation of the phase $\gamma[\lambda]$. This is provided by consideration of a *connection* (or, alternatively, of a *parallel transport*) on the Hopf bundle over the ray space. The Hopf bundle is the fibre bundle over ray space whose fibre over a ray consists of all *unit* vectors $|\psi\rangle$ in the ray. A *parallel transport* of a unit vector field $|\psi(s)\rangle$ along a curve λ in projective Hilbert space is, roughly speaking, a rule that associates to any vector $|\psi(0)\rangle$ a corresponding vector $|\psi(s)\rangle$ in the fibre over any other point of the curve. That is, it has to specify a *phase* for $|\psi(s)\rangle$. One such rule is provided by the condition[2]

$$\langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle = 0. \tag{8}$$

If (8) is not satisfied, then $|\psi(s)\rangle$ is picking up an additional phase during the (non-parallel) transport along the curve. In fact, if $\langle\varphi(s)|\frac{d}{ds}|\varphi(s)\rangle = 0$ and $|\psi(s)\rangle = e^{i\theta(s)}|\varphi(s)\rangle$, then

$$\begin{aligned}\langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle &= \langle\varphi(s)|e^{-i\theta(s)}\Big[i\frac{d\theta(s)}{ds}e^{i\theta(s)}|\varphi(s)\rangle + e^{i\theta(s)}\frac{d}{ds}|\varphi(s)\rangle\Big] = \\ &= i\frac{d\theta(s)}{ds}\langle\varphi(s)|\varphi(s)\rangle + \langle\varphi(s)|\frac{d}{ds}|\varphi(s)\rangle = \\ &= i\frac{d}{ds}\theta(s), \end{aligned} \tag{9}$$

so that this additional phase is precisely given by $\delta = -i\int_0^S\langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle ds$, that is (5). This justifies calling (5) a locally accumulated phase. Since in the definition (2) of $\gamma[\lambda]$ this phase is subtracted from the total phase (4), it is clear that $\gamma[\lambda]$ is the phase that the vector $|\psi(s)\rangle$ would have *globally* acquired if it had been parallel transported along the curve λ, and thus it is indeed a purely geometric phase. In fact, quite analogously to the case of Riemannian manifolds, this globally acquired geometric phase can be explained in terms of *curvature* of the ray space (see e. g. Page (1987)). This is more easily seen in the case of the AA phase, where λ is a closed curve, but is equally valid in the case of the AW phase, where one can close the curve λ by joining its endpoints through a geodesic (along which it can be shown that the geometric phase is zero), as in Samuel and Bhandari (1988).

3 Non-Invariance of Geometric Phase

Physical time t is not in general an appropriate parameter for a curve λ, since a mapping $t \mapsto \pi(|\psi(t)\rangle)$ need not be locally one-to-one. Nonetheless, it is

[2] If the fibre is not restricted to the unit vectors, one obtains the *line bundle* over projective Hilbert space. In this case, the real part of (8) becomes a non-trivial requirement that the *norm* of $|\psi(s)\rangle$ be constant during parallel transport. Considering the line bundle is more in tune with definition (1) as given by AW.

easy to show that if $\pi(|\psi(t)\rangle)$ is constant over a certain time interval, then the phase given by (2) is zero for that time interval. Thus (2) is an expression for the geometric phase $\gamma[\lambda]$ even when λ is expressed as a function of t:

$$e^{i\gamma[\lambda]} = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt\right\}, \qquad (10)$$

(Only in this case maybe should the phase corresponding to (5) be called the *dynamic phase.*) Further, the expression (2) is independent of the choice of curve in Hilbert space also when t is substituted for s in (6); and it is invariant under *time rescalings*, i.e. under strictly monotonical transformations $t \mapsto \tau$.

If λ is a function of time, we can consider the special case in which there is a curve $t \mapsto |\psi(t)\rangle$ in Hilbert space projecting to λ that is given by the Schrödinger evolution of a state $|\psi(0)\rangle$ with Hamiltonian H:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \qquad (11)$$

(where H will generally also be time-dependent). In this case, the geometric phase can be written as

$$e^{i\gamma[\lambda]} = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{\frac{i}{\hbar}\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt\right\}. \qquad (12)$$

Time rescaling invariance is manifest under the transformation $t \mapsto \tau$ if one considers that the Hamiltonian, in order to preserve covariance of the Schrödinger equation, transforms as $\tilde{H} = \frac{dt}{d\tau}H$. And since the geometric phase is a function of the path λ alone, we can understand it as a *universal* property of the dynamics, in the sense that if the curve λ is induced by a Schrödinger evolution, it will in fact be independent of the particular Hamiltonian that drives the system along the path λ. As we have seen, this phase is further geometric in the strong sense of being determined (indeed being a measure of) the curvature of projective Hilbert space. Notice that the expression (12) is not itself projective, but only coincides with (2) if one chooses the one curve in Hilbert space projecting to λ which is also a solution to the Schrödinger equation with the given H. If one takes (6) to be an *active* transformation, it will change the actual curve in Hilbert space, and while $\frac{d}{dt}$ will still be perfectly well-defined for that Hilbert-space curve, it will no longer be given by (11).

If the parameter t is physical time, one can enquire whether the geometric phase $\gamma[\lambda]$ is invariant under a number of (passive) *time-dependent unitary transformations* of the Hilbert space, such as *gauge* and *Galilei* transformations. (It is trivial to show that $\gamma[\lambda]$ is invariant under arbitrary time-*independent* unitary transformations.) The intuitive reason for *not* expecting geometric phases to be invariant in general under time-dependent unitary transformations is that the geometric phase is a function $\gamma[\lambda]$ of a path λ in projective Hilbert space, but such a path is clearly not an invariant object under general time-dependent unitary transformations, just as a time-dependent trajectory in *space* is not invariant under general *space–time* transformations. Indeed, in constructing an invariant geometric phase, we shall be pursuing the analogy with the space–time path. Typically, under such a transformation $U(t)$, a closed path $\lambda : [0,T] \to \mathcal{P}(\mathcal{H})$ in projective Hilbert space will be mapped to an *open* path, since generally $U(T) \neq U(0)$ and $U(T)\lambda(T) \neq U(0)\lambda(0)$. And even if we consider differences $\lambda - \mu$ of two open paths with the same endpoints,

$$\lambda \ : \ [0,T] \to \mathcal{P}(\mathcal{H}), \tag{13}$$

$$\mu \ : \ [0,T] \to \mathcal{P}(\mathcal{H}), \tag{14}$$

with

$$\lambda(0) = \mu(0), \quad \lambda(T) = \mu(T) \tag{15}$$

(i. e. $\lambda - \mu : [0,S] \to \mathcal{P}(\mathcal{H})$ proceeds along λ from $\lambda(0)$ to $\lambda(T) = \mu(T)$ and back along μ to $\mu(0) = \lambda(0)$), such a closed loop will be indeed mapped to a closed loop, but to a *different* one. Notice that along an open path not even the *total* phase is invariant. Instead for a difference $\lambda - \mu$ the total phase is invariant under arbitrary time-dependent unitary transformations, being directly related to the (observable) interference between the two paths λ and μ.[3] It is, indeed, only for such path differences (as emphasised by García de Polavieja (1997)) that the problem of the invariance of the geometric phase becomes interesting, or rather the problem of how to split invariantly the total phase into a geometric and a dynamic part.

Let us now look at specific time-dependent transformations. The case

[3] A projective expression for the total phase will be given below in Section 5.

of global gauge transformations is unproblematic. In fact, a global gauge transformation,

$$|\tilde{\psi}(t)\rangle := e^{iq\xi(t)/\hbar}|\psi(t)\rangle \tag{16}$$

(where q is the charge of the particle), induces a simultaneous transformation of the type (6) on all curves $t \in [0,T] \mapsto \pi(|\psi(t)\rangle$; and we have seen that (2) is invariant under such transformations. In this case, also expression (12) is invariant, since under the (passive) transformation (16), the Hamiltonian transforms as

$$\tilde{H} = e^{iq\xi(t)/\hbar} H e^{-iq\xi(t)/\hbar} - q\frac{d\xi(t)}{dt}. \tag{17}$$

This is because whenever we consider time-dependent unitary transformations $U : |\psi\rangle \mapsto |\tilde{\psi}\rangle = U|\psi\rangle$, the Leibniz rule gives

$$\frac{d}{dt}|\tilde{\psi}(t)\rangle = \Big(U\frac{d}{dt}U^{-1} + \frac{dU}{dt}U^{-1}\Big)|\tilde{\psi}(t)\rangle. \tag{18}$$

Thus, from (11) and (18), the Hamiltonian must transform as

$$\tilde{H} = UHU^{-1} + i\hbar\frac{dU}{dt}U^{-1}, \tag{19}$$

to ensure covariance of the Schrödinger equation. In particular, H transforms exactly as $i\hbar\frac{d}{dt}$, and one can use indifferently the representations (2) and (12) for the geometric phase. However, in the case of *local* gauge transformations and of *Galilei* transformations, the geometric phase is not invariant, as we can see as follows.

Take a (time-dependent) local gauge transformation as given by

$$|\tilde{\psi}(t)\rangle := e^{iq\xi(\mathbf{Q},t)/\hbar}|\psi(t)\rangle. \tag{20}$$

From (18) one has that

$$\begin{aligned} e^{i\gamma[\tilde{\lambda}]} &= \left(\frac{\langle\psi(0)|e^{-iq\xi(\mathbf{Q},0)/\hbar}e^{iq\xi(\mathbf{Q},T)/\hbar}|\psi(T)\rangle}{\langle\psi(T)|e^{-iq\xi(\mathbf{Q},T)/\hbar}e^{iq\xi(\mathbf{Q},0)/\hbar}|\psi(0)\rangle}\right)^{1/2} \\ &\quad\cdot \exp\left\{-\int_0^T \langle\psi(t)|\frac{d}{dt} + \frac{iq}{\hbar}\frac{\partial\xi(\mathbf{Q},t)}{\partial t}|\psi(t)\rangle dt\right\} = \end{aligned}$$

$$
\begin{aligned}
&= e^{i\gamma[\lambda]} \cdot \left(\frac{\langle\psi(0)|e^{-iq\xi(\mathbf{Q},0)/\hbar} e^{iq\xi(\mathbf{Q},T)/\hbar}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \right. \\
&\quad \left. \cdot \frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-iq\xi(\mathbf{Q},T)/\hbar} e^{iq\xi(\mathbf{Q},0)/\hbar}|\psi(0)\rangle} \right)^{1/2} \\
&\quad \cdot \exp\left\{ -\frac{iq}{\hbar} \int_0^T \langle\psi(t)| \frac{\partial\xi(\mathbf{Q},t)}{\partial t} |\psi(t)\rangle dt \right\}. \qquad (21)
\end{aligned}
$$

But now, while the middle factor on the right-hand side of (21) is a function of the endpoints of λ alone (in projective Hilbert space!), and so drops out if we consider closed loops, say $\lambda - \mu$, it is quite clear that $\int_0^T \langle\psi(t)|\frac{\partial\xi(\mathbf{Q},t)}{\partial t}|\psi(t)\rangle dt$ is a function of the actual path, so that in general for differences $\lambda - \mu$ the last factor in (21) will be different from one.[4]

The case of Galilei transformations has recently been treated by Sjöqvist, Brown and Carlsen (1997). SBC, as we shall henceforth call them, consider a system with Hamiltonian

$$
H = \frac{(\mathbf{P} - \frac{q}{c}\mathbf{A})^2}{2m} + V, \qquad (22)
$$

with a vector potential $\mathbf{A}$ and a scalar potential V, both generally functions of $\mathbf{Q}$ and t, and where again q denotes the charge of the particle. SBC take a passive Galilei boost to be implemented by the unitary transformation[5]

$$
\begin{aligned}
|\tilde{\psi}(t)\rangle &:= U(t)|\psi(t)\rangle := e^{-i\mathbf{v}(m\mathbf{Q}-\mathbf{P}t)/\hbar} = \\
&= e^{-im\mathbf{v}^2 t/2\hbar} e^{-im\mathbf{v}\cdot\mathbf{Q}/\hbar} e^{i\mathbf{v}\cdot\mathbf{P}t/\hbar}. \qquad (23)
\end{aligned}
$$

By using (12), (19), (22), (23) and the identities

$$
U^{-1}\mathbf{Q}U = \mathbf{Q} - \mathbf{v}t\mathbf{1},
$$

[4] Another way of seeing that the middle factor disappears is to notice that because of the projective nature of the geometric phase, the representations in Hilbert space of the two curves λ and μ can be chosen to have common endpoints. This amounts to choosing a single-valued representation in $\mathcal{H}$ of $\lambda - \mu$ seen as a closed loop in $\mathcal{P}(\mathcal{H})$, so that the geometric phase can be computed using (3). For a single path λ, it is clear that by varying $\frac{\partial\xi(\mathbf{Q},t)}{\partial t}$ one can fix the value of $\gamma[\lambda]$ quite arbitrarily.

[5] See e.g. Fonda and Ghirardi (1970, §2.5). Here, as below in (60), the rest frame and the moving frame coincide at $t = t' = 0$ (standard configuration). Finally, notice that for operators A and B that commute with their commutator, the identity $e^{A+B} = e^A e^B e^{-[A,B]/2}$ holds.

$$U^{-1}\mathbf{P}U \quad = \quad \mathbf{P} - m\mathbf{v}\mathbf{1}, \tag{24}$$

and

$$U^{-1}\frac{dU}{dt} = -\frac{i}{\hbar}\Big(\frac{m\mathbf{v}^2}{2} - \mathbf{v}\cdot\mathbf{P}\Big), \tag{25}$$

they then calculate $\gamma[\tilde{\lambda}]$ as

$$\begin{aligned} e^{i\gamma[\tilde{\lambda}]} \quad &= \quad e^{i\gamma[\lambda]}\cdot\left(\frac{\langle\psi(0)|e^{i\mathbf{v}\cdot\mathbf{P}T}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\cdot\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{v}\cdot\mathbf{P}T}|\psi(0)\rangle}\right)^{1/2} \\ &\quad\cdot\exp\left\{-\frac{i}{\hbar}\int_0^T\langle\psi(t)|\mathbf{v}\cdot\mathbf{P}|\psi(t)\rangle dt\right\}. \end{aligned} \tag{26}$$

Using Ehrenfest's theorem,

$$\langle\psi(t)|\mathbf{P}|\psi(t)\rangle = \frac{q}{c}\langle\psi(t)|\mathbf{A}|\psi(t)\rangle + m\frac{d}{dt}\langle\psi(t)|\mathbf{Q}|\psi(t)\rangle, \tag{27}$$

expression (26) becomes

$$\begin{aligned} e^{i\gamma[\tilde{\lambda}]} \quad &= \quad e^{i\gamma[\lambda]}\cdot\left(\frac{\langle\psi(0)|e^{i\mathbf{v}\cdot\mathbf{P}T}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\cdot\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{v}\cdot\mathbf{P}T}|\psi(0)\rangle}\right)^{1/2} \\ &\quad\cdot\exp\left\{-\frac{iq}{\hbar c}\int_0^T\langle\psi(t)|\mathbf{v}\cdot\mathbf{A}|\psi(t)\rangle dt\right\} \\ &\quad\cdot\exp\left\{-\frac{im}{\hbar}\Big[\langle\psi(T)|\mathbf{v}\cdot\mathbf{Q}|\psi(T)\rangle - \langle\psi(0)|\mathbf{v}\cdot\mathbf{Q}|\psi(0)\rangle\Big]\right\}. \end{aligned} \tag{28}$$

But now again, one sees that while the second and fourth factor on the right-hand side of (28) are (projective) functions of the endpoints of λ only, and thus drop out when considering differences $\lambda-\mu$, the factor containing $\int_0^T\langle\psi(t)|\mathbf{v}\cdot\mathbf{A}|\psi(t)\rangle dt \neq 0$ depends on the actual path, so that in general it will be different from one for differences $\lambda-\mu$. Thus, both for local gauge transformations and for Galilei transformations, geometric phases are generally *not* invariant, not even for closed loops.[6]

[6] These non-invariance results refute a claim by Kendrick (1992) to the effect that geometric phases would be invariant under *arbitrary* time-dependent unitary transformations $U(t)$. Taking Galilei transformations to fix the ideas, Kendrick in effect chooses a frame in which geometric phases are defined by means of (12), while in any other frame, they are given by a formula that essentially transforms back to the initial frame and explicilty depends on the transformation $U(t)$, that is, on knowledge of one's state of motion with respect to the preferred frame! Thus, although it allows one to compute the same *number* in different frames, Kendrick's definition of the geometric phase is plainly *not* invariant.

4 Previous Treatments of Non-Invariance

It was already seen very clearly by Aharonov and Anandan (1987) that geometric phases need not be invariant under time-dependent unitary transformations. In that paper and, in an improved form (which we shall more or less follow here), in Anandan and Aharonov (1988) they put forth a gauge-invariant definition of geometric phase, which is formally very similar to, if different in spirit from, the one we shall present below.

Anandan and Aharonov (1988) define a *gauge-invariant* state as:

$$|\hat{\psi}(t)\rangle := \exp\left\{\frac{iq}{\hbar}\int_{-\infty}^{t} A_0(t')dt'\right\}|\psi(t)\rangle, \tag{29}$$

where $A_0(t')$ is the electric potential. They then apply the usual definition of the geometric phase (here we take (2)) to the gauge-invariant state $|\hat{\psi}(t)\rangle$. Call the result $e^{i\alpha[\lambda]}$:

$$e^{i\alpha[\lambda]} := \left(\frac{\langle\hat{\psi}(0)|\hat{\psi}(T)\rangle}{\langle\hat{\psi}(T)|\hat{\psi}(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^T \langle\hat{\psi}(t)|\frac{d}{dt}|\hat{\psi}(t)\rangle dt\right\}. \tag{30}$$

This is manifestly gauge-invariant. AA apply their formula only to paths that are *cyclic* in the specific sense that $|\hat{\psi}(T)\rangle = e^{i\theta}|\hat{\psi}(0)\rangle$, but the above formula clearly provides a gauge-invariant definition also in the non-cyclic case. If $|\psi(t)\rangle$ satisfies the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \left[\frac{(\mathbf{P}-\frac{q}{c}\mathbf{A})^2}{2m} + qA_0\right]|\psi(t)\rangle, \tag{31}$$

then $|\hat{\psi}(t)\rangle$ satisfies

$$i\hbar\frac{d}{dt}|\hat{\psi}(t)\rangle = \frac{(\mathbf{P}-\frac{q}{c}\mathbf{A})^2}{2m}|\hat{\psi}(t)\rangle, \tag{32}$$

where $H_k := \frac{(\mathbf{P}-\frac{q}{c}\mathbf{A})^2}{2m}$ is the *kinetic part* of the Hamiltonian, so that (30) can also be written as

$$e^{i\alpha[\lambda]} = \left(\frac{\langle\hat{\psi}(0)|\hat{\psi}(T)\rangle}{\langle\hat{\psi}(T)|\hat{\psi}(0)\rangle}\right)^{1/2} \cdot \exp\left\{\frac{i}{\hbar}\int_0^T \langle\hat{\psi}(t)|H_k|\hat{\psi}(t)\rangle dt\right\}. \tag{33}$$

We shall construct our definition below to be numerically equivalent to (30) for *closed* loops (as those considered by AA), but with a quite different interpretation. Jeeva Anandan (1997) informs me of the possibility of basing a Galilei-invariant geometric phase on the definition of a Galilei-invariant state (analogous to (29)). However, this is no longer equivalent to the proposal below.

In an article prompted by SBC (1997), García de Polavieja (1997) proposes to consider only *phase differences* and to choose a special gauge in which $\mathbf{v}\cdot\mathbf{A}=0$ (for given $\mathbf{v}$: it is always possible to choose a gauge such that a particular component of the vector potential vanishes). This will make the third factor on the right-hand side of (28) vanish, so that the geometric phase around a closed loop $\lambda-\mu$ will be, indeed, invariant. A subtlety in García de Polavieja's treatment is that since the derivation of (28) assumes Ehrenfest's theorem (27), the vector potential $\mathbf{A}$ appearing in (28) must in fact be the one appearing in the Hamiltonian that drives the system along the path λ. If however λ and μ are both defined on $[0,T]$, then there is no guarantee that $\mathbf{v}\cdot\mathbf{A}=0$ may be gauged away along *both* paths. Instead, García de Polavieja considers the difference of two *successive* paths $\lambda:[0,T]\rightarrow\mathcal{P}(\mathcal{H})$ and $\mu:[T,T']\rightarrow\mathcal{P}(\mathcal{H})$ with common endpoints $\lambda(0)=\mu(T)$ and $\lambda(T)=\mu(T')$, and such that $\lambda(T)=\mu(T)$ (so that, in fact, the closed path is a *double* loop in projective Hilbert space). In this case, the geometric phase along $\lambda-\mu$ in the given gauge will be indeed invariant under all Galilei boosts in the direction of $\mathbf{v}$.

5 The Extended Geometric Phase

In this section and the next, we shall define a splitting of the total phase into two parts, analogous to the geometric phase and the locally accumulated phase, that is indeed gauge- and Galilei-invariant. We shall then see in Section 7 how this 'geometric' phase can further be associated with a notion of parallel transport on a *parametrised* projective Hilbert space $\mathcal{P}(\mathcal{H})_t$, by which I mean a fibre bundle over the time axis with fibres isomorphic to $\mathcal{P}(\mathcal{H})$. Then in Section 8 we shall sketch the classical analogue in Hamiltonian mechanics. Finally, in Section 9, we shall apply our newly defined phase to Berry's (1984) calculation of the Aharonov–Bohm effect for a box transported around a solenoid.

The easiest way to introduce our phase is to start with AA's proposal (30). The mapping (29) from $|\psi(t)\rangle$ to $|\hat{\psi}(t)\rangle$ has the form of a local gauge transformation. (It is the transformation to a gauge in which the electric potential vanishes.) Applying the transformation formula (21) for the geometric phase under local gauge transformations, we can write (30) in terms of the original states $|\psi(t)\rangle$ as:

$$e^{i\alpha[\lambda]} = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^T \langle\psi(t)|\frac{d}{dt} + \frac{iq}{\hbar}A_0|\psi(t)\rangle dt\right\}. \quad (34)$$

For a closed loop $\lambda - \mu$ we can write, analogously to (3):

$$\begin{aligned} \alpha[\lambda] &= i\oint_{\lambda-\mu} \langle\psi(t)|\frac{d}{dt} + \frac{iq}{\hbar}A_0|\psi(t)\rangle dt = \\ &= i\int_0^T \langle\psi(t)|\frac{d}{dt} + \frac{iq}{\hbar}A_0|\psi(t)\rangle dt \\ &\quad +i\int_0^T \langle\varphi(t)|\frac{d}{dt} + \frac{ie}{\hbar}A_0|\varphi(t)\rangle dt \end{aligned} \quad (35)$$

where $|\psi(t)\rangle$ and $|\varphi(t)\rangle$ are representations of λ and μ which agree at $t = 0$ and $t = T$. We shall presently check that (34), although not gauge-invariant for open paths (unlike (30)), is gauge-invariant for closed loops $\lambda - \mu$. It is clear that in this case it is numerically equal to AA's gauge-invariant phase.

In the special case in which the Hamiltonian is given by

$$H = \frac{(\mathbf{P} - \frac{q}{c}\mathbf{A})^2}{2m} + qA_0, \quad (36)$$

the phase (34) becomes

$$e^{i\alpha[\lambda]} = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^T \langle\psi(t)|H_k|\psi(t)\rangle dt\right\}, \quad (37)$$

where H_k is again the *kinetic part* of the Hamiltonian (36). The phase in the second factor on the right-hand side of (37), call it $\kappa[\lambda]$, may be aptly called the *kinetic phase* rather than the dynamic phase.

The phase (34) differs from (2) by a factor which is the exponential of the integral of an expectation value. Since expectation values are projective quantities, (37) is thus also *projective*, in the sense that it does not depend on the particular curve $t \mapsto |\psi(t)\rangle$ in Hilbert space used to define it. It is further *time rescaling* invariant, since it is clear that in order to maintain covariance of the Schrödinger equation A_0 transforms under $t \mapsto \tau$ to $\tilde{A}_0 = \frac{dt}{d\tau} A_0$.

Notice that, despite being independent of the curve in Hilbert space and of the choice of the time parameter, and thus being a function of the path λ, the phase $\alpha[\lambda]$, unlike $\gamma[\lambda]$, is not a function of λ alone. In fact, it need not be zero for a time interval in which $\pi(|\psi(t)\rangle)$ is constant. This precludes an interpretation in terms of the geometry of projective Hilbert space $\mathcal{P}(\mathcal{H})$ but, as we shall discuss in Section 7, it is readily understandable in terms of the analogous geometric structures on the parametrised ray space $\mathcal{P}(\mathcal{H})_t$. Also, the phase $\alpha[\lambda]$ truly depends on the *Hamiltonian* that drives the system along the path λ (and requires the existence of a Hamiltonian of the correct form (36), or (38) below). In this sense, $\alpha[\lambda]$ is no longer a universal aspect of the dynamics.[7]

If the Hamiltonian contains a further scalar potential, say V_0, besides the electric potential A_0,

$$H = \frac{(\mathbf{P} - \frac{q}{c}\mathbf{A})^2}{2m} + V, \tag{38}$$

with $V = qA_0 + V_0$, it is also quite natural to use the full potential V in the definition (34), so that it now reads

$$e^{i\beta[\lambda]} := \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle} \right)^{1/2} \cdot \exp\left\{ -\int_0^T \langle\psi(t)|\frac{d}{dt} + \frac{i}{\hbar} V |\psi(t)\rangle dt \right\}, \tag{39}$$

or

$$\alpha[\lambda] = i \oint_{\lambda-\mu} \langle\psi(t)|\frac{d}{dt} + \frac{i}{\hbar} V |\psi(t)\rangle dt \tag{40}$$

for a closed loop. If $V_0 \neq 0$, this ensures that (37) is valid. We shall see that the phase $\beta[\lambda]$ shares many of the invariance properties that hold for $\alpha[\lambda]$.

[7]Notice that it is only required that $|\psi(t)\rangle$ and $|\varphi(t)\rangle$ be any representations in $\mathcal{H}$ of paths in $\mathcal{P}(\mathcal{H})$ that arise through a Schrödinger evolution. Because $\alpha[\lambda]$ is projective, the actual representations of λ and μ need not (both) be solutions of the respective Schrödinger equations.

Finally, one should point out that if one defines a phase $\tau[\lambda]$ analogously to $\alpha[\lambda]$, but with the Hamiltonian H in place of the electric potential A_0,

$$e^{i\tau[\lambda]} := \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^T \langle\psi(t)|\frac{d}{dt} + \frac{i}{\hbar}H|\psi(t)\rangle dt\right\}, \quad (41)$$

then (since if $\frac{d}{dt}$ is given by the Schrödinger evolution, $\frac{d}{dt} + \frac{i}{\hbar}H = 0$) one has that $\tau[\lambda]$ is a projective representation of the *total phase*, which, as we have noticed in Section 2, is not projective when written as (4). The representation (41) allows us to consider, say, differences in total phases $\tau[\lambda - \mu]$ while freely choosing the representatives in $\mathcal{H}$ of the paths λ and μ.

6 Invariance Properties

We shall now derive the invariance properties of $\alpha[\lambda]$. Namely, we shall show that differences $\alpha[\lambda - \mu]$, where λ and μ have common endpoints in ray space, are invariant under all *gauge transformations* and *Galilei boosts* and under *transformations to linearly accelerated coordinate systems.*

Let a (global or) local gauge transformation be given by

$$|\tilde{\psi}(t)\rangle := U(t)|\psi(t)\rangle := e^{iq\xi(\mathbf{Q},t)/\hbar}|\psi(t)\rangle. \quad (42)$$

By (18) and (42), one has

$$\frac{d}{dt} = U\frac{d}{dt} + \frac{iq}{\hbar}\frac{d\xi(\mathbf{Q},t)}{dt}. \quad (43)$$

Further, it is well-known that under (42) the Hamiltonian (22) transforms as

$$\tilde{H} = \frac{(\mathbf{P} - q\nabla\xi(\mathbf{Q},t) - \frac{q}{c}\mathbf{A})^2}{2m} + V - q\frac{d\xi(\mathbf{Q},t)}{dt} \quad (44)$$

(in fact, this follows from (19)), so that in particular

$$\tilde{V} = V - q\frac{d\xi(\mathbf{Q},t)}{dt}. \quad (45)$$

If $V = qA_0 + V_0$ we can write

$$\tilde{A}_0 = A_0 - \frac{d\xi(\mathbf{Q},t)}{dt} \quad (46)$$

and

$$\tilde{V}_0 = V_0. \tag{47}$$

In fact, (46) is the standard transformation behaviour for the electric potential, and it is indeed obvious that $\frac{d\xi(\mathbf{Q},t)}{dt}$ must be part of the *electric* potential, since in (45) the particle couples to it through its charge q.

From (43) and (45) it follows that

$$\begin{aligned}
\frac{d}{dt} + \frac{iq}{\hbar}\tilde{A}_0 &= U\frac{d}{dt}U^{-1} + \frac{iq}{\hbar}\frac{d\xi(\mathbf{Q},t)}{dt} + \frac{iq}{\hbar}A_0 - \frac{iq}{\hbar}\frac{d\xi(\mathbf{Q},t)}{dt} = \\
&= U\frac{d}{dt}U^{-1} + \frac{iq}{\hbar}A_0 = \\
&= U\Big(\frac{d}{dt} + \frac{iq}{\hbar}A_0\Big)U^{-1}
\end{aligned} \tag{48}$$

(since A_0 is a function of $\mathbf{Q}$ and thus, by (42), commutes with U).

Inserting (48) into (39) we finally obtain

$$\begin{aligned}
e^{i\alpha[\tilde{\lambda}]} &= e^{i\alpha[\lambda]} \cdot \Bigg(\frac{\langle\psi(0)|e^{-iq\xi(\mathbf{Q},0)/\hbar}e^{iq\xi(\mathbf{Q},T)/\hbar}|\psi(t)\rangle}{\langle\psi(0)|\psi(T)\rangle} \\
&\quad \cdot \frac{\langle\psi(T)|e^{-iq\xi(\mathbf{Q},T)/\hbar}e^{iq\xi(\mathbf{Q},0)/\hbar}|\psi(0)\rangle}{\langle\psi(T)|\psi(0)\rangle}\Bigg)^{1/2},
\end{aligned} \tag{49}$$

where the second factor on the right-hand side is projective and a function only of the endpoints of λ, so that for two curves in ray space λ and μ with common endpoints one has

$$e^{i\alpha[\tilde{\lambda}-\tilde{\mu}]} = e^{i\alpha[\lambda-\mu]}. \tag{50}$$

Because of (45), V can be substituted for qA_0 throughout (48), and thus the analogous invariance under gauge transformations holds also for $\beta[\lambda-\mu]$.

In order to prove that $\alpha[\lambda-\mu]$ is invariant also under Galilei boosts, we note that under a Galilei boost (23), a Hamiltonian of the form (22) transforms as

$$\tilde{H} = \frac{(\mathbf{P} - \frac{q}{c}U\mathbf{A}U^{-1})^2}{2m} + U(V - \frac{q}{c}\mathbf{v}\cdot\mathbf{A})U^{-1}, \tag{51}$$

as can be shown for instance using (19), (24) and (25) (for details see Takagi (1991) and Brown and Holland (1997)). In particular, the scalar potential V transforms as

$$\tilde{V} = U(V - \frac{q}{c}\mathbf{v}\cdot\mathbf{A})U^{-1}. \tag{52}$$

Again, since the coupling constant appearing in $-\frac{q}{c}U\mathbf{v}\cdot\mathbf{A}U^{-1}$ is proportional to the charge q, this can be recognised as part of the electric potential with respect to the new frame, so we can write also

$$\tilde{A}_0 = U(A_0 - \frac{q}{c}\mathbf{v}\cdot\mathbf{A})U^{-1} \tag{53}$$

and

$$\tilde{V}_0 = UV_0U^{-1}. \tag{54}$$

Given that, comparing (2) and (39),

$$e^{i\alpha[\lambda]} = e^{i\gamma[\lambda]} \cdot \exp\left\{-\frac{iq}{\hbar}\int_0^T \langle\psi(t)|A_0|\psi(t)\rangle dt\right\}, \tag{55}$$

the transformation formula (28) for $\gamma[\lambda]$ yields:

$$\begin{aligned} e^{i\alpha[\tilde{\lambda}]} \quad = \quad & e^{i\gamma[\lambda]} \cdot \left(\frac{\langle\psi(0)|e^{i\mathbf{v}\cdot\mathbf{P}T}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \cdot \frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{v}\cdot\mathbf{P}T}|\psi(0)\rangle}\right)^{1/2} \\ & \cdot \exp\left\{-\frac{iq}{\hbar c}\int_0^T \langle\psi(t)|\mathbf{v}\cdot\mathbf{A}|\psi(t)\rangle dt\right\} \\ & \cdot \exp\left\{-\frac{im}{\hbar}\Big[\langle\psi(T)|\mathbf{v}\cdot\mathbf{Q}|\psi(T)\rangle - \langle\psi(0)|\mathbf{v}\cdot\mathbf{Q}|\psi(0)\rangle\Big]\right\} \\ & \cdot \exp\left\{-\frac{iq}{\hbar}\int_0^T \langle\tilde{\psi}(t)|\tilde{A}_0|\tilde{\psi}(t)\rangle dt\right\}, \end{aligned} \tag{56}$$

where now $|\tilde{\psi}(t)\rangle$ is given by (23).

From (53) we thus have

$$e^{i\alpha[\tilde{\lambda}]} \quad = \quad e^{i\gamma[\lambda]} \cdot \left(\frac{\langle\psi(0)|e^{i\mathbf{v}\cdot\mathbf{P}T}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \cdot \frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{v}\cdot\mathbf{P}T}|\psi(0)\rangle}\right)^{1/2}$$

$$
\begin{aligned}
&\cdot \exp\left\{-\frac{iq}{\hbar c}\int_0^T \langle\psi(t)|\mathbf{v}\cdot\mathbf{A}|\psi(t)\rangle dt\right\} \\
&\cdot \exp\left\{-\frac{im}{\hbar}\Big[\langle\psi(T)|\mathbf{v}\cdot\mathbf{Q}|\psi(T)\rangle - \langle\psi(0)|\mathbf{v}\cdot\mathbf{Q}|\psi(0)\rangle\Big]\right\} \\
&\cdot \exp\left\{-\frac{iq}{\hbar}\int_0^T \langle\psi(t)|A_0|\psi(t)\rangle dt\right\} \\
&\cdot \exp\left\{\frac{iq}{\hbar c}\int_0^T \langle\psi(t)|\mathbf{v}\cdot\mathbf{A}|\psi(t)\rangle dt\right\}, \qquad (57)
\end{aligned}
$$

or, again using (55), and noticing that the two factors containing the vector potential cancel out:

$$
\begin{aligned}
e^{i\alpha[\tilde{\lambda}]} &= e^{i\alpha[\lambda]}\cdot\left(\frac{\langle\psi(0)|e^{i\mathbf{v}\cdot\mathbf{P}T}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\cdot\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{v}\cdot\mathbf{P}T}|\psi(0)\rangle}\right)^{1/2} \\
&\quad\cdot \exp\left\{-\frac{im}{\hbar}\Big[\langle\psi(T)|\mathbf{v}\cdot\mathbf{Q}|\psi(T)\rangle - \langle\psi(0)|\mathbf{v}\cdot\mathbf{Q}|\psi(0)\rangle\Big]\right\}. \qquad (58)
\end{aligned}
$$

Again, given (52), V can be substituted throughout for qA_0. As before, the transformation behaviour of $\alpha[\lambda]$ depends only on the endpoints of λ, and one has

$$
e^{i\alpha[\tilde{\lambda}-\tilde{\mu}]} = e^{i\alpha[\lambda-\mu]} \qquad (59)
$$

for arbitrary Galilei boosts and any closed loops enclosed by curves λ and μ in projective Hilbert space. (The same is true of $\beta[\lambda-\mu]$.)

The invariant behaviour of $\alpha[\lambda-\mu]$ generalises further to transformations to *linearly accelerating* frames, as can be seen as follows. First of all, the unitary operator implementing a transformation to a linearly accelerated frame is given by

$$
U(t) := e^{-im\int_0^t \mathbf{v}(u)^2 du/2\hbar}e^{-im\mathbf{v}(t)\cdot\mathbf{Q}/\hbar}e^{i\mathbf{r}(t)\cdot\mathbf{P}/\hbar} \qquad (60)
$$

(compare (23)), where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are the position and velocity of the origin of the moving frame, respectively (see Takagi (1991, §3)).

SBC's (1997) derivation of the transformation behaviour (28) of $\gamma[\lambda]$ is virtually unaltered. It is enough to substitute a time-dependent velocity

$\mathbf{v}(t)$. One has then:

$$\begin{aligned} e^{i\gamma[\tilde{\lambda}]} &= e^{i\gamma[\lambda]} \cdot \left(\frac{\langle\psi(0)|e^{i\mathbf{r}(T)\cdot\mathbf{P}}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \cdot \frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{r}(T)\cdot\mathbf{P}}|\psi(0)\rangle} \right)^{1/2} \\ &\quad \cdot \exp\left\{ -\frac{i}{\hbar}\int_0^T \langle\psi(t)|\mathbf{v}(t)\cdot\mathbf{P}|\psi(t)\rangle dt \right\}. \end{aligned} \tag{61}$$

Using again (27), one obtains

$$\begin{aligned} e^{i\gamma[\tilde{\lambda}]} &= e^{i\gamma[\lambda]} \cdot \left(\frac{\langle\psi(0)|e^{i\mathbf{r}(T)\cdot\mathbf{P}}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \cdot \frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-i\mathbf{r}(T)\cdot\mathbf{P}}|\psi(0)\rangle} \right)^{1/2} \\ &\quad \cdot \exp\left\{ -\frac{iq}{\hbar c}\int_0^T \langle\psi(t)|\mathbf{v}(t)\cdot\mathbf{A}|\psi(t)\rangle dt \right\} \\ &\quad \cdot \exp\left\{ -\frac{im}{\hbar}\Big[\langle\psi(T)|\mathbf{v}(T)\cdot\mathbf{Q}|\psi(T)\rangle - \langle\psi(0)|\mathbf{v}(0)\cdot\mathbf{Q}|\psi(0)\rangle \Big] \right\}. \end{aligned} \tag{62}$$

The transformation behaviour of the Hamiltonian (22) under (60) is no longer given by (51), not even with a time-dependent $\mathbf{v}(t)$, but by

$$\tilde{H} = \frac{(\mathbf{P} - \frac{q}{c}U\mathbf{A}U^{-1})^2}{2m} + U\Big(V - \frac{q}{c}\mathbf{v}(t)\cdot\mathbf{A}\Big)U^{-1} - \mathbf{a}(t)\cdot\mathbf{Q}, \tag{63}$$

where $\mathbf{a}(t)$ is the generally non-zero *acceleration* of the moving frame (see again Takagi (1991, §3)). The additional potential $-\mathbf{a}(t)\mathbf{Q}$ is an *inertial* potential, which belongs to $\tilde{V}_0$:

$$\tilde{V}_0 = UV_0U^{-1} - \mathbf{a}(t)\mathbf{Q}, \tag{64}$$

while for the electric potential we still have the transformation behaviour

$$\tilde{a}_0 = U\Big(A_0 - \frac{q}{c}\mathbf{v}(t)\cdot\mathbf{A}\Big)U^{-1}. \tag{65}$$

Thus the invariance result for $\alpha[\lambda-\mu]$ generalises also to the case of linearly accelerated frames. Instead, in the transformation formula for $\beta[\lambda]$ there appears a new term,

$$\exp\left\{ \frac{i}{\hbar}\int_0^T \langle\psi(t)|\mathbf{a}(t)\cdot\mathbf{Q}|\psi(t)\rangle dt \right\}, \tag{66}$$

which does *not* depend only on the endpoints of λ, thus destroying in general the invariance of the phase differences.

Finally, we show that the representation (41) of the total phase for a path difference $\lambda - \mu$ is invariant under *arbitrary* time-dependent unitary transformations $U(t)$, as indeed it should.

Since

$$\tilde{H} = UHU^{-1} + i\hbar\frac{dU}{dt}U^{-1} \tag{67}$$

and

$$\frac{d}{dt} = U\frac{d}{dt}U^{-1} + \frac{dU}{dt}U^{-1} \tag{68}$$

(recall (18) and (19)), we have that

$$\frac{d}{dt} + \frac{i}{\hbar}\tilde{H} = U\Big(\frac{d}{dt} + \frac{i}{\hbar}H\Big)U^{-1}. \tag{69}$$

And thus, by the definition (41) of $\tau[\lambda]$,

$$e^{i\tau[\tilde{\lambda}]} = e^{i\tau[\lambda]} \cdot \left(\frac{\langle\psi(0)|U^{-1}(0)U(T)|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle} \cdot \frac{\langle\psi(T)|U^{-1}(T)U(0)|\psi(0)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2}. \tag{70}$$

Since the second factor on the right-hand side depends, yet again, only on the endpoints of λ in ray space, differences $\tau[\lambda] - \tau[\mu] = \tau[\lambda - \mu]$ (where λ and μ have common endpoints) are invariant under *arbitrary* time-dependent unitary transformations $U(t)$.

7 Geometric Interpretation

In the case in which $|\psi(t)\rangle$ is a solution of a Schrödinger equation, as we have seen, $\gamma[\lambda]$ is also given by

$$e^{i\gamma[\lambda]} = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{\frac{i}{\hbar}\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt\right\}, \tag{71}$$

so that subtracting the dynamic phase from the total phase, one indeed obtains a phase that is projective, independent of the particular Hamiltonian inducing the evolution, and geometric in the strong sense that it is given by the curvature of the projective Hilbert space.

In the case of the phase $\alpha[\lambda]$ introduced in Section 5, we have noted already that it is also projective and independent of time rescalings, but that it does not depend only on the path λ in ray space $\mathcal{P}(\mathcal{H})$. We shall now consider $\alpha[\lambda]$ instead in relation to a path, again denoted by λ, in what we might call *time–ray space*, that is the parametrised ray space $\mathcal{P}(\mathcal{H})_t$, a space in which the path λ becomes an invariant object, much like a world-line in relativistic space–time. The phase $\alpha[\lambda]$ will still not be independent of the Hamiltonian that drives the system along the path, but it will be interpretable as arising from a *Hamiltonian-dependent connection* on time–ray space.

In order to obtain a geometric characterisation of $\alpha[\lambda]$ we have to introduce on time–ray space a structure analogous to parallel transport in the Hopf bundle. We first construct such a fibre bundle, by defining the fibre over a point $(\pi, t) \in \mathcal{P}(\mathcal{H})_t$ as the set of unit vectors in the ray π:

$$\Big\{|\psi\rangle \Big| \, \|\psi\| = 1, \pi(|\psi\rangle) = \pi\Big\}. \tag{72}$$

A curve λ in time–ray space will now be given by a locally one-to-one mapping

$$\lambda : s \in [0, S] \mapsto \Big(\pi(|\psi(s)\rangle), t(s)\Big). \tag{73}$$

And a parallel transport in the bundle will clearly be given by a rule that fixes the phase of $|\psi(s)\rangle$.

The rule of parallel transport that yields $\alpha[\lambda]$ is in fact

$$\langle\psi(s)|\frac{d}{ds}|\psi(s)\rangle + \frac{iq}{\hbar}\langle\psi(s)|\frac{dt(s)}{ds} \cdot A_0(t(s))|\psi(s)\rangle = 0 \tag{74}$$

(or analogously for $\beta[\lambda]$ with $V(t(s))$ in the place of $qA_0(t(s))$). It is clear, by analogy to (9), that this provides us with a condition on the phase of $|\psi(s)\rangle$. We can thus use it to define a locally accumulated phase, as in (5), and a phase

$$e^{i\alpha[\lambda]} := \left(\frac{\langle\psi(0)|\psi(S)\rangle}{\langle\psi(S)|\psi(0)\rangle}\right)^{1/2} \cdot \exp\left\{-\int_0^S \langle\psi(s)|\frac{d}{ds} + \frac{iq}{\hbar}\frac{dt}{ds}A_0|\psi(s)\rangle ds\right\}, \tag{75}$$

again denoted by $\alpha[\lambda]$, which is manifestly projective and reparametrisation invariant. In fact, for the case in which $\frac{dt}{ds} = 1$, this is identical to the definition (39) of $\alpha[\lambda]$. Indeed, it is identical to (39) in the more general case in which $\frac{dt}{ds} \neq 0$ almost everywhere along the curve, which corresponds to a time rescaling $t \mapsto \tau = s$ (and possibly inverting the direction of time, as when going backwards along μ). Time rescaling invariance is explicitly built into the definition of (75) as a special case of reparametrisation invariance, while it is implicit in the definition (39). In fact, the transformation behaviour of A_0 is explicitly built into (75). Thus, (75) gives us an explicit interpretation of our phase $\alpha[\lambda]$ as a geometric phase defined in terms of a parallel transport on time–ray space. It becomes identical to (2) in the case in which $\frac{dt}{ds} = 0$ all along the curve, so that $\gamma[\lambda]$ can be interpreted as a special case of $\alpha[\lambda]$ for the case of parallel transport over time–ray space at any *fixed time*, which (as it should) coincides with parallel transport over ray space.

8 Classical Analogue

The geometric phase (2) has a classical analogue, as shown by Anandan (1988), in the *canonical invariant*

$$S_0 = \oint \mathbf{p} d\mathbf{q} \tag{76}$$

over a closed loop. In fact, this is the classical limit of the geometric phase for the case of coherent states.[8] To be precise, the analogue of (2) for open paths has been given by Sjöqvist and Hedström (1997) as

$$\frac{1}{2} \int \mathbf{p} d\mathbf{q} - \mathbf{q} d\mathbf{p}. \tag{77}$$

If we are interested only in loops of the form $\lambda - \mu$, however, we need to consider only (76).

[8]Notice also the analogy with the *adiabatic invariant*

$$I = \frac{1}{2\pi} \oint p\, dq$$

(Landau and Lifshitz, 1976, Section 49) **[check]**.

The quantity (76) is invariant under arbitrary canonical transformations, as long as these do not depend on *time*. In the more general case, one usually takes instead the quantity

$$S = \oint \mathbf{p} d\mathbf{q} - H dt, \tag{78}$$

where H is the Hamiltonian function of the system. This quantity is now invariant under arbitrary time-dependent canonical transformations. (In fact, (78) is the canonical invariant in the *extended* phase space, i. e. the phase space with the additional pair of conjugate variables $q_0 := t$ and $p_0 := -H$.) If (76) is the analogue of (2), it is easy to recognise (78) as the analogue of (41), i. e. of the *total* phase, which as we have seen in Section 6 is invariant under arbitrary time-dependent *unitary* transformations.

We can now ask whether the extended geometric phase $\alpha[\lambda]$ defined in (34) or (39) also has a classical analogue, which is invariant under canonical transformations implementing local gauge and Galilei transformations. And indeed, it has. We shall now illustrate the case of Galilei transformations.

Let the Hamiltonian function of the system be

$$H = \frac{(\mathbf{p} - \frac{q}{c}\mathbf{A})^2}{2m} + V, \tag{79}$$

where $V = qA_0 + V_0$, and consider a Galilei transformation

$$\mathbf{q} \;\rightarrow\; \tilde{\mathbf{q}} := \mathbf{q} - \mathbf{v}t, \tag{80}$$

$$\mathbf{p} \;\rightarrow\; \tilde{\mathbf{p}} := \mathbf{p} - m\mathbf{v}. \tag{81}$$

This transformation has a generating function, which can be written, say in terms of the variables $\mathbf{q}$ and $\tilde{\mathbf{p}}$, as

$$\Phi(\mathbf{q}, \tilde{\mathbf{p}}, t) = \tilde{\mathbf{p}}(\mathbf{q} - \mathbf{v}t) + m\mathbf{v}\mathbf{q}. \tag{82}$$

One then has, indeed (Landau and Lifshitz, 1976, Section 45),

$$p_i = \frac{\partial \Phi}{\partial q_i}, \qquad \tilde{q}_i = \frac{\partial \Phi}{\partial \tilde{p}_i}, \qquad \tilde{H} = H + \frac{\partial \Phi}{\partial t}. \tag{83}$$

The last of these equations describes how the Hamiltonian function has to transform in order for the Hamilton equations to be covariant, i. e. satisfied also with respect to the new coordinates. In this case, the Hamiltonian function transforms as:

$$\tilde{H} = H - \tilde{\mathbf{p}}\mathbf{v}. \tag{84}$$

Thus, from (79), (80) and (81),

$$\begin{aligned} H &= \frac{(\tilde{\mathbf{p}} + m\mathbf{v} - \frac{q}{c}\mathbf{A})^2}{2m} + V = \\ &= \frac{(\tilde{\mathbf{p}} - \frac{q}{c}\mathbf{A})^2}{2m} + \tilde{\mathbf{p}}\mathbf{v} + \frac{1}{2}mv^2 - \frac{q}{c}\mathbf{v}\mathbf{A} + V \end{aligned} \tag{85}$$

(where both $\mathbf{A}$ and V are understood as functions of $\tilde{\mathbf{q}} + \mathbf{v}t$), and, from this and (84),

$$\tilde{H} = \frac{(\tilde{\mathbf{p}} - \frac{q}{c}\mathbf{A})^2}{2m} + V - \frac{q}{c}\mathbf{v}\mathbf{A} + \frac{1}{2}mv^2. \tag{86}$$

The last term is a rescaling of energy, which can be safely ignored, while we see that the vector and scalar potentials transform as

$$\tilde{\mathbf{A}}(\tilde{\mathbf{q}}) = \mathbf{A}(\mathbf{q}), \tag{87}$$

$$\tilde{V}(\tilde{\mathbf{q}}) = V(\mathbf{q}) - \frac{q}{c}\mathbf{v}\mathbf{A}(\mathbf{q}), \tag{88}$$

or, indeed,

$$\tilde{\mathbf{A}} = \mathbf{A}, \tag{89}$$

$$\tilde{A}_0 = A_0 - \frac{\mathbf{v}}{c}\mathbf{A}, \tag{90}$$

$$\tilde{V}_0 = V_0 \tag{91}$$

(cf. (51)–(54)).

If we now consider the transformation behaviour of $\oint_{\lambda-\mu} \mathbf{p}d\mathbf{q}$ under (80)–(81), we obtain:

$$\begin{aligned} \oint_{\tilde{\lambda}-\tilde{\mu}} \tilde{\mathbf{p}}d\tilde{\mathbf{q}} &= \oint_{\lambda-\mu} (\mathbf{p} - m\mathbf{v})d(\mathbf{q} - \mathbf{v}t) = \\ &= \oint_{\lambda-\mu} \mathbf{p}d\mathbf{q} - \mathbf{p}\mathbf{v}dt - m\mathbf{v}d\mathbf{q} + mv^2 dt. \end{aligned} \tag{92}$$

Now, if λ and μ are solutions to the Hamilton equations, in particular with H given by (79), then $d\mathbf{q} = \dot{\mathbf{q}}dt$ and further

$$\mathbf{p} = m\dot{\mathbf{q}} + \frac{q}{c}\mathbf{A}. \tag{93}$$

Thus we have that

$$\oint_{\tilde{\lambda}-\tilde{\mu}} \tilde{\mathbf{p}} d\tilde{\mathbf{q}} = \oint_{\lambda-\mu} \mathbf{p} d\mathbf{q} - \frac{q}{c}\mathbf{v}\mathbf{A}dt - 2m\mathbf{v}\dot{\mathbf{q}}dt + mv^2 dt. \tag{94}$$

Since the integral is over a closed loop in extended phase space, the last two terms do not contribute and we obtain the final result:

$$\oint_{\tilde{\lambda}-\tilde{\mu}} \tilde{\mathbf{p}} d\tilde{\mathbf{q}} = \oint_{\lambda-\mu} \mathbf{p} d\mathbf{q} - \frac{q}{c}\mathbf{v}\mathbf{A}dt, \tag{95}$$

so that (76) is, indeed, not invariant in general under Galilei transformations.

From this and from (88) and (90), however, it is clear that the quantities

$$\oint_{\lambda-\mu} \mathbf{p} d\mathbf{q} - qA_0 dt \tag{96}$$

and

$$\oint_{\lambda-\mu} \mathbf{p} d\mathbf{q} - V dt \tag{97}$$

are both invariant under the Galilei transformation (80)–(81). We have thus found phase space loop integrals distinct from the canonical invariant (78) which are nevertheless also invariant under Galilei transformations. (As can be readily seen from the analogous calculations, the results for gauge transformations and for transformations to linearly accelerated frames are also valid in the classical case.)

9 Aharonov–Bohm Effect

Consideration of the classical analogue shows that, even if the geometric phase (2) is not (gauge- nor) Galilei-invariant, it surely has an important theoretical role to play, just as the canonical invariant (76). Non-invariance under gauge or Galilei transformations, however, raises the question of the observability of the geometric phase, as discussed by SBC (1997). In their discussion, SBC note how all experiments to date seem either to consider only the spin degrees of freedom — which makes them independent of spatially dependent transformations like gauge or Galilei transformations — or to consider new quantities which are analogues of the geometric phase for spin degrees of freedom (e. g. in describing interferometry experiments) — which has the same effect.

There is, however, in the literature a thought experiment that does, indeed, involve the geometric phase associated with the spatial degrees of freedom of a system, and in which the gauge- and frame-dependence will be manifested. This is Berry's derivation of the Aharonov–Bohm effect as the difference in geometric phases between a particle in a box at rest and a particle in a box which is transported around a solenoid and brought to coincide again with the first box. I wish to point out, as is evident already from Berry's treatment, that this calculation and thus the identification of the AB phase as a geometric phase depend on the chosen gauge (and frame). Instead, applying the definition of the extended geometric phase $\alpha[\lambda]$ to this obviously time-dependent problem yields an identification of the AB effect as geometric which is independent of gauge or frame.

Let us recall Berry's (1984) calculation, recast slightly in terms of what we have discussed above. Berry considers a box, or rather two boxes, one at rest and one transported from the same location as the first around a flux line and back, and such that the two boxes are never interpenetrated by the flux line. (The state of a particle will have to be localised within a box, so the potential walls are infinite.) We can write the Hamiltonian for the particle in such a box in the form

$$H = \frac{(\mathbf{P} - \frac{q}{c}\mathbf{A}(\mathbf{Q}))^2}{2m} + V(\mathbf{Q} - \mathbf{R}(t)), \tag{98}$$

where $\mathbf{R}(t) = \text{const}$ for the first box and equals a cyclic path in $\mathbf{R}$-space for the second box.

As Berry points out, the eigenvalues E_n of H do not depend on $\mathbf{R}$, nor on the presence or absence of the vector potential $\mathbf{A}$ (since the state of the particle is actually inside the box, the latter only induces an additional Dirac phase factor in the solution). Thus, denoting by $|n(\mathbf{R})\rangle$ the eigenstate of H (depending on $\mathbf{R}$) corresponding to E_n, we have:

$$H|n(\mathbf{R})\rangle = E_n|n(\mathbf{R})\rangle, \tag{99}$$

and

$$\langle \mathbf{r}|n(\mathbf{R})\rangle = \exp\left\{\frac{iq}{c}\int_{\mathbf{R}}^{\mathbf{r}'} d\mathbf{r}'\mathbf{A}(\mathbf{r}')\right\}\psi_n(\mathbf{r}-\mathbf{R}). \tag{100}$$

The geometric phase along the closed loop (in $\mathbf{R}$-space) given by the difference of the paths for the two boxes is calculated by Berry as

$$\gamma[C] = i\oint_C \langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle d\mathbf{R} \tag{101}$$

(which corresponds to (3) above). Further,

$$\begin{aligned}
\langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle &= \\
&= \iiint d^3\mathbf{r}\psi_n^*(\mathbf{r}-\mathbf{R})\left\{-\frac{iq}{\hbar}\mathbf{A}(\mathbf{R})\psi_n(\mathbf{r}-\mathbf{R}) + \nabla_{\mathbf{R}}\psi_n(\mathbf{r}-\mathbf{R})\right\} = \\
&= -\frac{iq}{\hbar}\mathbf{A}(\mathbf{R}),
\end{aligned} \tag{102}$$

assuming that ψ_n is normalised and real-valued. In fact,

$$\mathbf{0} = \nabla_{\mathbf{R}}(\psi_n^*\psi_n) = 2\mathrm{Re}\psi_n^*\nabla_{\mathbf{R}}\psi_n, \tag{103}$$

so the second term in the triple integral in (102) vanishes if $\psi_n^*\nabla_{\mathbf{R}}\psi_n$ has no imaginary part. This can always be achieved, but clearly depends on the choice of gauge. From (101) and (102) one has then:

$$\gamma[C] = \frac{iq}{\hbar}\int_C \mathbf{A}(\mathbf{R})d\mathbf{R} = \frac{iq}{\hbar}\Phi, \tag{104}$$

Φ denoting the flux of the magnetic field. Thus, the geometric phase is shown to be equal to the AB phase, and is, indeed, independent of n, so that the result is true whatever the state of the particle in the box.

One can check explicitly that Berry's definition (101) is the same as (3) above, if one takes $|\psi(t)\rangle = |n(\mathbf{R}(t))\rangle$, which in fact solves the Schrödinger equation with Hamiltonian (98) — despite the fact that Berry's definition is based on the notion of a path in the *parameter* space of the Hamiltonian (i. e. $\mathbf{R}$-space), while AA's definition which we are using is based on the notion of a path in projective Hilbert space. (As a matter of fact, as pointed out already by Berry, there is no need of adiabatic approximation in this case:

$|\psi(t)\rangle$ is a solution of the Schrödinger equation irrespective of how fast the box is being moved.) Thus, since Berry's $\gamma[C]$ is equal to $\gamma[\lambda-\mu]$ in the sense of AA (denoting by λ and μ the paths in projective Hilbert space of the two boxes, respectively), the result (104) depends both on the chosen gauge, as we have discussed quite in general in Section 3 and as just remarked above, and on the chosen frame, as follows from SBC's (1997) discussion, again as summarised in Section 3. For instance, one can generally find a Galilei transformation such that $\gamma[\tilde{C}] = 0$, so that the AB phase becomes entirely dynamic!

Such a case can now be prevented by use of the extended geometric phase α. In fact we have that for this case:

$$\begin{aligned}
\alpha[\lambda-\mu] &= \oint_{\lambda-\mu} \langle\psi(t)|\frac{d}{dt} + \frac{iq}{\hbar}A_0(\mathbf{Q}-\mathbf{R}(t))|\psi(t)\rangle dt = \\
&= \oint_{\lambda-\mu} \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt + \frac{iq}{\hbar}\oint_{\lambda-\mu}\langle\psi(t)|A_0(\mathbf{Q}-\mathbf{R}(t))|\psi(t)\rangle dt = \\
&= \gamma[\lambda-\mu] + \frac{iq}{c}\langle\psi(0)|A_0(\mathbf{Q}-\mathbf{R}(0))|\psi(0)\rangle \left\{\int_0^T dt - \int_0^T dt\right\} = \\
&= \gamma[C] = \\
&= \frac{iq}{\hbar}\Phi,
\end{aligned} \tag{105}$$

where we have used the fact that the energy of a particle stemming from the electric potential in either box (if any) is independent of $\mathbf{R}(t)$. (The same is true of $\beta[\lambda-\mu]$ and V.)

Thus, we have shown that our gauge and Galilei-invariant phase $\alpha[\lambda-\mu]$ is also equal to the AB phase in the case considered by Berry, thereby giving a gauge- and Galilei-invariant identification of the AB phase with a geometric-type phase.

10 Conclusion

My investigation was motivated by the quest for an invariant definition of geometric phase, and was carried out pursuing the analogy with space–time concepts. As to this second point, I have introduced the so-called time–ray space, where one could consider paths analogous to paths in space–time. I believe such a construction has independent interest. As to the first point, the phase I have constructed above — although retaining many of the geometric features of the AA phase — has lost the independence of the Hamiltonian. My definition of a connection in time–ray space depends itself on the Hamiltonian that drives the system along the path. This may be a disturbing feature of the phase α (although there may be some interest in considering the connection as thus dependent on the dynamics, which could be an interaction dynamics in the case of , say, a Born–Oppenheimer framework). On the other hand, the connection introduced on time–ray space may be related to the electromagnetic connection on space–time. Finally, the extended geometric phase is a phase quantity distinct from the total phase, which however like the total phase is invariant under gauge and Galilei transformations. The phase thus defined may be not geometric in the same sense as the original geometric phase, but its invariant behaviour makes it a real physical quantity.

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