As Revealing in the Breach as in the Observance: von Neumann's Uniqueness Theorem

John Earman
Dept. History and Philosophy of Science
University of Pittsburgh

von Neumann's celebrated uniqueness theorem is often advertised as demonstrating the essential uniqueness of representations (or realizations) of the Heisenberg commutation relations for a finite number N of degrees of freedom by showing that all such representations are unitarily equivalent to the Schrödinger representation and, thereby, securing the equivalence of Schrödinger wave mechanics to the matrix mechanics of Born, Heisenberg, and Jordan. But the theorem proves no such thing—even for finite N there are unitarily inequivalent representations of the Heisenberg commutation relations among which are physically interesting quantum phenomena, such as the Aharonov-Bohm effect; and von Neumann's own explanation of the equivalence of Schrödinger wave mechanics to the matrix mechanics makes no use of his uniqueness theorem. There are other loopholes and ways around the uniqueness theorem; but this does not constitute a criticism of von Neumann's theorem but is rather is a nod to the genius that produced a theorem as revealing in the breach as in the observance, for the exceptions illustrate (in the words of Asao Arai) how the universe uses inequivalent irreducible representations of the canonical commutation relations to produce "characteristic" quantum effects.

1 Introduction

Although it is common, if not quite standard, in the physics and mathematics literature to refer to the theorem demonstrating the essential uniqueness of the Schrödinger representation of the Weyl form of the canonical commutation relations as the Stone-von Neumann theorem, for reasons detailed in Section 2 I will be speaking of Stone's conjecture and von Neumann's uniqueness theorem.[1] Briefly, although Marshall Stone (1930) announced the theorem and gave a sketch of a proof strategy, he never produced an implementation of the strategy.[2] It was left to von Neumann (1931) to decisively settle the matter with an ingenious and elegant proof. Although von Neumann's theorem has been eclipsed by Mackey's more powerful imprimitivity results (see Mackey 1949), it remains, in the words of Gerald Folland (1989), "a very pretty argument that does not deserve the obscurity into which it has fallen" (p. 35).

von Neumann's uniqueness theorem appeared at a crucial junction in the development of 20th century physics when competing lines of research came together to produce what came to be known as the new quantum theory, and it is commonly thought that the importance of von Neumann's theorem is that it cemented the case that the matrix mechanics of Heisenberg, Born, and Jordan is equivalent to the wave mechanics of Schrödinger. Given von Neumann's Hilbert space construals of matrix mechanics and wave mechanics his uniqueness theorem would be sufficient to establish the equivalence if it established the essential uniqueness of the Schrödinger representation of the Heisenberg form of the canonical commutation relations (CCR). But von Neumann's theorem does no such thing. This, however, is not a defect of the theorem since no theorem can establish the impossible—there are many irreducible but unitarily inequivalent representations of the Heisenberg CCR, even for a finite number of degrees of freedom (indeed, for one degree of freedom). And as is made clear in *Mathematische Grundlagen der Quantenmechanik* (1932), von Neumann's Hilbert space construals of matrix mechanics and wave mechanics, without need to invoke his uniqueness theorem, already entail that the two are equivalent in a strong sense. The gap not covered by von Neumann's theorem, cases where a representation of the Heisenberg CCR does not produce a representation of the Weyl CCR, contains interesting and characteristically quantum phenomena. This together

[1] George Mackey, a Harvard colleague and friend of Marshall Stone, seems to have been the main source of the "Stone-von Neumann theorem" appellation (see Mackey 1949). Mackey is careful to say not that Stone proved the theorem but only that he "has stated and indicated a proof" of the theorem.

[2] What Stone *did* prove is a result that will be used repeatedly in what follows, viz. a self-adjoint operator exponentiates to produce a strongly continuous one-parameter group of unitary transformations; and conversely, a strongly continuous one-parameter group of transformations has a self-adjoint operator as its infinitessimal generator (Stone's theorem).

with other loopholes and exceptions does not constitute a criticism of von Neumann's theorem but is rather is a nod to the genius that produced a theorem as revealing in the breach as in the observance. These heterodox claims require quite a bit of explanation and justification which, perforce, necessitate delving into the technicalia of Hilbert space theory. As far as possible the technicalia are confined to Appendices.

There are two superb review articles discussing the mathematical and physical ramifications of von Neumann's theorem—Summers (2001) and Rosenberg (2004)—and I have little new to add to these reviews. Nevertheless, philosophers of physics may welcome a guide that points them to foundations issues that deserve more attention and draws some lessons that, while obvious to mathematical physicists, have not been sufficiently noted in the philosophical literature.

2 Stone's conjecture and von Neumann's uniqueness theorem

In the February 1, 1930 issue of *Proceedings of the National Academy Sciences* Marshall Stone (1930) announced a theorem asserting the uniqueness up to unitary equivalence of the Schrödinger representation among the irreducible representations of the Weyl form of the CCR. In fact, no proof of this proposition was given, only a sketch of an idea for a proof. The "principal difficulty," Stone wrote, lies in showing that certain operators are self-adjoint (s.a.). Once this is done the determination of the spectra of these operators "leads easily to the desired [unitary] transformation S [to the Schrödinger representation] by means of a device previously employed by J. v. Neumann in a rather different connection."[3]

The subsequent correspondence between Stone and von Neumann makes it clear that Stone's proof strategy did not come to fruition. von Neumann's letter to Stone dated October 8, 1930 begins:

> My dear Mr. Stone.
>
> Very many thanks for you kind letter. Having your approval, I will use in my paper the phrase I wrote to you, adding to it, as

[3]Here Stone refers to von Neumann (1929). von Neumann's article "Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren," was published in 1930 in *Mathematische Annalen*. [So why are the references typically to 1929?]

> you suggested, some indications about the difficulty existing in your way of proof.[4]

The letter continues with a detailed sketch of von Neumann's own uniqueness proof, published the following year in von Neumann (1931). In that article, after setting up the uniqueness problem using the Weyl form of the CCR, von Neumann comments:

> It remains to be shown that the only irreducible solutions[5] to Weyl's equations are Schrodinger's.[6] Stone gave a proof strategy for this, but so far no proof has been provided on this basis, as Mr. Stone has kindly informed me.[7]

Intriguingly, von Neumann's letter of October 8 mentions Stone's "objective to construct the operator $P^2 + Q^2$ of the 'oscillator'." Many years later Dixmier (1958) showed that if P and Q are both closed symmetric operators on a common dense domain $\mathcal{D}$ where they satisfy the Heisenberg CCR and if in addition $P^2 + Q^2$ restricted to $\mathcal{D}$ is essentially self-adjoint (e.s.a.) then P and Q are unitarily equivalent to the Schrödinger representation (see Section 5.3 below).

From the evidence of the published papers and von Neumann's letter to Stone it seems more appropriate to speak of Stone's conjecture and von Neumann's uniqueness proof rather than the Stone-von Neumann theorem. To confirm this conclusion it would be highly desirable to have Stone's side of the correspondence with von Neumann.[8]

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[4] The text of this letter is published in Redei (1922, p. 223) along with von Neumann's other letters to Stone.

[5] Here von Neumann adds a footnote defining irreducibility.

[6] Here a reference to Weyl (1928) and Stone (1930).

[7] "Es bliebe daber zu zeigen, das einzigen irreduziblen Lösungen der Weylschen Gleichungen die Schrödingerschen (d. h. die aus Anm) sind. Beweisansätze heirfür gab Stone (reference to Stone 1930) an, jedoch ist bischer ein Bewis auf dieser Grundlage, wie mir Herr Stone fruendlischt mitteilte, nicht erbracht worden." von Neumann (1931, 572).

[8] von Neumann's and Stone's gentlemanly handling of this affair helped to pave the way for friendly relations. The salutation "Dear Mr. Stone" and signature "John von Neumann" of 1930 gives way in later letters to Stone to "Dear Marshall" and "John". In 1930 von Neumann was engaged with G. D. Birkoff, Stone's Harvard colleague and mentor, in another, less amicable, priority issue regarding the ergodic theorem (see Redei 2022, pp. 32-35, and Zund 2002).

3 The place of von Neumann's uniqueness theorem in the history of quantum mechanics

The folklore, endorsed in many learned articles and books, begins by noting that in the 1920's there where two versions of the nascent new quantum theory, the matrix mechanics of Born, Heisenberg, and Jordan, and the wave mechanics of Schrödinger. Are they different theories or are they simply different mathematical presentations of two physically equivalent versions of the same theory? Heisenberg (1930), while suspecting that the answer was the latter, saw no proof:

> The fact that the particle picture [matrix mechanics] and the wave picture [Schrödinger wave mechanics] are two different aspects of one and the same physical reality forms the central problem of quantum theory.

Earlier Schrödinger (1926) had offered a "line of thought of a proof" (Der Gedankengand des Beweises) of their equivalence, but it was evidently not regarded as convincing. Supposedly, von Neumann's theorem settled the matter. But contrary to the folklore, this was neither the intent nor the implication of his theorem. Showing that any representation of the Heisenberg CCR is unitarily equivalent to the Schrödinger representation would be sufficient to establish the equivalence of matrix and wave mechanics. But von Neumann's theorem shows no such thing. This is not a defect of the theorem since no sound theorem can establish what is false. Thus, the unitary equivalence of any representation of the Heisenberg CCR to the Schrödinger representation cannot be a necessary condition for the equivalence of matrix and wave mechanics if, indeed, the two are equivalent.

A clue that von Neumann did not think of his uniqueness theorem as directed towards the issue of the equivalence of matrix mechanics and wave mechanics virtually shouts from his *Mathematische Grundlagen der Quantenmechanik* (1932) which was published the year following the appearance in print of his uniqueness proof: there is no use or mention in the book of the uniqueness theorem despite the fact that an announced goal of the book was to show that matrix mechanics and wave mechanics are equivalent in a very strong sense. This absence is repeated in the 1955 English edition of *Mathematical Foundations of Quantum Mechanics.*

Somewhat oversimplifying without distorting the upshot, in a nutshell von Neumann's strategy for accomplishing the announced goal is to argue that matrix mechanics and wave mechanics are part of one big happy family living together in different rooms of the home provided by Hilbert space. In more detail, first make the case that Hilbert space is the appropriate mathematical framework for developing a rigorous formulation of quantum mechanics. Second, argue that there are many concrete realizations of the axioms of Hilbert space, of which matrix mechanics and wave mechanics are two instances. Simplifying to the case of one degree of freedom and taking note of the fact that when the P and Q of Heisenberg's CCR are interpreted as matrices, the CCR cannot be satisfied by finite matrices (see Section 4 below), argue that the appropriate Hilbert space $\mathcal{H}_{MM}$ for matrix mechanics is $\ell^2(\mathbb{Z}_+)$, where the vectors are infinite sequences $(z_1, z_2, ...)$ of complex numbers such that $\sum\limits_{k=1}^{\infty} |z_k|^2$ is finite; linear operators construed as infinite matrices act by matrix multiplication on these vectors. Next, argue that the appropriate Hilbert space $\mathcal{H}_{WM}$ for wave mechanics is $L^2_{\mathbb{C}}(\mathbb{R})$, where the vectors are complex valued, square integrable functions on the configuration space $\mathbb{R}$;[9] linear operators act on these vectors in the way familiar from the differential calculus. Finally, show that $\mathcal{H}_{MM}$ and $\mathcal{H}_{WM}$ are isomorphic as Hilbert spaces and, thus, can be regarded as merely different concrete realizations of the same abstract Hilbert space. ($\mathcal{H}_{MM}$ and $\mathcal{H}_{WM}$ are both separable, and in fact all infinite dimensional separable Hilbert spaces are isomorphic as are all Hilbert spaces of the same finite dimension. Perhaps this was part of von Neumann's motivation for making separability of Hilbert space an axiom of the theory in *Mathematische Grundlagen*.)

In this way, von Neumann wrote, we realize a unified theory "independent of the accidents of the formal framework selected at the time, and exhibiting only the essential elements of quantum mechanics" (von Neumann 1955, 33). An isomorphism given by a unitary map $T : \mathcal{H}_{WM} \to \mathcal{H}_{MM}$ provides a translation scheme between realizations of P and Q as differential operators acting on wave functions in $L^2_{\mathbb{C}}(\mathbb{R})$ and realizations as infinite matrices acting on elements of $\ell^2(\mathbb{Z}_+)$. (For a linear operator A acting on $\mathcal{H}_{MM}$ the corresponding operator acting on $\mathcal{H}_{WM}$ is TAT^{-1}. Such a translation scheme preserves self-adjointness of operators, the spectra of these operators and their expectation values and, thus, arguably it preserves empirical content.)

[9] Actually, equivalence classes of such functions that differ by Lebesque measure zero.

Of course, such a translation scheme is not unique since an isomorphism between $\mathcal{H}_{WM}$ and $\mathcal{H}_{MM}$ can be implemented by different unitaries, but any two such translation schemes are unitarily equivalent. End of story, a story in which the von Neumann uniqueness theorem plays no role. (Actually, not quite the end of the story. Heisenberg found a representation of his CCR in terms of infinite matrices. One can wonder whether this particular representation is unitarily equivalent to Schrödinger's representation. If there are representations of the Heisenberg CCR inequivalent to the Schrödinger representation—as indeed there are—then inequivalent representations are present equally in matrix mechanics and wave mechanics alike. In matrix mechanics and wave mechanics alike the representations of the Heisenberg CCR break up into unitary equivalence classes, and the equivalence class structures are isomorphic.)

What issue then does the uniqueness theorem address? At the time, the process of quantization of a classical system was understood as putting the classical mechanics of the system in Hamiltonian form and then replacing the classical ps and qs with appropriate Hilbert space operators Ps and Qs, whether realized as infinite matrices acting on infinite sequences of complex numbers or differential operators acting on wave functions.[10] But appropriate in what sense? At a minimum, Heisenberg would have insisted, they must satisfy his CCR. But what if there are realizations of the CCR that are not unitarily equivalent? In that case the beautiful unification which von Neumann had produced at the abstract level threatens to shatter into physically inequivalent quantizations when applied to concrete systems. How can one know in advance which of the inequivalent quantizations to apply? Absent such knowledge an extra layer of unpredictability seems to be added to the statistical uncertainty encapsulated in the Heisenberg uncertainty relations, which follow from his CCR. (Call this the problem of ambiguity of quantization.)

One reaction to the ambiguity problem would be to squelch the ambiguity by lowering the bar for physical equivalence and accepting something less than unitary equivalence. In subsequent years this move was seriously considered for cases involving an infinite number of degrees of freedom where the von Neumann uniqueness theorem breaks down.[11] But for a finite num-

[10]The replacement may run into operator ordering problems, which will be ignored here.

[11]So-called weak equivalence has been offered as replacement for unitary equivalence. For a critical assessment see Summers (2001).

ber of degrees of freedom von Neumann's uniqueness theorem is supposed to remove the worry by showing that realizations of the CCR are all unitarily equivalent and, indeed, unitarily equivalent to the familiar Schrödinger realization. This the von Neumann theorem accomplishes for the Weyl form of the CCR but not for the Heisenberg form, which was the source of the worry. It not implausible that this distinction was not generally appreciated circa 1930—indeed, it was probably not generally appreciated that there was a possible gap between the two forms of the CCR since specific examples of operators pairs P, Q realizing the Heisenberg CCR but not realizing the Weyl CCR did not appear in the literature until the late 1950s—and the failure to appreciate the distinction may have fostered the mistaken idea that von Neumann's uniqueness theorem entailed the essential equivalence of matrix mechanics and wave mechanics.

4 The Heisenberg CCR and their uniqueness problem

At the beginning of "Die Eindeutigkeit der Schrödingerschen Operatoren" von Neumann complains that the formulation of the problem of the essential uniqueness of the Schrödinger realization of the Heisenberg form the canonical commutation relations (CCR) is not sufficiently precise. These relations read

$$
\begin{aligned}
P_k Q_l - Q_l P_k &= -iI\delta_{kl}, \qquad k, l = 1, 2, ..., N \\
P_k P_l - P_l P_k &= 0, \qquad Q_k Q_l - Q_l Q_k = 0
\end{aligned} \tag{1}
$$

where N is the number of degrees of freedom in the system.[12] Specifically the complaint is that the two sides of the first equation can be equated only if they have the same domains of definition; but whereas the rhs is defined on all of Hilbert space the lhs is not since it involves unbounded operators. von Neumann proposed to avoid the difficulty by switching to the Weyl form of the CCR; but in doing so he switched to what in retrospect is seen to be a different problem.

[12] Units have been chosen in which $\hbar = 1$ so that the $\hbar$ that normally appears on the rhs of the first equation is invisible.

Subsequent researchers proposed to stick with the original problem and make *it* more precise. As already mentioned, if the P's, Q's, and I stand for matrices then (1) cannot be satisfied by finite matrices, for the trace of the lhs of the first of the equations in (1) is zero for finite matrices while the trace of the rhs is non-zero. More generally, if (1) is realized by operators acting on a Hilbert space $\mathcal{H}$ and if the P's and Q's are s.a.—generally taken as a necessary condition for an operator to represent an "observable"—then not all of the P's and Q's can be bounded, as von Neumann recognized but did not bother to prove.[13] And so the P's and Q's must act on an infinite dimensional $\mathcal{H}$. Furthermore, since unbounded operators are not defined on all of this $\mathcal{H}$, (1) cannot be taken to mean that $(P_kQ_l - Q_lP_k)\psi = -i\psi$ for all $\psi \in \mathcal{H}$, and the best one can hope for is that (1) is satisfied in some appropriate sense on a dense subspace $\mathcal{D} \subset \mathcal{H}$.

With these facts in mind, the generally accepted proposal for what the Heisenberg canonical commutation relations (HCCR) assert in terms of Hilbert space operators is that

> The P_k and Q_l are s.a. operators on $\mathcal{H}$, and there is a dense $\mathcal{D} \subset \mathcal{H}$ such that
>
> (a) $\mathcal{D}$ is a common and invariant domain for the P's and the Q's, i.e. $\mathcal{D} \subset \cap_{k,l}(dom(P_k) \cap dom(Q_l))$, $P_k\mathcal{D} \subset \mathcal{D}$, $Q_l\mathcal{D} \subset \mathcal{D}$;
>
> (b) $\mathcal{D}$ is a core for the P_k and Q_l, i.e. the restrictions $P_k \downharpoonleft \mathcal{D}$ of P_k and $Q_l \downharpoonleft \mathcal{D}$ of Q_l to $\mathcal{D}$ are e.s.a.;[14]
>
> (c) for every $\psi \in \mathcal{D}$, $(P_kQ_l - Q_lP_k)\psi = -i\psi\delta_{kl}$, $(P_kP_l - P_lP_k)\psi = 0$, $(Q_kQ_l - Q_lQ_k)\psi = 0$.

The familiar Schrödinger representation or realization of the (HCCR) for N degrees of freedom[15] uses the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^N)$ and realizes the P's and the Q's as the operators

[13] Apparently the first proof to appear in print is due to Wieland (1949). See also Reed and Simon (1980, Example 2 p. 274).

[14] If A is a closed operator any $\mathcal{D} \subset dom(A)$ such that $\overline{A \downharpoonleft \mathcal{D}} = A$ is a core for A. If A is s.a. then $\mathcal{D}$ is a core for A iff $A \downharpoonleft \mathcal{D}$ is e.s.a., i.e. $\overline{A \downharpoonleft \mathcal{D}}$ is s.a. For a closable symmetric operator A the property of essential self-adjointness may also be defined as having a unique s.a. extension, namely, $\overline{A}$.

[15] "Degrees of freedom" is used ambiguously in the literature. In one sense it can be identified with the dimension of the configuration space of the system. So if the configuration space of a spinless particle is $\mathbb{R}^n$ and the system consists of N such particles then the

$$
\begin{aligned}
(Q_l^S\psi)(x_1, x_2, ..., x_N) &= x_l\psi(x_1, x_2, ..., x_N), \quad (x_1, x_2, ..., x_N) \in \mathbb{R}^N \qquad (2)\\
(P_k^S\psi)(x_1, x_2, ..., x_N) &= -i\frac{d}{dx_k}\psi(x_1, x_2, ..., x_N)
\end{aligned}
$$

Taking $\mathcal{D} = \mathcal{S}(\mathbb{R}^N)$ (the Schwartz space[16]), which is dense in $L^2_{\mathbb{C}}(\mathbb{R}^N)$, the Schrödinger P's and the Q's of (2) satisfy (a)-(c) of (HCCR). An explicit construction of a realization of (HCCR) in terms of infinite matrices acting on $\ell^2(\mathbb{Z}^N_+)$ (the Hilbert space of the N-fold direct product of infinite sequences of absolute square summable complex numbers) along with a proof of the unitary equivalence to the familiar Schrödinger realization can be found in Arai (2020, 89-96).

There are many other operator pairs P_k, Q_k that realize the (HCCR); indeed, there is an uncountable infinity of them. The issue then becomes: Are all of these mathematically different realizations nevertheless physically equivalent in some appropriate sense? If the criterion of physical equivalence is unitary equivalence (arguably the strongest criterion of physical equivalence) then there is an uncountable infinity of physically inequivalent realizations even for $N = 1$ (see Schmüdgen 1983b).[17]

Of course, many of these inequivalent realizations of the (HCCR)—perhaps all of them save those that are unitarily equivalent to the Schrödinger representation—have no physically interesting applications. But how can one know in advance that theoretical and experimental investigations will not someday reveal a noteworthy application of a non-Schrödinger equivalent realization?

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number of degrees of freedom is nN. The particle number can serve as proxy for degrees of freedom if the issue is finite vs. an infinite number of degrees of freedom. However, if one is thinking in terms of quantum fields rather than particles a different sense of degrees of freedom comes into play; see Section 6.2 below.

[16] Infinitely differentiable complex valued functions of rapid decrease.

[17] For $N = 1$ suppose P, Q acting on $\mathcal{H}$ and P', Q' acting on $\mathcal{H}'$ both realize (HCCR). The issue of unitary equivalence comes to this: Does there exist a unitary map $U : \mathcal{H} \to \mathcal{H}'$ such that $P' = UPU^{-1}$ and $Q' = UQU^{-1}$?

5 The Weyl form of the CCR and von Neumann's uniqueness theorem

5.1 The Weyl CCR

When von Neumann began thinking about the uniqueness problem the existence of inequivalent representations of (HCCR) was not known and, thus, there was hope that a uniqueness proof could be produced for this form of the CCR. His proof strategy was to employ a device first suggested by Weyl (1927, 1928) that avoids the finicky domain issues involved in a precise statement of the (HCCR) for unbounded operators. Exponentiating s.a P's and Q's acting on $\mathcal{H}$ produces continuous unitary groups[18] of bounded Hilbert space operators. In the case of one degree of freedom we have the one-parameter groups

$$U(s) := \exp(-isP), \quad V(a) := \exp(-itQ), \quad s, t \in \mathbb{R}, \tag{3}$$

and being bounded operators the group elements are defined on all of $\mathcal{H}$. Using the relation $[P, Q] = -iI$ and expanding the exponentials in (3) in power series, while ignoring issues of convergence of the series, yields the Weyl form of the CCR

$$U(s)V(s) = \exp(-ist)V(t)U(s), \quad s, t \in \mathbb{R}. \tag{4}$$

For N degrees of freedom (3) is generalized to

$$U(a) := \exp(-i\sum_{k=1}^{N} a_k P_k), \quad V(b) := \exp(-i\sum_{l=1}^{N} b_l Q_l), \quad a, b \in \mathbb{R}^N \tag{5}$$

where now $a = (a_1, a_2, ..., a_N)$ and $b = (b_1, b_2, ..., b_N)$. And again by a formal computation we conclude that $U(a)V(b) = \exp(-ia \cdot b)V(b)V(b), \quad a, b \in \mathbb{R}^N$ where $a \cdot b := \sum_{n=1}^{N} a_n b_n$. Let $U(\overline{a}_n)$ stand for the $a \in \mathbb{R}^N$ with the n-th component equal to $\overline{a}$ and all the other components equal to 0, and similarly with $V(\overline{b}_n)$. Then substituting the Schrödinger P_k and Q_k of (2) into (5),

[18]For unitary groups continuity in the strong and the weak operator topologies are equivalent, so we can speak simply of continuous unitary groups. The operator-valued map $t \ni \mathbb{R} \mapsto V(t)$ is strongly continuous at t_0 iff $\forall\psi \in \mathcal{H}$ $\lim_{t\to t_0} ||V(t)\psi - V(t_0)\psi|| = 0$.

$U(\overline{a}_n)$ acts as translation from the left by $\overline{a}$ and $V(\overline{b}_n)$ acts by multiplication by $\exp(i\overline{b}x_n)$:

$$\begin{aligned}(U(\overline{a}_n)\psi)(x_1, x_2, ..., x_N) &= \psi(x_1, x_2, .., x_n - \overline{a}, .., x_N), \quad \psi \in L^2_{\mathbb{C}}(\mathbb{R}^N) \\ (V(\overline{b}_n)\psi)(x_1, x_2, ..., x_N) &= \exp(i\overline{b}x_n)\psi(x_1, x_2, .., x_N).\end{aligned} \tag{6}$$

All of this prompts the formulation of the Weyl form (WCCR) of the CCR as

$$\begin{aligned}U(a)V(b) &= \exp(-ia \cdot b)V(b)V(b), \quad a, b \in \mathbb{R}^N \\ U(a_k)U(a_l) - U(a_l)U(a_k) &= 0,\ V(b_k)V(b_l) - V(b_l)V(b_k) = 0,\ 1 \leq k, l \leq N.\end{aligned} \tag{7}$$

Forgetting about how we arrived at the Weyl relations, take the $U(a)$ and $V(b)$ in (WCCR) not as defined by (5) but as standing for abstract continuous N-parameter unitary groups acting on a Hilbert space, and the P_k and Q_l are now defined respectively as the infinitesimal generators $P_k := i\left(\frac{\partial U(a)}{\partial a_k}\right)_{a_k=0}$ and $Q_l := i\left(\frac{\partial V(b)}{\partial b_l}\right)_{b_l=0}$ of $U(a)$ and $V(b)$. By Stone's theorem these generators are s.a. operators.

5.2 von Neumann's uniqueness theorem

Here I quote the version of von Neumann's theorem for one degree of freedom found in Reed and Simon (1980, p. 275):

> *Theorem*: Let $U(s)$ and $V(t)$, $s, t \in \mathbb{R}$, be continuous one-parameter unitary groups acting on a Hilbert space $\mathcal{H}$ and satisfying (WCCR) for $N = 1$. Then there are closed subspaces $\mathcal{H}_m \subset \mathcal{H}$ such that
>
> (a) $\mathcal{H} = \oplus_{m=1}^{M} \mathcal{H}_m$ for M a positive integer or ∞
>
> (b) $U(s) : \mathcal{H}_m \to \mathcal{H}_m$, $V(t) : \mathcal{H}_m \to \mathcal{H}_m$ for all $s, t \in \mathbb{R}$
>
> (c) For each m there is a unitary operator $T_m : \mathcal{H}_m \to L^2_{\mathbb{C}}(\mathbb{R})$ such that $T_m U(s) T_m^{-1}$ is translation to the left by s $((T_m U(s) T_m^{-1}\psi)(x) = \psi(x - s)$, $\psi \in \mathcal{H}_m$, $s \in \mathbb{R})$ and $T_m V(t) T_m^{-1}$ is multiplication by $\exp(-tx)$ $((T_m V(t) T_m^{-1}\psi)(x) = \exp(-tx)\psi(x)$, $\psi \in \mathcal{H}_m$, $s \in \mathbb{R})$.

Thus, for $N = 1$ every irreducible representation of the (WCCR) is unitarily equivalent to the Weyl form of the Schrödinger representation, and reducible representations are direct sums of irreducible representations, all unitarily equivalent to each other. As is often the case in the literature, Reed and Simon (1980) add to the hypotheses of the Theorem that $U(s)$ and $V(t)$ act on a separable $\mathcal{H}$, but for reasons to be discussed below it is unnecessary to add this as an additional assumption. An easy to follow outline of the main ideas of von Neumann's proof is given in Redei (2022, 16-18), and more detailed expositions can be found in Folland (1989), Blank, Exner, and Havliĉk (1994), and many other text books on quantum theory.

There is a natural generalization of von Neumann's theorem from the $N = 1$ case to any finite number of degrees of freedom. A corollary of the generalized theorem shows that satisfaction of the (WCCR) implies satisfaction of (HCCR):

> *Corollary.*[19] Let $U(a)$ and $V(b)$, $a, b \in \mathbb{R}^N$, be continuous N-parameter unitary groups acting on $\mathcal{H}$ satisfying the (WCCR), and let P_k, Q_l, $1 \leq k, l \leq N$, be the infinitesimal generators respectively of $U(a_k)$ and $V(b_l)$. Then the P_k and Q_l are s.a. operators on $\mathcal{H}$ and there is a dense $\mathcal{D} \subset \mathcal{H}$ such that
>
> (a) $\mathcal{D}$ is a common and invariant domain for the P_k and Q_l;
>
> (b) the restrictions of the P_k and the Q_l to $\mathcal{D}$ are e.s.a.;
>
> (c) for every $\psi \in \mathcal{D}$, $(P_kQ_l - Q_lP_k)\psi = -i\psi\delta_{kl}$, $(P_kP_l - P_lP_k)\psi = 0$, $(Q_kQ_l - Q_lQ_k)\psi = 0$.

5.3 The Heisenberg CCR vs. the Weyl CCR

The formal power series calculations mentioned in the preceding section are badly misleading since they falsely insinuate that satisfaction of the (HCCR) implies satisfaction of the (WCCR). The problem with the formal calculation lies in the implicit assumption that the power series converges, an assumption that is safe for bounded operators but unsafe for unbounded ones. It is possible that in the 1930s that physicists were misled by the formal calculations into thinking that the Weyl form of the CCR was just a way of making precise what was intended by the Heisenberg form of the CCR, the latter of which lends itself to a proof of the essential uniqueness of the Schrödinger

[19] This is the generalization to N degrees of freedom of the Corollary for one degree of freedom stated in Reed and Simon (1980, 275).

representation of the Weyl CCR. It is clear that Stone thought this was the case:

> The content of these permutation relations [the Heisenberg CCR] must be made precise by expressing them in terms of the one parameter groups of unitary transformations $U_\tau^{(k)}$ and $V_\tau^{(k)}$ generated by iP_k and iQ_k respectively. We have $U_\sigma^{(k)} V_\tau^{(k)} = e^{-i\sigma\tau} V_\tau^{(k)} U_\sigma^{(k)}$ (1930, 174).

In contrast von Neumann evidently thought that some justification was needed to move from the Heisenberg form of the CCR to the Weyl form. The justification he gives is a calculation he labels "formal," meaning that one does not worry about the domains of the operators.[20] This is curious since von Neumann points out that the P, Q satisfying the Heisenberg CCR cannot both be bounded, implying that one does have worry about the domains of the operators.

Even those chary of the formal calculations were unsure whether or not every solution of the (HCCR) is also a solution of the Weyl relations and, therefore, by von Neumann's theorem, all solutions of (HCCR) are unitarily equivalent. Thus, writing in 1958 Dixmier opined that "Unfortunately we do not know how to rigorously demonstrate the relation $U(s)V(t) = \exp(-ist)V(t)U(s)$ so that the question [of the essential uniqueness of representations of (HCCR)] remains open, to my knowledge" (Dixmier 1958, 264). It can be safely assumed that if the question was open to Dixmier's knowledge then it was open in the community at large. In any case, the first counterexample to appear in the literature was apparently due to Nelson (1959) (see Reed and Simon 1980, pp. 273-275).[21]

[20] See p. 571 of von Neumann (1931). Following Putnam's reconstruction (1967, 64-65) start from the Heisenberg relation $PQ - QP = -iI$ for one degree of freedom. Deduce that $PQ^n - Q^nP = -i(Q^n)'$ where n is any positive integer and the prime denotes differentiation with respect to Q. Conclude that for analytic functions f, $Pf(Q) - f(Q)P = -if'(Q)$. Choosing $f(Q) = \exp(isQ)$ gives $\exp(-isQ)P\exp(isQ) = P + sI$, and thus $\exp(-isQ)P^n \exp(isQ) = (P + sI)^n$ and $\exp(-isQ)g(P)\exp(isQ) = g(P + sI)$ for analytic g. Finally, choosing $g(P) = \exp(itP)$ gives $\exp(itP)\exp(isQ) = \exp(its)\exp(isQ)\exp(itP)$.

[21] Examples of representations satisfying the (HCCR) but not the (WCCR) are sometimes referred as belonging to the "Nelson phenomenon." Several such examples with physical applications will be considered below.

Thus, it seems likely that much of the quantum mechanics community was unaware until late in the game that although von Neumann's "Die Eindeutigkeit der Schrödingerschen Operatoren" settled the question of the essential uniqueness of the Schrödinger representation for the Weyl form of CCR with finite N it did not settle the matter for the Heisenberg form of the CCR. For example, Bogolubov, Logonov, and Todorov (1975, 558) opine that

> In the case of a system with a finite number of degrees of freedom any two irreducible representations of the commutation relations (21.28) [the Heisenberg CCR] by self-adjoint operators in a Hilbert space are unitarily equivalent (von Neumann (1931)).

What more does (WCCR) require over and above what (HCCR) requires? For two or more degrees of freedom one difference lies in the difference between weak and strong commutativity. Operators A and B acting on a Hilbert space $\mathcal{H}$ are said to weakly commute iff $[A, B]\psi = 0$ for all $\psi \in \mathcal{H}$ such that $\psi \in dom([A, B])$. For s.a A and B strong commutativity holds iff the spectral projections of A and B commute.

> *Lemma.* For s.a A and B strong commutativity holds iff $\exp(isA)\exp(itB) = \exp(itB)\exp(isA)$ for all $s, t \in \mathbb{R}$.[22]

Thus, whereas (HCCR) require only that the P_k weakly commute among themselves and similarly for the Q_k, the (WCCR) require that they strongly commute among themselves. As seen below (Section 6.1) this distinction is a crucial part of the mathematical basis of the Aharonov-Bohm effect.

Adding strong commutativity of the P_k among themselves and similarly for the Q_k to (HCCR) does not imply the satisfaction of (WCCR) for one degree of freedom since strong commutativity is trivial for $N = 1$ whereas even for $N = 1$ there are unitarily inequivalent representations of (HCCR). A sufficient condition to go from satisfaction of the (HCCR) to a satisfaction of the (WCCR) for any $1 \leq N < \infty$—and, thus, to the essential uniqueness of representations of (HCCR)—is that the restriction of $\sum_{k=1}^{N}(P_k^2 + Q_k^2)$ to a common dense domain of the P_k and the Q_k is e.s.a. (see Dixmier 1958, Nelson 1959, and Poulsen 1973). Other sufficient conditions are discussed in Putnam (1967, Ch. 4).

[22] See Theorem VIII.13 in Reed and Simon (1980, 271).

Since the Hamiltonian for an oscillator has the form $P^2 + cQ^2$, $c = const$, the last mentioned result suggests a strategy for closing the gap between (HCCR) and (WCCR). The strategy is illustrated for $N = 1$. First, require for a successful quantization of a classical system that the quantum Hamiltonian obtained from the classical Hamiltonian by substituting Hilbert space operators P, Q satisfying (HCCR) for the classical p, q are s.a. on $\mathcal{H}$ and e.s.a. when restricted to a common dense domain $\mathcal{D} \subset \mathcal{H}$ for P and Q. Second, prove that satisfaction of this requirement on quantization implies that the quantum operators P and Q satisfy the Weyl relations. The requirement on quantization is entirely reasonable in view of the fact that a Heisenberg or a Schrödinger dynamics for the system depends on a s.a. Hamiltonian operator. I am unaware of general results showing that this strategy works for a large class of Hamiltonians, e.g. Hamiltonians of the form $P^2 + V(Q)$ for analytic functions V or some other interesting class of functions.

In sum, while the von Neumann theorem gives a definitive positive answer to the question of the essential uniqueness of the representations of the Weyl form of the CCR for $N < \infty$, it gives only a partial and incomplete positive answer to the question of the essential uniqueness of representations of the Heisenberg form of CCR. The enterprise of finding conditions that fill the gap been (HCCR) and (WCCR) is a worthy one. But once it is seen that there are interesting and important physical applications that reside in the gap there arises the equally worthy enterprise of exploring the gap for signatures of effects that separate quantum physics from classical physics. It is to such examples I now turn.

6 Into the breach

Attitudes towards foundation issues in quantum physics have displayed dramatic shifts over the decades. A prime example concerns quantum entanglement which was initially viewed with consternation because, in Einstein's words, it seemed to embody "spooky action-at-a-distance." But in recent decades entanglement is seen as a resource to be exploited in quantum computing and quantum information theory.

A similar shift is underway regarding inequivalent representations of the CCR. The initial hope was to suppress them, but when suppression proved impossible they began to be perceived as a source of insight into quantum phenomena. This new attitude is most forcefully stated in Arai's thesis:

> The Universe uses inequivalent irreducible representations of CCR and/or CAR to create or produce "characteristic" quantum phenomena in which macroscopic objects may be involved. (Arai, 2020 vi)

If this is true how is the problem of the ambiguity of quantization to be handled? We shall see.

6.1 $N < \infty$

An implicit assumption behind using the Weyl form of the CCR, leading to the essential uniqueness of the Schrödinger representation, is that for a system with $N < \infty$ degrees of freedom the configuration space Ω of the system is $\mathbb{R}^N$; but when this assumption fails non-Schrödinger representations may emerge. For example, in the case of $N = 1$ the spectrum of the Schrödinger position operator Q^S, which acts as multiplication on elements of $L^2_{\mathbb{C}}(\mathbb{R})$, is $\mathbb{R}$. When Ω is topologically distinct from $\mathbb{R}$ or is a proper subset of $\mathbb{R}$ the spectrum of the position operator Q_Ω acting by multiplication on elements of $L^2_{\mathbb{C}}(\Omega)$ may not be equal to $\mathbb{R}$. Since unitary transformations preserve the spectra of s.a. operators, a representation of (HCCR) by operator pairs P_Ω, Q_Ω will not be equivalent to the Schrödinger representation P^S, Q^S.

Another way to see a hitch in applying the Weyl form the CCR in the case of $N = 1$ and a configuration space $\Omega \subsetneq \mathbb{R}$ with boundary $\partial\Omega \neq \emptyset$ focuses on momentum. The Weyl representation $U(s)$ of the action of the Schrödinger P^S as infinitessimal generator shifts the wave functions to the left by s for $s > 0$ and to the right by s for $s < 0$. For large enough s this action, loosely speaking, threatens to carry the wave function off the configuration space. If the unitarity of $U(s)$ is to be maintained for all $s \in \mathbb{R}$ a s.a. generator P_Ω other than the Schrödinger P^S is needed. In some instances there may be no suitable s.a. P_Ω, in which case the Weyl CCR are destroyed. In other instances there may be inequivalent s.a. P_Ωs corresponding to different boundary conditions at $\partial\Omega$ needed to ensure unitarity of $U(s)$ for all $s \in \mathbb{R}$. Simple concrete examples can be used to illustrate some of the possibilities.

(i) Particle on a circle. This example has applications in organic chemistry, e.g. in the free electron model of aromatic molecules. The configuration space of the system is a circle S^1 with circumference $L > 0$, and the Hilbert space is $L^2_{\mathbb{C}}(S^1)$. Alternatively, we can start with the Hilbert space

$L^2_{\mathbb{C}}([-L/2, L/2])$ and restrict to the subspace $L^2_{\mathbb{C}}([-L/2, L/2])^{S^1}$ of wave functions satisfying the periodic boundary condition $\psi(-L/2) = \psi(L/2)$. Define the position operator Q_L to act by multiplication on $L^2_{\mathbb{C}}([-L/2, L/2])$:

$$\begin{aligned} (Q_L\psi)(x) &:= x\psi(x),\ \psi \in dom(Q_L) \\ dom(Q_L) &:= \{\psi \in L^2_{\mathbb{C}}([-L/2, L/2]) : \textstyle\int |x\psi(x)|^2 dx < \infty\}. \end{aligned} \tag{8}$$

So defined Q_L is s.a. with continuous spectrum $[-L/2, L/2]$. Using a finite Fourier transform of Q_L Arai (2020, Sec. 1.14) defines a s.a. momentum operator P_L with the action

$$\begin{aligned} (P_L\psi)(x) &= -i\frac{d}{dx}\psi(x),\ \psi \in dom(P_L) \\ dom(P_L) &:= \{\psi \in L^2_{\mathbb{C}}([-L/2, L/2]) \cap AC[-L/2, L/2] : \\ \psi(-L/2) &= \psi(L/2),\ \frac{d}{dx}\psi(x) \in L^2_{\mathbb{C}}([-L/2, L/2])\ \} \end{aligned} \tag{9}$$

where $AC[-L/2, L/2]$ stands for the absolutely continuous functions on $[-L/2, L/2]$. This P_L is called a momentum operator with periodic boundary conditions since $dom(P_L)$ uses the boundary condition $\psi(-L/2) = \psi(L/2)$, but it could equally be called a momentum operator for the S^1 wave function space $L^2_{\mathbb{C}}([-L/2, L/2])^{S^1}$. It is proved that the pair Q_L, P_L give an irreducible representation of the (HCCR).

P_L has a pure discrete spectrum with eigenvalues proportional to n/L, $n \in \mathbb{Z}$, and thus the free particle Hamiltonian $H_L := P_L^2/2m$ has a pure discrete spectrum with eigenvalues proportional to n^2/L^2. The two-fold degeneracy in the energy eigenvalues for $n \neq 0$ is explained physically by the fact that each energy level can correspond either to a wave traveling clockwise or to a wave traveling anti-clockwise around the circle. Since the spectra of the s.a. P_L and H_L are different for different values of $L > 0$ it follows that P_L, Q_L,representations of (HCCR) with different values of L are unitarily inequivalent; and for all $L > 0$ they are unitarily inequivalent to the Schrödinger representation of a free particle since $spec(Q_L) \neq spec(Q^S)$ and $spec(P_L) \neq spec(P^S)$.

One might complain that this example doesn't deliver what was promised; namely, inequivalent representations of (HCCR) for one in the same physical system. For, arguably, systems with different values of L are different physical systems. Alternatively, if one accepts the inequivalent representations as representations of (HCCR) of the same system then the representations are distinguishable by a macroscopically measurable quantity L. And by measuring the value of L we can know ahead of time what momentum and energy spectra to predict. Either way the problem of the ambiguity of quantization is defused. This raises the hope that for finite N a similar moral can be applied to all physically interesting cases of inequivalent representations that escape von Neumann's uniqueness theorem. Whether or not that moral holds up depends on one's taste for what is physically interesting.

(ii) Particle on the half line. Here the configuration space is $[0, +\infty) \subset \mathbb{R}$. Taking the Hilbert space to be $L^2_{\mathbb{C}}([0, +\infty))$ the position operator $Q_{[0,+\infty)}$, defined per usual as acting on elements of $L^2_{\mathbb{C}}([0, +\infty))$ by multiplication, is s.a. The momentum operator $P_{[0,+\infty)} := -i\dfrac{d}{dx}$ with domain $C^1_0(0, \infty)$ (differentiable functions with compact support in $(0, \infty)$) is symmetric but not s.a.; indeed, this momentum operator has no s.a. extensions (see Arai 2020, Sec. 2.6.5), and so the pair $P_{[0,+\infty)}, Q_{[0,+\infty)}$ does not give a realization of the (HCCR). Nevertheless, $P^2_{[0,+\infty)} = -\dfrac{d^2}{dx^2}$ does have s.a. extensions, indeed, an uncountable infinity of them, corresponding to different boundary conditions at $x = 0$ (Reed and Simon 1980 144-145). The different—unitarily inequivalent—s.a. extensions of the free particle Hamiltonian operator $P^2_{[0,+\infty)}/2m$ produce different Schrödinger dynamics, yielding different phase changes as waves scatter off the potential barrier at $x = 0$ that we may imagine confines the particle to the half line (Reed and Simon op. cit.). So inequivalent representations for dynamics raise their hydra heads, even leaving aside the (HCCR). Here there is no helpful macroscopic parameter we can measure ahead of time to allow us to predict what boundary conditions will apply and what the resulting Schrödinger dynamics holds in store for a wave packet approaching $x = 0$ from the right.

It might be complained that this example is physically unrealistic. But it is no more unrealistic than many of the models in physics, including the model of a particle in a box that is standard fare in quantum mechanics text books.[23] Before considering that model we take up a variant of the present example.

(iii) Particle on a plane with a disk removed. Here the configuration space of a free particle is $\Omega_R := \mathbb{R}^2 \backslash D_R$ where $D_R := \{(x, y) : x_1^2 + x_2^2 \leq R\}$

[23]Indeed, one might say that the particle-in-a-box model is twice as unrealistic as the particle-on-the-half-line since the former involves two infinitely high potential barriers whereas the latter uses only one.

is a disk centered at the origin $(0, 0)$ of orthogonal coordinates x_1, x_2 for $\mathbb{R}^2$. Hirokowa (1997, 2000) shows that for $R > 0$ the momentum operators $P_{x_j,\Omega_R} := -i\dfrac{d}{dx_j}$, $j = 1, 2$, acting on $L^2_{\mathbb{C}}(\Omega_R)$ are symmetric and closed but not e.s.a. They have an uncountable number of s.a. extensions. Choose s.a. extensions $\widetilde{P}_{\Omega_R,x_j}$. Then the pairs $\widetilde{P}_{x_j,\Omega_R}, Q_{x_j,\Omega_R}$, where the position operators Q_{x_j,Ω_R} act by multiplication, gives an irreducible representation of (HCCR), and different pairs using different s.a. extensions give inequivalent representations. More interestingly, Hirokowa proves that none of the s.a. extensions satisfy (WCCR)! The (WCCR) are destroyed in this quantization.

Instead of using orthogonal coordinates on the plane one might choose to use coordinates x_{η_1}, x_{η_2} adapted to the streamlines of an incompressible and irrotational fluid flowing around the (missing) disk region. The momentum operators $P_{x_{\eta_j},\Omega_R}$, $j = 1, 2$, defined as the generators of shifts along the streamlines have unique s.a. extensions $\widetilde{P}_{x_{\eta_j},\Omega_R}$, and the pairs $\widetilde{P}_{x_{\eta_j},\Omega_R}, Q_{x_{\eta_j},\Omega_R}$, $j = 1, 2$, where the $Q_{x_{\eta_j},\Omega_R}$ act as multiplication by the streamline coordinates, can satisfy (WCCR)!

Can it really be that quantization depends on the choice of coordinates, adding another layer of ambiguity to quantizing a classical system? Or is there a unique "correct" choice of coordinates for quantizing the particle on a plane with a disk removed? And what are the criteria for recognizing correctness?

(iv) Particle on a finite interval $-L/2 \leq x \leq L/2$.

(a) No box. Define Q_L per usual as acting by multiplication on elements of $L^2_{\mathbb{C}}([-L/2, L/2])$. The operator $P_L = -i\dfrac{d}{dx}$ with dense domain $C_0^\infty(-L/2, L/2) \subset L^2_{\mathbb{C}}([-L/2, L/2])$ is closable, and its closure $\overline{P}_L$ is a symmetric operator. But $\overline{P}_L$ is not s.a. nor is it e.s.a. It does have s.a. extensions, in fact, an uncountable infinity of different extensions (see Arai 2020, 2.6.4(B)). Choose one of these s.a. extensions $\widetilde{\overline{P}}_L$. The pair $\widetilde{\overline{P}}_L, Q_L$ gives an irreducible representation of (HCCR). Any such pair is inequivalent to the standard Schrödinger pair P^S, Q^S since $spec(Q_L) = [-L/2, L/2]$ while $spec(Q^S) = \mathbb{R}$; and pairs that use different s.a. extensions $\widetilde{\overline{P}}_L$ give inequivalent representations of (HCCR). A different choice of an initial dense domain for P_L would be $\{\psi \in L^2_{\mathbb{C}}([-L/2, L/2]) : \psi \in AC[-L/2, L/2], \psi(-L/2) = 0 = \psi(L/2)\}$. As before, this symmetric operator has an uncountable infinity of different s.a. extensions (Reed and Simon 1980, VIII.3;

1975, X.1).

(b) Box. Here we imagine a particle that is confined to the interval $[-L/2, L/2]$ because it is trapped in a potential with $V(x) = 0$ for $-L/2 \leq x \leq L/2$ and $V(x) = \infty$ otherwise. The treatment of the particle in a box given in most textbooks is heuristic, showing no concern about domains of operators, self-adjointness, etc., and there is rarely any discussion of the satisfaction of the CCR. The textbook treatments typically posit that the Hamiltonian takes Schrödinger form $-\frac{1}{2m}\frac{d^2}{dx^2}+V(x)$. The time-independent wave equation for a particle-in-a-box is written as $-\frac{1}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$ for $-L/2 \leq x \leq L/2$ where E is the energy of the particle, which is by way of assuming that the Hamiltonian has a discrete spectrum. Imposing the boundary condition $\psi(-L/2) = \psi(L/2) = 0$ leads to an energy spectrum with eigenvalues proportional to n^2/L^2, now with $n \in \mathbb{Z}_+$. There is no degeneracy in the energy spectrum since the boundary condition entails that there are only standing waves rather than traveling waves.

A more careful analysis reveals that the heuristic treatment is essentially correct. The analysis is much like that for a particle on a circle with the extra condition $\psi(-L/2) = \psi(L/2) = 0$; this very restrictive form of periodic boundary conditions kills the traveling waves that exist for the particle on a circle. As in the case of a particle on a circle different values of L give rise to inequivalent represents of the (HCCR).

(v) Quantum time operators and the uncertainty relations.

Tom Pashby (2014) notes that if the formal manipulations leading from the Heisenberg CCR to the Weyl CCR were valid then the von Neumann uniqueness theorem could be used to give a "proof" of the following form of the non-existence of a "time operator" in QM: let H be a s.a. Hamiltonian operator whose spectrum is bounded from below—as one would expect for any physically realistic system. Then there is no s.a. quantum time operator T canonically conjugate to H wherein $[H, T] = -i$. For if there were such a T the von Neumann uniqueness theorem would imply that H, T are unitarily equivalent to the P, Q of a Schrödinger representation in which $sp(Q) = sp(P) = \mathbb{R}$. And since a unitary transformation preserves the spectra of s.a. operators this would mean that $sp(H) = \mathbb{R}$, contrary to assumption. However, there are known counterexamples to this non-existence claim; see Garrison and Wong (1970) who locate the source of the counterexamples in the difference between the Heisenberg and the Weyl form of the CCR. This is one of the earliest references to recognize that interesting physics lies in the gap between the two forms of the CCR. For a Hamiltonian H bounded from below the uniqueness theorem does supply a valid proof of the non-existence of a s.a. time covariant quantum operator $T(t) := U(t)TU(-t) = T(0) + tI$, where $U(t) = \exp(-iHt)$ (see Pashby 2014).

Garrison and Wong also point out the relevance of the difference between the Heisenberg and Weyl forms of the CCR for the uncertainty relations:

> For representations of the Weyl type, the commutator domain automatically includes all relevant physical states, and the conventional uncertainty relation follows. However, for representations of the Heisenberg type, the commutator domain may or may not contain all relevant physical states. In the latter case the uncertainty relation breaks down and ... [in some examples] the uncertainty relation is violated by some physical states. (Garrison and Wong 1970, 2248)

(vi) The Aharonov-Bohm effect.

The article by Aharonov and Bohm (1959) set off a furor that still reverberates today. Initially there was skepticism both about the reality of the eponymous Aharonov-Bohm (AB) effect and about claims of experimental verification. When the skepticism died down discussion shifted to a contentious debate about non-locality and the status of the electromagnetic potentials. Lost in the cacophony was a consideration of the mathematical basis of the AB effect and its connection to the CCR. It wasn't until nearly three decades after the publication of the Aharonov and Bohm's article that Reeh (1988) gave the first indication the AB effect can be considered to lie in the gap between the (HCCR) and the (WCCR) where unitary inequivalent representations lurk.

What makes the AB effect so puzzling and controversial is that it concerns cases where the configuration space of a charged particle is disjoint from regions where the magnetic field is non-zero. Classically the charged particle in such situation does not "feel" the magnetic field—it propagates independently of the strength of the magnetic field. But the Aharonov-Bohm analysis implies that the quantum particle does "feel" the magnetic field, as evidenced by in-principle detectable interference patterns that depend on the strength of the magnetic field. Realizing even an idealized detection apparatus involves cases where the configuration space of the charged particle is doubly connected, and here the example of particle moving on a plane with a disk removed can be pressed into service.

On $\mathbb{R}^2$ define a vector field $A(x_1, x_2) := (A_{x_1}(x_1, x_2), A_{x_2}(x_1, x_2))$ which may be singular inside the disk region D_R but is smooth on the exterior region $\Omega_R = \mathbb{R}^2 \backslash D_R$ and there satisfies

$$B(x_1, x_2) := \frac{\partial A_{x_2}(x_1, x_2)}{\partial x_1} - \frac{\partial A_{x_1}(x_1, x_2)}{\partial x_2} = 0. \tag{C}$$

With $A(x_1, x_2)$ interpreted as the vector potential of a magnetic field, condition (C) requires that the magnetic field $B(x_1, x_2)$ is zero outside D_R. A setup demonstrating the AB effect involves charged particles moving on a plane perpendicular to an infinitely long flux–carrying solenoid with cross section D_R. The particles are prevented from penetrating the solenoid by an infinitely high potential barrier so that their configuration space is Ω_R, and (C) is enforced by ensuring that no flux leaks from the solenoid. Together these conditions ensure that the configuration space Ω_R of the particle and regions where $B(x_1, x_2) \neq 0$ are disjoint. Nevertheless, charged particles passing the solenoid on opposite sides show interference effects that depend on the amount of flux in the solenoid, which is measured by the value of the integral of $A(x_1, x_2)$ around a closed curve encircling D_R. The mathematical basis of this effect lies in inequivalent representations of the (HCCR) and the failure of the non-Schrödinger momentum operators to strongly commute.

The limiting case where the disk has radius $R = 0$ and, thus, the configuration space $\Omega_0 := \mathbb{R}^2 \backslash (0, 0)$ of the charged particle is the plane with origin removed and the magnetic flux is reduced to a thread, was first investigated by Reeh (1988) and then in more generality by Arai (1995, 1997, 2020). Recalling that the vectors of an $L^2_{\mathbb{C}}$ Hilbert space are equivalence classes of wave functions where the equivalence relation is equality except for a set of points of measure zero, $L^2_{\mathbb{C}}(\Omega_0)$ has a natural identification with $L^2_{\mathbb{C}}(\mathbb{R}^2)$. Define the magnetic momentum operator as $P^A_{\Omega_0} := (P^A_{x_1,\Omega_0}, P^A_{x_2,\Omega_0})$ acting on $L^2_{\mathbb{C}}(\Omega_0)$, where the $P^A_{x_j,\Omega_0}$, $j = 1, 2$, are the closures of $P_{x_j} - qA_{x_j}$ acting on $L^2_{\mathbb{C}}(\mathbb{R}^2)$ and where $P_{x_j} := -i\dfrac{d}{dx_j}$ and q is the charge of the particle. With Q_{x_j,Ω_0} acting as multiplication, $P^A_{x_j,\Omega_0}$ and Q_{x_j,Ω_0}, $j = 1, 2$ are s.a. and they are e.s.a. on the common dense and invariant domain $\mathcal{D}$ consisting of C^∞ functions of compact support on Ω_0. The pair $P^A_{x_j,\Omega_0}, Q_{x_j,\Omega_0}$ gives an irreducible representation of the (HCCR) for $N = 2$. The $P^A_{x_j,\Omega_0}$ generate spatial shifts along the x_j-axes along with a phase shift that depends on the potential A. The unitary groups $\exp(isP^A_{x_1,\Omega_0})$ and $\exp(itP^A_{x_2,\Omega_0})$, $s,t \in \mathbb{R}$, have the commutation property that for any rectangular closed curve with sides lying along the x_1- and x_2-axes and having lengths respectively s and t

$$\exp(isP^A_{x_1,\Omega_0})\exp(itP^A_{x_2,\Omega_0}) = \exp(-iq\Phi^A_{s,t})\exp(itP^A_{x_2,\Omega_0})\exp(isP^A_{x_1,\Omega_0}) \quad (10)$$

where $\Phi^A_{s,t}$ is the integral of the vector field A around the closed curve and, thus, is equal to the amount of magnetic flux through a two-surface bounded by the curve (Arai 2020, Theorem 3.3). Thus, the $P^A_{x_j,\Omega_0}$ strongly commute, as required by the (WCCR) iff $q\Phi^A_{s,t}$ is an integer multiple of 2π for each $s,t \in \mathbb{R}$. Otherwise $P^A_{x_j,\Omega_0}, Q_{x_j,\Omega_0}$, $j = 1,2$, give an irreducible non-Schrödinger representation of the (HCCR). This is just the circumstance for the interference effects that witness the AB effect to manifest themselves. Using the streamline coordinates on $\Omega_R = \mathbb{R}^2 \backslash D_R$ for $R > 0$ Hirokawa (1997, 2000) extends these results to model the more realistic of case of a flux-carrying solenoid of finite non-zero radius.

The usual explanation of the AB effect is causal/dynamical, proceeding as follows: choose the appropriate Hamiltonian and solve the resulting Schrödinger equation for the wave packets of charged particles emitted from a common source and passing by the solenoid and thence to a detector screen; show that wave packets passing by the solenoid on different sides suffer different phase shifts, and use this difference to deduce the interference pattern manifested on the detector screen. A problem for this procedure arises at the first step of choosing the appropriate Hamiltonian. The natural choice is $H^A = ((P^A_{x_1,\Omega_0})^2 + (P^A_{x_2,\Omega_0})^2)/2m$. But as noted by Reeh H^A is *not* e.s.a. on the common dense and invariant domain $\mathcal{D}$ on which $P^A_{x_j,\Omega_0}$ and Q_{x_j,Ω_0}, $j = 1,2$, are e.s.a.![24] Here a choice of s.a. extension of H^A must be made in order to carry out the dynamical analysis, with the choice corresponding to a choice of what boundary conditions to impose on the wave functions at the surface of the solenoid. Intuitively, if the solenoid is well shielded so as to prevent wave packets from scattering off it then the choice of boundary conditions shouldn't have much effect on the prediction of the interference pattern on the detector screen, but this would need to be demonstrated.

[24]This can be inferred from Dixmier's theorem (see Section 5.3) and the facts that $(Q_{x_1,\Omega_0})^2 + (Q_{x_2,\Omega_0})^2$ is e.s.a. on the common and invariant dense domain and that if X and Y are both e.s.a. on a common invariant dense domain then so is $X+Y$.

The Arai result that the pairs $P^A_{x_j,\Omega_0}, Q_{x_j,\Omega_0}$, $j = 1, 2$, give an irreducible non-Schrödinger representation of the (HCCR) iff $q\Phi^A_{s,t}$ is not an integer multiple of 2π is "topological" in the sense that it is independent of the details of the dynamics (see Arai 2020, 165). Does this result then provide a non-dynamical explanation of the AB effect? The so-called deductive-nomological (DN) account of scientific explanation says that deduction from the applicable laws is sufficient for explanation. It would seem to follow from the Arai result that the existence of non-Schrödinger representations gives a non-dynamical DN explanation of the existence to the AB effect. The explanation of the details of the phase shift on the detector screen depends on the choice of the representation, and this choice is guided by the amount of flux in the solenoid, which is macroscopically determinable. One trouble here is that the proffered DN explanation works only if one can safely move from the premise about the integral of qA around a closed path surrounding to the solenoid to a conclusion about the differential phase shifts experienced by wave packets passing opposite sides of the solenoid, and justifying that inference requires a dynamical analysis, making the seemingly non-dynamical explanation parasitic on the dynamical explanation. In response it can be noted that the failure of the Weyl relations due the failure of the strong commutativity of $P^A_{x_1,\Omega_0}$ and $P^A_{x_2,\Omega_0}$ is the basis of the differential phase shifts. It seems that the two forms of explanation are mutually dependent on one another.

In any case, there is no need to see the two forms of explanation as competing. If explanation of a physical phenomenon means deepening our understanding of its origin and nature, then both the causal/dynamical approach and the deduction from inequivalent representations contribute to the explanation of the AB effect.

In closing this section I ask the reader to ponder: What other interesting quantum effects are waiting to be found among the non-Schrödinger representations of the (HCCR) in the $N < \infty$ regime that lurk in even more complicated configuration spaces?

6.2 $N = \infty$

The discussion of this topic is bedeviled by an ambiguity in the notion of degrees of freedom. For the moment let's stick to the original meaning where N stands for the number of particles times the dimension of the one-particle configuration space. von Neumann's uniqueness theorem appeared when quantum theory meant ordinary non-relativistic QM, understood to deal with systems having a finite number of degrees of freedom. The theorem made no claim to the effect that essential uniqueness would continue to hold when $N \to \infty$. Nevertheless, it seems to have been hoped that this would be the case, although in retrospect it should be clear that issues about the uniqueness of realizations of the (WCCR) are moot at the limit $N = \infty$; for to accommodate this case the Hilbert space must be non-separable[25] and because (as explained in the following section) an irreducible and continuous unitary representation of the Weyl relations is not possible on a non-separable Hilbert space. Oblivious to these facts various writers used toy examples to illustrate what goes wrong when $N \to \infty$, a favorite one involving rescaling transformations. Suppose that $\{P_k, Q_k\}_{k=1}^N$ satisfy (HCCR). Define rescaled momentum and position operators $P_k^{'} := P_k \exp(\lambda_k)$, $Q_k^{'} := Q_k \exp(-\lambda_k)$ for real numbers $\lambda_k > 0$. The $\{P_k', Q_k'\}_{k=1}^N$ also satisfy (HCCR), and although $\{P_k, Q_k\}_{k=1}^N$ and $\{P', Q_k'\}_{k=1}^N$ are unitarily equivalent for any $N < \infty$, $\{P_k, Q_k\}_{k=1}^\infty$ and $\{P', Q_k'\}_{k=1}^\infty$ are unitarily inequivalent (see Bogolubov et al. 1975, pp. 559-560).[26] Thus, the Weyl relations must fail in $N \to \infty$ limit since otherwise the corollary to the von Neumann uniqueness theorem would imply the essential uniqueness of realizations of the (HCCR).

The availability of inequivalent representations in quantum field theory (QFT) was revealed in a series of articles by Friedrichs, which were collected in a book (Friedrichs (1953)). Friedrichs emphasized so-called myriotic representations, inequivalent to standard zero-interaction Fock representation, in which the total particle number operator is not defined and there is no vacuum state (i.e. no vector that is annihilated by the destruction operators).[27] But what seems to have convinced physicists of the need to leave the familiar confines of the standard zero-interaction Fock representation was van Hove's (1952) analysis of the model of a neutral scalar field interacting with infinitely heavy, fixed point sources. van Hove found that if the interaction

[25] As in the case of an infinite tensor product of one particle Hilbert spaces of dim ≥ 2. See Appendix 3.

[26] The proof given there is incomplete. For finite N the rescaling is implemented by the unitary $U_N = \exp(-i \sum_{k=1}^N (P_k Q_k + Q_k P_k)\lambda_k)$. As $N \to \infty$ the canonical rescaling transformation remains well-defined for any choice of the λ_k, but $\sum_{k=1}^\infty (P_k Q_k + Q_k P_k)\lambda_k$ is not a well-defined operator unless $\lambda_k \to 0$ sufficiently rapidly as $k \to \infty$. So $U_\infty = \lim_{N\to\infty} U_N$ does not do the job. It remains to show that that there is no other unitary that does the job for $N = \infty$. Exercise: Complete the proof. For a rigorous proof, see Talagrand (2022, Appendix C).

[27] See the Appendix for a sketch of Fock-Cook space.

is treated as a perturbation, the perturbation calculations cannot be carried out in the free field Fock space:

> the stationary states of the field interacting with the sources are not linear combinations of the stationary states of the free field. The former are not contained in the Hilbert space spanned by the latter (they even turn out to be orthogonal to this space).[28]

Van Hove did not use the language of inequivalent representations, but that this was what was involved was emphasized by Haag (1955) and by Wightman and Schweber (1955). Following Haag (1955) the above toy rescaling example can be elaborated in the language of creation and annihilation operators used in Fock space. Starting from $\{P_k, Q_k\}$ satisfying the (HCCR) we can convert the Schrödinger representation in into a Bose-Fock space representation by defining creation and destruction operators by

$$a_k^\dagger := \frac{1}{\sqrt{2}}(P_k + iQ_k), \;\; a_k := \frac{1}{\sqrt{2}}(P_k - iQ_k) \tag{11}$$

The (HCCR) now appear in the Bose creation-destruction guise

$$\begin{aligned} a_k^\dagger a_l - a_l a_k^\dagger &= -\delta_{kl} I \\ a_k a_l - a_l a_k &= 0, \;\; a_k^\dagger a_l^\dagger - a_l^\dagger a_k^\dagger = 0 \end{aligned} \tag{12}$$

The occupation number operators are

$$N_k := a_k^\dagger a_k, \tag{13}$$

and the total number operator is $\mathcal{N} := \sum_{k=1}^\infty N_k$ when the limit exits, as it does in the standard zero-interaction Fock representation which describes an indefinitely large but not actually infinite number of particles. The vacuum state Φ_0 in this representation satisfies

$$a_k \Phi_0 = 0 \text{ for all } k \tag{14}$$

The creation and destruction operators $b_k^\dagger$ and b_k corresponding to the rescaled $\{P_k', Q_k'\}_{k=1}^N$ are related to the $a_k^\dagger$ and a_k corresponding to $\{P_k, Q_k\}_{k=1}^N$ by

[28] Translation from Sbisa (2020). This article contains a detailed exposition of van Hove's paper along with helpful discussion of the foundation issues raised by van Hove's model.

$$
\begin{aligned}
b_k &= \cosh\lambda_k \ a_k + \sinh\lambda_k \ a_k^\dagger \\
b_k^\dagger &= \sinh\lambda_k \ a_k + \cosh\lambda_k \ a_k^\dagger
\end{aligned} \tag{15}
$$

The $\{b_k^\dagger,\ b_k\}_{k=1}^N$ also satisfy the Bose creation-destruction CCR (12), and for $N < \infty$ are unitarily equivalent to the $\{a_k^\dagger,\ a_k\}_{k=1}^N$, the generator of the unitary transformation given by

$$
T := \frac{1}{2}\sum_{k=1}^N (a_k^\dagger a_k^\dagger - a_k a_k). \tag{16}
$$

But T is not a proper operator on the standard zero-interaction Fock space when $N = \infty$, and $\lim_{N\to\infty}\langle\psi_2, \exp(i\epsilon T)\psi_1\rangle = 0$ for any ψ_1, ψ_2 in the Fock space. Haag concludes that "there is no proper unitary transformation connecting the two operator systems [the $\{b_k^\dagger,\ b_k\}$ and the $\{a_k^\dagger,\ a_k\}$], i.e. these belong to inequivalent representations" of the CCR (Haag 1955, 20).[29] Note that the expectation value $\langle\Phi_0, N_k^b\Phi_0\rangle$ of the b-representation number operator $N_k^b := b_k^\dagger b_k$ in vacuum state Φ_0 of the a-representation diverges as $k \to \infty$ unless $\lambda_k \to 0$ sufficiently rapidly, so the b-representation can be said to be myriotic relative to the a-representation. But by itself this does not entail that the b-representation does not possess a vacuum state or a particle number operator. More precise statements of these matters are reserved to Appendix 1.

The alert reader will have noticed a potential confusion here. The index k in the original toy example was used used to record the number of particles or the number of degrees of freedom in the sense of the dimension of the total configuration space of the system. Under this reading, the example is not very useful in illustrating the need to leave the standard zero-interaction Fock representation for myriotic representations, for at $N = \infty$ neither $\{b_k^\dagger,\ b_k\}_{k=1}^\infty$ nor $\{a_k^\dagger,\ a_k\}_{k=1}^\infty$ give Fock space representations of the CCR. The point is that whereas Fock space is separable (assuming that the one-particle space from which the Fock space is built is separable) the accommodation of an actual infinity of particles requires a non-separable Hilbert space; for example, an infinite spin chain is properly described by an infinite tensor product of $2-$

[29] As with the Bogolubov et al. (1975, pp. 559-560) example mentioned above there is a gap between the premise that T is not a proper operator and the conclusion that there is no proper unitary transformation connecting the two operator systems.

dim Hilbert spaces producing a space of dimension 2^{χ_0}.[30] On the other hand, the index k on the creation and destruction operator of Fock space serves a different purpose. Let $\{e_k\}$ be a basis for the one-particle Hilbert space; the action of $a_k^\dagger$ (respectively a_k) is to create (respectively, destroy) a particle (or in the field interpretation, an excitation of the Bose field) in the state e_k. Once the confusion is dissolved the present example does serve a useful purpose. It shows that, if the one-particle Hilbert space is infinite dimensional, there are representations of the Bose-Fock CCR inequivalent to the standard zero-interaction Fock representation. And this is so even if the number of particles (or the number of degrees of freedom in the system in the sense of the dimension of the total configuration space) is finite.

When the dimension of the one-particle Hilbert space $\mathcal{H}$ is finite the von Neumann uniqueness theorem shows that irreducible regular Weyl representations of the CCR over $\mathcal{H}$ are all unitarily equivalent to one another and, thus, equivalent to the standard zero-interaction Fock representation, which in this context plays the role that the Schrödinger representation plays in ordinary QM (see Appendix 2). But one can wonder about the physical significance of this form of the uniqueness theorem for QFT—do we really have a true field if there are only finitely many linearly independent excitations?

Further clarification is found in Wightman and Schweber (1955), along with a recognition of how dynamics, as codified in the Hamiltonian of the system, guides the choice among the inequivalent representations of the CCR when the one-particle Hilbert space is infinite dimensional. Consider Hamiltonians of the form

$$H = \textstyle\sum_{k=1}^{\infty} \epsilon_k (a_k + \alpha_k)^\dagger (a_k + \alpha_k) \tag{17}$$

where the $\epsilon_k \geq \epsilon > 0$ and the α_k are respectively real and complex numbers and the $a_k^\dagger$, a_k satisfy the Bose creation-destruction CCR (12). Supposing that the $a_k^\dagger$, a_k possess a no-particle state, Wightman and Schweber find that the $(a_k+\alpha_k)^\dagger$, $(a_k+\alpha_k)$ (which provide a representation of the Bose-Fock CCR (12) if the $a_k^\dagger$, a_k do) possess a no-particle state $\widehat{\Phi}_0$ where $(a_k + \alpha_k)\widehat{\Phi}_0 = 0$ for all k iff $\sum_{k=1}^{\infty} |\alpha_k| < \infty$.[31] They conclude

[30] As we will see below, it follows from von Neumann's work that the infinite tensor product space is isomorphic to a direct sum of separable Hilbert spaces, but the sum is over an uncountable index set.

[31] This is a good place to attempt exercise C.5.2 of Talagrand (2022): Show that the representation $b_k = a_k + \alpha_k I$, $b_k^\dagger = a_k^\dagger + \alpha_k^* I$ of the Bose CCR is unitarily equivalent to the standard representation $a_k, a_k^\dagger$ iff $\sum_{k=1}^{\infty} |\alpha_k| < \infty$.

> This example shows that the "other" [non-zero interaction Fock] representations of the commutation relations are not pathological phenomena whose construction requires mathematical trickery. They occur in the most elementary examples in field theory (1955, 824).

To apply perturbation theory to this Hamiltonian they assume that the α_k are functions of the coupling constant g, with $a_k(g) = g\alpha_k(0)$. Perturbation theory assumes that to the no-particle state Φ_0 of the uncoupled system there corresponds a no-particle state Φ_g of the coupled system, which we have just seen is impossible when $\sum_{k=1}^{\infty} |\alpha_k| = \infty$. In this case

> to make the Hamiltonian into a well-defined operator, one is forced to choose a different representation of the $a_k + \alpha_k$ for each value of the coupling constant. (ibid.)

As if to reinforce the Wightman-Schweber morals Haag (1955) gives a heuristic argument showing that free scalar relativistic fields of different masses belong to inequivalent representations[32], and concludes that

> This may show that the "strange representations" of (12) [also our (12)] will almost inevitably turn up in any discussion in field theory. (Haag 1955, 21)

The example underscores the moral that defining "degrees of freedom" in terms of the number of particles or the dimension of the total configuration space is appropriate when one is thinking in terms of particles, but when fields are the basic entity and particles are conceived as field excitations, the dimension of the Hilbert space of field excitations (the one-particle Hilbert space in the case of Fock representations) is the appropriate measure.

Returning to the original meaning of N, inequivalent representations almost inevitably show up when the number of particles or the dimension of the total configuration space is infinite, e.g. in quantum statistical mechanics (QSM) in the thermodynamic limit in which the density, N (particle number)/V (volume), of a gas is held fixed as $N, V \to \infty$. In this setting the role of Gibbs equilibrium states of classical statistical mechanics is played by

[32] For a rigorous proof, see Blank et al. (1994, 12.3.6 Corollary). They take the one particle Hilbert space to be $L^2_{\mathbb{C}}(\mathbb{R}^3)$.

KMS states, and KMS states at different temperatures belong to inequivalent representations (Takesaki 1970).

Other characteristically quantum effects in which inequivalent representations are implicated when N (in one of the two senses distinguished above) $= \infty$ include the Casimir effect and Bose–Einstein condensation (see Arai 2018, 2020).

6.3 Non-regular representations

Sticking for simplicity to the $N = 1$ case, combine the $U(a)$ and $V(b)$ of the Weyl relations to form the Weyl group[33] $W(a, b) := \exp(-\frac{1}{2}ab)U(a)V(b)$ under the multiplication rule $W(a, b)W(c, d) = \exp(\frac{1}{2}(ad - bc))W(a + c, b + d)$. The Weyl form of the CCR used in von Neumann's uniqueness theorem demands a continuous unitary representation of the Weyl group, giving a Hilbert space realization of the additive group $\mathbb{R}^2$. Continuity (aka regularity) of the representation is desirable because it allows the P and Q to be recovered as the self-adjoint generators of the group which, by the Corollary to the the von Neumann uniqueness theorem, satisfy the (HCCR). But you can't always get what you want (or even what you need). Combining the fact that the Weyl group is a separable topological group with

> *Lemma* (Bekka and Harpe 2019): If a Hilbert space $\mathcal{H}$ carries a non-trivial, irreducible, and strongly continuous unitary representation of a separable topological group then $\mathcal{H}$ is separable.

it follows that if an irreducible representation of the Weyl group is carried by a non-separable $\mathcal{H}$ then that representation is not continuous. The lack of continuity opens space for representations inequivalent to the Schrödinger representation even for a finite number of degrees of freedom (indeed, even for one degree of freedom) (see Emch 1981).

Cavallero et al. (1999) have proved a generalization of the von Neumann uniqueness theorem for non-regular representations of the CCR by replacing the continuity assumption by a measurability condition. Applications of non-regular representations to QFT are discussed in Acerbi et al. (1993a), (1993b), (1993c).

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[33] More commonly referred to as the Heisenberg group.

Although an irreducible representation of the Weyl CCR on a non-separable $\mathcal{H}$ cannot be strongly continuous in both $U(a)$ and $V(b)$ it can be continuous in one of them: if it is strongly continuous in $U(a)$ (respectively $V(b)$) then by Stone's theorem the generator P of $U(a)$ (respectively the generator Q of $V(b)$) is a self-adjoint operator. These two representations—the momentum representation and the position representation—are unitarily inequivalent to one another and inequivalent to the Schrödinger representation. And there are other unitarily inequivalent representations, in fact an infinite array of them. An explicit construction in which Q but not P, or vice versa, has a complete set of orthonormal eigenvectors in a non-separable $\mathcal{H}$ is given in Halvorson (2001), and I briefly recapitulate his analysis here. The Hilbert space used is $\ell^2_{\mathbb{C}}(\mathbb{R})$, the vector space of square summable sequences of complex valued functions $\psi : \mathbb{R} \to \mathbb{C}$ with inner product $\langle \psi_1, \psi_2 \rangle := \sum_{x \in \mathbb{R}} \overline{\psi}_1(x)\psi_2(x)$. (Note that for each $\psi \in \ell^2_{\mathbb{C}}(\mathbb{R})$ the set of points $x \in \mathbb{R}$ at that $\psi(x) \neq 0$ is at most countably infinite.) An uncountable ON basis for $\ell^2_{\mathbb{C}}(\mathbb{R})$ is $\{\chi_\lambda : \lambda \in \mathbb{R}\}$ where χ_λ is the characteristic function of $\{\lambda\}$. Define the the action of the operator-valued map $b \ni \mathbb{R} \mapsto V(b)$ on this basis by $V(b)\chi_\lambda := \exp(ib\lambda)\chi_\lambda$. $V(b)$ extends uniquely to a unitary operator on $\ell^2_{\mathbb{C}}(\mathbb{R})$ which will be denoted by the same symbol. $V(b)$ is continuous and, thus, by Stone's theorem there is a s.a. generator Q, and $Q\chi_\lambda = \lambda\chi_\lambda$, i.e. the basis vectors of $\ell^2_{\mathbb{C}}(\mathbb{R})$ are eigenvectors of the position operator. Similarly define $U(a)$ by its action $U(a)\chi_\lambda := \chi_{\lambda - a}$. So defined $\{U(a) : a \in \mathbb{R}\}$, $\{V(b) : b \in \mathbb{R}\}$ give an irreducible representation of the (WCCR). However, the representation is non-regular because $U(a)$ is not continuous at $a = 0$, and there is no s.a. generator, i.e. no s.a. momentum operator. The construction can be reversed giving an inequivalent non-regular representation of the (WCCR) in which there is s.a. momentum operator with point eigenvalues but no s.a. position operator.

There is an obvious difficulty here in obtaining the quantum dynamics. For a classical system with Hamiltonian of the form $h(p, q) = p^2/2m + V(q)$ the usual procedure is replace p, q by self-adjoint operators P, Q satisfying the (HCCR) and then to exponentiate the resulting quantum Hamiltonian operator $H(P, Q)$ to produce a one-parameter unitary group $U(t) := \exp(-itH(P, Q))$ of time translations. In the present setting this procedure is foiled. Nevertheless, these types of non-regular representations have been put to work in loop quantum gravity (see Ashtekar et al. 2003, where the position representation on $\ell^2_{\mathbb{C}}(\mathbb{R})$ is called the polymer particle representation).

When von Neumann was proving his uniqueness theorem and composing *Mathematische Grundlagen der Quantenmechanik* non-separable Hilbert spaces were specifically excluded by an axiom of *Mathematische Grundlagen,* a move that facilitated his proof of the equivalence of matrix mechanics and wave mechanics as well as his uniqueness theorem. The exclusion was justified in the sense that none of the then known applications of the nascent quantum theory required anything but separable spaces. But the exclusion was a risky bet about what future applications would require. The bet seems to have largely paid off since, apart from possible applications in quantum gravity,[34] only highly idealized systems such as an infinite spin chain call for a non-separable Hilbert space. Ironically, von Neumann himself (see von Neumann 1939) provided the mathematical basis for such idealizations by defining and working out the mathematical properties of infinite tensor products of Hilbert spaces (see Appendix 3 for an outline). At the same time his analysis provides a route that leads, under appropriate conditions, to the conclusion that physical applications concern separable systems.

von Neumann's analysis shows that an infinite tensor product $\otimes_{a\in\mathcal{I}}\mathcal{H}_a$ of Hilbert spaces $\mathcal{H}_a$, which is non-separable when the dimension of the $\mathcal{H}_a$ is greater than 1, has a canonical direct sum decomposition $\oplus_{b\in\mathcal{J}}\mathcal{H}_b$ into separable $\mathcal{H}_b$ (called incomplete tensor product spaces—see Appendix 3). Suppose in addition that the von Neumann algebra of observables for the big system $\mathfrak{M}(\oplus_{b\in\mathcal{J}}\mathcal{H}_b)$ when restricted to any pair of subsystems $c \neq d$ reduces to the direct sum of the algebras of observables $\mathfrak{m}(\mathcal{H}_c)$ and $\mathfrak{m}(\mathcal{H}_d)$ of the individual systems, i.e. $\mathfrak{M}(\oplus_{b\in\mathcal{J}}\mathcal{H}_b)|_{\mathcal{H}_c\oplus\mathcal{H}_d} = \mathfrak{m}(\mathcal{H}_c) \oplus \mathfrak{m}(\mathcal{H}_d)$ for $c \neq d$. In the philosophers jargon, what we have are different possible non-interacting worlds, the description of each of which uses a separable Hilbert space. (The non-interaction is cashed out by noting that for $\psi_c, \psi_d \in \oplus_{b\in\mathcal{J}}\mathcal{H}_b$ having nonzero components only in $\mathcal{H}_c$ and $\mathcal{H}_d$ respectively, the superposition $\psi_c+\psi_d$ is not coherent, i.e. it gives a mixed state; and the transition probability $\langle\psi_c, A\psi_d\rangle_{\mathcal{H}_c\oplus\mathcal{H}_d} = 0$ for any $A \in \mathfrak{m}(\mathcal{H}_c) \oplus \mathfrak{m}(\mathcal{H}_d)$.) The non-separability of $\oplus_{b\in\mathcal{J}}\mathcal{H}_b$ results from pasting together the descriptions of all of the worlds; but insofar as a physical application focus an individual world, separability suffices. Needless to say the additional supposition about the structure of observables—which is characteristic of superselection rules—is not something

[34] And even here the need for non-separable Hilbert spaces is under dispute; contrast Fairbairn and Rovelli (2004) with Thiemann and Winkler (2001) and Sahlmann et al. (2001).

that is handed down from on high but must be justified on a case-by-case basis.

7 Conclusion

This review of the history and implications of von Neumann' uniqueness theorem brings two words to mind: mislabeled and misunderstood. The theorem is often labeled the Stone-von Neumann theorem; but although Stone deserves priority for announcing the theorem he never offered a proof, and von Neumann's elegant proof is uniquely his own. As for misunderstandings, the theorem is often credited with fulfilling the historically important role as demonstrating the equivalence of matrix mechanics and wave mechanics, but the theorem plays no role in the explanation of the equivalence von Neumann offered in *Mathematische Grundlagen der Quantenmechanik*. And the theorem is often billed as showing that for systems with a finite number N of degrees of freedom an irreducible representation of the CCR is essentially unique since any such representation is unitarily equivalent to the Schrödinger representation. But while the theorem does have this implication for the Weyl form of the CCR the implication does not hold for the Heisenberg form the CCR, which was what was at issue at the time. A representation of the Heisenberg CCR is not necessarily a representation of the Weyl CCR even for finite N, and in the gap lie interesting examples of inequivalent representations. Additionally, a potential misunderstanding also attends "finite N". In quantum mechanics for systems of particles the relevant distinction concerns finite vs. infinite number of particles or finite vs. infinite dimension of the configuration space of the particles. But for quantum fields the relevant distinction concerns finite vs. infinite dimension of the Hilbert space of field excitations (confusingly called the one-particle Hilbert space). When the dimension of the space of field excitations is infinite the representations of the Weyl CCR are not essentially unique.

Our review reveals a number of loopholes, exceptions, and ways around von Neumann's theorem. Ordinarily such an observation contains a implicit criticism, viz, a better, more powerful theorem is needed to close the loopholes. But von Neumann was no ordinary mathematician and his uniqueness theorem is no ordinary theorem of mathematical physics. There cannot be a better, more powerful theorem, for such a theorem would need to invoke assumptions that rule out theoretically possible quantum phenomena, a number of which have been experimentally confirmed. Indeed, studying the loopholes, exceptions, and ways around von Neumann's theorem is a good way to get an appreciation of how (as Arai puts it) the Universe uses inequivalent irreducible representations of CCR to produce characteristic quantum phenomena. The word "genius" is overused to the point of abuse. But it is no abuse to say that von Neumann's uniqueness theorem is touched by genius.

Appendix 1: Fock-Cook[35] space

Using a direct sum construction Fock space is designed to accommodate a *potential* rather than an *actual* infinity of particles. The direct sum $\oplus_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ space of the Hilbert spaces $\mathcal{H}_\alpha$ is defined for an index set $\mathcal{I}$ that may be finite, denumerable, or non-denumerable. It consists of vectors $\oplus_\vartheta := \oplus_{\alpha\in\mathcal{I}}\vartheta_\alpha$ defined by a family $\vartheta := \{\vartheta_\alpha\}$, $\alpha \in \mathcal{I}$ and $\vartheta_\alpha \in \mathcal{H}_\alpha$, provided that $\sum_{\alpha\in\mathcal{I}} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} < \infty$. The Fock space $\mathcal{F}(\mathcal{H})$ over the one-particle Hilbert space $\mathcal{H}$ is the direct sum space $\oplus_{n\in\mathbb{N}}\mathcal{H}_n$ where $\mathcal{H}_0 = \mathbb{C}$ (the no-particle or vacuum case), $\mathcal{H}_1 = \mathcal{H}$ (the one-particle case), and $\mathcal{H}_n = \mathcal{H}\otimes\mathcal{H}\otimes ...\otimes\mathcal{H}$, the n-fold tensor product of the one-particle space for $n \geq 2$ (the n-particle case). With an application to identical particles in mind, two subspaces of $\mathcal{F}(\mathcal{H})$ can be distinguished: the symmetric Fock space $\mathcal{F}_s(\mathcal{H}) = \oplus_{n\in\mathbb{N}}S_n\mathcal{H}_n$ (describing bosons) and the anti-symmetric Fock space $\mathcal{F}_a(\mathcal{H}) = \oplus_{n\in\mathbb{N}}A_n\mathcal{H}_n$ (describing fermions) where $S_n\mathcal{H}_n$ and $A_n\mathcal{H}_n$ stand respectively for symmetrized and anti-symmetrized tensor products for $n \geq 2$ while $S_n = A_n = \mathbb{I}_n$ for $n = 0, 1$. If, as is conventionally assumed, the one-particle Hilbert space $\mathcal{H}$ is separable, then $\mathcal{F}(\mathcal{H})$, $\mathcal{F}_s(\mathcal{H})$, and $\mathcal{F}_a(\mathcal{H})$ are all separable. Here we concentrate on the Bose Fock space $\mathcal{F}_s(\mathcal{H})$.

For $f_1, f_{2,...}, f_n \in \mathcal{F}_s(\mathcal{H})$ define the vector $\Psi_n \in \mathcal{F}_s(\mathcal{H})$ as $\{0, ..., S_n(f_1 \otimes f_2 \otimes ... \otimes f_n), 0, ..., 0\}$. The set of all such Ψ_n is denoted by $\mathcal{D}_s^{(n)}$ with $\mathcal{D}^{(0)} := \Phi_0 = 1 \oplus 0 \oplus 0...$ the vacuum vector, and $\mathcal{D}_s$ denotes the linear span of $\cup_{n=0}^{\infty}\mathcal{D}_s^{(n)}$, which is dense in $\mathcal{F}_s(\mathcal{H})$. The Fock-Cook creation and destruction operators $a^\dagger(f)$ and $a(f)$, which map $\mathcal{D}_s$ to $\mathcal{D}_s$, and respectively add a Bose particle (or better a Bose field excitation) in the state $f \in \mathcal{H}$, are defined as follows. $a(f) := 0 \oplus a_1(f) \oplus \sqrt{2}a_2(f) \oplus \sqrt{3}a_3(f) \oplus ...$ where the bounded operator $a_n(f) : \mathcal{H}_n \to \mathcal{H}_{n-1}$ acts $a_n(f)(g_1\otimes g_2\otimes ...\otimes g_n) = \langle g_1, f\rangle g_2\otimes ...\otimes g_n$. The adjoint $a^\dagger(f)$ of $a(f)$ has the form $a_1^\dagger(f) \oplus \sqrt{2}S_2a_2^\dagger(f) \oplus \sqrt{3}S_3a_3^\dagger(f) \oplus ...$ where $a_n^\dagger(f) : \mathcal{H}_{n-1} \to \mathcal{H}_n$ is defined by $a_n^\dagger(f)(g_1 \otimes g_2 \otimes ... \otimes g_{n-1}) = f \otimes g_1 \otimes g_2 \otimes ... \otimes g_{n-1}$. These operators satisfy the Bose-Fock form of the CCR

$$\begin{aligned}(a^\dagger(f)a(g) - a(g)a^\dagger(f))\Psi &= -\langle f, g\rangle\Psi, \quad \Psi \in \mathcal{D}_s, \;\; f, g \in \mathcal{H} \\ (a(f)a(g) - a(g)a(f))\Psi &= (a^\dagger(f)a^\dagger(g) - a^\dagger(g)a^\dagger(f))\Psi = 0\end{aligned}$$

With $\{e_k\}$ an ON basis for $\mathcal{H}$ set $a_k := a(e_k)$ and $a_k^\dagger := a\dagger(e_k)$, the Bose-Fock

[35] Cook (1953) gave a mathematically rigorous formulation of Fock space.

CCR then assume the guise

$$\begin{aligned}
(a_k^\dagger a_l - a_l a_k^\dagger)\Psi &= -\delta_{kl}\Psi, \quad \Psi \in \mathcal{D}_s, \quad f, g \in \mathcal{H} \\
(a_k a_l - a_l a_k)\Psi &= (a_k^\dagger a_l^\dagger - a_l^\dagger a_k^\dagger)\Psi = 0
\end{aligned}$$

Appendix 2: Weyl CCR for quantum fields and the von Neumann uniqueness theorem

With $\mathcal{H}$ a (complex) separable Hilbert space, a Weyl system (or Weyl representation of the CCR over $\mathcal{H}$) is a map $\mathcal{W}$ from $\mathcal{H}$ to unitary operators acting on a Hilbert space $\mathcal{K}$ such that

$$\mathcal{W}(f)\mathcal{W}(g) = \exp(-\frac{i}{2}\operatorname{Im}\langle f, g\rangle)\mathcal{W}(f+g), \quad \forall f, g \in \mathcal{H} \qquad \text{(A2.1)}$$

which implies that

$$\begin{aligned}
\mathcal{W}(f)\mathcal{W}(g) &= \exp(-i\operatorname{Im}\langle f, g\rangle))\mathcal{W}(g)\mathcal{W}(f), \quad \forall f, g \in \mathcal{H} \quad \text{(A2.2)} \\
\mathcal{W}(t_1 f)\mathcal{W}(t_2 f) &= \mathcal{W}((t_1 + t_2)f), \quad \forall t_1, t_2 \in \mathbb{R}, \forall f \in \mathcal{H} \\
\mathcal{W}^\dagger(f) &= \mathcal{W}(-f), \quad \mathcal{W}(0) = I, \quad \forall f \in \mathcal{H}
\end{aligned}$$

Such a representation is said to be regular if the unitary group $\mathcal{W}(tf)$, $t \in \mathbb{R}$, is continuous for each $f \in \mathcal{H}$. The s.a. group generator $R(f)$ of this group can be used to define creation and destruction operators $a^\dagger(f) := \frac{1}{\sqrt{2}}(R(f) - iR(if))$ and $a(f) := \frac{1}{\sqrt{2}}(R(f) + iR(if))$, which need not coincide with the Fock-Cook creation and destruction operators. If $\mathcal{H}_R \subset \mathcal{H}$ is a real linear subspace and $\mathcal{H} = \mathcal{H}_R \oplus i\mathcal{H}_R$, and if we define $U(f) := \mathcal{W}(if)$ and $V(g) := \mathcal{W}(g)$ for $f, g \in \mathcal{H}_R$ then the Weyl system assumes the form $U(f)V(g) = \exp(-i\operatorname{Re}\langle f, g\rangle)V(g)U(f)$; and if $\{e_k\}$ is an ON basis for $\mathcal{H}$ and if $\mathcal{H}_R$ is identified with the real linear span of the e_k then $U(se_j)V(te_k) = \exp(ist\delta_{jk}) = V(te_k)U(se_j)$ (see Chaikin 1967, 26-27).

With $\mathcal{H}$ the one-particle Hilbert space and $a^\dagger(f)$ and $a(f)$, $f \in \mathcal{H}$, respectively the Fock-Cook creation and destruction operators for a Bose-Fock space $F_s(H)$, define the field operator $\phi(f)$ as the closure of $\frac{1}{\sqrt{2}}(a(f) + a^\dagger(f))$. This field operator is s.a., and $\mathcal{H} \ni f \mapsto \mathcal{W}_0(f) := \exp(i\phi(f))$ is a regular Weyl system, unitarily equivalent to the standard zero-interaction Fock representation, and the set $\{\mathcal{W}(f) : f \in \mathcal{H}\}$ is irreducible (see Blank et al. 12.3.3 Theorem). On the field interpretation talk of creation or annihilation of particles in a state $f \in \mathcal{H}$ is construed as creation or annihilation of an excitation f in the Bose field. And in this setting the von Neumann uniqueness theorem asserts that if the number of linearly independent field excitations is finite ($\dim(\mathcal{H}) < \infty$) then all irreducible regular Weyl representations are unitarily equivalent and, thus, equivalent to the standard zero-interaction Fock-Cook representation, and a reducible regular representation is a direct sum of the irreducible ones (see Dereziński 2006).

In the standard zero-interaction Fock-Cook representation the vacuum vector Φ_0 is annihilated by all of the destruction operators $a(f)$, $f \in \mathcal{H}$, and the total number operator $\mathcal{N} := \sum_{k=1}^{\infty} N_k$, where $N_k := a_k^{\dagger} a_k$, exists as a densely defined operator. There is a sense in which these properties uniquely single out the standard zero-interaction Fock representation among the Weyl systems, but this sense has to be carefully specified. Chaikin (1967) shows that the sequence $\sum_{k=1}^{n} N_k$ can converge as $n \to \infty$ in Weyl systems unitarily inequivalent to the standard zero-interaction representation; and even in the standard zero-interaction representation there exists a basis $\{x_\gamma\}$ of $\mathcal{H}$ such that $\sum_{k=1}^{\infty} \langle N_k x_\gamma, x_\gamma \rangle = +\infty$. The technically correct statement is that the standard zero-interaction Fock representation is the unique up to unitary equivalence Weyl system for which a total particle number operator $\mathfrak{N}$ exists in the sense that $\mathfrak{N}$ is s.a., its spectrum is $\{0, 1, 2, ...\}$, and $\mathfrak{N} a(f) = a(f)(\mathfrak{N} + I)$, for all $f \in \mathcal{H}$ (Chaikin 1967, 1968).[36]

When the one-particle $\mathcal{H}$ is infinite dimensional the von Neumann uniqueness theorem is no longer valid, as shown by the example of the free relativistic scalar fields with positive pass $m > 0$, where the Weyl systems $\mathcal{W}_m(f)$ and $\mathcal{W}_{m'}(f)$ are not unitarily equivalent when $m \neq m'$ and $\dim(\mathcal{H}) = \infty$. (Blank et al. 12.3.5 Theorem and 12.3.6 Corollary). It follows that for at most one value of m does the Weyl system $\mathcal{W}_m(f)$ have a total number operator $\mathfrak{N}$ in Chaikin's sense.

Is there an analog in the present setting of the Nelson phenomenon for ordinary QM, where satisfaction of the Heisenberg CCR does not imply satisfaction of the Weyl CCR and there are unitarily inequivalent realizations of the Heisenberg CCR even for a finite number of degrees of freedom? More specifically, when $\dim(\mathcal{H}) < \infty$ can there exist self-adjoint operators $b^{\dagger}(f), b(f)$, which are e.s.a. on a common invariant dense domain $\mathcal{D} \subset \mathcal{F}_s(\mathcal{H})$

[36] The rigorous version of the last condition is that $\exp(it\mathfrak{N}) W(f) \exp(-it\mathfrak{N}) = W(\exp(it)f)$ for all $t \in \mathbb{R}$ and $f \in \mathcal{H}$.

satisfying the Bose-Fock CCR

$$\begin{aligned}(b^\dagger(f)b(g) - b(g)b^\dagger(f))\Psi &= -\langle f, g\rangle\Psi, \quad \Psi \in \mathcal{D}, \ \ f, g \in \mathcal{H} \\ (b(f)b(g) - b(g)b(f))\Psi &= (b^\dagger(f)b^\dagger(g) - b^\dagger(g)b^\dagger(f))\Psi = 0\end{aligned}$$

but $\mathcal{W}_b(f) := \exp(i\phi_b(f))$, where the field operator $\phi_b(f)$ is the closure of $\frac{1}{\sqrt{2}}(b(f) + b^\dagger(f))$, is not a regular Weyl system and the $b^\dagger(f), b(f)$, are not unitarily equivalent to the $a^\dagger(f), a(f)$, of the standard zero-interaction Bose-Cook representation? If the $b^\dagger(f), b(f)$ and $a^\dagger(f), a(f)$, are related by a Bogoliubov transformation then results of Shale (1967) imply a negative answer. But I am unaware of results supplying a general answer.

Appendix 3: von Neumann's infinite tensor product construction

A sequence $\xi := \{\xi_\alpha\}$, $\xi_\alpha \in \mathcal{H}_\alpha$ and $\alpha \in \mathcal{I}$, defines a C-vector $\otimes_\xi := \otimes_{\alpha\in\mathcal{I}}\xi_\alpha$ provided that $\Pi_{\alpha\in\mathcal{I}}||\xi_\alpha||_{\mathcal{H}_\alpha}$ converges.[37] The complete ITP Hilbert space $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ is constructed by forming finite linear combinations of C-vectors and completing in the norm derived from the inner product $\langle\otimes_\xi, \otimes_\zeta\rangle := \Pi_\alpha\langle\xi_\alpha, \zeta_\alpha\rangle_{\mathcal{H}_\alpha}$ of C-vectors $\otimes_\xi$ and $\otimes_\zeta$. If $\dim(\mathcal{H}_\alpha) = D$ for all α then $\dim(\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha) = D^{|\mathcal{I}|}$. In the simplest non-trivial case where $D = 2$ and $|\mathcal{I}| = \aleph_0$, as in the case of the infinite spin chain, $\dim(\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha) = 2^{\aleph_0}$.One of the main results of von Neumann's (1939) connects the ITP construction with the infinite direct sum construction. A Hilbert space $\mathcal{H}$ can be considered an internal direct sum if there is a family $\{\mathcal{H}_\beta\}$, $\beta \in \mathcal{J}$, of mutually orthogonal subspaces such that $\vee_{\beta\in\mathcal{J}}\mathcal{H}_\beta = \mathcal{H}$, for then $\mathcal{H}$ is isomorphic to $\oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta$ (see Kadison and Ringrose 1991, p. 124). von Neumann showed that an infinite tensor product space $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ has a canonical internal direct sum decomposition into what he called incomplete ITP's. A C_0-vector is a C-vector $\otimes_\xi$ such that $\sum_\alpha |\ ||\xi_\alpha||_{\mathcal{H}_\alpha} - 1|$ converges. Two C_0-vectors $\otimes_\xi$ and $\otimes_\zeta$ are said to be equivalent ($\xi \approx \zeta$) just in case $\sum_\alpha |\langle\xi_\alpha, \zeta_\alpha\rangle_{\mathcal{H}_\alpha} - 1|$ converges. It is shown that $\approx$ is in fact an equivalent relation; the equivalence class of ξ is denoted by $[\xi]$, and the set of equivalence classes is denoted by $\mathcal{S}$. For $[\xi] \in \mathcal{S}$ the Hilbert space $\mathcal{H}_{[\xi]}$ formed by taking the closure of finite

[37] When $\mathcal{I}$ is uncountable $\Pi_{\alpha\in\mathcal{I}}||\xi_\alpha||_{\mathcal{H}_\alpha}$ is understood as $\lim_F \Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha}$ where the F are finite subsets of $\mathcal{I}$, and $\lim_F \Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha} = L$ means that for any $\epsilon > 0$ there is a finite $F_0 \subset \mathcal{I}$ such that for any finite F with $\mathcal{I} \supset F \supset F_0$, $|\Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha} - L| < \epsilon$.

linear combinations of $\otimes_{\xi'}$'s with $\xi' \in [\xi]$ is an incomplete ITP.[38] The $\mathcal{H}_{[\xi]}$ are separable, and for any $[\xi] \in \mathcal{S}$ there is a $\xi^0 \in [\xi]$ such that $||\xi^0|| = 1$, and $\mathcal{H}_{[\xi]}$ is the closure of finite linear combinations of $\otimes_{\xi'}$'s such that $\xi' \in [\xi]$ and $\xi'_n = \xi^0_n$ for all but finitely many $n \in \mathbb{N}^+$. Further, the $\mathcal{H}_{[\xi]}$ are mutually orthogonal, for if $[\xi] \neq [\zeta]$ then $\langle \otimes_{\xi'}, \otimes_{\zeta'} \rangle = 0$ for all $\xi' \in [\xi]$ and $\zeta' \in [\zeta]$. von Neumann also showed that the closed set these incomplete ITP spaces determine is $\otimes_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha$ and, hence, $\otimes_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha = \oplus_{[\xi] \in \mathcal{S}} \mathcal{H}_{[\xi]}$. (von Neumann did not state the result in this format, presumably because he was not thinking in terms of infinite direct sums. He simply says that the complete ITP "splits up" into incomplete ITPs.)

[38] The infinite tensor product constructed in Bratelli and Robinson (1987, 144-145) corresponds to an incomplete ITP.

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