

The Conventionality of Real Valued Quantities

The representational theory of measurement provides a collection of results that specify the conditions under which an attribute admits of numerical representation. The original architects of the theory interpreted the formalism operationally and explicitly acknowledged that some aspects of their representations are conventional. There have been a number of recent efforts to reinterpret the formalism to arrive at a more metaphysically robust account of physical quantities. In this paper we argue that the conventional elements of the representations afforded by the representational theory of measurement require careful scrutiny as one moves toward such an interpretation. To illustrate why, we show that there is a sense in which the very number system in which one represents a physical quantity such as mass or length is conventional. We argue that this result does not undermine the project of reinterpreting the representational theory of measurement for metaphysical purposes in general, but it does undermine a certain class of inferences about the nature of physical quantities that some have been tempted to draw.

1. Introduction. The representational theory of measurement (RTM) characterizes the conditions under which qualitative attributes can be represented numerically.¹ This is achieved with representation theorems that demonstrate the existence of a homomorphism between two types of structure. The domain of the homomorphism is an empirical relational structure – a structure satisfying an axiomatization of the nature of the attribute. The codomain is a numerical relational structure – typically a set of numbers with a set of mathematical relations defined over it.

The interpretation of the empirical relational structures preferred by the original architects of RTM is expressly operationalist. On their view, an empirical relational structure is taken to consist of a set of objects that all instantiate the qualitative attribute in question together with a collection of operationally interpreted relations defined over the set. The numerical relational structure is typically taken to be the real numbers with the standard order structure and the standard notion of addition. The existence of a homomorphism between these two structures establishes a clear sense in which the relations over the set in the empirical relational structure can be represented by the order and addition relations obtaining over the real numbers.

There have been a number of recent efforts to reinterpret the formalism of RTM for the purposes of articulating a realist metaphysics of quantity.²

¹The canonical presentation of the theory can be found in (Krantz, Luce, Suppes, and Tversky 2007; Luce, Krantz, Suppes, and Tversky 2007; Suppes, Krantz, Luce, and Tversky 2007).

²See, for example, (Domotor and Batitsky 2008; Heilmann 2015; Wolff 2020).

We are quite sympathetic to what we take to be one central aspect of this project: reinterpreting the empirical relational structures of RTM as property spaces provides an apt characterization of the nature of the quantities arising in physical theories. There is another aspect of recent realist reinterpretations of RTM, however, which requires more careful scrutiny. In particular, a number of authors have recently advocated that in addition to reinterpreting the empirical relational structure, we ought to strengthen the representation theorems so that they establish the existence of an isomorphism rather than a homomorphism.

The move from homomorphism to isomorphism amounts to demanding that all of the structure of the number system in which the representation is sought can be projected back onto the empirical relational structure. The difficulty with this requirement is that certain aspects of the representations furnished by RTM are conventional. The original proponents of RTM were well aware of this, and on their operationalist interpretation of the formalism the conventionality of the representations is entirely unproblematic. Our worry is that the conventionality, together with the demand that the representations take the form of isomorphisms, leads to an unjustified pattern of inference concerning the nature of the empirical relational structure.

In order to make this worry explicit, we establish a new form of conventionality in the representations afforded by RTM. What we show is that there is an important sense in which the very number system in which one represents quantities like mass and length turns out to be conventional. More precisely, what we show is that positive closed extensive structures (PCES) are homomorphically representable in both the real and the rational numbers. If one conventionally selects the representation in the reals, and then demands isomorphism, this prejudices substantive questions about the nature of the qualitative attribute being represented. We argue that such questions can only be resolved through empirical investigation of the nature of the qualitative attribute, and hence conventionality together with the requirement of isomorphism can lead to unjustified inferences concerning the nature of physical quantities.

Our argument proceeds as follows. In Section 2, we recall the basic elements of RTM and its operationalist interpretation. We then discuss how to reinterpret the formalism in terms of property spaces. In the third section we prove a series of results concerning the numerical representability of positive closed extensive structures in both the real and rational numbers. In Section 4 we argue that these results demonstrate an important sense of conventionality in the choice of numerical structure for a certain class of quantities, and we discuss the impact of this conclusion for efforts to leverage the results of

the representational theory of measurement for the metaphysics of quantity.

2. Property space interpretations of RTM. According to the operational interpretation of RTM, the numerical representation of a physical quantity is achieved by first enumerating a set of objects A that have the quantity in question as one of their attributes. In the case of length we might take A to be a set of rigid rods of varying lengths. Such a collection of rods can be compared to one another and we introduce a comparison relation denoted by \succsim , such that for any $a, b \in A$, $a \succsim b$ iff the length of a is greater than that of b . We also define a concatenation relation, denoted by \circ , such that for any $a, b \in A$, $a \circ b$ can enter into the comparison relation with any $c \in A$. Taken together, $\langle A, \succsim, \circ \rangle$ forms the empirical relational structure.

The operationalist nature of the interpretation stems from the interpretation of \succsim and \circ as concrete comparison and concatenation procedures. In the case of length, to compare rods a and b we place them side by side with one end of each aligned with one another. If at the other end a extends past b , then $a \succ b$, if b extends past a then $b \succ a$, and if neither rod can be determined to extend past the other then we have $a \sim b$. Concatenation is interpreted as placing the rods end on end. If we compare c with a placed end on end with b and find that c extends past $a \circ b$, then we have $c \succ a \circ b$.

The sense of numerical representability employed in RTM is then given by the following definition:

Definition 2.1. *An empirical relational structure $\langle A, \succsim, \circ \rangle$ is homomorphically representable in a numerical relational structure $\langle B, \geq, + \rangle$ iff there exists a function $\phi : A \rightarrow B$ such that $\forall a, b \in A$:*

1. $a \succsim b$ iff $\phi(a) \geq \phi(b)$
2. $\phi(a \circ b) = \phi(a) + \phi(b)$

To generate a representation theorem, we need to articulate a particular empirical relational structure and a particular numerical relational structure. The numerical relational structure is typically taken to be $\langle \mathbb{R}, \geq, + \rangle$, or perhaps $\langle \mathbb{R}_+, \geq, + \rangle$. The precise nature of the empirical relational structure is articulated with axioms that specify the properties of \succsim and \circ . For quantities such as mass and length, the following empirical relational structure is employed:

Definition 2.2. *(Krantz, Luce, Suppes, and Tversky 2007, def. 3.1, p. 73). Let A be a nonempty set, let \succsim be a binary relation on A , and let \circ be a closed binary operation on A . An empirical relational structure $\langle A, \succsim, \circ \rangle$ is a positive closed extensive structure iff $\forall a, b, c, d \in A$, each of the following are satisfied:*

(Connectedness) Either $a \succsim b$ or $b \succsim a$.

(Transitivity) If $a \succsim b$ and $b \succsim c$, then $a \succsim c$.

(Weak associativity) $a \circ (b \circ c) \sim (a \circ b) \circ c$.

(Monotonicity) $a \succsim b$ iff $a \circ c \succsim b \circ c$ iff $c \circ a \succsim c \circ b$.

(Archimedean) There exists a positive integer n such that $na \succ b$, where na is defined inductively as $1a = a$, $(n + 1)a = na \circ a$.

(Positivity) $a \circ b \succ a$.

(Solvability) If $a \succ b$, then there exists $d \in A$ such that $a \sim b \circ d$.

The representation theorem then takes the following form:

Theorem 2.1. (Krantz, Luce, Suppes, and Tversky 2007, thm 3.1, p. 74, pp. 80–81). Let A be a non-empty set, let \succsim be a binary relation on A , and let \circ be a closed binary operation on A . Then the following are equivalent:

1. $\langle A, \succsim, \circ \rangle$ is a positive closed extensive structure.
2. $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$.

The operationalist interpretation of the empirical relational structure suited the purposes of the original architects of the theory, but it is deficient as the basis for a metaphysics of the quantities that appear in physical theories. To take just one example, a strict operationalist interpretation requires that there exists an actual rod of each length that we want to represent numerically. But of course, there are many possible lengths for which there is no corresponding rod. If one is interested in leveraging the results of the representational theory of measurement in support of more metaphysically committal views about the nature of the quantities, we need a new way to interpret the empirical relational structure.

In the metaphysics literature it has become common to treat quantities as determinable properties which admit of a collection of determinates as their possible values. One promising strategy for the reinterpretation of the empirical relational structure is to understand the underlying set as a property space. That is, we can take the set A in the empirical relational structure to be the set of all of the possible determinate values of the quantity and \succsim and \circ to be relations between determinate values of the quantity.

Reinterpreting the qualitative domain as a property space suggests additional constraints on the function ϕ . On an operationalist interpretation,

the order on A is not required to be antisymmetric, since there may be distinct objects a, b such that both $a \succsim b$ and $b \succsim a$. But the determinates of a given quantity are such that no distinct determinates may be tied with each other with respect to their ordering. So on a property space interpretation, it is reasonable to require that the order on A be antisymmetric. We thus characterize a property space by adding antisymmetry to the list of axioms given in Definition 2.2, resulting in the following definition:

Definition 2.3. *A positive closed extensive structure $\langle A, \succsim, \circ \rangle$ is simply ordered iff $\forall a, b \in A$, the following is satisfied:*

(Antisymmetry) If both $a \succsim b$ and $b \succsim a$, then a is (numerically) identical with b .

In combination with the two conditions on homomorphic representability given in Definition 2.1, the antisymmetry of the order on A implies that ϕ is injective. Realist adaptations of RTM often require that the homomorphism ϕ also be surjective, thereby replacing the notion of homomorphic representation with that of isomorphic representation.³ This is captured in the following strengthened notion of numerical representation:

Definition 2.4. *An empirical relational structure $\langle A, \succsim, \circ \rangle$ is isomorphically representable in a numerical relational structure $\langle B, \geq, + \rangle$ iff there exists a function $\phi : A \rightarrow B$ such that $\forall a, b \in A$:*

1. $a \succsim b$ iff $\phi(a) \geq \phi(b)$
2. $\phi(a \circ b) = \phi(a) + \phi(b)$
3. ϕ is a bijection.

As Wolff has it, requiring ϕ to be an isomorphism captures the realist commitment that “the rich structure of the numerical representation is projected back to the physical world” (Wolff 2020, p. 96).

We can now make our worry explicit. Once one moves to requiring isomorphic representation, some features of the representations which were unproblematic when the empirical relational structures were interpreted operationally start to look quite problematic when they are interpreted as property spaces. What we have in mind in particular are conventional elements of the representation. The original developers of RTM were explicit that the

³See, for example, (Domotor and Batitsky 2008; Wolff 2020). (Narens 2002) also employs isomorphic representation, though is concerned with meaningfulness and not (explicitly) with realism.

interpretation of the relations in the numerical relational structure were conventional. Alternative interpretations of those relations yield equally good numerical representations. We think there is a response available to advocates of isomorphic representation in the case of this sense of conventionality. However, in the next section we show that the number system itself is conventionally chosen. This raises serious problems for inferences that involve projecting structure from the numerical relational structure back to the empirical relational structure.

3. Numerical representability in \mathbb{R} and \mathbb{Q} . Theorem 2.1 establishes that a relational structure is a positive closed extensive structure iff it is homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$. In this section we show that the use of the reals as the underlying set in the numerical relational structure is a conventional choice. Our strategy is to determine a class of empirical relational structures that are homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$, and then to explore the constraints that homomorphic representability in $\langle \mathbb{Q}_+, \geq, + \rangle$ imposes on an empirical relational structure. To start, we first establish that there exist PCES that are not homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$.⁴

Theorem 3.1. *Not all positive closed extensive structures are homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$.*

While all PCES are homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$, not all PCES are homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$: being a PCES is not sufficient for being homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$. The following theorem demonstrates that any empirical relational structure that is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$ is also homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$.

Theorem 3.2. *Let A be a non-empty set, let \succsim be a binary relation on A , and let \circ be a closed binary operation on A . If $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$ then $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$.*

From Theorems 2.1 and 3.2, it follows that homomorphic representability of an empirical relational structure $\langle A, \succsim, \circ \rangle$ in $\langle \mathbb{Q}_+, \geq, + \rangle$ implies that $\langle A, \succsim, \circ \rangle$ is a PCES. Then from Theorem 3.1 it follows that, in addition to the axioms for PCES, some further constraint delineates those structures that are homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$. This constraint is captured by the following axiom:

⁴The proofs of the theorems in this section are given in the appendix.

Definition 3.1. A positive closed extensive structure is commensurable if it satisfies the following axiom:

(Commensurability) For every $a, b \in A$, there exists $c \in A$ and positive integers m, n such that $a \sim mc$ and $b \sim nc$.

A PCES is incommensurable if it does not satisfy commensurability.

Being a commensurable PCES is necessary and sufficient for being homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$:

Theorem 3.3. Let $\langle A, \succsim, \circ \rangle$ be a positive closed extensive structure. Then the following are equivalent:

1. $\langle A, \succsim, \circ \rangle$ is commensurable.
2. $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$.

Theorem 3.3 establishes that a representation theorem formulated in terms of necessary and sufficient conditions for homomorphic representation in $\langle \mathbb{Q}_+, \geq, + \rangle$ is available, provided that the represented PCES is stipulated to be commensurable. Theorem 2.1 establishes that $\langle \mathbb{R}_+, \geq, + \rangle$ can accommodate homomorphic representation of *any* PCES. This might be taken to suggest that $\langle \mathbb{R}_+, \geq, + \rangle$ is to be favored for being generally applicable to any PCES, whereas the applicability of $\langle \mathbb{Q}_+, \geq, + \rangle$ is limited to those PCES that are commensurable. It should be noted that this difference in scope of applicability is a result of the fact that $\langle \mathbb{R}_+, \geq, + \rangle$ contains $\langle \mathbb{Q}_+, \geq, + \rangle$ as a substructure, which implies that any homomorphism into $\langle \mathbb{Q}_+, \geq, + \rangle$ can be regarded as a homomorphism into $\langle \mathbb{R}_+, \geq, + \rangle$. Thus, any ERS that is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$ is also homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$.

The situation is different if we require isomorphic representation. Being isomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$ and being isomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$ are mutually exclusive. The general applicability of $\langle \mathbb{R}_+, \geq, + \rangle$ for representation of PCES as established by Theorem 2.1 does not survive the move from homomorphic to isomorphic representation. To demand that a PCES be isomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$ is effectively to stipulate that the PCES is incommensurable.

In the context of homomorphic representation, $\langle \mathbb{R}_+, \geq, + \rangle$ accommodates representation of both commensurable and incommensurable PCES. But isomorphic representation of a PCES in $\langle \mathbb{R}_+, \geq, + \rangle$ requires that the represented PCES be incommensurable, as established by the next result.

Theorem 3.4. Let $\langle A, \succsim, \circ \rangle$ be a positive, closed, extensive structure. If $\langle A, \succsim, \circ \rangle$ is isomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$, then $\langle A, \succsim, \circ \rangle$ is incommensurable.

Theorem 2.1 establishes that any PCES is homomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$, but this is not sufficient to determine whether the PCES is commensurable or incommensurable. If we demand a PCES to be isomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$, then it follows from Theorem 3.4 that we thereby indirectly demand the PCES to be incommensurable. The following corollary emphasizes this point.

Corollary 3.4.1. *No commensurable PCES is isomorphically representable in $\langle \mathbb{R}_+, \geq, + \rangle$.*

Similarly, if we demand a PCES to be isomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$, then we indirectly demand that it is commensurable:

Theorem 3.5. *Let $\langle A, \succ, \circ \rangle$ be a positive, closed, extensive structure. If $\langle A, \succ, \circ \rangle$ is isomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$, then $\langle A, \succ, \circ \rangle$ is commensurable.*

Corollary 3.5.1. *No incommensurable PCES is isomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$.*

Taken together, the results presented in this section establish a sense in which the use of the reals in representations of PCES is conventional. There exist PCES which are homomorphically representable just as well in the rationals as they are in the reals. Absent additional argumentation, we don't see any reason to prefer the representation in the reals to representation in the rationals. Defaulting to a representation based on one or the other of the number systems is to make a conventional choice. This conventionality combines with the move to isomorphic representation in a perhaps unexpected manner. If we naively demand that our conventional homomorphic representation in the reals can be extended to an isomorphic representation, we indirectly demand that the PCES exhibits incommensurability. If instead we conventionally start from the homomorphic representation in the rationals, and then demand isomorphism, we indirectly demand that the PCES exhibits commensurability. But conventional representational choices aren't the sort of thing that can tell us anything at all about the nature of the structure being represented, and so we think that the motivations for making the switch to isomorphic representation require further elaboration.

4. Conclusion. The results of the last section are not intended to undermine metaphysical readings of the representational theory of measurement. Rather, they provide reason to revisit what the commitments of the realist about quantities are supposed to be. In recent discussions, realism has become bound up with the demand for isomorphic representation. We propose

that realism about quantities need not be coupled with this demand. One can be a realist by taking a particular property space to capture the real structure of a physical quantity. That a quantity understood in this way can be represented in some number system isomorphically is an additional commitment that goes beyond realism.

5. Appendix.

5.1. Proofs of theorems

Proof of Theorem 3.1. An example of a positive closed extensive structure that is not homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$ is as follows: let $A = \mathbb{R}_+$, let \succsim be \geq (the usual non-strict order on \mathbb{R}_+), and let \circ be $+$ (the usual addition operation on \mathbb{R}_+).⁵ Showing that $\langle \mathbb{R}_+, \geq, + \rangle$ satisfies Definition 2.2 is straightforward. Suppose for contradiction that there exists a function $\phi : A \rightarrow \mathbb{Q}_+$ satisfying the two conditions given in Definition 2.1. Since $A = \mathbb{R}_+$, A is uncountable and, since \mathbb{Q}_+ is countable, we have that $|A| > |\mathbb{Q}_+|$. It follows that $\phi : A \rightarrow \mathbb{Q}_+$ is not injective, thus there exist $a, b \in A$ such that $a \neq b$ and $\phi(a) = \phi(b)$. Since \geq is a total order on \mathbb{R}_+ , we have that for every $x, y \in \mathbb{R}_+$, $x \neq y$ iff either $x > y$ or $y > x$ (where $>$ is the *strict* total order associated with \geq). There thus exist $a, b \in A$ such that either $a \succ b$ or $b \succ a$ and $\phi(a) = \phi(b)$. Suppose without loss of generality that $a \succ b$. From $\phi(a) = \phi(b)$ we have $\phi(b) \geq \phi(a)$ and thus, by Condition 1 from Definition 2.1, $b \succsim a$, a contradiction. \square

Proof of Theorem 3.2. Suppose that A is a non-empty set, \succsim is a binary relation on A , and that \circ is a closed binary operation on A . Suppose also that $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$. Then there exists a function $\phi : A \rightarrow \mathbb{Q}_+$ satisfying the two conditions given in Definition 2.1. Define the function $\psi : A \rightarrow \mathbb{R}_+$ such that $\psi(a) = \phi(a)$ for every $a \in A$. Since $\langle \mathbb{Q}_+, \geq, + \rangle$ is a substructure of $\langle \mathbb{R}_+, \geq, + \rangle$, ψ also satisfies the two conditions given in Definition 2.1. \square

Proof of Theorem 3.3. Let $\langle A, \succsim, \circ \rangle$ be a positive closed extensive structure. We first show that 1 implies 2. Suppose that $\langle A, \succsim, \circ \rangle$ is commensurable. Let $a, b \in A$, and let m, n be positive integers. By Commensurability, there exists

⁵Thanks to Neil Dewar for discussion of this counterexample.

$c \in A$ and positive integers m, n such that $a \sim mc$ and $b \sim nc$. It follows that

$$\begin{aligned}
na &\sim n(mc) && \text{by Lemma 5.5} \\
&\sim (nm)c && \text{by Lemma 5.2} \\
&= (mn)c \\
&\sim m(nc) && \text{by Lemma 5.2} \\
&\sim mb && \text{by Lemma 5.5}
\end{aligned}$$

Thus for all ordered pairs $(a, b) \in A \times A$ there exist positive integers n, m such that $na = mb$. For each ordered pair $(a, b) \in A \times A$, let $F_{(a,b)}$ be the set of all fractions $\frac{n}{m}$ such that $na = mb$. For any given pair (a, b) , if $\frac{n}{m}, \frac{n'}{m'} \in F_{(a,b)}$ we have that

$$\begin{array}{lll}
na \sim mb & n'a \sim m'b & \\
n'(na) \sim n'(mb) & n(m'b) \sim n(n'a) & \text{by Lemma 5.5} \\
(n'n)a \sim (n'm)b & (nm')b \sim (nn')a & \text{by Lemma 5.2} \\
= (mn')b & = (n'n)a &
\end{array}$$

Thus $(n'n)a \sim (mn')b$ and $(nm')b \sim (n'n)a$, so that $(mn')b \sim (nm')b$. By Lemma 5.7a), it follows that $mn' = nm'$. Therefore, for any $(a, b) \in A \times A$, all fractions in $F_{(a,b)}$ are equivalent, so that $F_{(a,b)}$ corresponds to a unique positive rational number representing the *ratio* of a to b , denoted $[a : b]$. To construct the function ϕ , choose an arbitrary $e \in A$. Then, for every $a \in A$, define the function ϕ as:

$$\phi(a) = [a : e]$$

We now show that ϕ satisfies both conditions stated in Definition 2.1.

Condition 1: $a \succsim b$ iff $\phi(a) \geq \phi(b)$. Let $a, b \in A$ with $a \succsim b$, and choose $n, m, n', m' \in \mathbb{Z}_{>0}$ so that $na \sim me$ and $n'b \sim m'e$. Then $\phi(a) = \frac{m}{n}$ and $\phi(b) = \frac{m'}{n'}$, and we have that

$$\begin{array}{lll}
na \sim me & n'b \sim m'e & \\
m'(na) \sim m'(me) & m(n'b) \sim m(m'e) & \text{by Lemmas 5.5} \\
(m'n)a \sim (m'm)e & (mn')b \sim (mm')e & \text{by Lemma 5.2}
\end{array}$$

Since $(mm')e = (m'm)e$, it follows that

$$(m'n)a \sim (mn')b$$

Now, since $a \succsim b$, we have that

$$\begin{aligned}
(mn')a &\succsim (mn')b && \text{by Lemma 5.5 and 5.6} \\
(mn')a &\succsim (m'n)a && \text{by (5.1)} \\
mn' &\geq m'n && \text{by Lemma 5.7} \\
\frac{m}{n} &\geq \frac{m'}{n'} \\
\phi(a) &\geq \phi(b)
\end{aligned}$$

Conversely, suppose that $\phi(a) \geq \phi(b)$. Then $\frac{n}{m} \geq \frac{n'}{m'}$ where $n, m, n', m' \in \mathbb{Z}_{>0}$ such that $na \sim me$ and $n'b \sim m'e$. Thus $m'(na) \sim (mn')b$, and the above inferences can be run in reverse, to arrive at $a \succsim b$.

Condition 2: $\phi(a \circ b) = \phi(a) + \phi(b)$. Let $\frac{n_a}{m_a} \in F_{(a,e)}$ and $\frac{n_b}{m_b} \in F_{(b,e)}$. Then $n_a e \sim m_a a$ and $n_b e \sim m_b b$. Define $m := m_a m_b$, $n := n_a m_b$, $n' := n_b m_a$. Then $\frac{n}{m} = \frac{n_a}{m_a} = [a : e] = \phi(a)$ and $\frac{n'}{m} = \frac{n_b}{m_b} = [b : e] = \phi(b)$, so that $ne \sim ma$ and $n'e \sim mb$. Then by Lemma 5.3, $ne \circ n'e \sim ma \circ mb$, and by Lemmas 5.1 and 5.4, $(n + n')e \sim m(a \circ b)$. Thus $\frac{n+n'}{m} \in F_{(a \circ b, e)}$. It follows that $\phi(a \circ b) = \frac{n+n'}{m} = \phi(a) + \phi(b)$.

Next, we show that 2 implies 1. Suppose that $\langle A, \succsim, \circ \rangle$ is homomorphically representable in $\langle \mathbb{Q}_+, \geq, + \rangle$. Let ϕ be a homomorphism of $\langle A, \succsim, \circ \rangle$ into $\langle \mathbb{Q}_+, \geq, + \rangle$. Let $a, b \in A$. We will show that there exists $c \in A$ such that $a \sim nc$ and $b \sim mc$. If $a \sim b$ then $n = m = 1$ and $c = a$ (or $c = b$), so suppose WLOG that $a \succ b$. By Archimedean, the set of integers n such that $a \succ nb$ is finite. Let $N(a, b)$ be the greatest integer such that $a \succ N(a, b)b$. Then by Solvability, there exists $c_0 \in A$ such that $a \sim N(a, b)b \circ c_0$, and $b \succ c_0$. If $c_0 \sim b$, then $a \sim [N(a, b) + 1]b$ and thus $n = N(a, b) + 1, m = 1, c = b$, in which case we are done. So suppose that $b \succ c_0$. Then there exists $c_1 \in A$ such that $b \sim N(b, c_0)c_0 \circ c_1$. If $c_1 \sim c_0$, then $b \sim [N(b, c_0) + 1]c_0$ and thus $n = N(a, b)[N(b, c_0) + 1], m = N(b, c_0) + 1, c = c_0$, in which case we are done. Supposing that $c_0 \succ c_1$, we may continue in this fashion:

$$\begin{aligned}
a &\sim N(a, b)b \circ c_0 \\
b &\sim N(b, c_0)c_0 \circ c_1 \\
c_0 &\sim N(c_0, c_1)c_1 \circ c_2 \\
&\vdots \\
c_n &\sim N(c_n, c_{n+1})c_{n+1} \circ c_{n+2} && \vdots
\end{aligned}$$

This process can be iterated for c_n as long as $c_{n+1} \succ c_{n+2}$. Once we arrive at a step for which $c_{n+1} \sim c_{n+2}$, we will have that $c_n \sim [N(c_n, c_{n+1}) + 1]c_{n+1}$ and thus that $a \sim N(a, b)b \circ N(c_0, c_1)c_1 \circ N(c_2, c_3)c_3 \circ \cdots \circ [N(c_n, c_{n+1}) + 1]c_{n+1}$. It follows that $a \sim [N(c_{n+1}, a) + 1]c_{n+1}$ and that $b \sim [N(c_{n+1}, b) + 1]c_{n+1}$. We now show that there exists n' such that $c_{n+1} \sim c_{n+2}$. First, we have that

$$\begin{aligned}
\phi(a) &= N(a, b)\phi(b) + \phi(c_0) \\
\phi(b) &= N(b, c_0)\phi(c_0) + \phi(c_1) \\
&\vdots \\
\phi(c_n) &= N(c_n, c_{n+1})\phi(c_{n+1}) + \phi(c_{n+2}) \\
&\vdots
\end{aligned} \tag{1}$$

It follows that

$$\begin{aligned}
\frac{\phi(a)}{\phi(b)} &= N(a, b) + \frac{\phi(c_0)}{\phi(b)} \\
\frac{\phi(b)}{\phi(c_0)} &= N(b, c_0) + \frac{\phi(c_1)}{\phi(c_0)} \\
&\vdots \\
\frac{\phi(c_n)}{\phi(c_{n+1})} &= N(c_n, c_{n+1}) + \frac{\phi(c_{n+2})}{\phi(c_{n+1})}
\end{aligned}$$

Notice that the second term of the RHS of each equation in this series is the reciprocal of the LHS of the subsequent equation. Thus, $\frac{\phi(a)}{\phi(b)}$ may be represented by the following continued fraction:

$$N(a, b) + \frac{1}{N(b, c_0) + \frac{1}{N(c_0, c_1) + \frac{1}{N(c_1, c_2) + \cdots}}}$$

A continued fraction is infinite iff its numerical value is irrational, and $\frac{\phi(a)}{\phi(b)} \in \mathbb{Q}_+$ since $\phi(a), \phi(b) \in \mathbb{Q}_+$. It follows that the continued fraction above is finite, and thus that the series of equations (1) terminates at $\phi(c_{n'})$, for some integer n' . Thus, the series of expressions terminates with $c_{n'}$, which implies that $c_{n'+1} \sim c_{n'+2}$. \square

Proof of Theorem 3.4. Let $\langle A, \succ, \circ \rangle$ be a positive, closed, extensive structure and let $\phi : A \rightarrow \mathbb{R}_+$ be an isomorphism from $\langle A, \succ, \circ \rangle$ to $\langle \mathbb{R}_+, \geq, + \rangle$. Since ϕ is bijective, $\forall x \in \mathbb{R}_+ \exists ! a \in A : \phi(a) = x$. Let $a, e \in A$ be such that

$\phi(a) = \sqrt{2}$ and $\phi(e) = 1$. Suppose for the sake of contradiction that there exists $c \in A$ and $m, n \in \mathbb{Z}_+$ such that $a \sim mc$ and $e \sim nc$. Then $na \sim me$, so that $\phi(na) = \phi(me)$ and thus $\phi(a) = \frac{m}{n}\phi(e) = (\frac{m}{n})(1) = \frac{m}{n}$. It follows that $\sqrt{2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}_+$, a contradiction. \square

Proof of Theorem 3.5. Let $\langle A, \succsim, \circ \rangle$ be a positive, closed, extensive structure and let $\phi : A \rightarrow \mathbb{Q}_+$ be an isomorphism from $\langle A, \succsim, \circ \rangle$ to $\langle \mathbb{Q}_+, \geq, + \rangle$. For all $a \in A$, $\phi(a) = \frac{n}{m}$ for some $n, m \in \mathbb{Z}_+$. Let a, b be arbitrary elements of A . We will show that there exist $c \in A$ and $m, n \in \mathbb{Z}_+$ such that $a \sim mc$ and $b \sim nc$. Let $n_a, m_a, n_b, m_b \in \mathbb{Z}_+$ such that $\phi(a) = \frac{n_a}{m_a}$ and $\phi(b) = \frac{n_b}{m_b}$. Then $\phi(a) + \phi(b) = \frac{m_b n_a + m_a n_b}{m_a m_b} = \phi(a \circ b)$. Since ϕ is bijective, $\forall x \in \mathbb{Q}_+ \exists! a \in A : \phi(a) = x$. Let $c \in A$ be such that $\phi(c) = \frac{1}{m_a m_b}$. Then $\phi(a) = n_a m_b \phi(c)$ and $\phi(b) = n_b m_a \phi(c)$. Then by Lemma 5.8, $\phi(a) = \phi(n_a m_b c)$ and $\phi(b) = \phi(n_b m_a c)$. Since ϕ preserves \succsim in \geq , it follows that $a \sim n_a m_b c$ and $b \sim n_b m_a c$. \square

5.2. Lemmas Let $a, b, c, d \in A$, and let m, n be positive integers.

Lemma 5.1. $ma \circ na \sim (m + n)a$

Lemma 5.2. $n(ma) \sim (nm)a$

Lemma 5.3. If $a \sim b$ and $c \sim d$, then $a \circ c \sim b \circ d$.

Lemma 5.4. $m(a \circ b) \sim ma \circ mb$

Lemma 5.5. For any positive integer n , $a \sim b$ iff $na \sim nb$.

Lemma 5.6. If $a \succ b$ and $c \succ d$ then $a \circ c \succ b \circ d$.

Lemma 5.7. Suppose that $a \sim b$. Then,

a) $na \sim mb$ iff $n = m$.

b) $na \succ mb$ iff $n > m$.

Lemma 5.8. Let $\phi : A \rightarrow B$ be a homomorphism of $\langle A, \succsim, \circ \rangle$ into $\langle B, \geq, + \rangle$, where $B = \mathbb{R}_+$ or $B = \mathbb{Q}_+$. Then for all $a \in A$ and all $n \in \mathbb{Z}_+$, $n\phi(a) = \phi(na)$.

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