Sequential measurements and the Kochen-Specker arguments

Gábor Hofer-Szabó[*]

Abstract

It will be shown that the Peres-Mermin square admits value-definite noncontextual hidden-variable models if the observables associated with the operators can be measured only sequentially but not simultaneously. Namely, sequential measurements allow for noncontextual models in which hidden states update between consecutive measurements. Two recent experiments realizing the Peres-Mermin square by sequential measurements will also be analyzed along with other hidden-variable models accounting for these experiments.

Keywords: sequential measurements, simultaneous measurements, Kochen-Specker theorem, Peres-Mermin square

1 Introduction

How can we justify that in every hidden (ontic) state two observables represented by commuting operators in quantum mechanics have joint values corresponding to the eigenvalues in one of the common eigenstates of the operators? Well, there is no other way than to measure the two observables *simultaneously* in various quantum states, eigenstates and non-eigenstates, and to check the joint outcomes directly. If the joint outcomes conform to these eigenvalues, then (assuming that quantum states are just distributions of hidden states) we can be pretty sure that in every hidden state the observables have just those joint values.

But what if the measurement of the observables can be performed only *sequentially*, that is only one after the other? In this case quantum mechanics tells us that the quantum state will update upon measurements according to the projection postulate and the subsequent outcomes will again conform to just the eigenvalues, irrespective of the order of the measurements. But does it mean that the observables have the joint values corresponding to those eigenvalues in each step of the measurement process? No, it does not. Generally one cannot draw a conclusion from diachronic evidences to synchronic facts. And indeed, as we will shortly see, one can easily cook up hidden-variable models for commuting operators such that the simultaneous joint values do not conform to the above eigenvalues, still the subsequent outcomes of the measurements do.

Why does this problem matter?

In the Kochen-Specker theorems one proves that there is no value assignment for certain tricky sets of operators such that the values assigned to any subset of mutually commuting operators

[*]Research Center for the Humanities, Budapest, email: szabo.gabor@btk.mta.hu

conform to the eigenvalues in one of the common eigenstates of the operators in the subset. These value assignments represent hidden states in a value-definite (outcome-deterministic) hidden-variable model and the values assigned to the commuting operators represent the joint values of the corresponding observables. Since the no-go theorems rule out such value assignments, they also rule out value-definite hidden-variable models.

These models are also called noncontextual. The reason for this is that the commuting operators are associated with simultaneous measurements which, if performed, yield just the outcomes corresponding to the above joint values. The model is noncontextual in the sense that the value of an observable and the outcome of the associated measurement is just the value assigned to the operator in the given hidden state, irrespective of whether a simultaneous measurement is performed or not (see also Hofer-Szabó, 2021a,b, 2022). In short, the no-go theorems show that there is no hidden-variable model which would assign values to observables which are independent of simultaneous measurements, that is which are noncontextual.

But what if the measurements realizing commuting operators cannot be performed simultaneously but only sequentially? As I claimed above and will show below, in this case the central assumption of the Kochen-Specker theorems—namely that the joint values assigned to commuting operators should conform to the eigenvalues in one of the common eigenstates of the operators—remains physically unjustified. In case of sequential measurements, the joint values can freely update in such a way that *the sequential outcomes do but the simultaneous joint values do not conform to these eigenvalues.* But if the joint values need not conform to the eigenvalue-constraint, the no-go theorems will not go through, hence opening the way for a noncontextual value-definite hidden-variable model.

Let me be clear already from the outset what kind of noncontextual value-definite hidden-variable models will be constructed. These models are value-definite in the sense that in every hidden state each observable has a definite value which is simply revealed if measured. The model is noncontextual in the usual sense: the value revealed by a measurement does not depend on whether other simultaneous measurements are performed or not. Now, the only fly in the ointment is that in each hidden state there are observables associated with commuting operators which have joint values *not* conforming to the eigenvalues in one of the common eigenstates of the operators. If these observables could be measured *simultaneously*, this anomaly would be revealed. But if they can be measured only *sequentially*, then the hidden state can change upon the measurements such that this anomaly remains concealed throughout the whole measurement process.

In some recent experiments devised to verify the Kochen-Specker arguments, simultaneous measurements are replaced, due to technological reasons, by sequential measurements. My paper is directed against these experiments. The central claim of the paper is the following: *if the operators featuring in a Kochen-Specker theorem are realized by measurements which can be performed only sequentially but not simultaneously, then the argument does not rule out noncontextual value-definite hidden-variable models in which the hidden state can update upon the subsequent measurements.*

This paper intends to contribute to the debate on the experimental testability of the Kochen-Specker theorems via *sequential* measurements. The question of empirical testability of the Kochen-Specker theorems is not new (see for example Held, 2022, Sec. 6 for the various experimental challenges). What is new, however, is the awareness of a kind of loophole in the argument provided by the experimental fact that the commuting operators are realized by sequential measurements. Whereas simultaneous measurements provide *synchronic constrains* on the possible hidden states which constraints then lead to the Kochen-Specker arguments, sequential measurements provide only *diachronic constrains*: the outcome statistics of the consecutive measurements need to conform to the projection postulate. This opens the way to construct hidden-variable models which avoid the synchronic constraints and satisfy the diachronic ones. And indeed, some authors (La Cour 2009, 2017) developed highly sophisticated hidden-variable models for various experimental tests with sequential measurements. Others (Gühne et al., 2010) derived generalized Kochen-Specker inequalities for sequential measurements not strictly subscribing to the projection postulate. Interestingly, the various models use different concepts of noncontextuality, some of which are stricter, some are weaker than the one used in this paper. In the Discussion, I will situate my approach in this wider context of noncontextual hidden-variable models for sequential tests of the Kochen-Specker theorems.

In the paper I will proceed as follows. In Section 2, as a warm-up exercise, I construct a noncontextual value-definite hidden-variable model for three commuting operators realized by sequential measurements. In Section 3, a similar model for the entire Peres-Mermin square will be constructed. In Section 4, I analyze two recent experiments realizing the Peres-Mermin argument by sequential measurements. In Section 5, the various concepts of noncontextuality used in the literature and the various sequential models will be compared. I conclude in Section 6.

2 A simple example

Consider the following three pairwise commuting self-adjoint operators:

$$A_1 = \sigma_x \otimes \sigma_x \qquad A_2 = \sigma_y \otimes \sigma_y \qquad A_3 = \sigma_z \otimes \sigma_z$$

where σ_x, σ_y, and σ_z are the Pauli operators on the two dimensional complex Hilbert space H_2. Suppose we realize the operators by three measurements M_i ($i = 1, 2, 3$) with outcomes $O_i^{\pm}$. The realization is successful if for each individual measurement M_i:

$$\langle \Psi | P_i^{\pm} \Psi \rangle = p_\Psi(O_i^{\pm} \mid M_i) \tag{1}$$

where $P_i^{\pm}$ denotes the eigenprojections of the operator A_i with eigenvalue ± 1 and p_Ψ denotes the probability of outcomes of measurements performed on a system prepared in quantum state $|\Psi\rangle$.

First, suppose that M_1, M_2 and M_3 can be *simultaneously* (jointly) performed. Denote the joint measurement by $M_1 \wedge M_2 \wedge M_3$ and the eight possible joint outcomes by $O_1^{\pm} \wedge O_2^{\pm} \wedge O_3^{\pm}$. For this simultaneous measurement quantum mechanics predicts the following distribution of the joint outcomes:

$$\langle \Psi | P_1^{\pm} P_2^{\pm} P_3^{\pm} \Psi \rangle \quad = \quad p_\Psi(O_1^{\pm} \wedge O_2^{\pm} \wedge O_3^{\pm} \mid M_1 \wedge M_2 \wedge M_3) \tag{2}$$

It turns out that only four of the eight joint outcomes have nonzero probability: those that correspond to the four common eigenvectors (Bell state vectors) of the three operators:

Eigenvectors and eigenstates	A_1	A_2	A_3
$\|\Psi^{+-+}\rangle = \frac{1}{\sqrt{2}}(\,\|00\rangle + \|11\rangle\,)$	$+1$	-1	$+1$
$\|\Psi^{-++}\rangle = \frac{1}{\sqrt{2}}(\,\|00\rangle - \|11\rangle\,)$	-1	$+1$	$+1$
$\|\Psi^{++-}\rangle = \frac{1}{\sqrt{2}}(\,\|01\rangle + \|10\rangle\,)$	$+1$	$+1$	-1
$\|\Psi^{---}\rangle = \frac{1}{\sqrt{2}}(\,\|01\rangle - \|10\rangle\,)$	-1	-1	-1

where $|0\rangle$ and $|1\rangle$ are the eigenvectors of σ_z with eigenvalue $+1$ and -1. This means that the simultaneous measurement $M_1 \wedge M_2 \wedge M_3$ can have only four possible joint outcome types in every quantum state:

$$O_1^+ \wedge O_2^- \wedge O_3^+ \quad , \qquad O_1^- \wedge O_2^+ \wedge O_3^+ \quad , \qquad O_1^+ \wedge O_2^+ \wedge O_3^- \quad , \qquad O_1^- \wedge O_2^- \wedge O_3^- \tag{3}$$

Consequently, only four possible joint outcome types are allowed in every hidden state.

Now, let (Λ, p) be a noncontextual value-definite hidden-variable model for a realization of A_1, A_2 and A_3 by the simultaneous measurements of M_1, M_2 and M_3. The set Λ is composed of four types of hidden states:

$$\Lambda = \Lambda^{+-+} \cup \Lambda^{-++} \cup \Lambda^{++-} \cup \Lambda^{---} \tag{4}$$

corresponding to the above four joint outcomes. For example, for a system in a hidden state in Λ^{+-+}, the outcome of the simultaneous measurement $M_1 \wedge M_2 \wedge M_3$ will be $O_1^+ \wedge O_2^- \wedge O_3^+$ and the outcome of the individual measurements M_1, M_2 and M_3 will be O_1^+, O_2^- and O_3^+, respectively. Thus, the model is value-definite and noncontextual. The probability of the four hidden state types for a system in quantum state $|\Psi\rangle$ is

$$\begin{aligned} p\left(\Lambda^{+-+}\right) &= |\langle\Psi|\Psi^{+-+}\rangle|^2 & (5)\\ p\left(\Lambda^{-++}\right) &= |\langle\Psi|\Psi^{-++}\rangle|^2 & (6)\\ p\left(\Lambda^{++-}\right) &= |\langle\Psi|\Psi^{++-}\rangle|^2 & (7)\\ p\left(\Lambda^{---}\right) &= |\langle\Psi|\Psi^{---}\rangle|^2 & (8) \end{aligned}$$

Since the eigenvectors form an orthonormal basis, the probabilities add up to 1. Furthermore, from

$$\begin{aligned} |\langle\Psi|\Psi^{+-+}\rangle|^2 &= \langle\Psi|P_1^+P_2^-P_3^+\Psi\rangle & (9)\\ |\langle\Psi|\Psi^{-++}\rangle|^2 &= \langle\Psi|P_1^-P_2^+P_3^+\Psi\rangle & (10)\\ |\langle\Psi|\Psi^{++-}\rangle|^2 &= \langle\Psi|P_1^+P_2^+P_3^-\Psi\rangle & (11)\\ |\langle\Psi|\Psi^{---}\rangle|^2 &= \langle\Psi|P_1^-P_2^-P_3^-\Psi\rangle & (12) \end{aligned}$$

and (2) it follows that

$$\begin{aligned} p\left(\Lambda^{+-+}\right) &= p_\Psi(O_1^+ \wedge O_2^- \wedge O_3^+ \,|\, M_1 \wedge M_2 \wedge M_3) & (13)\\ p\left(\Lambda^{-++}\right) &= p_\Psi(O_1^- \wedge O_2^+ \wedge O_3^+ \,|\, M_1 \wedge M_2 \wedge M_3) & (14)\\ p\left(\Lambda^{++-}\right) &= p_\Psi(O_1^+ \wedge O_2^+ \wedge O_3^- \,|\, M_1 \wedge M_2 \wedge M_3) & (15)\\ p\left(\Lambda^{---}\right) &= p_\Psi(O_1^- \wedge O_2^- \wedge O_3^- \,|\, M_1 \wedge M_2 \wedge M_3) & (16) \end{aligned}$$

Next, suppose that the three measurements realizing the operators cannot be performed simultaneously but only *sequentially*. This means that the constraint (2) does not apply to the measurements for the simple reason that the right hand side is not defined. There is, however, another constraint coming from the projection postulate. According to the projection postulate, upon performing the measurement M_i on the system and getting the outcome $O_i^{\pm}$, the quantum state $|\Psi\rangle$ will jump into the new state

$$|\Psi\rangle \quad \longrightarrow \quad |\Psi'\rangle = \frac{P_i^{\pm}\,|\Psi\rangle}{\langle\Psi|P_i^{\pm}\Psi\rangle^{\frac{1}{2}}} \tag{17}$$

Now, suppose we perform the three measurement M_1, M_2 and M_3 one after another and obtain an outcome for each measurement. The first measurement will send the system into a new quantum state, $|\Psi'\rangle$; the second measurement will send it further into another quantum state, $|\Psi''\rangle$ according to the projection postulate. We now ask: What is the probability that we obtain a given sequence of outcomes upon these sequential measurements? A simple calculation shows that this probability is just the quantum probability on the left hand side of (2):

$$p_\Psi(O_1^{\pm}\,|\,M_1)\cdot p_{\Psi'}(O_2^{\pm}\,|\,M_2)\cdot p_{\Psi''}(O_3^{\pm}\,|\,M_3) = \langle\Psi|P_1^{\pm}P_2^{\pm}P_3^{\pm}\Psi\rangle \tag{18}$$

It is easy to show that the right hand side of (18) remains the same even if the measurements are performed in a different order. Therefore, we introduce a general notation for an arbitrary sequence of measurements M_1, M_2 and M_3 performed on a system in quantum state $|\Psi\rangle$ with outcomes $O_1^{\pm}$, $O_2^{\pm}$ and $O_3^{\pm}$:

$$p_\Psi(O_1^{\pm}\!-\!O_2^{\pm}\!-\!O_3^{\pm}\,|\,M_1\!-\!M_2\!-\!M_3) = \langle\Psi|P_1^{\pm}P_2^{\pm}P_3^{\pm}\Psi\rangle \tag{19}$$

Note that while $M_1 \wedge M_2 \wedge M_3$ denoted the simultaneous measurement of M_1, M_2 and M_3, the term $M_1\!-\!M_2\!-\!M_3$ denotes a sequential measurement of M_1, M_2 and M_3 in any order.

(19) means that the sequential measurement $M_1\!-\!M_2\!-\!M_3$ can yield only those sequential outcomes which conform to the eigenvalues of one of the above four common eigenvectors. Moreover, the probabilities of these sequential outcomes will be just the probabilities in (9)-(12). Thus, (19) together with the projection postulate provide another interpretation of the term $\langle\Psi|P_1^{\pm}P_2^{\pm}P_3^{\pm}\Psi\rangle$: it will no longer represent the joint outcomes of the simultaneous measurement $M_1 \wedge M_2 \wedge M_3$, as in (2); rather it will represent the outcomes of the sequential measurement $M_1\!-\!M_2\!-\!M_3$, as in (19).

Now, what kind of noncontextual value-definite hidden-variable models are admitted if the operators A_1, A_2 and A_3 are realized not by simultaneous but by sequential measurements conforming to (19)?

Obviously, the previous noncontextual value-definite hidden-variable model is a hidden-variable model also for this sequential realization of the operators. Here measurements do not change the hidden state of the system. By performing a measurement and selecting out those runs which yield a given outcome, one changes only the distribution of the hidden states. This change will be consistent with the change of the quantum state via the projection postulate (17), that is the new distribution can be calculated by replacing $|\Psi\rangle$ with $|\Psi'\rangle$ in (5)-(8).

There is, however, another noncontextual value-definite hidden-variable model for this sequential realization of the operators where the joint values do *not* correspond to the above four eigenvectors. Let now Λ be composed of eight types of hidden states $\Lambda^{\pm\pm\pm}$: four corresponding to the above four joint outcomes and four corresponding to the joint outcomes with opposite signs. Let the probability of hidden state types for a system in quantum state $|\Psi\rangle$ be

$$p\left(\Lambda^{\pm\pm\pm}\right) \quad = \quad \langle\Psi|P_1^{\pm}\Psi\rangle \cdot \langle\Psi|P_2^{\pm}\Psi\rangle \cdot \langle\Psi|P_3^{\pm}\Psi\rangle \tag{20}$$

Note that in this model the joint outcome of the simultaneous measurement $M_1 \wedge M_2 \wedge M_3$ in the hidden state $\Lambda^{\pm\pm\pm}$ of the system *would* be $O_1^{\pm} \wedge O_2^{\pm} \wedge O_3^{\pm}$. However, M_1, M_2 and M_3 cannot be simultaneously measured, only sequentially and individually. If measured individually, the measurements provide the probabilities consistent with quantum mechanics:

$$\sum_{jk=\pm} p\left(\Lambda^{\pm jk}\right) = \langle\Psi|P_1^{\pm}\Psi\rangle = p_\Psi(O_1^{\pm} \mid M_1)$$

and similarly for $i = 2, 3$. The model is noncontextual since the observables associated with M_1, M_2 and M_3 have joint values, even if these measurements cannot be performed simultaneously.

Also note that the absence of simultaneous measurements is crucial: the hidden state type Λ^{+++}, for example, which was ruled out in the previous model since the joint outcome $O_1^{+} \wedge O_2^{+} \wedge O_3^{+}$ could never pop up for a simultaneous measurement, is *not* ruled out here. If the system is prepared in the quantum state $|\Psi\rangle = |00\rangle$, for example, then the probability of the type Λ^{+++} is nonzero:

$$p\left(\Lambda^{+++}\right) = \langle 00|P_1^{+}00\rangle \cdot \langle 00|P_2^{+}00\rangle \cdot \langle 00|P_3^{+}00\rangle = \frac{1}{4}$$

Our task is now to make the model consistent also diachronically with respect to the sequential measurements. In other words, we need to introduce a change in the distribution of the hidden states which is consistent with (19). One can reach this goal by ensuring that the distribution of hidden states upon every measurement conforms to the updated quantum state. If the change of the distribution $p \to p' \to p'' \to ...$ of the hidden states upon a sequence of measurements is consistent with the change of the quantum states $|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to ...$ as governed by the projection postulate—that is probabilities relate to the quantum states via (20) in every step of the measurement process, then (18) and hence (19) will hold trivially.

There are many stochastic transition processes which satisfy this requirement (see La Cour, 2009, 2017; Kleinmann et al., 2011; Cabello et al., 2018). Here is the presumably simplest (and admittedly a least realistic) one: upon performing the measurement M_i on the system and getting the outcome $O_i^{\pm}$, let each hidden state in $\Lambda^{\pm\pm\pm}$ jump into a new hidden state in $\Lambda'^{\pm\pm\pm}$:

$$\lambda \in \Lambda^{\pm\pm\pm} \quad \xrightarrow{p'} \quad \lambda' \in \Lambda'^{\pm\pm\pm} \tag{21}$$

with probability $p'\left(\Lambda'^{\pm\pm\pm}\right)$—that is with the probability of the new hidden state type $\Lambda'^{\pm\pm\pm}$ calculated by (20) with $|\Psi\rangle$ replaced by $|\Psi'\rangle$.

As an example, consider a system in a hidden state in Λ^{+++} which is measured sequentially by M_1, M_2 and M_3. The outcome of M_1 will be O_1^{+} and the hidden state will jump into one of the four types Λ^{+jk} with probability $\langle\Psi'|P_2^{j}\Psi'\rangle \cdot \langle\Psi'|P_3^{k}\Psi'\rangle$, where

$$|\Psi'\rangle = \frac{P_1^+ |\Psi\rangle}{\langle\Psi|P_1^+\Psi\rangle^{\frac{1}{2}}}$$

Suppose the system remains in Λ^{+++} after the first measurement. If we perform the second measurement M_2 on this system, the outcome will be O_2^+ and the system will jump into Λ^{++-} with probability 1 (since the new projected state $|\Psi''\rangle$ is just $|\Psi^{++-}\rangle$.) In this new hidden state, the third measurement M_3 will give the outcome O_3^-.

This example highlights a general rule which the jumps need to follow. Independent of which hidden state the system starts from, after two consecutive measurements it will land in one of the four states in (4) corresponding to the joint outcomes. This must be so since repeating any of the two measurements, the outcome needs to be same as before; and performing the third measurement, the outcome needs to be one of (3). Thus, hidden states not in (4) are washed out after two sequential measurements. We come back to this point in the Discussion.

Also note that the probability p' of the transition between the old and new hidden states in (21) depends on the new quantum state $|\Psi'\rangle$ via (20). But this new quantum state $|\Psi'\rangle$ is determined by the old quantum state $|\Psi\rangle$, the measurement M_i and the outcome $O_i^{\pm}$ via (17). Therefore, to correctly govern the transition, the hidden-variable model needs to incorporate also the quantum states $|\Psi\rangle$. Thus, the hidden states will be of the form $\{\lambda, |\Psi\rangle\}$, where $\lambda \in \Lambda$ and $|\Psi\rangle \in H_2$. In short, the model will be Ψ*-ontic* (see Harrigan and Spekkens, 2010).

In the next section, I construct a similar noncontextual value-definite hidden-variable model with stochastic transitions for the sequential realization of the operators in the Peres-Mermin square.

3 The Peres-Mermin square

The Peres-Mermin square (Peres, 1990; Mermin 1993) is the following 3×3 matrix of self-adjoint operators:

$$\begin{array}{lll}
A_{11} = \sigma_z \otimes I & A_{12} = I \otimes \sigma_z & A_{13} = \sigma_z \otimes \sigma_z \\
A_{21} = I \otimes \sigma_x & A_{22} = \sigma_x \otimes I & A_{23} = \sigma_x \otimes \sigma_x \\
A_{31} = \sigma_z \otimes \sigma_x & A_{32} = \sigma_x \otimes \sigma_z & A_{33} = \sigma_y \otimes \sigma_y
\end{array}$$

where I is the unit operator on H_2. Each operator in the matrix has two eigenvalues, ± 1, and are arranged in such a way that two operators are commuting if and only if they are in the same row or in the same column. The three operators in the third column are just the three commuting operators in the previous section.

A realization (interpretation) of the Peres-Mermin square is a unique association of operators $\{A_{ij}\}$ $i, j = 1, 2, 3$ in the matrix with real-world measurements $\{M_{ij}\}$. Suppose that the measurements realizing commuting operators cannot be simultaneously performed but only sequentially. In other words, instead of performing the joint measurement $M_{1j} \wedge M_{2j} \wedge M_{3j}$ one can only perform the sequential measurements $M_{1j} - M_{2j} - M_{3j}$. Similarly, instead of performing the joint measurement $M_{i1} \wedge M_{i2} \wedge M_{i3}$ one can only perform the sequential measurements

$M_{i1} - M_{i2} - M_{i3}$. The realization is empirically adequate if (1) and (19) hold for the individual and sequential measurements.

I will construct now a noncontextual value-definite hidden-variable model for this realization of the Peres-Mermin square. The model will recover the outcome statistics of both the individual measurements and the sequential measurements. Still, the joint values of the observables corresponding to the commuting triples of operators will *not* correspond to the eigenvalues in one of the common eigenstates of these operators. Consequently, the usual constraints on the valuations leading to the Kochen-Specker contradiction will not apply. The model is not ruled out by the Kochen-Specker arguments.

Let $\varepsilon = (\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{33})$ be a vector such that $\varepsilon_{ij} = \pm 1$. Let $P_{ij}^{\varepsilon_{ij}}$ denote the eigenprojection of the operator A_{ij} with eigenvalue ε_{ij} and let $|\Psi\rangle$ be the quantum state of the system. The hidden-variable model consist of hidden states $\{\lambda, |\Psi\rangle\} \in \Lambda \times H_2$ where Λ is composed of 2^9 hidden state types:

$$\Lambda \quad = \bigcup_{\varepsilon \in \{-1,+1\}^9} \Lambda^\varepsilon$$

such that $\lambda \in \Lambda^\varepsilon$ if and only if the outcome of the measurement M_{ij} would be $O_{ij}^{\varepsilon_{ij}}$ for $i, j = 1, 2, 3$. The probability of the hidden state type Λ^ε is

$$p(\Lambda^\varepsilon) \quad = \quad \prod_{ij} p_{ij}^{\varepsilon_{ij}} \qquad \text{where} \quad p_{ij}^{\varepsilon_{ij}} = \langle \Psi | P_{ij}^{\varepsilon_{ij}} \Psi \rangle \tag{22}$$

The probability of those hidden states which provide the outcome $O_{ij}^{\pm}$ for the measurement M_{ij} is

$$\sum_{\varepsilon:\, \varepsilon_{ij} = \pm 1} p(\Lambda^\varepsilon) = p_{ij}^{\pm 1} \tag{23}$$

The probabilities are normalized:

$$\sum_{\varepsilon} p(\Lambda^\varepsilon) = 1$$

Now, according to the projection postulate (17), upon performing the measurement M_{ij} on the system and getting the outcome $O_{ij}^{\pm}$, the quantum state $|\Psi\rangle$ jumps into the new state

$$|\Psi\rangle \quad \longrightarrow \quad |\Psi'\rangle = \frac{P_{ij}^{\pm} |\Psi\rangle}{\langle \Psi | P_{ij}^{\pm} \Psi \rangle^{\frac{1}{2}}}$$

For the model to be consistent, the new probability distribution of the hidden states after the measurement should be given again by (22) with $|\Psi\rangle$ replaced by $|\Psi'\rangle$. This can be guaranteed again by the following simple stochastic transition process between the old and new hidden states: upon performing the measurement M_{ij} and getting the outcome $O_{ij}^{\pm}$, each hidden state in Λ^ε jumps into the new hidden state type Λ'^ε with probability $p(\Lambda'^\varepsilon)$. This transition guarantees that the distribution of hidden states will co-vary with the quantum state in tune with the projection postulate. Specifically, the probability of those new hidden states which provide again the outcome $O_{ij}^{\pm}$ if we repeat the measurement M_{ij} will be

$$\sum_{\varepsilon:\, \varepsilon_{ij} = \pm 1} p(\Lambda'^\varepsilon) = 1$$

since $\langle\Psi'|\, P_{ij}^{\pm 1}\, |\Psi'\rangle = 1$.

Note that the probability $p(\Lambda'^{\varepsilon})$ of the transition

$$\lambda \in \Lambda^{\varepsilon} \quad \xrightarrow{p'} \quad \lambda' \in \Lambda'^{\varepsilon} \tag{24}$$

does not depend on the old probability $p(\Lambda^{\varepsilon})$ but it does depend on $|\Psi'\rangle$ which is determined by the old quantum state $|\Psi\rangle$, the measurement M_{ij} and the outcome $O_{ij}^{\pm}$. This is why the hidden-variable model needs to incorporate also the quantum state $|\Psi\rangle$ which makes the model Ψ-ontic. There is a division of labor in the model: the λ-part of the hidden states $\{\lambda, |\Psi\rangle\}$ ensures that the individual measurements have definite outcomes; the $|\Psi\rangle$-part governs (stochastically) the update of the hidden states upon measurements. Note that since no measurements can be simultaneously performed, the model does not make a difference between measurements realized by commuting and noncommuting operators: it treats them alike.

To sum up, if the Peres-Mermin square is not realized by simultaneous but only by sequential measurements, then it does not rule out a value-definite (and Ψ-ontic) hidden variable model respecting also the projection postulate.

4 Two recent experiments with sequential measurements

In a recent experiment, Kirchmair et al. (2009) realized the Peres-Mermin operators by sequential measurements performed on pairs of $^{40}Ca^{+}$ ions. In the experiment, trapped ions were prepared in a two-qubit quantum state by laser–ion interactions. The observable associated with the operator σ_z was realized by two different energy levels of the ions. This observable was measured by electron shelving projecting onto these eigenstates. The measurement of the other eight observables in the Peres-Mermin square was reduced to the measurement of the observable represented by σ_z by applying a suitable unitary transformation to the quantum state before this measurement and its inverse after this measurement. Thus, the association of the operators and measurements was unique: each of the 9 operators was associated with a different (quantum non-demolition) measurement.

The aim of the experiment was to test the violation of the Peres-Mermin inequality

$$\langle A_{11}A_{12}A_{13}\rangle + \langle A_{21}A_{22}A_{23}\rangle + \langle A_{31}A_{32}A_{33}\rangle + \langle A_{11}A_{21}A_{31}\rangle + \langle A_{21}A_{22}A_{32}\rangle - \langle A_{13}A_{23}A_{33}\rangle \leqslant 4 \tag{25}$$

derived by Cabello (2008) as a constraint on the Peres-Mermin square to have a noncontextual value-definite hidden variable model. For the right hand side of (25), quantum mechanics predicts 6 in any quantum state which violates the inequality. The experiment of Kirchmair et al. confirmed this prediction by obtaining the result 5.46 for the singlet state.

The violation of (25), however, does not rule out a noncontextual value-definite hidden variable model the Peres-Mermin square since the measurements realizing the three operators in a row or column are not simultaneously performed. Just consider the model developed in the previous section. For every quantum state, (22) provides the probability distribution of the hidden state types which returns the outcome statistics of the individual measurements via (23). Upon performing a measurement and obtaining an outcome, the quantum state updates in tune with the projection postulate and the hidden states stochastically jump into another type with the probability of this new type. The expectation value of the subsequent measurement of three observables in a row or column will provide just the six expressions on the right hand side of (25) leading to the violation of the inequality. Still, the model is noncontextual in every step of the measurement process: the nine observables have joint values at each moment.

The experiment of Kirchmair et al. has been further developed and carried out by photons by Liu et al. (2016). In this experiment, two entangled photons were distributed between two spatially separated parties, Alice and Bob. Both photons encoded two qubits, one in the spatial and another in the polarization mode. Thus, the photon pair was in a four-qubit quantum state. Alice performed three sequential measurements on her photon and Bob performed one single measurement on his photon. The sequential measurements of Alice realized one of the three rows or columns of the Peres-Mermin square using beam splitters, half-wave plates, beam displacers and phase compensators. Bob's single measurement realized one of the Peres-Mermin operator. Now, in tune with quantum mechanics, if Bob chose a measurement which was identical with the second or third measurement in Alice's sequence of measurements, then there was a perfect correlation or anticorrelation between their outcomes.

The aim of the experiment was to verify the violation of a generalization by Cabello's (2010) of the Peres-Mermin inequality (25) where the perfect correlation or anticorrelation terms were added. The experiment proved the violation of this generalized Peres-Mermin inequality.

However, similarly to the experiment of Kirchmair et al., the experiment of Liu et al. does not rule out noncontextual value-definite hidden variable models. A sophisticated model for the experiment of Liu et al. was given by La Cour (2017). But more simple-minded models can also be given. Here we just sketch how it goes: Quantum mechanics determines the quantum state of the system after each measurement via the projection postulate. Use (22) to establish the probability of the hidden states of Alice in every step of the measurement process and use the perfect correlation and anticorrelation between the outcomes of Alice's and Bob's measurement to establish the probability of the hidden states of Bob.

Instead of continuing with other more recent experiments (see i.e. Leupold et al., 2018) let me make once more explicit the crucial difference between simultaneous and sequential measurements. If a set of measurements can be performed only sequentially, then only one measurement can be performed at a time on the system. In this case the hidden variable model needs to return only the outcome statistics of the *individual* measurements. If, however, the measurements can be performed also simultaneously, then the hidden variable model needs to provide the statistics of both the *individual* and the *simultaneous* measurements; moreover, to be noncontextual, it needs to yield the same outcomes.

5 Discussion

With this paper, I intend to contribute to the debate on the experimental testability of the Kochen-Specker theorems via *sequential* measurements. One central question of this debate is whether the standard notion of noncontextuality is applicable in case of sequential measurements or one needs to adapt the concept to these new experimental conditions.

Some authors opt for the second alternative. Gühne et al. (2010, Def. 2), for example, calls a hidden variable model noncontextual if in a hidden state the outcome of a measurement does not depend on whether another compatible measurement—represented by a commuting operator—is measured *before* it, *simultaneously* with it, or *after* it. In other words, if we perform a measurement on a system in a given hidden state and obtain an outcome, this outcome would be the same had we performed a compatible measurement or even a whole sequence of compatible measurements before or jointly with or after it. For compatible measurements, the outcomes are fixed once and for all and are not sensitive to the order of measurements.

In their paper, Gühne et al. analyze the additional assumptions leading to the violation of Kochen-Specker inequalities in case of sequential measurements. One such assumption is what they call "compatibility loophole". The authors investigate the possibility of abandoning *perfect compatibility,* that is, to allow for a measurement to provide, at least sometimes, different outcomes in a sequence of compatible measurements. They show how this *imperfect compatibility* can lead to different modifications of the Kochen-Specker inequalities; compare these inequalities with real-world experiments; and construct various contextual hidden variable models.

The present paper differs from that of Gühne et al. in two important points. First, the model developed in Section 3 satisfies perfect compatibility. The stochastic transition (24) was explicitly designed such that it tracks the transformation of the wave function under the projection postulate. Upon any sequence of measurements corresponding to a given row or column of the Peres-Mermin square, the outcomes will always conform to the one of the four common eigenstates of the three commuting operators in that row or column. I also showed that since after two consecutive measurements the quantum state will be projected onto one of the common eigenstates, the hidden state of the system will be the state corresponding to these joint outcomes. Only after performing a measurement realizing a *non*-commuting operator can the system leave this hidden state.

Second, I opted for the first alternative in the above dilemma and sticked with the traditional definition of noncontextuality. I called a hidden variable model noncontextual if the observables associated with the commuting operators have joint values in every hidden state. These joint values can be revealed only by *simultaneous* measurements and the model is noncontextual only if these values do not depend on whether the simultaneous measurements are performed or not. Noncontextuality, in my understanding, does not include that measurements cannot alter the hidden state of the system and hence cannot alter the outcome of a subsequent measurement represented by a commuting operator. In a common eigenstate of two operators this is certainly the case: both measurements have a fixed value and these values remain the same no matter how many times and in what order we perform the measurements. But generally, in a non-eigenstate noncontextuality as defined by Gühne et al. seems to be too strong: observables can well have joint values at each time which values update for every new measurement.

La Cour (2009) also rejects the definition of noncontextuality as defined by Gühne et al. As he writes: "In the broadest sense, a measurement of an observable is said to be noncontextual if the outcome of the measurement does not depend upon which other compatible observables are measured subsequently, simultaneously, or previously... A better definition of a noncontextual measurement, then, would require only that the joint statistics of commuting observables be unchanged by the details of how they are measured." (p. 012102-1) And he goes on and constructs a value-definite, noncontextual hidden-variable model which reproduces the quantum statistics of the Mermin-Peres square. The model is highly sophisticated. It specifies the change of the hidden states upon sequential and simultaneous measurements; satisfies perfect compatibility;

and provides a model for certain recent real-world photon and neutron interferometry experiments. The restrictions of the standard Kochen-Specker theorems are avoided in the same way as in this paper: by allowing for the hidden states to change during the measurements.

La Cour's definition of noncontextuality, however, is different from the concept of noncontextuality used in this paper. Noncontextuality in La Cour is a statistical feature of the model in the sense that one associates a single random variable with each operator in the Mermin-Peres square and reproduces the quantum statistics with different probability measures corresponding to the different experiments. Although La Cour's model provides a deterministic mechanism for the update of hidden states upon measurements, his model qualities as *contextual* in our terminology. The reason for that lies in the difference how the two models treat the *order* of measurement outputs and measurement interactions. In our model a given hidden state *first* determines the measurement outcome and *then* updates due to the measurement interaction. In La Cour's model the order is just the opposite: an initial hidden state *first* transforms into a new state depending on the chosen measurement and *then* this new hidden state determines the outcomes. Obviously, one can argue for either order—being both experimentally inaccessible (see Conclusions). Interestingly, however, La Cour's choice of order makes his model contextual, at least with respect to our terminology. Namely, in La Cour two compatible measurements, say, M_{11} and $M_{11} \wedge M_{12} \wedge M_{13}$ will take an initial hidden state λ into two different new states λ' and λ'' which *then* can yield different outcomes. La Cour does not consider this feature contextually. He writes: "As discussed previously, this is not a violation of noncontextuality but merely a reflection of the possible dependence of a particular outcome on the experimental procedure." (p. 012102-1) In our model, however, λ first determines the outcome for both measurements and then updates according the projection postulate. Since the outcome for M_{11} and the would-be outcome[1] for $M_{11} \wedge M_{12} \wedge M_{13}$ (with respect to M_{11}), we call the model noncontextual. This shows that the order of output and update during measurements is strongly connected to the concept of noncontextuality. To make La Cour's model noncontextual with respect to our terminology, one should also demand that compatible measurements drive the hidden states into the same new state (or at least into such different states for which the outcome of the common part of the measurements—in this case M_{11}—is the same).

Another difference is that at certain points of his paper, La cour seems to make concessions to the traditional wording by referring to the dependence of the random variables or the probability measures on the commuting sets as "effective" or "apparent contextuality" and explaining this contextuality by the interaction of the system with the measurement apparatus. We, however, reserve the term "contextual" only to the dependence of a measurement outcome in a given hidden state on whether another measurement is performed on the same system at the same time. As stressed above, I think that it is worth discerning 1) the *robustness of the system to respond* in a definite way to a measurement when simultaneous measurements are also performed and 2) the *robustness of the hidden state to change* when it interacts with a measurement apparatus.

[1]Note that the joint measurement $M_{11} \wedge M_{12} \wedge M_{13}$ cannot be performed; but if it could, it would yield a definite outcome in every hidden state.

6 Conclusions

In the paper I argued that the Peres-Mermin square does not rule out a value-definite noncontextual hidden variable model if the observables associated with the operators cannot be measured simultaneously but only sequentially. To highlight this claim, I constructed such a model for any realization of the Peres-Mermin square by sequential measurements.

I would like to conclude with two remarks.

1. The stochastic transition of the hidden states upon measurement is *not necessarily local* in the sense that if M_i and M_j are measurements on two spacelike separated subsystems, then upon measuring M_i and obtaining an outcome, the hidden state λ^ε can jump into a hidden state λ'^ε for which the outcome of measurement M_j will be different from that in λ^ε. But this nonlocal character of the transition is a different feature of the theory than noncontextuality as the robustness of the outcome of a measurement against simultaneous measurements. The aim of this paper is not to dispute the claim that quantum mechanics does not admit local hidden variable models—we know this from Bell's inequalities. The aim is simply to argue that the Kochen-Specker arguments do not rule out a value-definite noncontextual hidden variable model if the observables associated with the operators can be measured only sequentially.

2. When are two measurements simultaneous and when are they subsequent? What time difference is needed for two measurements to be sequential? Well, this question cannot be answered a priori. It depends on the nature of the interaction between the system and the measuring apparatus. Still, there is a conceptual difference between simultaneous and subsequent measurements. In case of sequential measurements, the first measurement has time, so to say, to alter the hidden state and hence to influence the system's response to a second measurement. In the experiments analyzed above this is clearly the case: photons when entering a measurement apparatus and when leaving it need not be in the same hidden state due to their interaction with the apparatus. Consequently, they can enter the subsequent measurement apparatus in a new hidden state. As said above, I do not call this phenomenon contextuality. Contextuality occurs only when a measurement has a direct causal influence on the outcome of another measurement without previously altering the hidden state of the system. And our best way to ensure this, is to perform the two measurements simultaneously.

   The sceptic might respond: Simultaneous and sequential measurements are on a par since neither simultaneous measurements can completely rule out that the hidden states update between the measurements. Since measurements take time, it can well happen that the system's interaction with the one apparatus happens much faster than with the other and hence the hidden state can update between the two measurements. In this case, an updating noncontextual hidden variable model could be provided also for such simultaneous measurements. This is true. But while this model would be based on a speculative and empirically inaccessible order of interactions of otherwise simultaneous measurements, the updating models provided for sequential measurements are in tune with the observed sequence of interactions of the consecutive measurements.

Acknowledgements. This work has been supported by the Friedrich Wilhelm Bessel Research Award of the Alexander von Humboldt Foundation, the Hungarian National Research, Development and Innovation Office, K-134275. I thank Ádan Cabello for valuable discussions.

References

- J. S. Bell, "On the problem of hidden variables in quantum mechanics," *Reviews of Modern Physics*, 38, 447-452 (1966) reprinted in J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, (Cambridge: Cambridge University Press, 2004).
- A. Cabello, "Experimentally testable state-independent quantum contextuality," *Phys. Rev. Lett.*, 101, 210401 (2008).
- A. Cabello, "Proposal for revealing quantum nonlocality via local contextuality," *Phys. Rev. Lett.*, 104, 220401 (2010).
- A. Cabello, M. Gu, O. Gühne, and Z. Xu, "Optimal classical simulation of state-independent quantum contextuality," *Phys. Rev. Lett.* 120, 130401 (2018).
- O. Gühne, M. Kleinmann, A. Cabello, J. Larsson, G. Kirchmair, F. Zähringer, R. Gerritsma, and C. F. Roos, "Compatibility and noncontextuality for sequential measurements," *Phys. Rev. A,* 81, 022121 (2010).
- N. Harrigan, and R. Spekkens, "Einstein, incompleteness, and the epistemic view of quantum states," *Found. Phys.*, 40, 125–157 (2010).
- C. Held, "The Kochen-Specker theorem," *The Stanford Encyclopedia of Philosophy*, (2022)
  URL = https://plato.stanford.edu/entries/kochen-specker/#question.
- G. Hofer-Szabó, "Commutativity, comeasurability, and contextuality in the Kochen-Specker arguments," *Phil. Sci.*, 88, 483-510 (2021a).
- G. Hofer-Szabó, "Three noncontextual hidden variable models for the Peres-Mermin square," *Eur. J. Phil. Sci.*, 11, 30 (2021b).
- G. Hofer-Szabó, "Two concepts of noncontextuality in quantum mechanics," *Stud. Hist. Phil. Sci*, 93, 21-29 (2022).
- G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt and C. F. Roos, "State-independent experimental test of quantum contextuality," *Nature Letters*, 460, 494-498 (2009*).*
- M. Kleinmann, O. Gühne, J. Portillo, J. Larsson and A. Cabello, "Memory cost of quantum contextuality," *New J. Phys.* 13 113011 (2011).
- B. Liu, X. Hu, J. Chen, Y. Huang, Y. Han, C. Li, G. Guo, and A. Cabello, "Nonlocality from local contextuality," *Phys. Rev. Lett.*, 117, 220402 (2016).
- B. R. La Cour, "Quantum contextuality in the Mermin-Peres square: A hidden-variable perspective," *Phys. Rev. A*, 79, 012102 (2009) .
- B. R. La Cour, "Local hidden-variable model for a recent experimental test of quantum nonlocality and local contextuality," *Phys. Lett. A*, 381, 2230-2234 (2017).
- D. Mermin, "Hidden variables and the two theorems of John Bell," *Rev. Mod. Phys.*, 65 (3), 803-815 (1993).
- A. Peres, "Incompatible results of quantum measurements," *Phys. Lett. A*, 151, 107-108 (1990).
- F. M. Leupold, M. Malinowski, C. Zhang, V. Negnevitsky, J. Alonso, and J. P. Home, "Sustained state-independent quantum contextual correlations from a single ion," *Phys. Rev. Lett.* 120*,* 180401 (2018).