Equivalent Axiomatizations of Euclidean Geometry

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Abstract

I give six different *first-order mathematicized axiomatic systems*, expressing that physical space is Euclidean, and prove their equivalence.

Contents

1 Axiomatic Euclidean Geometry

There are multiple *equivalent* axiomatizations, or formulations, of the claim that ordinary physical space is three dimensional and Euclidean.

In order to make this claim precise, we need to be clear about the *primitive* notions used. This means that, at the very start, we need to specify certain *physical primitives*—a *signature*. Here, we shall begin with the physical synthetic signature for Euclidean geometry:

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$$\sigma_{phys} = \{\texttt{point}, \mathbf{B}, \equiv\} \tag{1}$$

with the following physical meanings:

Physical Primitives

`point`	$\texttt{point}(p)$ means "p is a point".
$\mathbf{B}$	$\mathbf{B}(p, q, r)$ means "the point q lies between points p and r".
$\equiv$	$pq \equiv rs$ means "the segment pq has the same length as (is congruent to) the segment rs".

Notice that we do not assume any sort of metric or distance function (rather, we shall prove that these exist). We then introduce a system of axioms about betweenness and congruence expressing what seem to us to be basic physical properties of these relations. These axioms are, of course, descendants of Euclid's axioms (Euclid (1956)), and are more directly descended from Hilbert's axioms for Euclidean geometry, given in Hilbert (1899). They were then more concisely formulated and simplifed by Alfred Tarski:

Definition 1.1. The non-logical axioms of $\mathsf{EG}(3)$ in $L(\sigma)$ are the following eleven:

Synthetic Euclidean geometry (three dimensions)

E1.	$\mathbf{B}$-Identity	$\mathbf{B}(p, q, p) \to p = q$.
E2.	$\equiv$-Identity	$pq \equiv rr \to p = q$.
E3.	$\equiv$-Transitivity	$pq \equiv rs \wedge pq \equiv tu \to rs \equiv tu$.
E4.	$\equiv$-Reflexivity	$pq \equiv qp$.
E5.	$\equiv$-Extension	$\exists r\, (\mathbf{B}(p, q, r) \wedge qr \equiv su)$.
E6.	Pasch	$\mathbf{B}(p, q, r) \wedge \mathbf{B}(s, u, r) \to \exists x\, (\mathbf{B}(q, x, s) \wedge \mathbf{B}(u, x, p))$.
E7.	Euclid	$\mathbf{B}(a, d, t) \wedge \mathbf{B}(b, d, c) \wedge a \neq d \to \exists x\, \exists y\, (\mathbf{B}(a, b, x) \wedge \mathbf{B}(a, c, y) \wedge \mathbf{B}(x, t, y))$
E8.	5-Segment	$p \neq q \wedge \mathbf{B}(p, q, r) \wedge \mathbf{B}(p', q', r') \wedge pq \equiv p'q' \wedge qr \equiv q'r' \wedge\ ps \equiv p's' \wedge qs \equiv q's' \to rs \equiv r's'$.
E9.	Lower Dimension	There exist four points which are not coplanar.
E10.	Upper Dimension	Any five points are in the same 3-dimensional space.
E11.	Continuity Axiom	$[\exists r\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathbf{B}(r, p, q)] \to \exists s\, (\forall p \in \mathsf{X}_1)\, (\forall q \in \mathsf{X}_2)\, \mathbf{B}(p, s, q)$

The original source of this axiomatization is Tarski (1959) and Tarski & Givant (1999). See Tarski (1959), pp. 19–20, for a formulation of the first-order two-dimensional theory, with twelve axioms and one axiom scheme (for continuity); and Tarski & Givant (1999) for a simplification down to ten axioms and one axiom scheme (for continuity). (The axioms E9 and E10 are called the "lower dimension" and "upper dimension" axioms, and can be formulated solely using the $\mathbf{B}$ predicate.)

Tarski's system in Tarski (1959) is denoted $\mathcal{E}_2$. This is first-order ("elementary") and is the two-dimensional theory. On the other hand, our $\mathsf{EG}(3)$ is second-order in the sense that it has quantifiers over points and *sets* of points. This may, of course, be considered a *first-order theory*, with separate sorts for points and sets. Indeed, that is how it is treated below.

The sole second-order axiom of $\mathsf{EG}(3)$ is the second-order *Continuity Axiom*, axiom E11. This is, more or less, a geometrical rephrasing of the Cut axiom for real numbers. If one replaces the Continuity Axiom by the Continuity *Axiom Scheme*, one obtains a first-order theory that I call $\mathsf{EG}_0(3)$, and which in Tarski's terminology is $\mathcal{E}_3$. Modifying the upper and lower dimension axioms (to those for two dimensions) yields $\mathcal{E}_2$.

Tarski proves two important meta-theorems about the axiom system $\mathcal{E}_2$:[1]

Theorem 1 (Tarski (1959))	A structure M is a model of $\mathcal{E}_2$ if and only if it is isomorphic to $(F^2, B_{F^2}, \equiv_{F^2})$, where F is a real-closed field.
Theorem 2 (Tarski (1959))	$\mathcal{E}_2$ is complete.

The first of these is called "the representation theorem" for $\mathcal{E}_2$. But, as noted, we are interested in the second-order axiom system, $\mathsf{EG}(3)$, which is the one relevant to *physics*. The meta-theorem Theorem 1 (Tarski (1959)) then, under suitable modifications (replacing the continuity scheme with the continuity axiom; modifying the upper and lower dimension axioms), yields the following representation theorem for $\mathsf{EG}(3)$:

Theorem 1.1. A structure M is a *full model* of $\mathsf{EG}(3)$ if and only if it is isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$.

It is clear that Theorem 1 (Tarski (1959)) and Theorem 1.1 are, in fact, provable *inside* a suitable ambient meta-theory, such as ZF set theory, and we shall exploit this fact below, in proving the *equivalence* of the synthetic theory $\mathsf{EG}(3)$ and the "representational equivalent" (listed (3) in Theorem 5.1).

2 Ambient Set Theory

In addition to $\mathsf{EG}(3)$, I wish to give *five further axiomatizations* and prove that these are equivalent, modulo "ambient set theory". In other words, each equivalence is proved assuming a background "base theory".

This base theory is "presupposed", in the sense explained in the 1960 monograph *Foundations of Geometry*, by Karol Borsuk and Wanda Szmielew:

[1] Tarski's work on this area took place at The University of Warsaw in the 1920s. Due to serious personal difficulties Tarski encountered—including getting a job in an increasingly antisemitic environment, and the Holocaust, in which almost all of Tarski's family were murdered at Auschwitz (by a stroke of luck, Tarski escaped from Poland in August 1939, thanks to an invitation to visit Harvard, from W.V. Quine)—it did not, however, get published until much later. The details of these events are described in the biography Feferman & Feferman (2004) by Tarski's student Solomon Feferman and Anita Feferman.

> In constructing an axiomatic theory T, we usually make use of other axiomatic theories which are *presupposed* in the following sense: all the primitive notions in the presupposed theory are included in the system of primitive notions of T, and all the axioms of those theories are included in the axiom system of T. Mathematical theories presuppose as a rule mathematical logic and usually also set theory (to a larger or smaller extent). In developing geometry in this book we presuppose mathematical logic, set theory and the arithmetic of the real numbers (which can either be treated as an independent theory or can be constructed as a portion of set theory). An axiomatic treatment of these theories can be found in various special works. (Borsuk & Szmielew (1960): 6–7)

The set theorist Andreas Blass has commented on the general situation as follows:[2]

> Mathematicians generally reason in a theory T which (up to possible minor variations between individual mathematicians) can be described as follows. It is a *many-sorted first-order theory*. The sorts include numbers (natural, real, complex), sets, ordered pairs and other tuples, functions, manifolds, projective spaces, Hilbert spaces, and whatnot. There are axioms asserting the basic properties of these and the relations between them. . . . This theory T, large and unwieldy though it is, can be interpreted in far simpler-looking theories. ZFC, with its single sort and single primitive predicate, is the main example of such a simpler theory. (I've left large categories out of T in order to make this literally true, but Feferman has shown how to interpret most of category theory, including large categories, in a conservative extension of ZFC.) (Blass (2012))

As Blass notes, "this theory T . . . can be interpreted in far simpler-looking theories. ZFC, with its single sort and single primitive predicate, is the main example of such a simpler theory".

Here, I explicitly give this ambient set theory: it is denoted AM.

Definition 2.1. The syntax of AM is given as follows:

Syntax

Sorts	$\Sigma = \{\texttt{atom}, \texttt{class}, \texttt{global}\}$
Signature	$\sigma = \{\texttt{point}, \mathbf{B}, \equiv, \in\}$
`point`	$\texttt{atom} \Rightarrow \texttt{bool}$
$\mathbf{B}$	$\texttt{atom} \Rightarrow \texttt{atom} \Rightarrow \texttt{atom} \Rightarrow \texttt{bool}$
$\equiv$	$\texttt{atom} \Rightarrow \texttt{atom} \Rightarrow \texttt{atom} \Rightarrow \texttt{atom} \Rightarrow \texttt{bool}$
$\in$	$\texttt{global} \Rightarrow \texttt{global} \Rightarrow \texttt{bool}$

Here, I have specified three sorts, `atom`, `class`, `global`, and if we are very precise, we would assign definitive variables to each sort. We have four basic predicates, as listed.

[2] Blass's comment is by no means off-beat. It is endorsed by the world's most talented mathematician, the Fields Medallist Terence Tao, and linked to at Professor Tao's famous web-blog.

Each predicate has a "sort declaration", which tells you the predicate's arity, and how its argument places are completed. I use a simple but very effective means of declaring these, by copying the type-theoretic terminology of the theorem prover Isabelle. This uses a "pretend sort", `bool`, to express the predicate's arity. Note that the membership predicate $\in$ is assumed, and its sort declaration is: $\texttt{global} \Rightarrow \texttt{global} \Rightarrow \texttt{bool}$.

Our language is $L(\sigma)$. The first-order three-sorted language over the signature σ and sorts Σ.

Definition 2.2. The axioms of AM, in the language $L(\sigma)$, are:[3]

Base theory for applied mathematics: AM

Theory	AM
Partition	$\texttt{atom}(x) \leftrightarrow \neg\texttt{class}(x)$
Atoms	$\texttt{atom}(x) \to (\texttt{empty}(x) \wedge \texttt{El}(x))$
Extensionality	$\forall x\,(x \in X \leftrightarrow x \in Y) \to X = Y$
Comprehension	$\exists X\,\forall x\,(x \in X \leftrightarrow (\texttt{El}(x) \wedge \varphi(x)))$
Empty set	$\varnothing \in U.$
Pairing	$x, y \in U \to \{x, y\} \in U$
Union	$X \in U \to \bigcup X \in U$
Power	$X \in U \to \mathcal{P}(X) \in U$
Infinity	$(\exists X \in U)\,\texttt{Inductive}(X)$
Replacement	$\texttt{Fun}(F) \wedge \texttt{Dom}(F) \in U \to \texttt{Ran}(F) \in U$
Choice	$(\forall y \in X)\,(\texttt{set}(y) \wedge y \neq \varnothing) \to (\exists F : X \to \bigcup X)\,(\forall y \in X)\,(F(y) \in y)$
Foundation	$\forall A\,(A \neq \varnothing \to (\exists B \in A)(B \cap A = \varnothing)).$

There are eleven axioms, and one axiom scheme. All known mathematics, and all mathematics needed for theoretical physics, can be developed from AM. There are a couple of redundancies here, as set theory aficionados will note. Replacement implies Separation, which implies Empty Set. Also, Replacement with Power set implies Pairing.

I shall next assume that all the standard definitions of mathematical notions are already given, as may be found in any set theory textbook.[4]

[3] This system AM (for "applied mathematics") is a small variation of that given in Jean Rubin's monograph Rubin (1967). It is, more or less, Morse-Kelley class theory with urelements. It proves all the theorems of ZFC, and a bit more, because its class comprehension axiom is impredicative. In an earlier paper Ketland (2021), I gave a system called ZFCA (for "ZFC with atoms (urelements)"). Being a class theory, AM is simpler to formulate and proves all the theorems of ZFCA.

[4] I mean, e.g.: "$x \subseteq y$", "$x \cup y$", "$x \cap y$", "the ordered pair (x, y)", "$\mathcal{P}(A)$", "$\bigcup A$", "R is a relation", "F is a function from A to B", "$x \in \mathbb{N}$", "$x \in \mathbb{Z}$", "$x \in \mathbb{Q}$", "$x \in \mathbb{R}$", etc. Definitions for these are given in, e.g., Halmos (1974), Enderton (1977), Drake & Singh (1996) (in particular, "Appendix: Chapter 10: Some Basic Definitions"), Hrbacek & Jech (1999). A fairly formal list may be found in Quine (1940): 323–324. Also Quine (1969): 333–341. Usually, the definition of $\mathbb{N}$ is simply ω (the finite ordinals), with $+$ and $\times$ defined on ω using The Recursion Theorem. The definitions for "$x \in \mathbb{Z}$", "$x \in \mathbb{Q}$", "$x \in \mathbb{R}$", and their operations and relations, are summarized in Enderton (1977): 121.

3 Example: The Axiom of Choice and Equivalents

Before getting on to the formulation of the *physical* equivalents, I mention briefly a series of equivalences well-known to all mathematicians. These concern Zermelo's Axiom of Choice. Here I shall follow the very nice presentation in Chapter 5 ("The Axiom of Choice") from Machover (1996).

Theorem 3.1. The following five statements are equivalent modulo ZF set theory:

(1) Choice	For each family of non-empty sets there is a choice function F.
(2) WOT	Every set can be well-ordered.
(3) Tukey-Teichmuller	Let $\mathcal{F}$ be a family of sets. Let $\mathcal{F}$ be of finite character. Then, for each $A \in \mathcal{F}$, there exists some $B \in \mathcal{F}$, with $A \subseteq B$, and B is maximal in $\mathcal{F}$ with respect to $\subseteq$.
(4) Hausdorff Maximality	Let P partial order and let $\mathtt{Chain}(P)$ be the set of chains in P. Then, for every $C \in \mathtt{Chain}(P)$, there is a $C' \in \mathtt{Chain}(P)$ which is maximal in $\mathtt{Chain}(P)$ with respect to inclusion.
(5) Kuratowski-Zorn	Let P be a partial order. Suppose each chain in P has an upper bound. Then P has a maximal element.

Proof. I direct the reader to Machover (1996), Chapter 5. □

These five examples by no means exhaust the list of equivalents of the Axiom of Choice. There is a vast number of equivalents of Choice which have many applications throughout analysis, algebra and topology. See, for example, Drake & Singh (1996), Chapter 5, or Hrbacek & Jech (1999), Chapter 10. There are whole books devoted to the topic: e.g., Rubin & Rubin (1985). Equivalents of Choice are the tip of a huge iceberg in mathematical logic.

4 Definitions

We now return to the *physics* case.

We shall begin with the synthetic (but second-order) axiom system $\mathsf{EG}(3)$, with its the physical primitives "point", "between" and "congruent". And we shall exhibit five further equivalents of $\mathsf{EG}(3)$. That is, statements which are inter-derivable with each other, and with $\mathsf{EG}(3)$, modulo the base AM. These statements introduce further mathematical machinery (by *explicit definition*). But note that this is no different from what happens with the equivalents of Choice listed above. These equivalences for Euclidean are proven modulo the fixed base theory, AM. But this is no different from what happens with the equivalents of Choice listed above.

For example, one of the equivalents states:

(3) There is a Cartesian chart on $(\mathbb{P}, B, \equiv)$.

Another states:

(6) There is a Riemannian manifold (M, g) isometric to standard Euclidean space on $\mathbb{R}^3$ and which, in a suitable sense, "represents" $(\mathbb{P}, B, \equiv)$.

The equivalent theories given below are theories in $L(\sigma)$. To stress, this is a *first-order language*. It has the usual Boolean connectives, some basic predicates (including $=$ and $\in$, and the physical ones in σ_{phys}), along with variables and quantifiers. The underlying logical axiom system is simply the usual one for first-order logic with identity, amended slightly because it has sorts.[5] From time to time, I will use "dedicated variables", which simulate sorts. This is standard mathematical practice.

Definition 4.1. We explicitly define:

$$\begin{aligned}
\mathbb{P} &:= \{p \mid \mathtt{point}(p)\} && (2)\\
B &:= \{(p,q,r) \in \mathbb{P}^3 \mid \mathbf{B}(p,q,r)\} && (3)\\
\equiv &:= \{(p,q,r,s) \in \mathbb{P}^4 \mid pq \equiv rs\} && (4)
\end{aligned}$$

So, $\mathbb{P}$ is the set of physical points, B is the physical betweenness relation on $\mathbb{P}$, and $\equiv$ is the physical congruence relation on $\mathbb{P}$. (We conflate the name of the relation with the predicate itself. This conflation is harmless.)

Definition 4.2. The applied mathematics base theory AM is extended with two axioms:

$$\mathsf{AM}^* := \mathsf{AM} + (\mathbb{P} \subseteq \mathsf{Atom}) + \mathtt{set}(\mathbb{P}). \qquad (5)$$

In other words, the points are *atoms* (i.e., physical urelements), and the class of points is a *set*. We need the second axiom to ensure that we can define *satisfaction* for the restricted sublanguage with quantifiers over points, and sets of points.

In order to formulate our (equivalent) physical axioms, we need a long series of explicit definitions. I must stress that these are *definitions*, not axioms. We have already given the axioms. These are AM, and $\mathbb{P} \subseteq \mathsf{Atom}$ and $\mathtt{set}(\mathbb{P})$.

Definition 4.3. We explicitly define, for $x, y, z, u \in \mathbb{R}^3$:

$$\begin{aligned}
B_{\mathbb{R}^3}(x,y,z) &:= (\exists \lambda \in [0,1])\,((y-x) = \lambda(z-x)) && (6)\\
\Delta_{\mathbb{R}^3}(x,y) &:= \sqrt{\sum_{i=1}^{3}(x_i - y_i)^2} && (7)\\
xy \equiv_{\mathbb{R}^3} zu &:= \Delta_{\mathbb{R}^3}(x,y) = \Delta_{\mathbb{R}^3}(z,u) && (8)
\end{aligned}$$

$B_{\mathbb{R}^3}$ is the coordinate betweenness relation: $B_{\mathbb{R}^3}(x,y,z)$ holds exactly if the point y lies between the points x and z. $\Delta_{\mathbb{R}^3}$ is the coordinate distance function. And $\equiv_{\mathbb{R}^3}$ is

[5] See, for example, Machover (1996), p. 114 for the propositional logic sector, and pp. 176–177 for the first-order logic sector. Or Drake & Singh (1996), Chapter 2. See Manzano (1996) for the smallish changes to accommodate sorts.

the coordinate congruence relation: $xy \equiv_{\mathbb{R}^3} zu$ holds exactly if the segment xy has the same length as the segment xu.

The *standard standard coordinate structure* for Euclidean space is then:

$$\mathbb{EG}(3) \quad := \quad (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{9}$$

Definition 4.4. We explicitly define:

$$\begin{aligned} \texttt{EucTranMap}(h, \alpha, d, R) \ := \ & \mathbb{R}^3 \overset{h}{\cong} \mathbb{R}^3 \wedge \alpha \in \mathbb{R} \wedge \alpha > 0 \wedge d \in \mathbb{R}^3 \wedge R \in O(3) \\ & \wedge (\forall x \in \mathbb{R}^3)\,(h(x) = \alpha R(x) + d) \end{aligned} \tag{10}$$

This can be read:

> "$h : \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean transition map, with parameters α, d, R"

Its definition states that h is a bijection from $\mathbb{R}^3$ to $\mathbb{R}^3$, that α is a positive real, that d is a vector in $\mathbb{R}^3$ and R is a rotation matrix in $O(3)$. In particular, if h is such a transition map with parameters α, d, R, then, for any point $x \in \mathbb{R}^3$:

$$h(x) = \alpha R(x) + d \tag{11}$$

We next define the set of these transition maps:

$$E(3) \quad := \quad \{h \mid \exists \alpha\, \exists d\, \exists R\, \texttt{EucTranMap}(h, \alpha, d, R)\} \tag{12}$$

It is clear that $E(3)$ is a Lie group. The following is the *automorphism theorem* for the coordinate structure $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$.

Theorem 4.1. $\mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})) = E(3)$.

Proof. First, we show: $E(3) \subseteq \mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}))$. Let $h \in E(3)$. So, there exists $\alpha > 0$, $R \in O(3)$, $d \in \mathbb{R}^3$ such that, for all $x \in \mathbb{R}^3$:

$$h(x) = \alpha R(x) + d \tag{13}$$

We claim:

$$B_{\mathbb{R}^3}(h(x), h(y), h(z)) \quad \leftrightarrow \quad B_{\mathbb{R}^3}(x, y, z) \tag{14}$$

$$h(x)h(y) \equiv_{\mathbb{R}^3} h(z)h(u) \quad \leftrightarrow \quad xy \equiv_{\mathbb{R}^3} zu \tag{15}$$

Both are these are fairly straightforward to verify.

Next, we need to show $\text{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})) \subseteq E(3)$. This is more difficult. Let $h \in \text{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}))$. So, h is a bijection $\mathbb{R}^3 \to \mathbb{R}^3$ and,

$$B_{\mathbb{R}^3}(h(x), h(y), h(z)) \leftrightarrow B_{\mathbb{R}^3}(x, y, z) \tag{16}$$

$$h(x)h(y) \equiv_{\mathbb{R}^3} h(z)h(u) \leftrightarrow xy \equiv_{\mathbb{R}^3} zu \tag{17}$$

The first condition (16) implies that h preserves straight lines, collinearity and parallelism. From this, we obtain that there exists a $GL(3)$ matrix A and a displacement d such that:

$$h(x) = A(x) + d \tag{18}$$

In other words, h is an affine map ($h \in \text{Aff}(3)$). The hard part is to show that $A = \alpha R$, for some dilation $\alpha > 0$ and some rotation $R \in O(3)$. This conclusion is obtained by studying four basic points in $\mathbb{R}^3$: $O = (0, 0, 0)$; $X = (1, 0, 0)$; $Y = (0, 1, 0)$; $Z = (0, 0, 1)$; and examining what the condition (17) implies for these.

□

Definition 4.5. Let $\Phi : \mathbb{P} \to \mathbb{R}^3$ be a function. We say that Φ is a *Cartesian chart* if and only if Φ is an isomorphism from $(\mathbb{P}, B, \equiv)$ to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. We define a simplifying formula $\texttt{Cart}(\Phi, \mathbb{P}, B, \equiv)$ to express this:

$$\texttt{Cart}(\Phi, \mathbb{P}, B, \equiv) \quad := \quad (\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{19}$$

Definition 4.6. We next define four formulas to simplify the axioms:

$$\texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta) \quad := \quad (\forall p, q, r \in \mathbb{P})\,(\mathbf{B}(p, q, r) \leftrightarrow \delta(p, r) = \delta(p, q) + \delta(q, r)) \tag{20}$$

$$\texttt{Def}_2(\mathbb{P}, \equiv, \delta) \quad := \quad (\forall p, q, r, s \in \mathbb{P})\,(pq \equiv rs \leftrightarrow \delta(p, q) = \delta(r, s)) \tag{21}$$

$$\texttt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C}) \quad := \quad (\forall f \in \mathcal{C})\,[f(B) = B_{\mathbb{R}^3}] \tag{22}$$

$$\texttt{Rep}(\equiv, \equiv_{\mathbb{R}^3}, \mathcal{C}) \quad := \quad (\forall f \in \mathcal{C})\,[f(\equiv) = \equiv_{\mathbb{R}^3}] \tag{23}$$

For example, the formula $\texttt{Def}_2(\mathbb{P}, \equiv, \delta)$ states that, for any elements p, q, r, s of $\mathbb{P}$, the relation $\equiv$ holds for p, q, r, s just if the value $\delta(p, q)$ is equal to the value $\delta(r, s)$. Roughly speaking, $\texttt{Def}_2(\mathbb{P}, \equiv, \delta)$ says that $\equiv$ is definable using δ. And, roughly speaking, the formula $\texttt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C})$ says that if we take the *image* $f(B)$ of the physical betweenness relation B, under any chart $f \in \mathcal{C}$, we get $B_{\mathbb{R}^3}$.

Next, we express David Wallace's "Kleinian" notion of a "*G-structured space*":

> Suppose that G is a group of bijections of $\mathbb{R}^N$. Then a *G-structured space* is a set $\mathbb{P}$ together with a nonempty collection $\mathcal{C}$ of bijections from $\mathbb{P}$ to $\mathbb{R}^N$ (the 'coordinatisations' of $\mathbb{P}$), such that if $f \in C$, then $f' \in C$ iff $f \circ (f')^{-1} \in G$. This is, in effect, a form of the definition of geometry in Klein's famous *Erlangen* program (Klein (1892)). (Wallace (2019): 127)

Definition 4.7. $(X, \mathcal{C})$ is a G-structured space if and only if:

(i)	G is a *group* of bijections $\mathbb{R}^3 \to \mathbb{R}^3$.
(ii)	$\mathcal{C}$ is a non-empty collection of bijections $X \to \mathbb{R}^3$.
(iii)	for any $f \in \mathcal{C}$, we have $f' \in \mathcal{C} \leftrightarrow f' \circ f^{-1} \in G$.

Definition 4.8. The definition of $\texttt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C})$ is:

$$\texttt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C}) := \quad \mathcal{C} \neq \varnothing \wedge (\forall f \in \mathcal{C}) \; [\mathbb{P} \overset{f}{\cong} \mathbb{R}^3 \wedge \forall f'(f' \in \mathcal{C} \leftrightarrow f' \circ f^{-1} \in E(3))] \quad (24)$$

Finally, we shall mention an equivalent using differential geometry. In general, modern discussions of *spacetime geometry* generally develop the mathematics using the machinery of differential geometry.[6] The basic notions are that of a topological manifold M, a differentiable manifold M, and the notion of vector fields and tensor fields on M. To save space, I assume it is known what an atlas $\mathcal{C}$ is, what a smooth 3-dimensional manifold $(M, \mathcal{C})$ is, what a flat Riemannian tensor field g on M is, and also what a what a torsion-free connection ∇ is.

Definition 4.9. The *canonical Euclidean space* on the manifold $\mathbb{R}^3$ is $(\mathbb{R}^3, g^{euc})$, where

$$g^{euc}(\partial_i, \partial_j) = \delta_{ij}, \quad (25)$$

wrt coordinates given by the standard identity chart on $\mathbb{R}^3$.

Definition 4.10. A *Euclidean space* is a Riemannian manifold (M, g) isometric to the canonical Euclidean space $(\mathbb{R}^3, g^{euc})$.

Definition 4.11. We first define the formula:

$$\texttt{EuclSpace}(M, g) \quad := \quad \texttt{Diff}(M, \mathbb{R}^3) \wedge \texttt{Euc}(M, g) \quad (26)$$

where $\texttt{Diff}(M, \mathbb{R}^3)$ expresses that M is diffeomorphic to $\mathbb{R}^4$, and $\texttt{Euc}(M, g)$ expresses the isometry condition.

This is nothing more than a transcription of the usual semi-formal Definition 4.10.

Definition 4.12. We next define four formulas expressing how a Euclidean space (M, g) "represents" a synthetic Euclidean model $(\mathbb{P}, B, \equiv)$. The first is:

$$\Psi_1(\mathcal{C}_0, \mathcal{C}, M, \mathbb{P}, B, \equiv) := [\mathcal{C}_0 = \{\Phi : \mathbb{P} \to \mathbb{R}^3 \mid (\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})\} \\ \wedge \mathcal{C}_0 \neq \varnothing \wedge \texttt{MaxExt}(\mathcal{C}, \mathcal{C}_0) \wedge M = (\mathbb{P}, \mathcal{C})] \quad (27)$$

6 See Schutz (1980). See Malament (2009) and Malament (2012) for very clear and detailed expositions.

Here $\texttt{MaxExt}(\mathcal{C}, \mathcal{C}_0)$ is an abbreviation for "$\mathcal{C}$ is the maximal atlas extending $\mathcal{C}_0$".

The second and third formulas are as follows:

$$\Psi_2(M, \nabla, \mathbb{P}, B) \;:=\; \texttt{TFC}(\nabla, M) \wedge \mathcal{L}_{\mathbb{P},B} = \{\texttt{ran}\,\gamma \mid \gamma \in \texttt{Geod}(\nabla)\} \tag{28}$$

$$\Psi_3(M, g, \mathbb{P}, \equiv) \;:=\; (\forall p, q, r, s \in \mathbb{P})\,(pq \equiv rs \leftrightarrow L_g(p, q) = L_g(r, s)) \tag{29}$$

Here, $\texttt{TFC}(\nabla, M)$ is a formula saying "∇ is a torsion-free connection on M". $\mathcal{L}_{\mathbb{P},B}$ is defined as "the set of straight lines in $\mathbb{P}$, according to the physical betweenness relation B". $\texttt{Geod}(\nabla)$ is defined as "the set of curves which are geodesics of ∇". And $L_g(p, q)$ is defined as "the length, relative to the tensor g, of some geodesic γ from p to q".

So, $\Psi_2(M, \nabla, \mathbb{P}, B)$ expresses how the connection ∇ is determined by the system of straight lines given by B; and $\Psi_3(M, g, \mathbb{P}, \equiv)$ expresses how the metric g is determined by the requirement that lengths determined by g correspond to the physical congruence relation $\equiv$. (In fact, it is not uniquely determined.)

Finally, the overall representation condition is given as follows:

$$\texttt{Rep}(M, g, \mathbb{P}, B, \equiv) \;:=\; \exists \mathcal{C}_0\, \exists \mathcal{C}\, \exists \nabla\, [\Psi_1(\mathcal{C}_0, \mathcal{C}, M, \mathbb{P}, B, \equiv) \wedge \Psi_2(M, \nabla, \mathbb{P}, B) \wedge \Psi_3(M, g, \mathbb{P}, B, \equiv)] \tag{30}$$

5 Main Theorem

Theorem 5.1. The following six statements are equivalent modulo the base theory AM^*:

(1) Synthetic	$\mathsf{EG}(3)$
(2) Semantical	$(\mathbb{P}, B, \equiv) \models_2 \ulcorner \mathsf{EG}(3) \urcorner$
(3) Representational	$(\exists \Phi : \mathbb{P} \to \mathbb{R}^3)\,[(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})]$
(4) Metrical	$(\exists \delta : \mathbb{P}^2 \to \mathbb{R}^+_0)\,[(\mathbb{P}, \delta) \cong (\mathbb{R}^3, \Delta_{\mathbb{R}^3}) \wedge \texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta) \wedge \texttt{Def}_2(\mathbb{P}, \equiv, \delta)]$
(5) Kleinian	$\exists \mathcal{C}\,[\texttt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C}) \wedge \texttt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C}) \wedge \texttt{Rep}(\equiv, \equiv_{\mathbb{R}^3}, \mathcal{C})]$
(6) Diff Geometrical	$\exists M\, \exists g\, [\texttt{EuclSpace}(M, g) \wedge \texttt{Rep}(M, g, \mathbb{P}, B, \equiv)]$

Proof. (1) $\Leftrightarrow$ (2). Essentially, this is nothing more than disquotational equivalence:

$$\texttt{True}(\ulcorner \varphi \urcorner) \leftrightarrow \varphi \tag{31}$$

The details require us to formally define the satisfaction relation for the restricted sublanguage of $\mathsf{EG}(3)$. Call this $L(\mathsf{EG}(3))$. This is a *second-order language*, and a sublanguage of the much more expressive *first-order language* $L(\sigma)$![7] $L(\mathsf{EG}(3))$ has variables

[7] How can a *second-order* language be a sublanguage of a *first-order* language? That sounds weird.

(say $p_1, p_2, \dots$) for points, and variables (say $\mathsf{X}_1, \mathsf{X}_2, \dots$) for sets of points, and the predicates `point`, **B** and $\equiv$. One can define *satisfaction* $\models_2$ for this sublanguage $L(\mathsf{EG}(3))$ *inside* the theory AM^*. Then, we obtain, for any sentence φ of $L(\mathsf{EG}(3))$,

$$\mathsf{AM}^* \vdash \varphi \leftrightarrow [(\mathbb{P}, B, \equiv) \models_2 \ulcorner \varphi \urcorner] \tag{32}$$

And hence:

$$\mathsf{AM}^* \vdash \mathsf{EG}(3) \leftrightarrow [(\mathbb{P}, B, \equiv) \models_2 \ulcorner \mathsf{EG}(3) \urcorner] \tag{33}$$

This verifies (1) $\Leftrightarrow$ (2).

□

With a bit of use/mention abuse, this can be expressed:

(E) $\mathsf{EG}(3)$ if and only if $(\mathbb{P}, B, \equiv)$ satisfies $\mathsf{EG}(3)$

This is analogous to the equivalence between:

(i) Every even number over 2 is a sum of two primes.
(ii) $(\mathbb{N}, 0, S, +, \times)$ satisfies $(\forall n > 1)\, \exists p_1\, \exists p_2\, (\mathtt{prime}(p_1) \wedge \mathtt{prime}(p_2) \wedge 2n = p_1 + p_2)$.

Proof. (1) $\Leftrightarrow$ (3). This amounts to the Representation Theorem for $\mathsf{EG}(3)$. This theorem is essentially David Hilbert's, from Hilbert (1899). Hilbert's axioms, though, are a bit complicated, and were significantly simplified by Alfred Tarski and coworkers, leading to our axiom system $\mathsf{EG}(3)$. The main result is our Theorem 1.1 above.

A sketch of proof the theorem for the two-dimensional first-order theory (which I'll call $\mathsf{EG}_0(2)$; Tarski calls it $\mathcal{E}_2$) is given in Tarski (1959), Theorem 1. In fact, this theorem can certainly be proved *inside* our base theory AM^* and says:[8]

(i) M is a model of $\mathsf{EG}_0(2)$ if and only if M is isomorphic to $(F^2, B_{F^2}, \equiv_{F^2})$, where F is a real-closed field.

If one modifies the Continuity Axiom *Scheme* to the second-order Continuity *Axiom*, one gets the theory I call $\mathsf{EG}(2)$. Then the Representation Theorem states:

Well, the atomic $\in$-formulas of the second-order language are of the form $p_i \in \mathsf{X}_j$, with that specific restriction. But the first-order language has $x \in y$, for any variables x, y. To see this in practice, consider second-order arithmetic PA_2. This can be translated into first-order ZFC. In fact, n-order arithmetic PA_n can be translated in Zermelo set theory Z. In fact, the *limit* $\bigcup_n \mathsf{PA}_n$ can be translated into Z too. This limit is equivalent to Russell's Simple Type Theory, STT (simplified by Frank Ramsey, in Ramsey (1926)). It is also essentially the same as Gödel's system denoted P in Gödel (1931).

8 The first-order theory $\mathsf{EG}_0(2)$ has the striking property, discovered by Tarski, that it is (negation)-*complete*. The proof of this follows from the fact that one can interpret $\mathsf{EG}_0(2)$ into the theory RCF of the real-closed ordered field of real numbers, and Tarski had already obtained a proof that RCF is complete, using the method known as "quantifier elimination". See Tarski (1948).

(ii) M is a full model of $\mathsf{EG}(2)$ if and only if M is isomorphic to $(\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2})$

If we modify the dimension axioms, we get $\mathsf{EG}(3)$, and then obtain Theorem 1.1:

(iii) M is a full model of $\mathsf{EG}(3)$ if and only if M is isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$

Hence, for $(\mathbb{P}, B, \equiv)$, we obtain:

(iv) $(\mathbb{P}, B, \equiv)$ is a full model of $\mathsf{EG}(3)$ if and only if $(\mathbb{P}, B, \equiv)$ is isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$

And by the "disquotation" trick above (which is, by the way, also due to Tarski: Tarski (1936)) this reduces to:

(v) $\mathsf{EG}(3)$ if and only if $(\mathbb{P}, B, \equiv)$ is isomorphic to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$

This verifies (1) $\Leftrightarrow$ (3).

□

So, we can express Euclidean geometry at "the ground level", as it were, via the axioms $\mathsf{EG}(3)$, or as a semantical claim, "$(\mathbb{P}, B, \equiv)$ *satisfies* the axioms $\mathsf{EG}(3)$". These are equivalent. They can be derived from each other.

Proof. (3) $\Rightarrow$ (4). Let us assume (Representational) holds, and hence there exists an isomorphism from the *physical structure* $(\mathbb{P}, B, \equiv)$ to the *mathematical coordinate structure* $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. Now fix some isomorphism, Φ say. Then, we have the two representation conditions, for points $p, q, r, s \in \mathbb{P}$:

$$\mathbf{B}(p,q,r) \quad \leftrightarrow \quad B_{\mathbb{R}^3}(\Phi(p), \Phi(q), \Phi(r)) \tag{34}$$

$$pq \equiv rs \quad \leftrightarrow \quad \Phi(p)\Phi(q) \equiv_{\mathbb{R}^3} \Phi(r)\Phi(s) \tag{35}$$

Let us explicitly define a function:

$$\delta : \mathbb{P}^2 \to \mathbb{R}^+ \tag{36}$$

pointwise, by:

$$\delta(p,q) := \Delta_{\mathbb{R}^3}(\Phi(p), \Phi(q)) \tag{37}$$

Hence, by construction:

$$(\mathbb{P}, \delta) \overset{\Phi}{\cong} (\mathbb{R}^3, \Delta_{\mathbb{R}^3}). \tag{38}$$

What is more, since $(\mathbb{R}^3, \Delta_{\mathbb{R}^3})$ is the usual metric space on $\mathbb{R}^3$, it follows that $(\mathbb{P}, \delta)$ is an isomorphic copy, and so also a metric space. Now the coordinate betweenness relation

$B_{\mathbb{R}^3}$ inside $\mathbb{R}^3$, and the coordinate congruence relation $\equiv_{\mathbb{R}^3}$ inside $\mathbb{R}^3$, satisfy, for any points $x \in \mathbb{R}^3, y \in \mathbb{R}^3, z \in \mathbb{R}^3, u \in \mathbb{R}^3$:

$$B_{\mathbb{R}^3}(x,y,z) \quad \leftrightarrow \quad \Delta_{\mathbb{R}^3}(x,z) = \Delta_{\mathbb{R}^3}(x,y) + \Delta_{\mathbb{R}^3}(y,z) \tag{39}$$
$$xy \equiv_{\mathbb{R}^3} zu \quad \leftrightarrow \quad \Delta_{\mathbb{R}^3}(x,y) = \Delta_{\mathbb{R}^3}(z,u) \tag{40}$$

We wish to prove the definability formulas $\texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta)$ and $\texttt{Def}_2(\mathbb{P}, \equiv, \delta)$. That is, we claim:

$$(\forall p, q, r \in \mathbb{P}) \quad (\mathbf{B}(p,q,r) \leftrightarrow (\delta(p,r) = \delta(p,q) + \delta(q,r))) \tag{41}$$
$$(\forall p, q, r, s \in \mathbb{P}) \quad (pq \equiv rs \leftrightarrow \delta(p,q) = \delta(r,s)) \tag{42}$$

So, to prove (41), we reason like this:

$$\begin{aligned} \mathbf{B}(p,q,r) \quad &\Leftrightarrow \quad B_{\mathbb{R}^3}(\Phi(p), \Phi(q), \Phi(r)) &(43)\\ &\Leftrightarrow \quad \Delta_{\mathbb{R}^3}(\Phi(p), \Phi(r)) = \Delta_{\mathbb{R}^3}(\Phi(p), \Phi(q)) + \Delta_{\mathbb{R}^3}(\Phi(q), \Phi(r)) &(44)\\ &\Leftrightarrow \quad \delta(p,r) = \delta(p,q) + \delta(q,r) &(45)\end{aligned}$$

Likewise, to prove (42):

$$\begin{aligned} pq \equiv rs \quad &\Leftrightarrow \quad \Phi(p)\Phi(q) \equiv_{\mathbb{R}^3} \Phi(r)\Phi(s) &(46)\\ &\Leftrightarrow \quad \Delta_{\mathbb{R}^3}(\Phi(p), \Phi(q)) = \Delta_{\mathbb{R}^3}(\Phi(r), \Phi(s)) &(47)\\ &\Leftrightarrow \quad \delta(p,q) = \delta(r,s) &(48)\end{aligned}$$

This verifies (3) $\Rightarrow$ (4). □

It's worth noting that the defined metric δ on $\mathbb{P}$ is not unique. If $\alpha > 0$, then $\alpha\delta$ does the trick too. For we have, for any $\alpha > 0$,

$$\delta(p,q) = \delta(r,s) \quad \leftrightarrow \quad \alpha\delta(p,q) = \alpha\delta(r,s) \tag{49}$$

This is "gauge equivalence".

Proof. (4) $\Rightarrow$ (3). We suppose (Metrical) holds. That is,

$$(\exists \delta : \mathbb{P}^2 \to \mathbb{R}^+_0)\,[(\mathbb{P}, \delta) \cong (\mathbb{R}^3, \Delta_{\mathbb{R}^3}) \wedge \texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta) \wedge \texttt{Def}_2(\mathbb{P}, \equiv, \delta)] \tag{50}$$

So, we have a function $\delta : \mathbb{P}^2 \to \mathbb{R}^+_0$ such that $(\mathbb{P}, \delta)$ is isomorphic to the standard metric space $(\mathbb{R}^3, \Delta_{\mathbb{R}^3})$. Moreover, $\texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta)$ and $\texttt{Def}_2(\mathbb{P}, \equiv, \delta)$ hold. These translate to the following. For points $p, q, r, s \in \mathbb{P}$, we have:

$$\mathbf{B}(p,q,r) \quad \leftrightarrow \quad \delta(p,r) = \delta(p,q) + \delta(q,r) \tag{51}$$
$$pq \equiv rs \quad \leftrightarrow \quad \delta(p,q) = \delta(r,s) \tag{52}$$

From these assumptions we can *prove* the eleven axioms, $\mathsf{EG}(3)$, governing the *physical* primitives $\mathbf{B}$ and $\equiv$. For example, consider the axioms E1 and E2:

$$\text{(E1)} \quad \mathbf{B}(p,q,p) \to p = q \tag{53}$$
$$\text{(E2)} \quad pq \equiv rr \to p = q \tag{54}$$

For E1, suppose $\mathbf{B}(p,q,p)$. Then, $\delta(p,p) = \delta(p,q) + \delta(q,p)$. But $\delta(p,p) = 0$. And $\delta(p,q) = \delta(q,p)$. Hence, $0 = 2\delta(p,q)$. Hence, $\delta(p,q) = 0$. Hence, $p = q$. Likewise, for E2, suppose $pq \equiv rr$. Then $\delta(p,q) = \delta(r,r)$. But $\delta(r,r) = 0$. Hence, $\delta(p,q) = 0$, and therefore $p = q$. The proofs of the other axioms of $\mathsf{EG}(3)$ are similar, but the details are tedious and messy.

But given $\mathsf{EG}(3)$, we can now prove the Representation Theorem. We obtain: $(\exists \Phi : \mathbb{P} \to \mathbb{R}^3)\, [(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})]$. This is (3), as claimed. □

Proof. (3) ⇒ (5). Suppose that there exists an isomorphism Φ from the synthetic physical structure $(\mathbb{P}, B, \equiv)$ to the standard coordinate structure $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. Let

$$\mathcal{C} \quad := \quad \{f \mid (\mathbb{P}, B, \equiv) \overset{f}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})\} \tag{55}$$

By our assumption, $\mathcal{C} \neq \varnothing$. Because each element of $\mathcal{C}$ is indeed an isomorphism, it follows that $\mathtt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C})$ and $\mathtt{Rep}(\equiv, \equiv_{\mathbb{R}^3}, \mathcal{C})$. It remains to prove $\mathtt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C})$. This is the claim that $\mathcal{C}$ turns $\mathbb{P}$ into what Wallace calls "an $E(3)$-space". More exactly:

$$\mathcal{C} \neq \varnothing \wedge (\forall f \in \mathcal{C})\ [\mathbb{P} \overset{f}{\cong} \mathbb{R}^3 \wedge \forall f'(f' \in \mathcal{C} \leftrightarrow f' \circ f^{-1} \in E(3))] \tag{56}$$

We already have $\mathcal{C} \neq \varnothing$. Moreover, if $f \in \mathcal{C}$, then $f : \mathbb{P} \to \mathbb{R}^3$ is a bijection. It remains to prove that if $f \in \mathcal{C}$, then $f' \in \mathcal{C}$ if and only if $f' \circ f^{-1} \in E(3)$. Suppose $f \in \mathcal{C}$.

First, we recall Theorem 4.1:

$$\mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})) = E(3) \tag{57}$$

Now suppose $f' \in \mathcal{C}$. We claim: $f' \circ f^{-1} \in E(3)$. Our assumptions unwind to:

$$(\mathbb{P}, B, \equiv) \overset{f}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{58}$$
$$(\mathbb{P}, B, \equiv) \overset{f'}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{59}$$

Hence, by composition of isomorphisms:

$$f' \circ f^{-1} \in \mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})) \tag{60}$$

By equation (57), we conclude:

$$f' \circ f^{-1} \in E(3) \tag{61}$$

Conversely, suppose $f' \circ f^{-1} \in E(3)$. Let $h = f' \circ f^{-1}$. Then, $h \in \mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}))$. So, $f' = h \circ f$. Since $f \in \mathcal{C}$ and $h \in \mathrm{Aut}((\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}))$, we have

$$(\mathbb{P}, B, \equiv) \overset{f}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{62}$$

$$(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \overset{h}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{63}$$

It follows that

$$(\mathbb{P}, B, \equiv) \overset{f'}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{64}$$

$$\tag{65}$$

I.e., $f' \in \mathcal{C}$, as desired.

And hence, $\texttt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C})$, as claimed. This is (5). □

Proof. (5) ⇒ (3). Suppose (5) Kleinian holds. We have a non-empty chart system $\mathcal{C}$ which is indeed an $E(3)$-chart system on the physical points $\mathbb{P}$, and also $\texttt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C})$ and $\texttt{Rep}(\equiv, \equiv_{\mathbb{R}^3}, \mathcal{C})$ both hold. That is:

(a) The physical betweenness relation $B \subseteq \mathbb{P}^3$ is represented by $B_{\mathbb{R}^3}$, for any $f \in \mathcal{C}$.
(b) The physical congruence relation $\equiv \subseteq \mathbb{P}^4$ is represented by $\equiv_{\mathbb{R}^3}$, for any $f \in \mathcal{C}$.

We claim that (Representational) holds. Let $\Phi \in \mathcal{C}$ be any chart. Consequently, Φ is a bijection $\mathbb{P} \to \mathbb{R}^3$. From (a) and (b), we obtain: for points $p, q, r, s \in \mathbb{P}$:

$$(p, q, r) \in B \quad \leftrightarrow \quad (\Phi(p), \Phi(q), \Phi(r)) \in B_{\mathbb{R}^3} \tag{66}$$

$$pq \equiv rs \quad \leftrightarrow \quad \Phi(p), \Phi(q), \Phi(r), \Phi(s)) \in \equiv_{\mathbb{R}^3} \tag{67}$$

Hence, Φ is an isomorphism from $(\mathbb{P}, B, \equiv)$ to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. This is (3). □

Finally, we move to the differential geometry formulation (6).

Proof. (6) $\Rightarrow$ (3). This is the easy direction, for the statement (3) is pretty much built into the representation condition $\texttt{Rep}(M, g, \mathbb{P}, B, \equiv)$. So, suppose (6) holds. We have a Euclidean space (M, g), and we have: $\texttt{Rep}(M, g, \mathbb{P}, B, \equiv)$. So, unwinding, we have $\Psi_1(\mathcal{C}_0, \mathcal{C}, M, \mathbb{P}, B, \equiv)$, and hence:

$$\mathcal{C}_0 = \{\Phi : \mathbb{P} \to \mathbb{R}^3 \mid (\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})\} \wedge \mathcal{C}_0 \neq \varnothing \tag{68}$$

along with $\texttt{MaxExt}(\mathcal{C}, \mathcal{C}_0)$, and $M = (\mathbb{P}, \mathcal{C})$. Therefore, there exists some $\Phi : \mathbb{P} \to \mathbb{R}^3$, such that $(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. This is (3). □

Proof. (3) $\Rightarrow$ (6). This is harder, but only because we have to keep checking that the definitions work. It fills up about five pages from here. It can chunked into four main parts, I, II, III and IV:

(Part I) Given the existence of at least one Cartesian chart $(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$, we define $\mathcal{C}_0 := \{\Phi : \mathbb{P} \to \mathbb{R}^3 \mid (\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})\}$, and prove that this is a smooth atlas on $\mathbb{P}$. We extend $\mathcal{C}_0$ to a maximal atlas, $\mathcal{C}$. Then $M = (\mathbb{P}, \mathcal{C})$ is a manifold diffeomorphic to $\mathbb{R}^3$ (as a manifold).

(Part II) Given B, we define the set $\mathcal{L}_{\mathbb{P},B}$ of straight lines in $(\mathbb{P}, B, \equiv)$. Thus uniquely fixes a torsion-free connection ∇ satisfying the condition that, for any smooth curve γ, its image $\texttt{ran}\,\gamma$ is in $\mathcal{L}_{\mathbb{P},B}$ if and only if $\gamma \in \texttt{Geod}(\nabla)$.

(Part III) Fix a Cartesian chart Φ. We define g to be the tensor on M with components $diag(1,1,1)$ relative to Φ. We define $L_g(p,q)$ to be the "length" $\int_0^1 d\lambda \sqrt{g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu}$, for some geodesic $\gamma \in \texttt{Geod}(\nabla)$. We prove that $L_g(p,q) = \Delta_\Phi(p,q)$. Then we prove that $L_g(p,q) = L_g(r,s) \leftrightarrow pq \equiv rs$.

(Part IV) We have that M is isomorphic to $\mathbb{R}^3$ and that g is isometric with the canonical metric g^{euc} on $\mathbb{R}^3$. This implies that (M, g) is Euclidean space.

The details, then, are:
(Part I). Let's suppose (3) holds. Hence, we have at least one bijection

$$\Phi : \mathbb{P} \to \mathbb{R}^3 \tag{69}$$

such that

$$(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3}) \tag{70}$$

Thus, our synthetic system $(\mathbb{P}, B, \equiv)$ is isomorphic to the standard coordinate structure for Euclidean space $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. We now have to "extract", by definitions, from the physical system $(\mathbb{P}, B, \equiv)$ the differential geometry "equivalent", (M, g) and show that it is a standard Euclidean space.

From $(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$, we have two isomorphism conditions: for any $p, q, r, s \in \mathbb{P}$,

$$\mathbb{B}(p,q,r) \quad \leftrightarrow \quad (\Phi(p),\Phi(q),\Phi(r)) \in B_{\mathbb{R}^3} \tag{71}$$
$$pq \equiv rs \quad \leftrightarrow \quad (\Phi(p),\Phi(q),\Phi(r),\Phi(s)) \in \equiv_{\mathbb{R}^3} \tag{72}$$

Let us define the distance function, given Φ:

$$\Delta_\Phi(p,q) := \sqrt{\sum_{i=1}^{3}(\Phi^i(p) - \Phi^i(q))^2} \tag{73}$$

Then, using (72) and the definition of $\equiv_{\mathbb{R}^3}$:

$$pq \equiv rs \quad \leftrightarrow \quad \Delta_\Phi(p,q) = \Delta_\Phi(r,s) \tag{74}$$

We now construct the chart system on $\mathbb{P}$. We have at least one isomorphism Φ from $(\mathbb{P}, B, \equiv)$ to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$, and so let us define:

$$\mathcal{C}_0 \quad := \quad \{\Phi : \mathbb{P} \to \mathbb{R}^3 \mid (\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})\} \tag{75}$$

One may check that $\mathcal{C}_0$ is a *smooth atlas* (on $\mathbb{P}$, not $\mathbb{R}^3$—we are doing *physics*). This comes down to figuring out what the transitions maps are and proving that they are smooth. Well, the transition maps are bijections $f : \mathbb{R}^3 \to \mathbb{R}^3$ and as we know from the automorphism Theorem 4.1 above, these functions are elements of $E(3)$: they have the form, for any $x \in \mathbb{R}^3$,

$$f(x) = \alpha R(x) + d \tag{76}$$

where $\alpha \in \mathbb{R}$ ($\alpha > 0$) is the dilation parameter, $R \in O(3)$ is the rotation matrix, and $d \in \mathbb{R}^3$ is the linear displacement. It is clear that these bijections are smooth. Let $\mathcal{C}$ be the maximal extension of $\mathcal{C}_0$. Define:

$$M := (\mathbb{P}, \mathcal{C}) \tag{77}$$

Hence $(\mathbb{P}, \mathcal{C})$ is a three-dimensional smooth manifold. The smooth atlas $\mathcal{C}_0$ has a subatlas $\mathcal{C}_* = \{(\mathbb{P}, \Phi)\}$ consisting of a single chart. And the manifold $\mathbb{R}^3$ has a subatlas $\mathcal{C}_{\mathbb{R}^3} = \{(\mathbb{R}^3, Id_{\mathbb{R}^3})\}$. Then the mapping $\Phi : \mathbb{P} \to \mathbb{R}^3$ is a *diffeomorphism* from $(\mathbb{P}, \mathcal{C}_*)$ to $(\mathbb{R}^3, \mathcal{C}_{\mathbb{R}^3})$. So, M is diffeomorphic to $\mathbb{R}^3$ (as a manifold). Checking back on the definitions above, the reader will see that we have now proved the claim: $\Psi_1(\mathcal{C}_0, \mathcal{C}, M, \mathbb{P}, B, \equiv)$.

(Part II). Next, we need to define a torsion-free connection ∇ on M in terms of our basic physical primitive B. Roughly, ∇ is going to be the unique torsion free connection on M whose geodesics are exactly the straight lines of the betweenness relation B on $\mathbb{P}$. We shall call this set of straight lines $\mathcal{L}_{\mathbb{P},B}$. So, the defining condition for ∇ is that $\mathcal{L}_{\mathbb{P},B}$ is equal to the set of images of curves $\gamma \in \texttt{Geod}(\nabla)$.

Given the betweenness relation B on $\mathbb{P}$, we define the collinearity relation:

$$\texttt{col}(p,q,r) \quad := \quad \mathbf{B}(p,q,r) \vee \mathbf{B}(q,r,p) \vee \mathbf{B}(r,p,q) \tag{78}$$

And, supposing $p \neq q$, we define:

$$\ell(p,q) \quad := \quad \{r \in \mathbb{P} \mid \texttt{col}(p,q,r)\} \tag{79}$$

So, $\ell(p,q)$ is the "straight line containing p and q". Define:

$$\mathcal{L}_{B,\mathbb{P}} \quad := \quad \{\ell(p,q) \mid p,q \in \mathbb{P} \wedge p \neq q\} \tag{80}$$

So, $\mathcal{L}_{B,\mathbb{P}}$ is the set of straight lines in $\mathbb{P}$. Now working in our standard chart Φ, we can show that there exists a set N of nicely parametrized curves $\gamma : \mathbb{R} \to \mathbb{P}$, such that, we have:[9]

$$(i) \quad (\forall \gamma \in N)\,(\texttt{ran}\,\gamma \in \mathcal{L}_{B,\mathbb{P}} \leftrightarrow (\exists x,y \in \mathbb{R}^3)\,(\forall \lambda \in \mathbb{R})(\Phi(\gamma(\lambda)) = x + \lambda(y-x)) \tag{81}$$

$$(ii) \quad \ell \in \mathcal{L}_{B,\mathbb{P}} \to (\exists \gamma \in N)\,(\ell = \texttt{ran}\,\gamma) \tag{82}$$

And hence, for each $\gamma \in N$:

$$\texttt{ran}\,\gamma \in \mathcal{L}_{B,\mathbb{P}} \quad \leftrightarrow \quad \bigwedge_{\beta=1}^{3} (\forall \lambda \in \mathbb{R})\,\ddot{\gamma}^{\beta}(\lambda) = 0 \tag{83}$$

Supposing ∇ is a connection on M, we have, in any coordinates:

$$\nabla_{\mu} \frac{\partial}{\partial x^{\nu}} = \Gamma^{\beta}_{\mu\nu} \frac{\partial}{\partial x^{\beta}} \tag{84}$$

where the (3x3x3) numbers $\Gamma^{\beta}_{\mu\nu}$ are the connection coefficients.

And, in any coordinates, a curve γ is a geodesic of ∇ just if:

$$\ddot{\gamma}^{\beta} = \Gamma^{\beta}_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} \tag{85}$$

[9] I.e., if x, y are distinct points in $\mathbb{R}^3$, the *straight line* $\ell(x,y)$ through x and y can be taken as the range of the "nicely parametrized" curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, where: $\gamma(\lambda) = x + \lambda(y-x)$.

So, let ∇ be the connection with, for standard coordinate system Φ, vanishing connection coefficients:

$$\Gamma^{\beta}_{\mu\nu} = 0 \tag{86}$$

So:

$$\gamma \in \texttt{Geod}(\nabla) \quad \leftrightarrow \quad \bigwedge_{\beta=1}^{3} (\forall \lambda \in \mathbb{R})\, \ddot{\gamma}^{\beta}(\lambda) = 0 \tag{87}$$

Therefore, by combining equation (83) and equation (87), we obtain: for any $\gamma \in N$:

$$\texttt{ran}\, \gamma \in \mathcal{L}_{B,\mathbb{P}} \quad \leftrightarrow \quad \gamma \in \texttt{Geod}(\nabla) \tag{88}$$

This (vanishing) connection ∇ is torsion-free. So we have proved that there exists a torsion-free connection ∇ on M such that

$$\mathcal{L}_{\mathbb{P},B} = \{\texttt{ran}\, \gamma \mid \gamma \in \texttt{Geod}(\nabla)\}. \tag{89}$$

So, we have proved $\Psi_2(M, \nabla, \mathbb{P}, B)$.

(Part III). Next, we wish to define a metric tensor g, such that the condition for equal length of segments pq and rs, relative to g (i.e., the length $L_g(p, q)$ of the geodesic (straight line) from p to q, determined by g, is equal to the length $L_g(r, s)$ of the geodesic (straight line) from r to s, determined by g) corresponds exactly to the physical condition $pq \equiv rs$.

Supposing g is a metric tensor on M, we define $L_g(p, q)$ as follows. Given any pair p, q of distinct points, there exists exactly one straight line $\ell(p, q) \in \mathcal{L}_{\mathbb{P},B}$. Consequently, up to reparametrization, there is exactly one smooth curve $\gamma : [0, 1] \to \mathbb{P}$, with $\gamma(0) = p$ and $\gamma(1) = q$, which is a *geodesic* from p to q. Then:

$$L_g(p, q) \quad := \quad \int_0^1 d\lambda \sqrt{g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}} \tag{90}$$

One may check that this quantity $L_g(p, q)$ is coordinate invariant and parametrization invariant. So, to calculate $L_g(p, q)$ we choose nice coordinates (any Cartesian chart $\Phi \in \mathcal{C}_0$). If $p = q$, the integral is 0. So, assume $p \neq q$. Up to reparametrization, there is a unique geodesic γ from p to q. Assuming $p \neq q$, we can choose a nice parametrization for γ, namely:

$$\gamma^{\mu}(\lambda) = \Phi^{\mu}(\gamma(\lambda)) = \Phi^{\mu}(p) + \lambda(\Phi^{\mu}(q) - \Phi^{\mu}(p)) \tag{91}$$

This implies:

$$\dot{\gamma}^{\mu}(\lambda) = \Phi^{\mu}(q) - \Phi^{\mu}(p) \tag{92}$$

Hence (Einstein summation convention in place),

$$g_{\mu\nu}\dot{\gamma}\mu\dot{\gamma}\nu = g_{\mu\nu}(\Phi^{\mu}(q) - \Phi^{\mu}(p))(\Phi^{\nu}(q) - \Phi^{\nu}(p)) \tag{93}$$

Now let g be defined, in coordinates Φ, by:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{94}$$

In particular, since $g_{\mu\nu} = diag(1,1,1)$, we can calculate the quantity $L_g(p,q)$:

$$\int_0^1 d\lambda \sqrt{g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}} = \left(\sqrt{\sum_{\mu=1}^{3}(\Phi^{\mu}(q) - \Phi^{\mu}(p))^2}\right)\int_0^1 d\lambda \tag{95}$$

$$= \Delta_{\Phi}(p,q) \tag{96}$$

In the second line, we have used the definition (73) of Δ_{Φ}. And so,

$$L_g(p,q) = L_g(r,s) \quad \Leftrightarrow \quad \Delta_{\Phi}(p,q) = \Delta_{\Phi}(r,s) \tag{97}$$

$$\Leftrightarrow \quad pq \equiv rs \tag{98}$$

Where, in the second line, we used (74). Summarizing where we are, we have now verified:

$$\mathcal{L}_{\mathbb{P},B} = \{\texttt{ran}\,\gamma \mid \gamma \in \texttt{Geod}(\nabla)\} \tag{99}$$

$$(\forall p,q,r,s \in \mathbb{P})\,(pq \equiv rs \leftrightarrow L_g(p,q) = L_g(r,s)) \tag{100}$$

That is, we have verified $\Psi_2(M,\nabla,\mathbb{P},B)$ and $\Psi_3(M,g,\mathbb{P},\equiv)$. Hence, since we have verified $\Psi_1(\mathcal{C}_0,\mathcal{C},M,\mathbb{P},B,\equiv)$, we have thereby verified the representation condition:

$$\texttt{Rep}(M,g,\mathbb{P},B,\equiv). \tag{101}$$

(Part IV). It remains to prove $\texttt{EuclSpace}(M,g)$. First, M is indeed diffeomorphic to $\mathbb{R}^3$. So, we need to show that g (on M) is isometric to g^{euc} (on $\mathbb{R}^3$). I.e., we need a diffeomorphism $\phi : M \to \mathbb{R}^3$ such that:

$$g = \phi^* g^{euc} \tag{102}$$

holds. ($\phi^* g^{euc}$ is pullback of g^{euc} under ϕ.) But this is immediate from the definition (94). For the chart Φ is a diffeomorphism from M to $\mathbb{R}^3$ such that $g = \Phi^* g^{euc}$. So, we have $\texttt{EuclSpace}(M, g)$ and $\texttt{Rep}(M, g, \mathbb{P}, B, \equiv)$. And this is (6). □

It's perhaps worth adding that the metric tensor g we defined is *not unique*. For any $\alpha > 0$, the metric tensor αg does the trick too. This has components $diag(\alpha, \alpha, \alpha)$, with respect to the Cartesian chart Φ we keep fixed throughout. But so what? The metric αg nonetheless yields *precisely the same congruence relation* on the physical point set $\mathbb{P}$: namely, $\equiv$. That is, the physical betweenness and congruence relations do not determine a *unique metric tensor*, but rather a class of metric tensors, related by a non-zero positive dilation coefficient, α.

6 Equivalent Axiomatizations

In summary, Theorem 5.1 gives *six mathematically equivalent first-order axiomatizations* of the claim that ordinary physical space is Euclidean, with respect to physical betweenness and physical congruence. Recall that our *physical* signature is:

$$\sigma_{phys} = \{\texttt{point}, \mathbf{B}, \equiv\} \tag{103}$$

And our *mathematicized* signature is:

$$\sigma = \{\texttt{point}, \mathbf{B}, \equiv, \in\} \tag{104}$$

The overall (first-order) language is $L(\sigma)$. And the equivalent axiomatizations, in $L(\sigma)$, are:

Equivalent axiomatizations

(1) Synthetic	$\mathsf{EG}(3)$
(2) Semantical	$(\mathbb{P}, B, \equiv) \models_2 \ulcorner \mathsf{EG}(3) \urcorner$
(3) Representational	$(\exists \Phi : \mathbb{P} \to \mathbb{R}^3)\, [(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})]$
(4) Metrical	$(\exists \delta : \mathbb{P}^2 \to \mathbb{R}^+_0)\, [(\mathbb{P}, \delta) \cong (\mathbb{R}^3, \Delta_{\mathbb{R}^3}) \wedge \texttt{Def}_1(\mathbb{P}, \mathbf{B}, \delta) \wedge \texttt{Def}_2(\mathbb{P}, \equiv, \delta)]$
(5) Kleinian	$\exists \mathcal{C}\, [\texttt{Klein}_{E(3)}(\mathbb{P}, \mathcal{C}) \wedge \texttt{Rep}(B, B_{\mathbb{R}^3}, \mathcal{C}) \wedge \texttt{Rep}(\equiv, \equiv_{\mathbb{R}^3}, \mathcal{C})]$
(6) Diff geometrical	$\exists M\, \exists g\, [\texttt{EuclSpace}(M, g) \wedge \texttt{Rep}(M, g, \mathbb{P}, B, \equiv)]$

These are *physical axioms*, which are *inter-derivable in an ambient first-order set theory*. This is perfectly analogous to the inter-derivability of Choice and its equivalents. Except that these are *physical axioms*.

7 Definitional Equivalents

In the above results, the physical signature remains fixed, and we proved (1) $\Leftrightarrow$ (2), and so on, modulo the base AM^*. What if we change the physical signature?

Let us now modify the *physical* signatures as follows:

Definition 7.1. We define two new physical signatures:

$$\sigma'_{phys} := \{\texttt{point}, \mathbf{B}, \equiv, \mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\} \quad (105)$$
$$\sigma''_{phys} := \{\texttt{point}, \varphi_1, \varphi_2, \varphi_3\} \quad (106)$$

In the first, we have the three symbols, $\texttt{point}, \mathbf{B}, \equiv$ of σ_{phys}, and four new symbols: we declare that $\mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are constants, of the sort $\texttt{atom}$ (and indeed denote points). In the second, we keep $\texttt{point}$, but drop $\mathbf{B}$ and $\equiv$, and add three new symbols, φ_i. We shall declare that each symbol φ_i is *function symbol*, of the sort $\texttt{atom} \Rightarrow \texttt{class}$ (and indeed denote functions from points to reals).

Definition 7.2. In the language $L(\sigma')$, we define a four-place formula:

$$\begin{aligned}\text{Eucl-3-frame}(p,q,r,s) := {} & p,q,r,s \in \mathbb{P} \wedge p \neq q \wedge p \neq r \wedge p \neq s \wedge q \neq r \wedge q \neq s \wedge r \neq s \\ & \wedge (pq \equiv pr \wedge pq \perp pr) \wedge (pq \equiv ps \wedge pq \perp ps) \wedge (pr \equiv ps \wedge pr \perp ps) \quad (107)\end{aligned}$$

This expresses that we have four distinct points p, q, r, s, forming an orthogonal frame, with p at the centre, and each "leg" pq, pr and ps having the same length. In Burgess & Rosen (1997), John Burgess calls this a system of "benchmarks". The detailed proof of the Representation Theorem for $\mathsf{EG}(3)$ in fact shows that, given any such Euclidean 3-frame, p, q, r, s, we have a unique Cartesian chart Φ meeting the conditions that $\Phi(p) = (0,0,0), \Phi(q) = (1,0,0), \Phi(r) = (0,1,0)$ and $\Phi(s) = (0,0,1)$.

Definition 7.3. Let $\mathsf{EG}^+(3)$ be the theory in $L(\sigma')$ whose axioms are:[10]

$$\mathsf{AM}^*(\sigma') + \mathsf{EG}(3) + \text{Eucl-3-frame}(\mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \quad (108)$$

Definition 7.4. In $L(\sigma'')$, the formula $\texttt{Bij}(\varphi_i)$ is defined as follows:

$$\texttt{Bij}(\varphi_i) := (\forall p, q \in \mathbb{P})\,(\varphi_i(p) = \varphi_i(q) \to p = q) \wedge (\forall x \in \mathbb{R})\,(\exists p \in \mathbb{P})\,(x = \varphi_i(p)) \quad (109)$$

Definition 7.5. The theory $\mathsf{ACG}(3)$ of analytic coordinate geometry in three dimensions is the theory in $L(\sigma'')$ whose axioms are:

$$\mathsf{AM}^*(\sigma'') + \bigwedge_i \texttt{Bij}(\varphi_i) \quad (110)$$

In other words, each axiom $\texttt{Bij}(\varphi_i)$ just states that the corresponding coordinate function denoted by φ_i is a bijection.

[10] $\mathsf{AM}^*(\sigma')$ is the base theory, now in the language $L(\sigma')$.

We can then prove the following definitional equivalence:

Theorem 7.1. $\mathsf{EG}^+(3)$ is definitionally equivalent to $\mathsf{ACG}(3)$.

Proof. Working in analytic coordinate geometry $\mathsf{ACG}(3)$, we have that each φ_i is a bijection from $\mathbb{P}$ to $\mathbb{R}$. Let us define a function $\Phi : \mathbb{P} \to \mathbb{R}^3$ by:

$$\Phi(p) := (\varphi_1(p), \varphi_2(p), \varphi_3(p)) \tag{111}$$

Since each φ_i is a bijection, it follows that $\Phi : \mathbb{P} \to \mathbb{R}^3$ is a bijection.

Next we add the six explicit definitions:

(D1)	$p = \mathbf{O}$	$:=$	$\Phi(p) = (0,0,0)$
(D2)	$p = \mathbf{X}$	$:=$	$\Phi(p) = (1,0,0)$
(D3)	$p = \mathbf{Y}$	$:=$	$\Phi(p) = (0,1,0)$
(D4)	$p = \mathbf{Z}$	$:=$	$\Phi(p) = (0,0,1)$
(D5)	$\mathbf{B}(p,q,r)$	$:=$	$B_{\mathbb{R}^3}(\Phi(p), \Phi(q), \Phi(r))$
(D6)	$pq \equiv rs$	$:=$	$\Phi(p)\Phi(q) \equiv_{\mathbb{R}^3} \Phi(r)\Phi(s)$.

Let $\mathsf{Df1}$ be the conjunction of these six definitions. Let $B = \{(p,q,r) \in \mathbb{P}^3 \mid \mathbf{B}(p,q,r)\}$, and let $\equiv = \{(p,q,r,s) \in \mathbb{P}^4 \mid pq \equiv rs\}$. So, B is now a three-place relation on $\mathbb{P}$ and $\equiv$ is a four-place relation on $\mathbb{P}$. The fact that $\Phi : \mathbb{P} \to \mathbb{R}^3$, along with (D5) and (D6), implies that Φ is an isomorphism from $(\mathbb{P}, B, \equiv)$ to $(\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. Hence we have an equivalent of $\mathsf{EG}(3)$, via Theorem 5.1, (3). So, from these definitions (working in $\mathsf{ACG}(3)$), we can prove the synthetic Euclidean axioms $\mathsf{EG}(3)$. What is more, we have $\Phi(\mathbf{O}) = (0,0,0)$, $\Phi(\mathbf{X}) = (1,0,0)$, $\Phi(\mathbf{Y}) = (0,1,0)$ and $\Phi(\mathbf{Z}) = (0,0,1)$. From these, we obtain $\mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{P}$. We can easily prove that the segment $(0,0,0)$ to $(1,0,0)$ has the same length as the segment $(0,0,0)$ to $(0,1,0)$. This yields $\mathbf{OX} \equiv \mathbf{OY}$. And we can easily prove that the segment $(0,0,0)$ to $(1,0,0)$ is perpendicular to the segment $(0,0,0)$ to $(0,1,0)$. This yields $\mathbf{OX} \perp \mathbf{OY}$. In this way, we prove the frame axiom Eucl-3-frame$(\mathbf{O}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$, as desired.

Conversely, as we know, working in $\mathsf{EG}(3)$, we can prove item (3) in Theorem 5.1: there exists $\Phi : \mathbb{P} \to \mathbb{R}^3$ such that $(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^3, B_{\mathbb{R}^3}, \equiv_{\mathbb{R}^3})$. Examining the details of this proof, we can prove that such a Φ exists such that $\Phi(\mathbf{O}) = (0,0,0)$ and $\Phi(\mathbf{X}) = (1,0,0)$ and $\Phi(\mathbf{Y}) = (0,1,0)$ and $\Phi(\mathbf{Z}) = (0,0,1)$. Let us add the three definitions:

(D1)	$\varphi_1(p)$	$:=$	$\Phi^1(p)$
(D2)	$\varphi_2(p)$	$:=$	$\Phi^2(p)$
(D3)	$\varphi_3(p)$	$:=$	$\Phi^3(p)$

Let $\mathsf{Df2}$ be the conjunction of these three definitions. From these definitions (working in $\mathsf{EG}^+(3)$), we can prove that each φ_i is a bijection: $\mathtt{Bij}(\varphi_1)$, $\mathtt{Bij}(\varphi_2)$ and $\mathtt{Bij}(\varphi_3)$. So, we can prove the axioms of analytic coordinate geometry $\mathsf{ACG}(3)$.

We have therefore shown that the two theories, suitably extended, are logically equivalent:

$$\mathsf{EG}^+(3) + \mathsf{Df2} \quad \dashv\vdash \quad \mathsf{ACG}(3) + \mathsf{Df1} \tag{112}$$

So, $\mathsf{EG}^+(3)$ and $\mathsf{ACG}(3)$ have a common definitional extension. Hence, they are definitionally equivalent.

□

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