Cluster Decomposition and Two Senses of Isolability

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In the framework of quantum field theory, one finds multiple load-bearing locality and causality conditions. One of the most important is the cluster decomposition principle, which requires that scattering experiments conducted at large spatial separation have statistically independent results. The principle grounds a number of features of quantum field theory, especially the structure of scattering theory. However, the statistical independence required by cluster decomposition is in tension with the long-range correlations characteristic of entangled states. In this paper, we argue that cluster decomposition is best stated as a condition on the dynamics of a quantum field theory, not directly as a statistical independence condition. This redefinition avoids the tension with entanglement while better capturing the physical significance of cluster decomposition and the role it plays in the structure of quantum field theory.

1. Introduction. The cluster decomposition principle is intended to secure the unremarkable fact that the long-run statistics of experimental outcomes in our laboratories do not depend on events in distant regions of the universe; what happens in the accelerator tunnel at Fermilab is independent of experiments simultaneously taking place at CERN.[1] This seemingly innocuous constraint turns out to play a deep and wide-ranging role in the architecture of quantum field theory. Indeed, one very influential approach to the foundations of quantum field theory has it that "the whole formalism of fields, particles, and antiparticles seems to be an inevitable consequence of Lorentz invariance, quantum mechanics, and cluster decomposition" (Weinberg, 1999, p. 244).[2] Furthermore, its conceptual implications sit at the intersection of a number of topics in quantum field theory that have drawn the attention of philosophers: the analysis of locality and causality conditions,[3] the localizability of states,[4] and the nature of entanglement in quantum field theory,[5] among other topics.

One typically encounters cluster decomposition formulated as a constraint on two distinct mathematical objects: vacuum expectation values (VEVs)

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[1]The cluster decompostion principle is also sometimes called "the cluster property" in the literature. We use these terms interchangeably.

[2]See (Weinberg, 1995) and (Duncan, 2012) for textbook presentations of quantum field theory structured according to this perspective.

[3]See Summers (1990); Rédei and Summers (2002); Butterfield (2007); Summers (2009); Ruetsche (2011, chapter 5.3); Earman and Valente (2014)

[4]See Fleming and Butterfield (1999); Halvorson (2001); Wallace (2006); Wallace and Timpson (2010); Swanson (2020)

[5]See Redhead (1995); Clifton and Halvorson (2001); Valente (2013); Lam (2013); Earman (2015); Jaksland (2020)

of operators or S-matrix elements.[6] In each case, it is stated as a statistical independence condition.[7] As a condition on VEVs, cluster decomposition requires that if $\{x_i\}$ and $\{y_i\}$ are sets of spacetime points, and all of the points $\{x_i\}$ are translated an arbitrarily large spacelike distance away from all of the points $\{y_i\}$, then the VEV

$$\langle \Omega \,|\, \varphi(x_1) \cdots \varphi(x_n)\, \varphi(y_1) \cdots \varphi(y_m) \,|\, \Omega \rangle$$

factorizes into the product

$$\langle \Omega \,|\, \varphi(x_1) \cdots \varphi(x_n) \,|\, \Omega \rangle \, \langle \Omega \,|\, \varphi(y_1) \cdots \varphi(y_m) \,|\, \Omega \rangle$$

This requires that in the vacuum state $|\Omega\rangle$ of the theory, the outcomes of measurements performed on the state of the field at the points $\{x_i\}$ become uncorrelated with outcomes of measurements performed on the state of the field at the points $\{y_i\}$ as the two sets of spacetime points become separated by an arbitrarily large spacelike distance.[8]

As a condition on S-matrix elements, cluster decomposition requires that two scattering processes $\alpha \to \alpha'$ and $\beta \to \beta'$ be statistically independent when one is translated an arbitrarily large spacelike distance from the other. This translates into the requirement that the scattering amplitude for the total process

$$\langle \alpha', \beta' \,|\, S \,|\, \alpha, \beta \rangle$$

factorizes into the product

$$\langle \alpha' \,|\, S \,|\, \alpha \rangle \, \langle \beta' \,|\, S \,|\, \beta' \rangle$$

as the "cluster" of particles β is translated an arbitrarily large spacelike distance away from the "cluster" of particles α.

Understanding cluster decomposition as a statistical independence condition in one of the forms just stated runs into complications. Most pressingly, it makes the property state-dependent. VEVs and S-matrix elements are functions of both operators *and* states, and whether the statistical independence is satisfied will depend on the nature of the vacuum state in which one calculates VEVs or which basis of in and out states one uses to define the S-matrix. As we discuss in Section 5, scattering processes involving entangled states will not factorize at large spacelike separation, which entails that an S-matrix defined

[6] It is also sometimes expressed as a condition on algebras of observables. We discuss this version of the principle in Section 4.

[7] Our goal at this stage is just to capture the basic idea of these conditions. See Section 4 for more precise formulations of the principle.

[8] Our reference to the state of the field at a spacetime point, or in a bounded spacetime region, should be read loosely. Defining strictly localized states is impossible in quantum field theory, as a consequence of the Reeh-Schlieder theorem. The extent to which cluster decomposition does, and does not, secure a notion of localized states is an issue to which we return several times.

using a basis of entangled states will not satisfy cluster decomposition. This state-dependence sits uncomfortably with the foundational role that cluster decomposition is supposed to play in the architecture of quantum field theory and, we argue, obscures its real physical significance. A primary aim of this paper is to offer an understanding of cluster decomposition that is better aligned with its physical significance and the role that it plays in quantum field theory.

The paper proceeds as follows. In Section 2, we sketch the historical development of cluster decomposition and its integration into quantum field theory. This history is interesting in its own right, reflecting the combined efforts of different communities of physicists in the 1950s and 60s. It also provides a useful introduction to some of the different formal and conceptual roles that cluster decomposition plays in quantum field theory. In Section 3, we survey a variety of reasons that have been offered for requiring quantum field theories to satisfy cluster decomposition. Our goal is to extract a common theme: we argue that each of them treats cluster decomposition as grounding an ability to isolate subsystems, in one form or another. In Section 4, we review how cluster decomposition constrains three types of mathematical objects – VEVs, scattering amplitudes, and algebras of observables. In Section 5, we consider two phenomena that produce states that prevent VEVs and S-matrix elements from factorizing in accord with cluster decomposition: spontaneous symmetry breaking and entanglement. We consider them side-by-side because while the state-dependence of VEVs arising from spontaneous symmetry breaking is innocuous and can be addressed straightforwardly, it is illuminating to see that the state-dependence arising from entangled states cannot be resolved in the same way. This discussion naturally raises the question of how the cluster decomposition of VEVs is compatible with the fact that vacuum states in quantum field theory are entangled over arbitrarily long distances. After answering this question, in Section 6 we argue that stating cluster decomposition as a statistical independence condition conflates two senses in which one could isolate a subsystem, and that this conflation is responsible for the incompatibility with entanglement. To resolve this, one should redefine cluster decomposition as a condition purely on the dynamics of a quantum field theory. This restatement of cluster decomposition still ensures that subsystems are isolable in the sense necessary for it to play the various physical roles described in Section 3, but does not require the second, unnecessary sense of isolability that leads to the tension with entangled states.

2. The Origins of Cluster Decomposition. A version of the cluster property was first introduced into quantum field theory in 1957, when the factorization of vacuum expectation values at large spacelike separation was discussed by both Rudolf Haag and Arthur Wightman at a well-attended conference on the mathematical properties of quantum field theory in Lille (Deheuvels and Michel, 1959).[9] While Wightman's discussion of this factorization was brief and focused primarily on its relationship to the mathematical question of whether VEVs are tempered distributions in the spacetime coordinates, Haag attributed it a more central physical significance.

It was at Lille that Haag first introduced the idea of formulating quantum field theory using algebras of local observables, and a version of cluster decomposition was introduced as one of the axioms of the new framework. A few years prior to the conference in Lille, a powerful formalism for scattering theory had been introduced by Harry Lehmann, Kurt Symanzik, and Wolfhart Zimmermann (Lehmann et al., 1955). The LSZ formalism required assuming an 'asymptotic condition' which stated, roughly speaking, that in the asymptotic past and the asymptotic future, the field operators of an interacting quantum field theory behave as if the theory were non-interacting. The influence of the LSZ formalism is reflected in both Haag and Wightman's talks at Lille. Wightman proposed a preliminary set of axioms that should be satisfied by any mathematically rigorous quantum field theory, and included the asymptotic condition as one such axiom. Haag, however, introduced cluster decomposition as an axiom with the primary aim of supplanting the asymptotic condition with a more physically transparent condition "believed to be more fundamental" from which the asymptotic condition could then be derived as a consequence (Haag, 2010). According to Klaus Fredenhagen (2010), the replacement of the asymptotic condition with an assumption that VEVs satisfied the cluster decomposition property was "the revolutionary new idea" at the heart of Haag's proposal to formulate quantum field theory in terms of algebras of local observables.

After the conference in Lille, cluster decomposition was quickly integrated into mathematically rigorous work on quantum field theory and a number of interesting connections were discovered.[10] Haag (1958) began the development of a rigorous framework for scattering theory that incorporated cluster decomposition as an axiom and several years later David Ruelle (1962) strengthened the framework considerably, producing the Haag-Ruelle formalism for scattering theory. Ruelle was able to build on several papers which showed that despite Haag's adoption of cluster decomposition as an axiom of quantum field theory, it could actually be derived as a theorem from other, weaker assumptions like Lorentz invariance and the non-negativity of the energy spectrum (Araki, 1960; Jost and Hepp, 1962). During the same period, results obtained by Hepp et al. (1961) and Borchers (1962) jointly demonstrated that the VEVs of a quantum field theory satisfy cluster decomposition if and only

[9]An English translation of Haag's talk was published as (Haag, 2010) and a list of attendees at the Lille conference is reproduced in (Fredenhagen, 2010).

[10]It seems the term "cluster decomposition" was introduced by Araki (1960), though widespread adoption doesn't seem to have been immediate. Ruelle (1962) refers to the factorization of VEVs at large spacelike separation as a "spacelike asymptotic condition" and Araki et al. (1962) refer to "clusters" but do not use the term "cluster decomposition".

if the vacuum state of the theory is unique.[11]

At the same time that cluster decomposition of VEVs was being explored in axiomatic quantum field theory, cluster decomposition of the S-matrix was independently introduced into the burgeoning program of S-Matrix Theory.[12] Eyvind Wichmann and James Crichton argued that cluster decomposition was so physically sensible that it should be satisfied by *any* physically meaningful S-matrix and, in the spirit of S-Matrix Theory, formulated conditions ensuring the cluster decomposition of an S-matrix that were not tied to the mathematical formalism of quantum field theory (Wichmann and Crichton, 1963).[13] Although Wichmann and Crichton were aware of the related work taking place in axiomatic quantum field theory, they reported being unsure of how it might relate to their discussion of the cluster decomposition of the S-matrix (Wichmann and Crichton, 1963, fn. 16).

It seems to be primarily from its formulation in S-matrix theory that the cluster decomposition property made its way into the mainstream of perturbative quantum field theory. Of particular significance in this regard was a paper of Steven Weinberg (1979). He presented an argument that "although individual quantum field theories have of course a good deal of content, quantum field theory itself has no content beyond analyticity, unitarity, cluster decomposition, and symmetry" (Weinberg, 1979, p. 329). The inclusion of cluster decomposition as a core principle of quantum field theory was not common among mainstream high energy theorists at the time, but during Weinberg's time at Berkeley in the 1960s his colleague Wichmann had emphasized its importance to him and the lesson had stuck (Weinberg, 2021). Largely as a result of Weinberg structuring his enormously influential textbook (Weinberg, 1995) around an elaboration of this 1979 argument – that the structure of quantum field theory follows naturally (if not unavoidably) from combining quantum mechanics, Poincaré invariance, and cluster decomposition – over the intervening years mainstream high energy theory has come to recognize cluster decomposition as a foundational component in the architecture of quantum field theory.

3. Motivating Cluster Decomposition. The reasons for requiring cluster decomposition that one finds in the physics literature range from treating it as reflecting a contingent fact about the range of forces in our universe to declaring it a precondition for the possibility of experimental science. In this section identify two common themes that emerge from these apparently

[11] A useful presentation of this early mathematically rigorous work on cluster decomposition can be found in (Wightman, 1963).

[12] Also at this time a related property, *asymptotic abelianness*, was introduced into quantum statistical mechanics and was quickly recognized to have connections with the cluster decomposition of VEVs (Doplicher et al., 1966; Ruelle, 1966; Kastler and Robinson, 1966). We discuss this property briefly in Section 4.

[13] See (Taylor, 1966) for some important dotting of i's and crossing of t's omitted by Wichmann and Crichton. Cluster decomposition was ultimately included as an "axiom" of S-Matrix Theory (Eden et al., 1966, p. 12 & chapter 4.2).

disparate motivations. We return to these themes in Section 6 to argue that defining cluster decomposition as a purely dynamical constraint captures the role it plays in the structure of quantum field theory better than defining it as a statistical independence condition.

3.1. Precondition for Physical Theorizing. The most striking justification for imposing cluster decomposition is a transcendental one: it is sometimes claimed to be a precondition for the possibility of experimental science. For instance, Weinberg motivates requiring the cluster decomposition of the S-matrix this way:

> It is one of the fundamental principles of physics (indeed, of all science) that experiments that are sufficiently separated in space have unrelated results.... If this principle were not valid, then we could never make any predictions about any experiment without knowing everything about the universe (Weinberg, 1995, p. 177).

While emphasizing the central role that cluster decomposition plays in the structure of quantum field theory, Tony Duncan echoes this motivation:

> The characteristic phenomena of relativistic field theory only appear once we insist on the third principle: clustering, i.e., the factorization of the S-matrix containing the scattering amplitude for an arbitrary process as the product of two independent amplitudes in the event of two spatially far separated scattering subprocesses. This principle, which seems intuitively obvious, is surely a precondition for the success of experimental science. It relieves us of the obligation to specify completely the state of the entire world outside the laboratory prior to a correct interpretation of the results of an experiment (Duncan, 2012, p. 58).[14]

In the same spirit, many years earlier one finds Hartle and Taylor (1969, §III) saying that cluster decomposition "may be regarded as an essential requirement of any reasonable physical theory" and Steinmann (1966, p. 757) claiming that "[w]ithout an assumption of this type physics is clearly impossible."

In each of these presentations, they are describing cluster decomposition as a condition on an S-matrix, i.e., as the requirement that the S-matrix element

$$\langle \alpha', \beta' \,|\, S \,|\, \alpha, \beta \rangle$$

factorizes into the product

$$\langle \alpha' \,|\, S \,|\, \alpha \rangle \, \langle \beta' \,|\, S \,|\, \beta' \rangle$$

[14]The first two principles to which Duncan implicitly refers are quantum mechanics and Poincaré invariance.

when the scattering process $\beta \to \beta'$ is translated an arbitrarily large spacelike distance away from the $\alpha \to \alpha'$ process. As noted above, an S-matrix element is determined by two things: the scattering operator S and the initial and final states of the scattering process. Despite this, both Weinberg and Duncan, like their predecessors in S-matrix Theory, treat cluster decomposition as a state-independent property. They argue that imposing certain requirements on the Hamiltonian alone is sufficient to ensure the factorization of the S-matrix.[15]

How should one understand the claim that cluster decomposition is a precondition for the possibility of experimental science? We think it reflects the idea that experimental science requires the ability to identify effectively isolated subsystems and model their behavior independently from their broader environment. If it were not possible to isolate systems in this way – if one could not isolate a sample of graphene in their laboratory, make predictions for its behavior, and experimentally intervene on it without worrying about the state of every other laboratory on campus – then experimental science would become exponentially more difficult, if not impossible. In the next subsection, we will see that a similar reason is often given for requiring the cluster decomposition of VEVs.

3.2. Localization. Another reason for adopting cluster decomposition one often encounters is that the cluster decomposition of VEVs enables us to assign approximately localized states to a quantum field in distinct bounded spacetime regions, as long as those spacetime regions are sufficiently spacelike separated. This understanding of the physical import of cluster decomposition was essential in Haag's initial statement of the principle in Lille (Haag, 2010). The physical idea is described clearly by Huzihiro Araki in one of the earliest statements of cluster decomposition:[16]

> The physical idea behind this assumption is the following. Consider a simple case where $x_1 \cdots x_p$ are concentrated in a finite region A, $x_{p+1} \cdots x_n$ are concentrated in a finite region B, and the distance between A and B tends to infinity. We imagine that the change, caused by $\varphi(x)$ on the state on which $\varphi(x)$ operates is concentrated "near the point x" and therefore that the state $\varphi(x_1) \cdots \varphi(x_p)\Psi_0$ is only slightly different from the vacuum except "near the region A" while the state $\varphi(x_{p+1}) \cdots \varphi(x_n)\Psi_0$ is only slightly different from the vacuum except "near the region B." Hence as the distance between A and B tends to infinity in the vacuum expectation value $(\Psi_0, \varphi(x_1) \cdots \varphi(x_n)\Psi_0)$, $\varphi(x_1) \cdots \varphi(x_p)$ sees an approximate vacuum on its right while $\varphi(x_{p+1}) \cdots \varphi(x_n)$ sees an approximate

[15] We discuss these conditions in detail in Section 4.

[16] Ruelle also took this to be its physical significance in his seminal paper establishing Haag-Ruelle scattering theory: "The physical meaning of the [cluster decomposition] theorem becomes clear. . . the state $\Phi(x)$ is asymptotically localizable in the sense of Knight" (Ruelle, 1962, p. 155).

> vacuum on the left and therefore, $(\Psi_0, \varphi(x_1) \cdots \varphi(x_n)\Psi_0)$ tends to $(\Psi_0, \varphi(x_1) \cdots \varphi(x_p)\Psi_0)(\Psi_0, \varphi(x_{p+1}) \cdots \varphi(x_n)\Psi_0)$. From this one can conclude (by induction) that [the connected correlation function] tends to zero as the distance between A and B tends to infinity. Repeating this kind of heuristic argument, we arrive at a cluster decomposition property (Araki, 1960, 261).[17]

As Araki describes, cluster decomposition allows us to define what Haag (1996, §II.5.3) calls "qualitatively" or "essentially" localized states: states of a quantum field that differ significantly from the vacuum state only within some bounded spacetime region O. This is notable because it is a result of some foundational significance that, as a consequence of the Reeh-Schlieder theorem, one cannot define "strictly" localized states in quantum field theory, i.e., states of a quantum field that differ from the vacuum *at all* only within some bounded spacetime region O. To the extent that the ability to decompose the world into independent subsystems upon which we can perform interventions is a precondition for the possibility of experimental science, one might have worried that the inability to define strictly localized states threatens the scientific enterprise. However, the cluster decomposition of VEVs demonstrates how the inability to define "strictly" localized states can coexist with the ability to define the "essentially" localized states that seem necessary for doing experimental science.[18]

Although not formulated as a constraint on the S-matrix itself, the cluster decomposition of VEVs does have a deep connection to scattering theory.[19] Recall that Haag's initial motivation for introducing cluster decomposition was that it was a "more fundamental" principle from which the asymptotic condition of the LSZ formalism could be derived. The LSZ asymptotic condition, roughly speaking, states that in the asymptotic past and the asymptotic future the field operators in a fully interacting theory behave like free fields. Excited states of the field at asymptotic times can then be interpreted as describing multiple isolated, non-interacting particles. These are the initial and final states that one typically uses to define the S-matrix. In fact, it is only because the cluster property allows the definition of "essentially" or "qualitatively" localized states of a quantum field, in the sense described by Araki, that one can interpret these states of the field as describing multiple non-interacting particles.

3.3. Range of Forces. Another commonly invoked reason to believe that a quantum field theory should satisfy cluster decomposition concerns the "short

[17]Strocchi gives a similar diagnosis of its physical significance: "The physical meaning of the cluster property is that the ground state reacts locally to local operations...In a certain sense, this condition neutralizes the non-local content of the ground state..." (Strocchi, 2008, p. 99).

[18]We consider some further subtleties concerning cluster decomposition and the localizability of states in Section 5.

[19]We give a more formal discussion of this connection in Section 4.

range" of interparticle forces. If there are no correlations between particles in the initial state then all correlations between particles in the final state must be due to mutual interaction during the scattering process. If the interaction decays sufficiently quickly with distance then there won't be any interaction between clusters that have been translated a large spacelike distance from one another, so in the final state there will not be any correlations between particles in distinct clusters. This ensures that the two scattering processes will be statistically independent. Cluster decomposition simply encodes this observed, contingent fact about the world into the structure of quantum field theory.

In fact, this reasoning played a central role in Wichmann and Crichton's original introduction of cluster decomposition into S-Matrix Theory:

> The conditions we wish to impose derive from the idea that the interparticle interactions are of short range; therefore, the outcome of a scattering event involving two particles that are close to each other at some time does not depend on the presence of other particles very far away. To dramatize the situation we may say that the presence of particles on the moon must not affect the outcome of events in a bubble chamber on the earth (Wichmann and Crichton, 1963, p. 2788).

Following Wichmann and Crichton, the connection of cluster decomposition with short-range forces was common in the S-Matrix Theory tradition; for examples, see Eden et al. (1966, pp. 190–91) or Taylor (1966).

We should clarify the meaning of "short-range" forces. It typically refers to interactions that are mediated by a massive particle, which entails that the range of the force decays exponentially with distance. Interactions mediated by a massless particle – as in quantum electrodynamics, for example – will decay polynomially with distance. Cluster decomposition can still be satisfied in such theories, but the conclusions one can draw on the basis of its satisfaction are more complicated. For instance, to return to the connection between the cluster decomposition of VEVs and the existence of asymptotic states of multiple isolated, non-interacting particles, in quantum field theories with interactions mediated by massless particles the former may not be sufficient to secure the existence of the latter. As a result one may not be able to define an S-matrix for the theory (Strocchi, 2013, §6.3; Duncan, 2012, chapter 19.1).

Indeed, one of the virtues of understanding cluster decomposition as encoding this contingent fact about the prevalance of short-range interactions into the structure of quantum field theory is that it provides a physically transparent explanation of the existence of initial and final scattering states describing multiple isolated, non-interacting particles. For example, Tom Banks explains cluster decomposition this way:

> It is an experimental fact that there exist (approximately) stable single-particle states in the world, as well as states of multiple

> particles at large relative space-like distances, which behave, to a very good approximation, like free particles. Interactions fall off at large spatial separation, and the *cluster property* of QFT provides a neat mathematical explanation of this.... If all particles are massive, the falloff is exponential, whereas if there is no mass gap we expect power-law falloff. In this case, a variety of behaviors is possible, and there is not always a scattering theory (Banks, 2008, chapter 3.7).[20]

In fact, Wichmann and Crichton already noted this more foundational role played by the assumption of short-range forces in the context of S-Matrix Theory:

> The basic assumption of S-matrix theory is that the interactions between the particles are, in some sense, of short range, and because of this property of the interaction it is possible to describe a state either in terms of an initial asymptotic configuration of noninteracting particles or in terms of a final asymptotic configuration of noninteracting particles. In the asymptotic limits, the particles behave like noninteracting particles simply because their mean separations tend to infinity and hence the interactions become ineffective (Wichmann and Crichton, 1963, p. 2788).

We previously saw that the clustering of VEVs was necessary to secure the existence of initial and final scattering states describing multiple isolated, non-interacting particles. We have now seen that interactions must fall off rapidly with distance to ensure the existence of such states. It is unsurprising, then, that many have taken the presence of short-ranged interactions to provide the physical underpinning for the cluster decomposition of VEVs. For example, after deriving a bound on the rate at which VEVs must factorize at spacelike separation, Araki says:

> It is possible to give a physical interpretation for such an exponential clustering property by Yukawa's theory. Namely, the correlation at a distance R is interpreted as being induced by an energy exchange (in Yukawa's theory, a particle exchange) and its effective radius is determined by (and hence the correlation tends to zero beyond the distance equal to) the reciprocal of the exchanged energy (Araki, 1999, p. 92).[21]

[20] Note that if one read Banks strictly here – we are not – his order of explanation would be that cluster decomposition somehow explains why interactions decay with distance, rather than vice-versa.

[21] The exponential clustering property is the statement that for any two operators A and B $\langle \Omega \,|\, A(0)\, B(R) \,|\, \Omega \rangle \rightarrow \langle \Omega \,|\, A(0) \,|\, \Omega \rangle \langle \Omega \,|\, B(R) \,|\, \Omega \rangle$ at a rate proportional to e^{-mR} where m is the mass of the lightest particle mediating their interactions.

Raymond Streater and Wightman (1964, p. 113) make a similar connection, describing the cluster decomposition of VEVs as a requirement that "when two systems at points x and y become separated by a large space-like distance, the interaction between them falls off to zero."

There are two things to note about this reasoning. The first is that it seems to raise a puzzle about the behavior of VEVs in *free* quantum field theories. If the decay of the correlations between operators at large spacelike separation is because of the short-range of interactions, the implication is that those correlations are present at all because of interactions between degrees of freedom in the two spatially distant regions. One might naturally expect, then, that in a free quantum field theory all correlation functions would be zero, i.e., that $\langle\Omega\,|\,\varphi(x)\varphi(y)\,|\,\Omega\rangle = 0$ for all distinct spacetime points x and y. But that expectation is wrong. Correlation functions in a free quantum field theory are non-zero and behave like VEVs in a field theory with interactions: they decay with distance, exponentially for a theory with a mass gap and polynomially for a theory of massless particles.[22]

The second is that justifying cluster decomposition by appeal to the short-range of (most) real-world interactions might strike the reader as a far cry from the transcendental arguments from Weinberg and Duncan with which we opened this section. It may now sound like the inclusion of cluster decomposition in quantum field theory merely reflects a contingent empirical fact about the real world: we just happen to live in a world where interactions fall off fairly rapidly with distance and our quantum field theories should reflect that. We think there is something to this attitude, but it is also true that one could not define even "essentially" localized states of a quantum field, nor define initial and final scattering states, nor treat scattering processes involving particles at large spatial separation as distinct, if interactions did not fall off sufficiently rapidly with distance. So insofar as there is a case to be made that the ability to decompose the world into effectively isolated subsystems upon which we can independently intervene is a precondition for the possibility of experimental science, there is equally a case to be made that interactions that decay sufficiently rapidly with distance are such a precondition as well.

3.4. No Superluminal Signaling. Yet another natural way to motivate cluster decomposition is as a "no-signaling" condition of the sort that have been considered at length in discussions of Bell-type theorems. For example, Duncan says that cluster decomposition must be satisfied to ensure that a quantum field theory will have "sensible long range behavior, purged of bizarre action-at-a-distance effects" and immediately clarifies that:

> By "action-at-a-distance" effects, we do *not* refer here to the psychologically unsettling effects involving non-local transitions in entangled wavefunctions, commonly referred to as the "EPR paradox", but to *physically observable* non-local phenomena: namely,

[22]We return to this issue in Section 5.

> those leading to superluminal transmission of physical signals (Duncan, 2012, chapter 3, fn. 5).

Returning later to the role of cluster decomposition in purging the theory of "action-at-a-distance" effects, Duncan reiterates that:

> Einstein's original use of the term "spukhafte Fernwirkung" referred to the peculiar (from the classical standpoint) statistical correlations of entangled quantum states, as in the famous EPR effect. These correlations, while perhaps psychologically disturbing, do not lead to the physically unacceptable action-at-a-distance effect of the kind discussed here (Duncan, 2012, chapter 5.4, fn. 14).

In both statements, Duncan is at pains to clarify that cluster decomposition *is not* incompatible with the long-range correlations characteristic of entangled systems. Cluster decomposition is supposed to be perfectly compatible with "outcome-outcome" dependence, to borrow some terminology familiar from discussions of Bell-type theorems. What cluster decomposition *is* supposed to prohibit are superluminal signals – what is sometimes called "act-outcome" dependence. It rules out quantum field theories in which interventions one can perform locally, such as injecting a beam of protons into the accelerator tunnel at CERN, superluminally affect the predicted cross sections of experiments at Fermilab.

Jonathan Bain explains the significance of cluster decomposition similarly, stating that:

> [Cluster decomposition] serves the same purpose for the S-matrix as micro-causality does for fields: both are locality constraints that prohibit causal influences from propagating between space-like separated regions of spacetime (Bain, 1998, pp. 7–8).[23]

Over decades of analyses of Bell-type theorems, much has been said about the motivation for prohibiting superluminal signals and whether they are compelling.[24] Our aim is not to contribute to those discussions. Rather, we want to emphasize a conceptual relationship between the understanding of cluster decomposition as a no-signalling condition and the understandings we have encountered previously.

One frequently invoked reason for prohibiting superluminal signals is an alleged incompatibility with relativistic spacetime structure. Whether one thinks this incompatibility is genuine depends on what one means by "compatibility with relativistic spacetime structure" and "superluminal signal." In the remarks by Duncan and Bain, the answers are straightforward: a

[23] See also Bain (2016, §2.1) – especially §2.1.3 – for additional discussion of the sense in which cluster decomposition can be understood as a no-signaling condition.

[24] See, for example, Maudlin (2011, chapter 4).

"superluminal signal" can be sent from spacetime region O to a spacelike separated region O' if an event confined to the region O can produce a change in the statistical distribution of measurement results in the region O'. And a quantum field theory is "compatible with relativistic spacetime structure" if its space of states transforms under an irreducible representation of the Poincaré group.

Given this understanding, the thought that cluster decomposition is necessary to ensure compatibility with relativistic spacetime structure (by way of prohibiting superluminal signals) is a non-starter. In fact, Duncan shows that a large class of quantum theories that transform appropriately under the Poincaré group – and are thus compatible with relativistic spacetime structure – will *also* allow for superluminal signaling. Cluster decomposition is then imposed to restrict this class of theories to only those that do not allow action-at-a-distance effects (Duncan, 2012, chapters 5–6). If we understand cluster decomposition as a no-signaling condition, one cannot justify requiring a quantum field theory to satisfy it on the grounds that it is necessary to ensure compatibility with relativistic spacetime structure.

We think that there is a better motivation for cluster decomposition, understood as a no-signaling condition. In a quantum theory that allows superluminal signaling, there is a sense in which no system can ever be treated as isolated from any other. Suppose Ada prepares an experiment in her laboratory. To make predictions for the outcome of that experiment she needs to know the state of her quantum system and be able to predict its dynamical evolution under the specified experimental conditions. If superluminal signaling is possible, then events in regions of the universe very far from her laboratory can change the state of her system in a way that affects the *statistical distribution* of the results of her experiment.[25] She cannot specify a state for her system, nor how it will evolve, without knowing about the state of the universe in regions far beyond her laboratory.

Understanding the problem with superluminal signals this way connects to several motivations for cluster decomposition we have seen already. In particular, we see again that cluster decomposition allows us to decompose the world into effectively isolated subsystems whose states can be described without referring to other physical systems. We can then make experimental predictions and perform various local interventions, including measurements, on those subsystems without knowing anything at all about the state of the universe in distant spatial regions. Understood this way, the importance of prohibiting superluminal signals seems much deeper than merely preserving compatibility with relativistic spacetime structure; the presence of superluminal signals would seem to endanger our ability to define isolated subsystems and make experimental predictions at all.

[25] If Ada's system is entangled with a system at spacelike separation then events in regions of the universe spacelike separated from Ada's could change the *state* of her system, but cannot change the statistical distribution of her experimental results.

3.5. Common Themes. We have considered four of the most commonly encountered reasons for requiring quantum field theories to satisfy cluster decomposition. At first glance, they form a disparate collection. They apparently attribute different degrees of modal significance to cluster decomposition, from a precondition for the possibility of experimental science to a contingent fact about the range of forces in our world; they constrain different mathematical objects, VEVs and the S-matrix; and they give different accounts of its physical significance, from securing the "essential" localizability of states of a quantum field to ensuring that cross sections for widely separated scattering processes are statistically independent.

Nevertheless, there are two important common themes shared between all of them. The first is a belief that cluster decomposition, understood as a statistical independence condition, is a state-independent constraint: appropriate conditions on the dynamics alone are sufficient to ensure that it is satisfied. In Section 5 we argue that this is incorrect: conditions on a Hamiltonian alone are not sufficient to ensure that an S-matrix factorizes appropriately. Whether cluster decomposition, as it is standardly articulated, is satisfied depends not only on properties of the dynamics, but also on the states used define the S-matrix. We argue in Section 6 that one virtue of recasting cluster decomposition as a purely dynamical constraint is that it does make the condition truly state-independent.

The second theme is that cluster decomposition is intended to secure the decomposition of a larger physical system, like a quantum field or a collection of particles, into effectively isolated subsystems to which one can assign independent states, upon which one can perform local experimental interventions, and whose measured properties are independent of any physical systems localized at large spatial separation. In Section 6 we argue that a second virtue of recasting cluster decomposition as a purely dynamical condition, rather than a statistical independence condition on the S-matrix, is that this captures the sense of isolability that these different motivations aim to secure more satisfactorily than the standard definition. As we discuss in Section 6, the standard definition requires both this desired sense of isolability *and* a related, but distinct and unnecessary, sense in which a system might be called isolable.

4. Mathematical Expressions of Cluster Decomposition. Cluster decomposition can be stated as a condition on three different types of mathematical object: VEVs, S-matrices, or algebras of observables. Here we briefly review these statements and their logical relationships.

The cluster decomposition of VEVs requires that if $\vec{a}$ is spacelike, then

$$\langle\Omega\,|\,\varphi(x_1)\cdots\varphi(x_n)\,\varphi(y_1+\lambda\vec{a})\cdots\varphi(y_m+\lambda\vec{a})\,|\,\Omega\rangle$$

factorizes into

$$\langle\Omega\,|\,\varphi(x_1)\cdots\varphi(x_n)\,|\,\Omega\rangle\,\langle\Omega\,|\,\varphi(y_1)\cdots\varphi(y_m)\,|\,\Omega\rangle$$

as $\lambda \to \infty$.[26] In axiomatic settings, it has long been known that one can prove VEV clustering under a variety of distinct assumptions, all rather weak. For example, it has been shown that[27]

- Assuming a mass gap, microcausality, and Lorentz invariance, VEVs factorize at a rate faster than any polynomial (Araki, 1960).
- Assuming a mass gap, microcausality, and translation invariance, VEVs factorize at a rate faster than any polynomial (Ruelle, 1962).
- Assuming a mass gap and Lorentz invariance, but not microcausality, VEVs factorize at a rate faster than any polynomial (Jost and Hepp, 1962).
- Assuming a mass gap, microcausality, and translation invariance, VEVs factorize at a rate faster than any polynomial. Dropping the assumption of a mass gap still gives a (different) superpolynomial rate of convergence (Araki et al., 1962).

The strongest bound to date was given well after this flurry of activity in the 1960s when Fredenhagen (1985), assuming microcausality and a mass gap, proved that VEVs factorize strictly exponentially, not merely superpolynomially. In perturbative quantum field theory one can sketch a proof along the same lines, assuming non-negativity of the energy and microcausality, that shows that VEVs factorize exponentially in the presence of a mass gap and polynomially for massless particles (Brown, 1992, section 6.4).

The cluster decomposition of S-matrices and the cluster decomposition of VEVs are intimately related. We will touch on several aspects of that connection here after a review of the cluster decomposition of the S-matrix. Recall that the cluster decomposition of the S-matrix is meant to secure the fact that if one considers two "clusters" of particles in states $|\alpha\rangle$ and $|\beta\rangle$, and the β cluster is translated an arbitrarily large spacelike distance away from the α cluster, then those two scattering events should be statistically independent:

$$\langle \alpha', \beta' \,|\, S \,|\, \alpha, \beta \rangle \to \langle \alpha' \,|\, S \,|\, \alpha \rangle \, \langle \beta' \,|\, S \,|\, \beta \rangle$$

This places several important constraints on the structure of a scattering amplitude. First, recall that the *connected* part of an $N \to M$ scattering amplitude describes all N particles in the initial state scattering among themselves. This

[26] One can also decompose VEVs into *connected* and *disconnected* parts and equivalently state cluster decomposition as requiring that the *connected* part of a VEV vanish when any one of the coordinates is translated an arbitrarily large spacelike distance from the others (e.g., (Haag, 1958), (Duncan, 2012, chapter 9.2)). This has the virtue of mirroring the decomposition of a scattering amplitude into connected and disconnected parts, with cluster decomposition constraining the connected part of a VEV.

[27] All of these proofs also make assumptions about the test functions used to define the quantum fields, typically (at least) that they are smooth and have compact support.

contrasts with the various *disconnected* parts of an amplitude which correspond to processes where two subsets of particles scatter among themselves, but there is no interaction between the subsets. Cluster decomposition requires that a total scattering amplitude

$$\langle q_m, \ldots, q_1 \,|\, S \,|\, p_1, \ldots, p_n \rangle = S_{q_m \cdots q_1 \,,\, p_1 \cdots p_n}$$

decompose into a sum of connected and disconnected parts.[28] For example, it requires that the total amplitude for $3 \to 3$ scattering decompose as

$$S_{q_3\, q_2\, q_1 \,,\, p_1\, p_2\, p_3} = S^C_{q_3\, q_2\, q_1 \,,\, p_1\, p_2\, p_n} + S^C_{q_2\, q_1 \,,\, p_1\, p_2}\, S^C_{q_3 \,,\, p_3} + \mathcal{P}_{123}$$

$$+ S^C_{q_1 \,,\, p_1}\, S^C_{q_2 \,,\, p_2}\, S^C_{q_3 \,,\, p_3} + \mathcal{P}_{123}$$

where $\mathcal{P}_{123}$ indicates repetitions of the previous term with permutations of the indices.

This makes it apparent that requiring cluster decomposition of the total amplitude translates into a constraint on its *connected parts*: the connected amplitude $S^C_{q_m \cdots q_1 \,,\, p_1 \cdots p_n}$ must vanish when any one or more of the particles is translated an arbitrarily large spacelike distance away from the others. This is necessary for the total amplitude to factorize appropriately.

This, in turn, requires that the connected parts of a scattering amplitude can only contain a single delta function enforcing *total* energy-momentum conservation $\delta(E_q - E_p)\, \delta^3(q_m + \cdots + q_1 - p_1 - \cdots\; p_n)$. They cannot contain additional delta functions enforcing conservation of energy-momentum for any subset of particles, i.e., delta functions like $\delta(E_{q_2} + E_{q_1} - E_{p_1} + E_{p_2})\, \delta^3(q_2 + q_1 - p_1 - p_2)$. (The various *disconnected* parts of an amplitude, like $S^C_{q_2\, q_1 \,,\, p_1\, p_2}\, S^C_{q_3 \,,\, p_3}$ in the amplitude above, will of course contain multiple energy-momentum delta functions: one conserving total energy-momentum for each connected subprocess.)

The problem comes specifically from any additional delta function enforcing momentum conservation: it prevents the connected part of the amplitude from responding appropriately to the action of the translation operator. To see this, consider the Fourier transform of the connected contribution to an $N \to N$ scattering amplitude:

$$S^C_{x'_n \cdots x'_1 \,,\, x_1 \cdots x_n} = \int d^3q_n \cdots d^3q_1\, d^3p_1 \cdots d^3p_n$$
$$S^C_{q_n \cdots q_1 \,,\, p_1 \cdots p_n}\, e^{i(\sum q_i x'_i - \sum p_i x_i)}$$

Cluster decomposition requires this connected amplitude to vanish whenever any subset of particles is translated an arbitrarily large spacelike distance

[28] See Duncan (2012, chapter 6) or Weinberg (1995, chapter 4) for extensive discussion of what follows.

away from the others. However, this cannot happen if $S^C_{q_n \cdots q_1, p_1 \cdots p_n}$ contains a delta function that conserves momenta of only a subset of particles.

To see this, suppose we want to translate particles initially localized around x_1, x_2 an arbitrarily large spacelike distance away from the others and compute the amplitude for them to be found localized around x'_1, x'_2 after the scattering. We act with the translation operator $e^{i\lambda\vec{a}(q_1+q_2-p_1-p_2)}$ and consider the limit

$$\lim_{\lambda\to\infty} S^C_{x'_n \cdots (x'_2+\lambda\vec{a})\,(x'_1+\lambda\vec{a})\,,\,(x_1+\lambda\vec{a})\,(x_2+\lambda\vec{a}) \cdots x_n}$$

Cluster decomposition is satisfied only if this connected part of the amplitude vanishes as $\lambda \to \infty$, but this cannot happen if $S^C_{q_n \cdots q_1, p_1 \cdots p_n}$ contains a delta function that conserves momenta of particles 1 and 2. To see this, suppose

$$S^C_{q_n \cdots q_1, p_1 \cdots p_n} = \delta^3(q_1 + q_2 - p_1 - p_2)\, \mathcal{A}^C_{q_n \cdots q_1, p_1 \cdots p_n}$$

Then we have

$$\begin{gathered}\lim_{\lambda\to\infty} S^C_{x'_n \cdots (x'_2+\lambda\vec{a})\,(x'_1+\lambda\vec{a})\,,\,(x_1+\lambda\vec{a})\,(x_2+\lambda\vec{a}) \cdots x_n} \\ = \lim_{\lambda\to\infty} \int d^3q_n \cdots d^3q_1\, d^3p_1 \cdots d^3p_n \\ e^{i\lambda\vec{a}(q_1+q_2-p_1-p_2)}\, \delta^3(q_1 + q_2 - p_1 - p_2)\; \mathcal{A}^C_{q_n \cdots q_1, p_1 \cdots p_n}\, e^{i(\sum q_i x'_i - \sum p_i x_i)}\end{gathered}$$

The momentum conservation delta function appearing in $S^C_{q_n \cdots q_1, p_1 \cdots p_n}$ kills the action of the translation operator, eliminating any dependence on $\lambda\vec{a}$ on the RHS. The RHS is unchanged in the $\lambda \to \infty$ limit and the LHS – the connected amplitude in position space – cannot vanish, preventing the S-matrix from satisfying cluster decomposition.[29]

Weinberg (1964) identified a condition on a Hamiltonian that is typically presented as sufficient *and necessary* for a quantum field theory to satisfy cluster decomposition.[30] It formalizes the physical intuition that for an S-matrix to satisfy cluster decomposition, the interactions between scattered particles must fall off with distance.

[29]Essentially the same argument is given in (Duncan, 2012, chapter 6.1) using wavepackets, which is a bit more physically satisfactory.

[30]The claim that this condition is necessary is worth clarifying, since (Weinberg, 1995, chapter 4.4) introduces it by stating only that an S-matrix satisfies cluster decomposition "if (*and as far as I know*, only if)" the Hamiltonian satisfies this condition. Weinberg is temporarily treating it as an open question whether quantum field theory is the appropriate framework for constructing a unitary S-matrix that is both Poincaré invariant and satisfies cluster decomposition. His parenthetical remark has to be understood in that context: as far as he knows, the only way to write down a Hamiltonian that that produces such an S-matrix is to use quantum field theory (i.e., to write it as a polynomial in local field operators and their derivatives). However, working *within* quantum field theory, it is obvious that the stated condition on the Hamiltonian is understood to be both sufficient and necessary for the corresponding S-matrix to satisfy cluster decomposition.

First, note that *any* linear operator acting on a Fock space can be written as a sum of products of creation and annihilation operators (Weinberg, 1995, chapter 4.2). Thus one can always write a Hamiltonian in the form

$$H = \sum_{n,m} \int d^3p'_m \cdots d^3p'_1 \, d^3p_1 \cdots d^3p_n$$
$$h_{nm}(p'_m \cdots p'_1 \, , \, p_1 \cdots p_n) \, a^\dagger(p'_m) \cdots a^\dagger(p'_1) \, a(p_1) \cdots a(p_n)$$

The condition that guarantees cluster decomposition of the S-matrix is simply that, just like the scattering amplitudes, the coefficients h_{nm} must be proportional to a single delta function conserving *total* energy-momentum but cannot contain any delta functions conserving momenta among any subset of the scattered particles. This rules out interaction potentials that conserve momenta of a subset of $K < N$ particles in an $N \to N$ scattering process, for example.

This constrains the rate of fall-off of interactions in the following way. Any physically satisfactory Hamiltonian will be invariant under spatial translations, i.e., it must remain unchanged by a uniform shift of all spatial coordinates. This ensures two things: that it is a function of differences of spatial coordinates and that it is proportional to an overall momentum conservation delta function $\delta^3(p'_m + \cdots + p'_1 - p_n - \cdots - p_1)$. The only way the Hamiltonian could contain additional, partial momentum conservation delta functions is if it were also invariant under a uniform shift in some *subset* of the spatial coordinates; this would require that the interaction remain constant as that subset of particles is translated to spacelike infinity. The prohibition on partial momentum conservation delta functions thus amounts to a requirement that the interaction between any two subsets of particles fall to zero as one subset of particles is translated an arbitrarily large spacelike distance from the other.[31] In quantum field theory, all of this is secured automatically by constructing Hamiltonians out of polynomials of local field operators and their derivatives.

We now turn to some aspects of the relationship between VEV and S-matrix clustering. VEVs and S-matrices are connected by the LSZ reduction formula. It allows one to compute S-matrix elements by computing (time-ordered) VEVs:

[31]The matrix elements of the Hamiltonian between Fock states break up into connected and disconnected contributions, just like S-matrix elements (Duncan, 2012, chapter 6.2). The prohibition on partial momentum conservation delta functions ensures that the *connected* part of those Hamiltonian matrix elements goes to zero when any subset of particles is translated an arbitrarily large spacelike distance from the others.

$$\prod_{i=1}^{m}\int d^4x_i\, e^{-ik_ix_i}\prod_{j=1}^{n}d^4y_j\, e^{-ip_jy_j}\,\langle\Omega\,|\,\mathcal{T}\left\{\varphi(x_1)\cdots\varphi(x_m)\,\varphi(y_1)\cdots\varphi(y_n)\right\}|\,\Omega\rangle$$
$$=\left(\prod_{i=1}^{m}\frac{i\sqrt{Z}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{i\sqrt{Z}}{p_j^2-m^2}\right)\langle p_n\cdots p_1\,|\,S\,|\,k_1\cdots k_m\rangle$$

From this it follows immediately that in any theory that satisfies the assumptions of the LSZ reduction formula, a set of VEVs satisfy cluster decomposition *if and only if* the corresponding S-matrix elements do.[32] This means that a Hamiltonian generates an S-matrix that satisfies cluster decomposition if and only if that Hamiltonian also has a ground state in which VEVs cluster, a point to which we will return in Section 6.

There is also a more foundational relationship between the cluster decomposition of VEVs and an S-matrix: the former is a necessary condition for the latter to exist at all. Specifically, to establish the existence of states of multiple non-interacting particles in the Hilbert space of an interacting quantum field theory – the initial and final states of a scattering process – requires that VEVs of that theory satisfy cluster decomposition. We will sketch an outline of this relationship, but the basic physical intuition is that any N particles with wavefunctions at $t = 0$ that are essentially localized around spatial points $x_1, \ldots, x_n$ and momenta $p_1, \ldots, p_n$ (where the supports of the localization of the different particles in position and momentum space may overlap) should become spatially isolated at early and late times $t \to \infty$. Suppose, for example, that all N wavepackets have non-overlapping support in momentum space at $t = 0$ – they are propagating in different directions. If they propagate in different directions for an infinite amount of time, the result is N wavepackets whose position space supports are also essentially non-overlapping at early and late times and thus correspond to non-interacting particles. This is the basic physical idea that allows one to turn the independence of wavepackets at large spacelike separation, secured by cluster decomposition of VEVs, into an independence of wavepackets at early and late times. This, in turn, allows one to prove that the Hilbert space of the *fully interacting* theory contains, as asymptotic states at early and late times, the familiar initial and final states of multiple non-interacting particles used to define the S-matrix.[33]

[32] The qualifier about the applicability of the LSZ reduction formula is important. Non-perturbatively, it means this does not hold for theories with massless particles because such theories do not have a well-defined S-matrix (though see Dybalski (2017) for recent progress and Buchholz and Dybalski (2023) for a more general status report). Perturbatively, however, the claim holds for massless theories in the following two senses: (1) it holds for IR-regulated S-matrix elements, and (2) via the Kinoshita-Lee-Nauenberg theorem, it holds if one replaces "S-matrix elements" with "inclusive quantities", such as scattering cross sections, that depend only on $|S_{\beta\alpha}|^2$. See Weinberg (1995, chapter 13) for details of the KLN theorem and Frye et al. (2019) for some subtleties, and Miller (2021) for philosophical discussion.

[33] See Haag (1996, chapter II.3.1) for a concise description of the basic physical strategy.

This early and late time behavior for particles is secured by proving the LSZ asymptotic condition: that a field Φ evolving in accord with the full Hamiltonian "behaves like" a free field $\varphi_{in/out}$ as $t \to \pm\infty$. Crucially, this behavior includes creating single particles of mass m when acting on the vacuum state. This is the content of the LSZ asymptotic condition, which states that as a weak limit[34]

$$\lim_{t\to\pm\infty} \Phi(t) \to \varphi_{in/out}$$

where the field operators $\varphi_{in/out}$ satisfy the Klein-Gordon equation, but the field operator Φ – evolving in according with the full, interacting Hamiltonian of the theory – obviously does not. (Recall that proving the LSZ asymptotic condition from more fundamental principles was Haag's original motivation for introducing cluster decomposition in Lille in 1957.) The full derivation of the LSZ asymptotic condition is lengthy and technical, drawing on the full structure of Haag-Ruelle scattering theory. However, the basic strategy – and the role of the cluster property – are simpler to convey, so we will sketch the basic ideas here and refer the reader to Duncan (2012, chapter 9.1-4) or Strocchi (2013, chapter 6) for more expanded presentations.

Begin with a field $\Phi(x,t)$ evolving in accord with the full interacting Hamiltonian. While a free field acts on the vacuum to create only 1-particle states, the action of $\Phi(x,t)$ on the vacuum can create 1-particle or multiparticle states. We would like to extract the "part" of $\Phi(x,t)$ that creates 1-particle states of mass m. One can do that in two steps: first, one smears $\Phi(x,t)$ with a function $f(x)$ chosen specifically so that its Fourier transform, $\tilde{f}(p)$, has support only in a small neighborhood of the mass shell $p^2 = m^2$ and is identically zero outside that neighborhood. This defines a field

$$\varphi_1(x,t) = \int d^4x\, f(x)\, \Phi(x,t)$$

which acts on the vacuum to create only 1-particle states. For the second step, one smears $\varphi_1(x,t)$ with a solution of the Klein-Gordon equation

$$g(x,t) = \int \frac{d^3p}{2E}\, \tilde{g}(\vec{p})\, e^{i(p\cdot x - Et)}$$

where $\tilde{g}(\vec{p})$ is a smooth function that decays faster than any polynomial at large momenta, which ensures that the time derivative of $g(x,t)$ does too. From this we can define a field

Note that for wavepackets with overlapping support in momentum space, the physical intuition is somewhat different – the spreading of the wavepacket entails that the amplitude for finding the two particles in any two particular nearby regions goes to zero as $t \to \pm\infty$ – but cluster decomposition of VEVs remains essential.

[34]That is, this is a statement about the limiting behavior of matrix elements $\langle \Psi_\beta \,|\, \Phi \,|\, \Psi_\alpha \rangle$ taken between a set of asymptotic states.

$$\varphi_{1,g}(t) = -i \int d^3x\, g(x,t)\, \frac{\partial}{\partial t}\varphi_1(x,t) - \frac{\partial}{\partial t} g(x,t)\, \varphi_1(x,t)$$

This field $\varphi_{1,g}(t)$ is the "part" of $\Phi(x,t)$ that creates only 1-particle states from the vacuum; it is the analogue of a creation operator in the full interacting theory. A useful feature of the field $\varphi_{1,g}(t)$ is that the 1-particle states $\varphi_{1,g}\,|\Omega\rangle$ that it creates are time-independent, even though $\varphi_{1,g}(t)$ itself is not. This means the momentum-space wavefunctions of these 1-particle states have a simple form:

$$\langle \vec{p}\,|\,\varphi_{1,g}(t)\,|\,\Omega\rangle = \psi_{1,g}(\vec{p}) \propto \tilde{g}(\vec{p})\,\tilde{f}(\vec{p})$$

These wavefunctions (i) are non-zero only near $p^2 = m^2$ due to the presence of $\tilde{f}(\vec{p})$ and (ii) they have the asymptotic $t \to \pm\infty$ behavior of a solution to the Klein-Gordon equation due to the presence of $\tilde{g}(\vec{p})$. One can show that this entails, at early and late times, that these wavefunctions decay no slower than $|t|^{-3/2}$, and for fairly generic conditions on the support of $\tilde{g}(\vec{p})$, they will decay faster than any power of t. This simple asymptotic behavior is extremely useful because it is precisely the $t \to \pm\infty$ behavior of states of the form

$$|\Psi, t\rangle = \varphi_{1,g_1}(t) \,\cdots\, \varphi_{1,g_n}(t)\,|\Omega\rangle$$

that one is interested in when attempting to construct scattering states.

The construction of those states in Haag-Ruelle scattering theory now follows from two things: the asymptotic behavior of solutions $g(x,t)$ of the Klein-Gordon equation and the cluster decomposition of VEVs. Roughly speaking, the constraints they jointly impose on timelike and spacelike asymptotic behavior entail that only states describing multiple time-independent, spatially isolated, 1-particle wavefunctions survive as $t \to \pm\infty$. More specifically, they allow one to show that the time-dependent state[35]

$$|\Psi, t\rangle = \varphi_{1,g_1}(t) \,\cdots\, \varphi_{1,g_n}(t)\,|\Omega\rangle$$

converges as $t \to \pm\infty$, in the fully interacting Hilbert space, to the familiar time-independent scattering state

$$|\Psi\rangle_{in/out} = \int d^3p_1 \,\ldots\, d^3p_n\, \psi_{1,g_1} \cdots\, \psi_{1,g_n}\, |p_1, \,\ldots,\, p_n\rangle_{in/out}$$

This is a state of N spatially isolated particles, with non-overlapping supports in position space. The set of all such states at $t \to -\infty$ provide a basis for a Hilbert space $\mathcal{H}_{in}$, while the states at $t \to +\infty$ provide a basis for $\mathcal{H}_{out}$. The scattering operator S can then be defined as a map from the space of initial states $\mathcal{H}_{in}$ to the space of final states $\mathcal{H}_{out}$. If one makes the standard assumption of *asymptotic completeness* – that the full Hilbert space of the

[35]The 1-particle states $\varphi_{1,g}(t)\,|\Omega\rangle$ are time-independent, but this is no longer true for states created by multiple applications of fields $\varphi_{1,g_i}(t)$.

interacting theory is $\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}$ – then S is a unitary operator. The cluster decomposition of the S-matrix then follows straightforwardly from the cluster decomposition of VEVs; this can be shown either via the LSZ reduction formula, as noted above, or more directly as in (Duncan, 2012, pp. 277-78).

Pushing forward with the same strategy, relying essentially on the timelike asymptotic behavior of solutions to the Klein-Gordon equation and the cluster property, finally allows one to establish the LSZ asymptotic condition. This establishes that the field $\Phi(x, t)$ evolving in accord with the full Hamiltonian asymptotically approaches the free field $\varphi_{in} = \varphi_{out}$ as $t \to \pm\infty$. Neither the existence of asymptotic scattering states nor the LSZ asymptotic condition can be proven without the cluster decomposition of VEVs; as Strocchi (2013, chapter 2.3) emphasizes, "the cluster property is the really crucial and physically essential property for the existence of the asymptotic states and of the S-matrix."

Finally, one can state a physically intuitive cluster decomposition property directly at the level of the algebra of observables. The property is *asymptotic abelianness*, which states that any two observables A, B in an algebra $\mathfrak{A}$ commute in the limit in which one is translated an arbitrarily large spacelike distance from the other:

$$\lim_{|a|\to\infty} \left[A,\, U(\vec{a})BU(\vec{a})^{-1}\right]$$

This is particularly natural in algebraic quantum field theory, where such algebras are the fundamental mathematical objects of the theory, but it is also valuable and physically transparent in a more standard presentation of quantum field theory that works within a particular representation of that algebra in terms of field operators.

The basic physical idea is that even if A and B are localized around timelike-related points, a large enough translation in a spacelike direction will eventually put B outside the lightcone of A, at which point microcausality ensures that $[A,\, B] = 0$. We said that this property is also useful in standard approaches to quantum field theory which take field operators as the primary objects of interest. As one example, this property is often relied upon in analyses of the cluster property of VEVs, when it is assumed that any two field operators commute inside a time-ordered VEV once their spatial coordinates have been translated an arbitrarily large spacelike distance away from one another.

5. Cluster Decomposition, Entanglement, and the Vacuum. One might naturally expect that cluster decomposition must be a state-independent condition, given its centrality in the structure of quantum field theory. Surely no physically reasonable states could endanger such a central structural pillar of quantum field theory? In fact, neither the cluster decomposition of VEVs nor of the S-matrix is state-independent. However, one well-known source of state-dependence for the cluster decomposition of VEVs serves a positive function in quantum field theory: it acts as a selection principle for physically reasonable ground states. Specifically, one identifies physical vacuum states, at least in part, by requiring that VEVs in that vacuum state satisfy cluster decomposition.

It is less widely appreciated that even within a physically reasonable vacuum state, the cluster decomposition of the S-matrix remains state-dependent. This stems from the failure of scattering amplitudes to cluster if there is entanglement in the initial state of the scattering process. A basis of entangled states is a completely unobjectionable basis for a Fock space, but the S-matrix in this basis will not satisfy cluster decomposition. Attempting to recast this as a selection principle for physically acceptable scattering states or as an argument for a preferred basis for the Fock space, analogous to how one resolves the state-dependence of VEV clustering, is untenable. Instead, we argue in Section 6 that this is best resolved by redefining cluster decomposition as a dynamical constraint.

5.1. State Dependence: Vacuum State. It has been known since the early 1960s that VEVs in a quantum field theory satisfy cluster decomposition if and only if the Hilbert space contains a unique vacuum state (Hepp et al., 1961; Borchers, 1962). An early example illustrating the connection was offered by E. C. G. Sudarshan and Korkut Bardacki (Wightman, 1963, Section 1.3). Their example was somewhat artificial but turned out to be prescient.

They considered two quantum field theories describing an uncharged scalar field. Each is characterized by a Hilbert space $\mathcal{H}_i$, field operators $\varphi_i(x)$, an irreducible representation of the Poincaré group $U_i(a, \Lambda)$ acting on $\mathcal{H}_i$, and a vacuum state $|\omega_i\rangle$. They then constructed a new quantum field theory by taking their direct sum, producing a quantum field theory with a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, field operators $\Phi(x) = \varphi_1(x) \oplus \varphi_2(x)$, and a reducible representation of the Poincaré group $U(a, \Lambda) = U_1(a, \Lambda) \oplus U_2(a, \Lambda)$ acting on $\mathcal{H}$. Importantly, there is no unique vacuum state for this new theory. Instead, there is a set of states of the form $|\Omega\rangle = \alpha_1|\omega_1\rangle + \alpha_2|\omega_2\rangle$, with $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Each of these is a ground state of the Hamiltonian of the theory and is invariant under the action of the reducible representation $U(a, \Lambda)$ of the Poincaré group. This set of vacua occupies a two-dimensional subspace of the Hilbert space $\mathcal{H}$, not the one-dimensional subspace associated with a unique vacuum state.

A simple calculation shows that even if the VEVs $\langle \omega_1 \,|\, \varphi_1(x_1) \ldots \varphi_1(x_j) \,|\, \omega_1 \rangle$ and $\langle \omega_2 \,|\, \varphi_2(x_{j+1}) \ldots \varphi_2(x_n) \,|\, \omega_2 \rangle$ satisfy cluster decomposition, the VEV

$$\langle \Omega \,|\, \Phi(x_1) \ldots \Phi(x_j)\, \Phi(x_{j+1}) \ldots \Phi(x_n) \,|\, \Omega \rangle$$

will generally not. It will cluster if and only if $\alpha_1 = 0$ or $\alpha_1 = 1$, i.e., if and only if $|\Omega\rangle$ is equal to one of the original unique vacua $|\omega_1\rangle$ or $|\omega_2\rangle$.

This rather artificial example was prescient because an essentially identical situation arises in quantum field theories with spontaneous symmetry breaking (SSB), one of the great theoretical developments of the 1960s. In the broken phase of a theory with SSB, the full Hilbert space decomposes into a direct sum of superselection sectors $\mathcal{H} = \bigoplus_j \mathcal{H}_j$ with each superselection sector $\mathcal{H}_j$ equipped with a unique vacuum state $|\omega_j\rangle$.[36] These vacuum states are orthogonal and the matrix element of any local operator $A(x)$ taken between two distinct vacua $|\omega_j\rangle$ and $|\omega_k\rangle$ vanishes. These vacuum states are degenerate in the sense that they are each ground states of the Hamiltonian of the theory, but are distinguished by the VEV of some other operator $\mathcal{O}(x)$. The VEV $\langle\omega_j\,|\,O(x)\,|\,\omega_j\rangle$ that distinguishes states that break the symmetry of interest is called the order parameter for that symmetry. The requirement that VEVs satisfy cluster decomposition again acts as a selection principle, explaining why the individual vacuum states $|\omega_j\rangle$ that break the symmetry are physically satisfactory vacua, while linear combinations of those vacua $|\Omega\rangle = \sum_j \alpha_j\,|\omega_j\rangle$ that preseve the symmetry are not.

The general point can be illustrated with a simple example (Duncan, 2012, pp. 503–505). Consider a theory of an uncharged scalar field $\varphi(x)$ governed by a Hamiltonian that is invariant under the $\mathbb{Z}_2$ symmetry $\varphi(x) \to -\varphi(x)$. The familar φ^4 theory with a "wrong-sign" mass term provides an example:

$$H = \int d^3x\ \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\varphi(x))^2 - \frac{1}{2}m^2\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4$$

This Hamiltonian has two vacuum states $|\omega_+\rangle$ and $|\omega_-\rangle$. The field $\varphi(x)$ has a non-zero VEV in each of these two vacua:

$$\langle\omega_+\,|\,\varphi(x)\,|\,\omega_+\rangle = v \quad \text{and} \quad \langle\omega_-\,|\,\varphi(x)\,|\,\omega_-\rangle = -v$$

In each vacuum state, the symmetry $\varphi \to -\varphi$ obeyed by the Hamiltonian is broken:

$$\langle\omega_\pm\,|\,\varphi(x)\,|\,\omega_\pm\rangle \neq \langle\omega_\pm\,|\,-\varphi(x)\,|\,\omega_\pm\rangle$$

The vacuum states are orthogonal, and the fact that $\langle\omega_\pm\,|\,A(x)\,|\,\omega_\mp\rangle = 0$ for all local operators $A(x)$ means that no local operator can turn $|\pm v\rangle$ into $|\mp v\rangle$. The Hilbert space for the theory thus decomposes into the direct sum of two superselection sectors $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with $|\omega_+\rangle$ the vacuum state of $\mathcal{H}_+$ and $|\omega_-\rangle$ the vacuum state of $\mathcal{H}_-$.

Why not simply avoid breaking the $\mathbb{Z}_2$ symmetry by taking the vacuum state of the theory to be $|\Omega\rangle = \alpha_+\,|\omega_+\rangle + \alpha_-\,|\omega_-\rangle$ with $|\alpha_+|^2 + |\alpha_-|^2 = 1$? This is also a ground state of the Hamiltonian – it yields the same expectation value for the Hamiltonian as each of the degenerate vacua $|\omega_+\rangle$ and $|\omega_-\rangle$ – *and* it preserves the symmetry $\varphi \to -\varphi$ obeyed by the Hamiltonian. The reason $|\Omega\rangle$ is not a physically acceptable vacuum state is that VEVs in this state do not satisfy cluster decomposition.

[36] In the language of algebraic quantum field theory, the superselection sectors $\mathcal{H}_j$ are unitarily inequivalent irreducible GNS representations corresponding to distinct states ω_j on the $C^\star$-algebra defining the theory.

To do so, VEVs taken in the state $|\Omega\rangle$ would have to satisfy

$$\lim_{\lambda\to\infty} \langle\Omega|\varphi(x)\varphi(y+\lambda\vec{a})|\Omega\rangle - \langle\Omega|\varphi(x)|\Omega\rangle\,\langle\Omega|\varphi(y)|\Omega\rangle = 0$$

Instead, a simple calculation reveals that

$$\langle\Omega|\varphi(x)\varphi(y+\lambda\vec{a})|\Omega\rangle = v^2$$

while

$$\langle\Omega|\varphi(x)|\Omega\rangle\,\langle\Omega|\varphi(y)|\Omega\rangle = v^2(1-2\alpha)^2$$

Combining the two results, one sees that cluster decomposition is satisfied if and only if $\alpha = 1$ or $\alpha = 0$, i.e., if and only if the vacuum state of the theory is either $|\omega_+\rangle$ or $|\omega_-\rangle$.

Thus the cluster decomposition of VEVs is state-dependent – it is only satisfied in particular vacuum states. However, as long as the Hamiltonian has ground states in which VEVs do cluster, this state-dependence is straightforwardly resolved by treating the requirement that VEVs satisfy cluster decomposition as a selection criteria for physical vacuum states.[37] One can then define the necessary ingredients for a scattering theory as outlined in Section 4.

5.2. State Dependence: Entanglement. This role played by cluster decomposition in identifying physical vacua is well known. However, it seems less widely appreciated that even if one has identified a vacuum state in which VEVs satisfy cluster decomposition, the corresponding Fock space contains states for which the scattering amplitudes *do not* factorize. Indeed, one can actually form a basis for the Fock space with such states. This becomes apparent when one considers scattering processes with entangled initial states.

Entangled states don't show up in textbook discussions of scattering theory, but preparing and scattering particles in a partially or fully entangled state is a perfectly physical process.[38] Of course, distant scattering experiments whose initial states are entangled will have correlated results. The corresponding scattering amplitude will not factorize when any subset of particles are translated an arbitrarily large spacelike distance away from the others.

Here is a simple example. Recall that the space of asymptotic states of a quantum field $\mathcal{H} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$ is a Fock space, that is, a direct sum of n-fold

[37]The requirement that correlation functions in the ground state satisfy cluster decomposition also selects physical ground states in statistical physics (Parisi, 1988, Section 2.2; Denef, 2012, Section 2.2.2). A system in the broken phase is no longer ergodic on the full state space and the equivalence between phase averages and time averages can only be restored by choosing one of the two symmetry-breaking ground states for the system. Indeed, Ruelle (1969, chapter 6.3) just calls clustering ground states "ergodic states".

[38]In fact, considerable attention has recently been devoted to various aspects of the scattering behavior of such states. For a sample of this literature, see Seki et al. (2015); Peschanski and Seki (2016); Grignani and Semenoff (2017); Fan et al. (2017); Kharzeev and Levin (2017); Peschanski and Seki (2019); Araujo et al. (2019); Faleiro et al. (2020).

tensor products of the subspace $\mathcal{H}^1$ containing one-particle states of the field:

$$\mathcal{H} = \mathbb{C} \oplus \mathcal{H}^1 \oplus \left(\mathcal{H}^1 \otimes \mathcal{H}^1\right) \oplus \left(\mathcal{H}^1 \otimes \mathcal{H}^1 \otimes \mathcal{H}^1\right) \oplus \cdots$$

In a theory containing multiple quantum fields φ_j, φ_k, the full Hilbert space of asymptotic states is just the tensor product of the Hilbert spaces associated with each field:

$$\mathcal{H}_{jk} = \mathcal{H}_j \otimes \mathcal{H}_k$$

Now consider an elastic $3 \to 3$ scattering of distinguishable particles (the generalization to generic $N \to M$ scattering is straightforward). Label distinguishable particles as A, B, and C. Each particle is associated with a quantum field Φ_a, Φ_b, or Φ_c, and asymptotic states of those quantum fields form the Fock spaces $\mathcal{H}_a$, $\mathcal{H}_b$, and $\mathcal{H}_c$.[39]

The full Hilbert space of asymptotic states is thus

$$\mathcal{H}_{abc} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$$

For our purposes we can restrict our attention to asymptotic states containing only a single particle each of species A, B, and C, so we consider only the subspace

$$\mathcal{H}^1_{abc} = \mathcal{H}^1_a \otimes \mathcal{H}^1_b \otimes \mathcal{H}^1_c$$

and scattering amplitudes with

$$\mathcal{H}^1_{abc} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$$

States in $\mathcal{H}^1_{abc}$ have the form

$$|k, p, q\rangle_{abc} = |k; \sigma_a\rangle_a \otimes |p; \sigma_b\rangle_b \otimes |q; \sigma_c\rangle_c$$

where the σ_i represent the state of any other quantum numbers the particle might have, like spin or flavor. We will ignore them and consider only entangled momenta, but of course a similar failure of factorization could arise from entanglement between those degrees of freedom as well.

Now consider S-matrix elements involving partially or fully entangled initial states

$$|i\rangle_{abc} = a_1 |k_1, p_1, q_1\rangle_{abc} \pm a_2 |k_2, p_2, q_2\rangle_{abc}$$

For example, suppose that in the initial state, the momenta of particles A and B are entangled with each other, while particle C is not entangled with either one.[40] The three particles are in a partially entangled state of the form

$$|\Psi\rangle_{abc} = (a_1 |k, p\rangle_{ab} + a_2 |p, k\rangle_{ab}) \otimes |q\rangle_c$$

[39]Despite the notation, the Φ_i need not be scalar fields.

[40]One could prepare such a state by preparing C independently and letting A and B be products of a decay process, like $\pi^0 \to e^+ e^-$.

Suppose that B and C scatter, while A is translated an arbitrarily large spacelike distance away and undergoes no scattering, as in Figure 1. The scattering amplitude for $A \to A$ will not be statistically independent from the amplitude for $BC \to BC$.

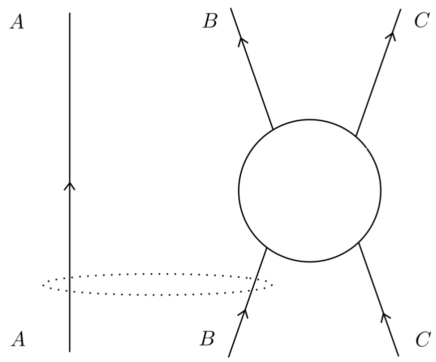

Figure 1: Particles A and B are initially entangled, particle A propagates freely, and particles B and C scatter.

At the risk of belaboring the point, the physical situation is the following. In the state $|\Psi\rangle_{abc}$ particles A and B do not have individually determinate momentum states, due to their entanglement, while particle C initially has momentum q. Particle A is translated a large spacelike distance away from particles B and C, propagates freely, and hits a detector that records its momentum. This measurement will *determine* the momentum of particle A, since it previously could not be assigned a momentum due to its entanglement with particle B. The possible results of this measurement are k and p. Over the same time interval, particles B and C undergo a scattering process. That process conserves total momentum, so the possible final states for particles B and C will have total momentum of either $k + q$ or $p + q$.

As soon as the momentum of particle A is registered by the detector, the total momentum of the final state of particles B and C is instantly determined. Suppose the momentum of particle A is measured to be k. That instantly determines, with certainty, that particles B and C will be in a final state with total momentum $p + q$ – the amplitude for any process that takes particles B and C into a final state with momentum $k+q$ is zero. Prior to the measurement of particle A's momentum, the amplitude for the outcome of the B and C scattering is spread over two regions of $\mathcal{H}$: one region of final states with $p+q$ and another of final states with $k + q$. The measurement of the momentum of particle A instantly bunches up all of the amplitude for the B and C scattering into the $p+q$ region of $\mathcal{H}$. Of course this all holds, *mutatis mutandis*, if instead the detector registers the momentum of particle A to be p. To borrow some terminology familiar from discussion of Bell-type theorems, the two scattering processes exhibit "outcome-outcome" dependence and the $3 \to 3$ amplitude does not factorize into a product of statistically independent $1 \to 1$ and $2 \to 2$ amplitudes.

Now, cluster decomposition does not require that *every* scattering amplitude between *arbitrary* states factorize; it requires only that the S-matrix does, and the S-matrix is the set of matrix elements $\langle \Psi_\beta \,|\, S \,|\, \Psi_\alpha \rangle$ taken between states $|\Psi_\alpha\rangle, |\Psi_\beta\rangle$ of a particular basis. It is not the set of matrix elements between all states in $\mathcal{H}_{in/out}$. Insofar as cluster decomposition is a condition on an S-matrix, then, what really matters is basis-independence, not state-independence. One might therefore hope that the cluster decomposition of the S-matrix holds in all bases, even if the analogous factorization property doesn't hold for all states.

However, it is easy to show that the state-dependence described above translates into a basis-dependence; simply choose an entangled basis for the Fock space $\mathcal{H}_{in/out}$. For example, for a fermionic field, one can exploit the analogy between fermionic modes and qubits and construct an entangled basis for the Fock space by starting with a GHZ-type state[41]

$$|\Psi_{GHZ}\rangle = \alpha\,|0,\,0,\,\cdots\,0,\,\cdots\rangle + \beta\,|1,\,1,\,\cdots\,1,\,\cdots\rangle$$

One can then construct an entangled basis for the momentum representation of the Fock space by toggling modes, in direct analogy to the standard procedure for constructing an entangled basis for the Hilbert space of N qubits by starting from a GHZ state. An analogous construction generates an entangled basis for a bosonic Fock space.

For the reasons outlined above, an S-matrix in this basis will not satisfy cluster decomposition *even if* the Hamiltonian of the theory satisfies the condition discussed in Section 4 that is typically deemed sufficient (and necessary) for cluster decomposition. This simply reflects the fact that an S-matrix element $\langle\Psi_\beta\,|\,S\,|\,\Psi_\alpha\rangle$ is a function of the scattering operator and the basis states $\{|\Psi_\alpha\rangle\}$, so cluster decomposition can be violated if either one produces long-range correlations. Note that while we considered distinguishable particles for the extended example above, that was not essential; here we have a basis for the Fock space associated with a *single* fermionic field – i.e., for a system of indistinguishable particles – in which the S-matrix does not satisfy cluster decomposition.

Of course, the S-matrix in an entangled basis doesn't factorize because the initial states don't factorize when one subsystem is translated an arbitrarily large spacelike distance away from the others. In fact, this failure of the initial state to factorize means that certain constraints on S-matrix elements that were said to follow from cluster decomposition in Section 4 can't even be stated in this basis. In particular, recall that cluster decomposition requires that S-matrix elements decompose into a sum of connected and disconnected parts:

$$S_{q_3\,q_2\,q_1\,,\,p_1\,p_2\,p_3} = S^C_{q_3\,q_2\,q_1\,,\,p_1\,p_2\,p_n} + S^C_{q_2\,q_1\,,\,p_1\,p_2}\,S^C_{q_3\,,\,p_3} + \mathcal{P}_{123}$$

$$+S^C_{q_1\,,\,p_1}\,S^C_{q_2\,,\,p_2}\,S^C_{q_3\,,\,p_3} + \mathcal{P}_{123}$$

Stating this condition assumes that each of the three particles can be assigned *some* determinate momenta in the initial state. But this is false, in general,

[41]We consider a spinless fermion and entanglement between momentum modes for simplicity, but nothing hinges on this.

for a system of N particles whose momenta are entangled.

Cluster decomposition is thus incompatible with entanglement in the initial state of a scattering process. Cluster decomposition of the S-matrix fails – indeed, cannot even be sensibly stated – when the S-matrix is defined using a basis of entangled states. This is because even though it is universally described as requiring only an ability to *dynamically* isolate subsystems from one another, by making it a condition on the S-matrix it *also* introduces a requirement on states: that those subsystems be isolated from one another in the sense of being unentangled. The conflation of these two senses of isolability, and how to recast cluster decomposition in a way that cleaves them, is discussed in Section 6.

5.3. Vacuum Entanglement. Our discussion of entangled states raises two puzzles. It is well known that the vacuum state of a quantum field theory is entangled; one can maximally violate the Bell inequalities in such a state.[42] The first puzzle: why is the entanglement of the vacuum state compatible with VEVs satisfying cluster decomposition, while entanglement in an initial state in a scattering process prevents the analogous factorization of the scattering amplitude? The second puzzle: the standard basis for a Fock space consists of separable states of N particles, but such states are prepared by exciting local, entangled regions of the vacuum state; one might worry that the entanglement of the vacuum should be transferred to entanglement between the created particles. Given the incompatibility of cluster decomposition and entangled initial states, that raises the question of how cluster decomposition can *ever* be satisfied.

The beginnings of an answer were given by Redhead (1995).[43] He employed the tools of algebraic quantum field theory in his analysis: one assigns to every open, bounded spacetime region U a von Neumann algebra $\mathfrak{A}(U)$ of bounded operators, and self-adjoint elements of the algebra $\mathfrak{A}(U)$ correspond to observables that can be measured in the spacetime region U. For any open, bounded spacetime region V that is spacelike separated from U, all elements of the algebra $\mathfrak{A}(V)$ commute with all elements of the algebra $\mathfrak{A}(U)$. For a quantum field in its vacuum state $|\Omega\rangle$ Redhead considers arbitrary spacelike separated regions U and V and shows that for any nontrivial projection operator P contained in the algebra $\mathfrak{A}(U)$, there exists some nontrivial projection operator Q in the algebra $\mathfrak{A}(V)$ such that the two projectors are maximally correlated in the vacuum state. That is, Redhead shows that for all $\epsilon > 0$ and

[42]See Summers (2011, Section 4) for a summary discussion of Bell inequality violation in the quantum field theory vacuum, with many citations to original papers. See Casini and Huerta (2009), Headrick (2019), and Casini and Huerta (2021) for an overview of entanglement in the quantum field theory vacuum with a focus on properties of entanglement entropy.

[43]See Clifton et al. (1998, sections II-III) for a clear and concise presentation of Redhead's argument and some related mathematical and conceptual issues.

any P in $\mathfrak{A}(U)$, there exists some Q in $\mathfrak{A}(V)$ such that[44]

$$\Pr_{\Omega}(P = 1 \mid Q = 1) > 1 - \varepsilon$$

The significance of this result is that in the vacuum state, for any chosen observable in spacetime region U (such as the projector P), there is always some observable one could measure in spacetime region V (such as the projector Q) and some particular result one could get for that measurement in V, that would determine with certainty the result of measuring the chosen observable in U.

These long-range correlations reflect the fact that the vacuum state of a quantum field theory is highly entangled. They also seem *prima facie* incompatible with the requirement that VEVs satisfy cluster decomposition, but Redhead demonstates this is not right: these two properties of the vacuum state can coexist. The cost of this coexistence is that correlations between measurements of observables in U and measurements of observables in V must fall off exponentially with the spacelike distance $d(U, V)$ between the two regions. Redhead shows that this exponential falloff imposes an upper bound on the probability of getting the measurement result for the observable in V that would determine, with certainty, the result of measuring the chosen observable in U. The key fact is that this upper bound has to decrease exponentially as the two spacetime regions U and V are taken to ever-greater spacelike separation. For example, if we take a projection operator P in $\mathfrak{A}(U)$ and its maximally correlated partner Q as the observables, Redhead shows that the probability of getting the result $Q = 1$ that would determine with certainty that $P = 1$ must satisfy the following bound:

$$\Pr_{\Omega}(Q = 1) \leq e^{-2md(U,V)} \frac{\Pr_{\Omega}(P = 1)}{(1 - \Pr_{\Omega}(P = 1))^2}$$

where m is the mass of the particle associated with the field and $d(U, V)$ is the minimum spacelike distance between spacetime regions U and V. The result establishes that for a fixed probability $\Pr_{\Omega}(P = 1)$, the probability that a measurement of the maximally correlated projector will give the outcome $Q = 1$ falls off exponentially with the distance between U and V.

This goes a long way toward solving our puzzle. Unlike the correlations between momentum or spin measurements in an EPR-type state, which are independent of the distance between the subsystems, correlations between measurements performed on different spacetime regions of the field in its vacuum state fall off exponentially with the spacelike distance between those regions. The rate of this falloff is what makes the entanglement of the vacuum state consistent with the requirement that VEVs taken in that vacuum state

[44] The conditional probability $\Pr_{\Omega}(P = 1 \mid Q = 1)$ can be equivalently written as $\langle \Omega \mid P\, Q \mid \Omega \rangle / \langle \Omega \mid Q \mid \Omega \rangle$. This follows from the definition of conditional probability and the fact that $\Pr_{\Psi}(P_j = 1) = \langle \Psi \mid P_j \mid \Psi \rangle$ for any state $|\Psi\rangle$ and any projection operator P_j.

satisfy cluster decomposition. However, it raises a further question: *why* do these correlations fall off with distance? Redhead has shown that compatibility with cluster decomposition requires that they do, but what is the physical explanation for that behavior?

The answer, and the final piece of the puzzle, lies in the origin of vacuum entanglement. It is well known that a free quantum field is mathematically equivalent to an infinite set of coupled harmonic oscillators. The source of the coupling is the gradient term $(\nabla\varphi)^2$ in the Hamiltonian, which couples harmonic oscillators at neighboring spatial points. This is the only term in the Hamiltonian that couples operators at different spatial points, and it is the source of the entanglement in the vacuum state. We will use a free massive scalar field as our example, but this physical explanation of the origin of vacuum correlations applies equally well to theories with perturbative interactions and higher-spin fields.[45]

The essential features can all be gleaned from two coupled harmonic oscillators (Srednicki, 1993). The Hamiltonian is

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_a(x_1^2 + x_2^2) + k_b(x_1 - x_2)^2\right]$$

One typically transforms this into a system of decoupled harmonic oscillators by re-expressing it in the basis of its normal modes:

$$H = \frac{1}{2}\left[p_+^2 + \omega_+ x_+^2 + p_-^2 + \omega_- x_-^2\right]$$

with $x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm x_2)$, $\omega_+^2 = k_a$, and $\omega_-^2 = k_a + 2k_b^2$. This amounts to a choice of how to divide the system of oscillators up into subsystems: each normal mode of the coupled oscillators is treated as a subsystem, and we no longer keep track of each individual oscillator. In this basis, the ground state wavefunction of the system is separable:

$$\Psi_0(x_+, x_-) = \Psi_0(x_+)\Psi_0(x_-) = \frac{1}{\sqrt{\pi}(\omega_+\omega_-)^{1/4}}\exp\left[-\frac{1}{4}(\omega_+ x_+ + \omega_- x_-)\right]$$

However, there is no requirement that we individuate subsystems this way. If we keep track of the individual oscillators rather than the normal modes of the coupled system, we find that the ground state of the position degrees of

[45] It applies to Abelian and non-Abelian gauge fields because such Hamiltonians also contain gradient-squared terms, although there is no free non-Abelian gauge theory and the interaction potential generically *grows* linearly with distance so the physical argument below for how the decay of correlations in the vacuum is related to their origin will not hold for such theories. Indeed, there is good reason to expect VEVs should not satisfy cluster decomposition in those theories (Wilson, 1974; Strocchi, 1978; Lowdon, 2016). For fermions, the discretization of the Hamiltonian for Dirac or Majorana fermions still couples operators at neighboring lattice sites, even though it is linear in derivatives (Casini and Huerta, 2009, sections 3.1.8 & 3.2.3).

freedom of the two oscillators is entangled:

$$\Psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}(\omega_+\omega_-)^{1/4}} \exp\left[-\frac{1}{8}(\omega_+(x_1+x_2)^2 + \omega_-(x_1+x_2)^2)\right]$$
$$\neq \Psi_0(x_1)\Psi_0(x_2)$$

A little algebra reveals that the source of the entanglement in this ground state is the coupling between the two oscillators, here encoded in a term of the form $x_1 x_2(\omega_+ - \omega_-)$.

This diagnosis of the physical origin of the entanglement of the ground state for two coupled harmonic oscillators generalizes directly to the case of the free massive scalar field. Consider N coupled harmonic oscillators:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j$$

where p_i and x_i represent the momentum and position of the i^{th} oscillator and the matrix K_{ij} couples oscillators at neighboring points. The ground state of the position degrees of freedom of the N oscillator system is again entangled:

$$\Psi_0(x_1, x_2, \ldots, x_N) \propto \exp\left[-\frac{1}{2}(\mathbf{x}^T \cdot \sqrt{K} \cdot \mathbf{x})\right]$$

where the non-factorizability of the ground state again comes from the coupling between oscillators at neighboring points.

It is easy connect this to quantum field theory. After a little algebra and a suggestive switch of notation, the Hamiltonian becomes

$$H = \frac{1}{2}\sum_{i=1}^{N} \pi_i + \frac{1}{2}\sum_{i,j=1}^{N} (\varphi_i^2 + \varphi_j^2) + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \varphi_i K_{ij} \varphi_j$$

where π_i and φ_i now represent the momentum and position operators associated with the i^{th} oscillator. Compare this to the discretized Hamiltonian for a free massive scalar field:

$$H = \frac{1}{2}a^3 \sum_{i=1}^{N}\left[\pi(x_i)^2 + \sum_{n=1}^{3}\left(\frac{\varphi(x_i) - \varphi(x_i + a\hat{n})}{a}\right)^2 + m^2\varphi(x_i)^2\right]$$

where x_i labels lattice sites, a is the lattice spacing, and $\hat{n}$ is a unit vector along each lattice link.

Our interest is in the terms

$$\left(\frac{\varphi(x_i) - \varphi(x_i + a\hat{n})}{a}\right)^2$$

that come from discretizing $(\nabla\varphi)^2$ in the continuum Hamiltonian. These are the only terms that couple operators at neighboring spatial points. Multiplying them out gives

$$H = \frac{1}{2}a^3 \sum_i^N \Bigg[\pi(x_i)^2 + \frac{1}{a^2}\sum_{n=1}^{3}\left[\varphi(x_i)^2 + \varphi(x_i + a\hat{n})^2\right] - \frac{1}{a^2}\sum_{n=1}^{3} 2\varphi(x_i)\,\varphi(x_i + a\hat{n}) + m^2\varphi(x_i)^2 \Bigg]$$

This exhibits the same coupling between operators at neighboring spatial points as the Hamiltonian for N harmonic oscillators with a particular choice of the matrix K_{ij}.

In the continuum limit,

$$\varphi(x_i) \to \varphi(x), \qquad \pi(x_i) \to \pi(x), \quad \text{and} \quad a^{d-1}\sum_{i=1}^{N} \to \int d^{d-1}x$$

and the finite-difference terms that couple operators at neighboring points are replaced by $(\nabla\varphi)^2$. One recovers the familiar Hamiltonian

$$H = \frac{1}{2}\int d^3x\ \pi(x,t)^2 + (\nabla\varphi(x,t))^2 + m^2\varphi(x,t)^2$$

When we keep track of the field operators at individual spacetime points, rather than the normal modes of the entire field, the ground state wavefunctional of this Hamiltonian is entangled. As we just saw, this is due to the coupling of field operators $\varphi(x)$ and $\varphi(y)$ at adjacent spatial points.

The coupling between fields at neighboring spatial points explains why a change in the state of the field in a spacetime region U cannot be confined to that region: the effect of exciting the field in U will spread throughout the field due to the coupling between the oscillators. This is a physically intuitive way of understanding why one cannot define strictly localized states for spacetime regions U and V in a quantum field theory: the state of the field in region V can never be perfectly insulated from changes in the state of the field in U.

This also gives us an answer to our second puzzle. The $(\nabla\varphi)^2$ term couples field operators only at neighboring spatial points, so one expects that the effect of exciting the field in region U should be *mostly* confined to the region U, and the effect it has on the state of the field in region V should fall off with the distance between the two regions. Indeed, this effect falls off exponentially for massive fields and polynomially for massless fields just as cluster decomposition requires. For example, suppose the particle associated with Ψ has mass $\sim 0.5\,\text{MeV}$. The overlap between the state of the field $\langle\Omega|\Psi(x)$ produced by exciting the field around the spacetime point x and the state $\Psi(y)|\Omega\rangle$ produced by exciting the field around spacetime point y is given by $\langle\Omega\,|\,\Psi(x)\Psi(y)\,|\,\Omega\rangle$, and the quantity $\langle\Omega\,|\,\Psi(x)\Psi(y)\,|\,\Omega\rangle - \langle\Omega\,|\,\Psi(x)\,|\,\Omega\rangle\langle\Omega\,|\,\Psi(y)\,|\,\Omega\rangle$ captures the degree to which the state $\Psi(y)|\Omega\rangle$ differs from the vacuum $|\Omega\rangle$ for an observer localized around the point x. This second quantity is much smaller than 10^{-400} once points x and y are separated by even $\sim 1\,\mathrm{nm}$.[46] For an observer localized around x, there is essentially no difference between the field around y being in its vacuum state or in the state $\Psi(y)|\Omega\rangle$; any change in the state of the field at y is essentially localized to the spacetime region immediately around y. This is what underlies the idea we encountered in Section 3 that cluster decomposition secures the approximate, or "essential", localization of states.

It is worth commenting briefly on this notion of "essentially" localized states. If one excites the field in a spacetime region U, the effect of that excitation on the field in region V will become vanishingly small as V is translated by a large spacelike distance; to a very good approximation, as far as the state of the field in V is concerned, it will be as if the field in U is in its vacuum state. Qualitatively, *but only qualitatively*, this amounts to an ability to factorize the global state of the field into a product of states assigned to its subregions. For example, Haag says that cluster decomposition of VEVs means that "it is meaningful to define a 'product state vector' $\Psi = \psi_1 \otimes^t \psi_2 \ldots$ although in general no tensor product between vectors of $\mathcal{H}$ with values in $\mathcal{H}$ is defined, such a product becomes meaningful between states which are localized far apart at a particular time" (Haag, 1996, II.3.1).[47] This ensures that, despite the entanglement of the vacuum state, one can still define initial and final states describing multiple independent, essentially localized particles, just as in the sketch of the Haag-Ruelle derivation of the LSZ asymptotic condition in Section 4.

However, one cannot factorize the global state exactly. The ability to exploit the *extremely* small but non-vanishing effects that acting in U can have on the state of the field in region V is precisely what underlies the Reeh–Schlieder theorem (see, e.g., Witten (2018, section 2)). This is what leads Haag (1996, Theorem 5.3.1), for example, to offer the reminder that while cluster decomposition does secure the localization of states in this qualitative sense, "the concept of *localized states*, if used in a more than qualitative sense, must be handled with care." The origin of the entanglement of the vacuum state offers a physically intuitive explanation of why VEVs satisfy cluster decomposition, despite the vacuum being an entangled state, while also explaining the origin of the miniscule, but extremely important, long-range correlations that underlie the Reeh–Schlieder theorem.

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[46] If the theory is not in a symmetry breaking phase then $\langle\Omega\,|\,\Psi\,|\,\Omega\rangle = 0$ and this becomes the statement that the amplitude for finding the state $\Psi(y)|\Omega\rangle$ in the state $\Psi(x)|\Omega\rangle$ goes to zero exponentially in the distance between x and y.

[47] Similarly, Weinberg invokes the same idea in justifying the use of "product states" in scattering theory, explaining that "physical states before and after the collision consist of particles that are so far apart that they are effectively non-interacting, so they can be described as direct products of the one-particle states" (Weinberg, 1995, p. 107).

6. Two Senses of Isolability How should we understand the physical significance of cluster decomposition? How should we understand its role in the structure of quantum field theory?

In Section 3, we encountered a number of ostensibly distinct roles that cluster decomposition is supposed to play in quantum field theory: it is necessary for the possibility of experimental science; it allows for the definition of "essentially" localized states of a quantum field; it reflects the short range of forces in the actual world; it prohibits superluminal signaling. At the heart of each of these motivations was a recognition of the importance of being able to decompose a larger physical system, like a quantum field or a collection of particles, into effectively isolated subsystems for the purposes of description, intervention, and measurement.

It is crucial to distinguish two senses in which one might be unable to isolate a quantum system. One the one hand, one might not be able to isolate a system in a bounded spacetime region U from *dynamical influences* coming from events in spatially distant regions V. Dynamical influences of this sort have at least two characteristic features: (i) events in V can change the statistical distribution of results of measurements performed on the system in region U and, as a result, (ii) one cannot specify an independent state for a system in region U, but instead must include information about the state of the universe in other, perhaps distant regions of the universe V. In that case, one cannot specify the state of a system in U (a laboratory, for example), extract predictions for experiments from that state assignment, and test those predictions with an experiment performed in U without including the state of all of the systems in regions outside the lab that could potential dynamically influence the system in U. An inability to dynamically isolate subystems endangers the possibility of locally preparing and describing systems and performing experimental measurements; it really is unclear how experimental science could take place in a universe with widespread failures of isolability in this sense.

On the other hand, one might not be able to isolate a quantum system in U from systems in distant regions V due to *entanglement*. In that case, (i) one *can* independently specify a state for the system in region U without knowing about the state of the universe in a (perhaps distant) region V with which the system in U is entangled. Furthermore, (ii) influences from V *cannot* change the statistical distribution of results of measurements performed on the system in U; events in V can change the *state* of a system in U, but not the statistical distribution of results of measurements performed in U. In fact, this bears importantly on (i): one cannot specify a *pure* state for a system in region U if it is entangled with a system in V, but one can always assign a mixed state to the system in region U.[48] This mixed state will furnish predictions

[48]One might worry that this basic feature of nonrelativistic quantum mechanics is somehow altered in quantum field theory, but Clifton and Halvorson (2001) show that worry is mistaken. Note, also, that if Ada doesn't know that her system is entangled with Bob's

for the results of experiments, and those predictions (and results) are entirely independent of anything happening outside of the region U. The inability to isolate a system due to entanglement may have many surprising consequences, but it does not endanger our ability to do experimental science.

Rather famously, Einstein is often read as conflating these two senses of isolability in a passage written in 1948:

> If one asks what is characteristic of the realm of physical ideas independently of the quantum theory, then above all the following attracts our attention: the concepts of physics refer to a real external world, i.e., ideas are posited of things that claim a 'real existence' independent of the perceiving subject (bodies, fields, etc.)... it appears to be essential for this arrangement of the things introduced in physics that, at a specific time, these things claim an existence independent of one another, insofar as these things 'lie in different parts of space'. Without such an assumption of the mutually independent existence (the 'being-thus') of spatially distant things, an assumption which originates in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such a clean separation.... For the relative independence of spatially distant things (A and B), this idea is characteristic: an external influence on A has no immediate effect on B; this is known as the 'principle of local action', which is applied consistently only in field theory. The complete suspension of this basic principle would make impossible the idea of the existence of (quasi-)closed systems and, thereby, the establishment of empirically testable laws in the sense familiar to us (Einstein, 1948, pp. 321–322) (transl. (Howard, 1985, pp. 187–188)).

Einstein apparently worried that the inability to isolate subsystems in different spatial regions *from entanglement* and assign them independent states would make experimental science impossible. Of course, he was mistaken; he failed to distinguish the consequences of being unable to isolate subsystems from external dynamical influences, which arguably would make experimental science impossible, from the consequences of being unable to isolate subsystems due to entanglement, which does not.

The same conflation is encoded in the standard statement of cluster decomposition as a constraint on the S-matrix. For the S-matrix to factorize

and (falsely) assigns it a pure state, it will make no difference to the predictions she makes, the interventions she can perform, nor the results of her measurements in U. For example, Ada will make the same predictions if she (falsely) assigns her system the pure state $|A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ that she would if she assigned it the mixed state obtained by tracing Bob's subsystem out of the (true) state $|AB\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$. So Ada truly doesn't need to know anything about regions outside of U.

as one subsystem is translated a large spacelike distance from the others, both the Hamiltonian *and* the states must be well behaved. More precisely, the Hamiltonian must not contain any delta functions that conserve momenta among a subset of particles *and* the states used to define the S-matrix must be separable. The Hamiltonian condition ensures that as the two subsystems become increasingly separated, they will become dynamically isolated. This arguably is a precondition for the possibility of experimental science; if it failed generically, a subsystem of particles translated an arbitrarily large spacelike distance away from Ada's lab could influence the statistical distribution of her measurement results. Forces would no longer fall off with distance; subsystems could no longer be treated as effectively localized; superluminal signals could be sent; scattering in the accelerator tunnel at Fermilab could not be modeled independently of events taking place at CERN.

The requirement of the separability of the states used to define the S-matrix, however, is inessential. Its failure would not endanger any of the roles that cluster decomposition is supposed to play in the structure of quantum field theory; indeed, recall the assurances from Duncan in Section 3 that the behaviors that cluster decomposition is intended to rule out should be perfectly compatible with entanglement. Suppose the Hilbert space of a quantum field theory, *per impossible*, contained *only* entangled states; that would make no difference for whether forces fall off with distance, whether superluminal signals could be sent, whether one could assign subsystems essentially localized states, and so on. It certainly would not threaten the possibility of experimental science. That the use of an entangled basis would nevertheless invalidate cluster decomposition is simply an artefact of stating cluster decomposition as a factorizability condition for the S-matrix.

This conflation of these two senses of isolability in the standard statement of cluster decomposition has several drawbacks. It obscures the physical meaning and structural import of cluster decomposition: its formal statement is misaligned with the roles it is supposed to play in quantum field theory. It suggests that an ability to isolate subsystems from entanglement is somehow essential for cluster decomposition to secure the structural features of quantum field theory that it does, when that is actually a purely dynamical matter. It is rather unsatisfactory that cluster decomposition, stated as a factorizability condition for the S-matrix, can fail without threatening any of the structural features of quantum field theory that it supports. But this is exactly what happens for a theory whose Hamiltonian is well-behaved, but whose S-matrix is defined in an entangled basis. Worse, it makes cluster decomposition a basis-dependent statement. The condition is typically presented as one of the pillars on which quantum field theory is built, even a precondition for the possibility of experimental science; certainly the possibility of quantum field theory, or of *experimental science itself*, is not a basis-dependent matter.

We propose redefining cluster decomposition to avoid these drawbacks. Specifically, we propose that cluster decomposition should *not* be defined as a condition requiring the factorization of an S-matrix. Instead, it should be identified with the condition on the Hamiltonian that we discussed in Section 4, originally introduced by Weinberg (1964): *a quantum field theory should be said to satisfy cluster decomposition if and only if its Hamiltonian can be written in the form*

$$H = \sum_{n,m} \int d^3p'_m \cdots d^3p'_1 \, d^3p_1 \cdots d^3p_n$$
$$h_{nm}(p'_m \cdots p'_1 \,,\, p_1 \cdots p_n) \, a^\dagger(p'_m) \cdots a^\dagger(p'_1) \, a(p_1) \cdots a(p_n)$$

where the coefficients h_{nm} are proportional to only a single delta function conserving total energy-momentum. In quantum field theory, recall, this can be accomplished simply by constructing a Hamiltonian as a polynomial of local field operators and their derivatives. This condition is typically presented as sufficient and necessary for cluster decomposition, understood as the factorization of the S-matrix. However, as we saw in Section 5, that is not correct: a Hamiltonian that satisfies this condition can still produce an S-matrix that violates cluster decomposition if one uses a basis of entangled states to define the S-matrix.

This redefinition is, in a sense, a small change: we propose simply *identifying* cluster decomposition with the condition that is standardly (mistakenly) taken to be sufficient and necessary for the factorization of an S-matrix (i.e., what is standardly called cluster decomposition). Nevertheless, recasting cluster decomposition in this way has a number of virtues. The isolability of subsystems that cluster decomposition is meant to secure is *dynamical* isolability: by identifying cluster decomposition with a condition on the Hamiltonian, it becomes a constraint on only the dynamical behavior of a quantum field theory without introducing any irrelevant dependence on states. Furthermore, it renders cluster decomposition basis-independent and avoids any conflict with entanglement; the condition on the Hamiltonian that is typically (but, as we have seen, incorrectly) presented as sufficient and necessary for a quantum field theory to satisfy cluster decomposition is now *truly* sufficient and necessary (albeit by definition). Factorizability properties of the S-matrix are now entailed by cluster decomposition, not identified with it; in a separable basis, the S-matrix will factorize and in an entangled basis, it will not. As long as the Hamiltonian does not *create* correlations between initially separable particles that are localized at spacelike separation, none of the structural features of quantum field theory that cluster decomposition is intended to support will be endangered. It is no longer true that cluster decomposition, now stated as a condition on the Hamiltonian, could fail without threatening any of the structural features of quantum field theory that it supports.

It also emphasizes the origin of the cluster decomposition of VEVs. The vacuum state of a quantum field theory is, after all, the ground state of its Hamiltonian: the structure of the Hamiltonian thus determines the behavior of the vacuum. And indeed, there are multiple ways to see that a Hamiltonian that can be written in the form above has a ground state in which VEVs factorize when one subsystem is translated a large spacelike distance from the others. The most direct route is via the LSZ reduction formula: a scattering amplitude on the RHS of the reduction formula will factorize when a subset of particles localized around $x_1, x_2, \cdots, x_n$ are translated an arbitrarily large spacelike distance from all others if and only if the VEV on the LHS also factorizes when the same translation is applied to the corresponding field operators. Thus if a Hamiltonian generates an S-matrix that factorizes appropriately, its ground state necessarily does the same. More physically, we also saw that the Hamiltonian determines the structure of correlations in its ground state. The kinetic term couples field operators at neighboring points, which generates the correlations; the mass spectrum of the Hamiltonian (specifically, whether the spectrum has a mass gap) determines whether those correlations decay exponentially or polynomially along spacelike directions.

In short, redefining cluster decomposition as we propose allows us to recover ordinary practice on the back a definition of cluster decomposition that, compared to the standard statement, is both more physically transparent and better reflects the structural role that it plays in the architecture of quantum field theory. Insofar as the standard definition of cluster decomposition shapes the structure of quantum field theory, it is due to the dynamical constraints it imposes. A satisfactory definition should reflect that.

7. Conclusion. We have argued that the standard statement of one of the central pillars of quantum field theory, the cluster decomposition principle, is unsatisfactory and proposed a redefinition. Our argument proceeded by demonstrating that the factorization of the S-matrix under large spacelike translations of a subset of particles – what is standardly called cluster decomposition – can fail for two reasons: the Hamiltonian of the theory can be badly behaved *or* there can be entanglement between particles in the states used to define the S-matrix. We argued that the second source of failure is, in a sense, spurious: it does not endanger any of the structural features of quantum field theory that cluster decomposition is typically taken to ground. These structural features, broadly speaking, all concern senses in which a physical system can be decomposed into essentially isolated subsystems that one can assign independent states, upon which one can perform local interventions, and for which the statistical distributions of measurement results are independent of events elsewhere in the universe. Those features depend only on the Hamiltonian of the theory being well-behaved, in the sense of Section 4. Accordingly, our proposed redefinition of cluster decomposition identifies it with a dynamical condition alone. This allows us to recover ordinary practice while relying on a definition of cluster decomposition that, compared to the standard statement, is both more physically transparent and better reflects the structural role that it plays in the architecture of quantum field theory.

References

Araki, H. (1960). On asymptotic behavior of vacuum expectation values at large space-like separation. *Annals of Physics*, 11(2):260–274.

Araki, H. (1999). *Mathematical Theory of Quantum Fields*. Oxford University Press.

Araki, H., Hepp, K., and Ruelle, D. (1962). On the asymptotic behaviour of Wightman functions in spacelike directions. *Helvetica Physica Acta*, 35(3):164–174.

Araujo, J. B., Hiller, B., da Paz, I., Ferreira Jr., M. M., Sampaio, M., and Costa, H. (2019). Measuring qed cross sections via entanglement. *Physical Review D*, 100(10):105018.

Bain, J. (1998). Weinberg on QFT: Demonstrative induction and underdetermination. *Synthese*, 117(1):1–30.

Bain, J. (2016). *CPT Invariance and the Spin–Statistics Connection*. Oxford University Press.

Banks, T. (2008). *Modern Quantum Field Theory*. Cambridge University Press.

Borchers, H.-J. (1962). On structure of the algebra of field operators. *Il Nuovo Cimento (1955-1965)*, 24(2):214–236.

Brown, L. S. (1992). *Quantum Field Theory*. Cambridge University Press, Cambridge.

Buchholz, D. and Dybalski, W. (2023). Scattering in relativistic quantum field theory: basic concepts, tools, and results. *arXiv preprint:math-ph/0509047*.

Butterfield, J. (2007). Stochastic Einstein locality revisited. *The British Journal for the Philosophy of Science*, 58(4):805–867.

Casini, H. and Huerta, M. (2009). Entanglement entropy in free quantum field theory. *Journal of Physics A: Mathematical and Theoretical*, 42(50):504007.

Casini, H. and Huerta, M. (2021). TASI lectures on entanglement in quantum field theory. *arXiv preprint:2201.13310*.

Clifton, R., Feldman, D. V., Halvorson, H., Redhead, M., and Wilce, A. (1998). Superentangled states. *Physical Review A*, 58(1):135.

Clifton, R. and Halvorson, H. (2001). Entanglement and open systems in algebraic quantum field theory. *Studies in History and Philosophy of Modern Physics*, 32(1):1–31.

Deheuvels, R. and Michel, L. E. (1959). *Les problèmes mathématiques de la théorie quantique des champs*, volume 75. Colloques internationaux du CNRS, Paris.

Denef, F. (2012). TASI lectures on complex structures. In *String Theory and its Applications: TASI 2010 From meV to the Planck Scale*, pages 407–512. World Scientific.

Doplicher, S., Kastler, D., and Robinson, D. W. (1966). Covariance algebras in field theory and statistical mechanics. *Communications in Mathematical Physics*, 3(1):1–28.

Duncan, A. (2012). *The Conceptual Framework of Quantum Field Theory*. Oxford University Press, Oxford.

Dybalski, W. (2017). From faddeev–kulish to lsz. towards a non-perturbative description of colliding electrons. *Nuclear Physics B*, 925:455–469.

Earman, J. (2015). Some puzzles and unresolved issues about quantum entanglement. *Erkenntnis*, 80(2):303–337.

Earman, J. and Valente, G. (2014). Relativistic causality in algebraic quantum field theory. *International Studies in the Philosophy of Science*, 28(1):1–48.

Eden, R. J., Landshoff, P. V., Olive, D. I., and Polkinghorne, J. C. (1966). *The Analytic S-Matrix*. Cambridge University Press, London.

Einstein, A. (1948). Quanten-mechanik und wirklichkeit. *Dialectica*, 2(3-4):320–324.

Faleiro, R., Pavão, R., Costa, H. A., Hiller, B., Blin, A. H., and Sampaio, M. (2020). Perturbative approach to entanglement generation in qft using the s matrix. *Journal of Physics A: Mathematical and Theoretical*, 53(36):365301.

Fan, J., Deng, Y., and Huang, Y.-C. (2017). Variation of entanglement entropy and mutual information in fermion-fermion scattering. *Physical Review D*, 95(6):065017.

Fleming, G. and Butterfield, J. (1999). Strange positions. In Butterfield, J. and Pagonis, C., editors, *From Physics to Philosophy*, pages 108–165. Cambridge University Press.

Fredenhagen, K. (1985). A remark on the cluster theorem. *Communications in Mathematical Physics*, 97:461–463.

Fredenhagen, K. (2010). Lille 1957: The birth of the concept of local algebras of observables. *The European Physical Journal H*, 35:239–241.

Frye, C., Hannesdottir, H., Paul, N., Schwartz, M. D., and Yan, K. (2019). Infrared finiteness and forward scattering. *Physical Review D*, 99(5):056015.

Grignani, G. and Semenoff, G. W. (2017). Scattering and momentum space entanglement. *Physics Letters B*, 772:699–702.

Haag, R. (1957/2010). Discussion of the 'axioms' and the asymptotic properties of a local field theory with composite particles. *The European Physical Journal H*, 35:243–253.

Haag, R. (1958). Quantum field theories with composite particles and asymptotic conditions. *Physical Review*, 112:669–673.

Haag, R. (1996). *Local Quantum Physics*. Springer, 2nd edition.

Halvorson, H. (2001). Reeh–Schlieder defeats Newton–Wigner: On alternative localization schemes in relativistic quantum field theory. *Philosophy of Science*, 68(1):111–133.

Hartle, J. B. and Taylor, J. R. (1969). Quantum mechanics of paraparticles. *Physical Review*, 178(5):2043–51.

Headrick, M. (2019). Lectures on entanglement entropy in field theory and holography. *arXiv preprint:1907.08126*.

Hepp, K., Jost, R., Ruelle, D., and Steinmann, O. (1961). Necessary restriction on Wightman functions. *Helvetica Physica Acta*, 34(VI-VIII):542–544.

Howard, D. (1985). Einstein on locality and separability. *Studies in History and Philosophy of Science Part A*, 16(3):171–201.

Jaksland, R. (2020). Entanglement as the world-making relation: Distance from entanglement. *Synthese*, pages 1–33.

Jost, R. and Hepp, K. (1962). Über die Matrixelemente des Translationsoperators. *Helvetica Physica Acta*, 35(34):34–46.

Kastler, D. and Robinson, D. W. (1966). Invariant states in statistical mechanics. *Communications in Mathematical Physics*, 3(3):151–180.

Kharzeev, D. E. and Levin, E. M. (2017). Deep inelastic scattering as a probe of entanglement. *Physical Review D*, 95(11):114008.

Lam, V. (2013). The entanglement structure of quantum field systems. *International Studies in the Philosophy of Science*, 27(1):59–72.

Lehmann, H., Symanzik, K., and Zimmermann, W. (1955). Zur Formulierung quantisierter Feldtheorien. *Il Nuovo Cimento (1955-1965)*, 1:205–225.

Lowdon, P. (2016). Conditions on the violation of the cluster decomposition property in QCD. *Journal of Mathematical Physics*, 57:102302.

Maudlin, T. (2011). *Quantum Non-Locality and Relativity: Third Edition)*. Wiley-Blackwell.

Miller, M. E. (2021). Infrared cancellation and measurement. *Philosophy of Science*, 88(5):1125–1136.

Parisi, G. (1988). *Statistical Field Theory*. Addison-Wesley, Redwood City, CA.

Peschanski, R. and Seki, S. (2016). Entanglement entropy of scattering particles. *Physics Letters B*, 758:89–92.

Peschanski, R. and Seki, S. (2019). Evaluation of entanglement entropy in high energy elastic scattering. *Physical Review D*, 100(7):076012.

Rédei, M. and Summers, S. J. (2002). Local primitive causality and the common cause principle in quantum field theory. *Foundations of Physics*, 32(3):335–355.

Redhead, M. L. G. (1995). More ado about nothing. *Foundations of Physics*, 25(1):123–137.

Ruelle, D. (1962). On the asymptotic condition in quantum field theory. *Helvetica Physica Acta*, 35(3):147–163.

Ruelle, D. (1966). States of physical systems. *Communications in Mathematical Physics*, 3:133–150.

Ruelle, D. (1969). *Statistical Mechanics: Rigorous Results*. W. A. Benjamin.

Ruetsche, L. (2011). *Interpreting Quantum Theories*. Oxford University Press.

Seki, S., Park, I., and Sin, S.-J. (2015). Variation of entanglement entropy in scattering process. *Physics Letters B*, 743:147–153.

Srednicki, M. (1993). Entropy and area. *Physical Review Letters*, 71(5):666.

Steinmann, O. (1966). Symmetrization postulate and cluster property. *Il Nuovo Cimento A (1965 - 1970)*, 44(3):755–767.

Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics, and All That*. W. A. Benjamin.

Strocchi, F. (1978). Local and covariant gauge quantum field theories. cluster property, superselection rules, and the infrared problem. *Physical Review D*, 17(8):2010.

Strocchi, F. (2008). *Symmetry Breaking: second edition*. Springer.

Strocchi, F. (2013). *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*. Oxford University Press.

Summers, S. J. (1990). On the independence of local algebras in quantum field theory. *Reviews in Mathematical Physics*, 2(02):201–247.

Summers, S. J. (2009). Subsystems and independence in relativistic microscopic physics. *Studies in History and Philosophy of Modern Physics*, 40(2):133–141.

Summers, S. J. (2011). Yet more ado about nothing: the remarkable relativistic vacuum state. In Halvorson, H., editor, *Deep Beauty*, pages 317–341. Cambridge University Press.

Swanson, N. (2020). How to be a relativistic spacetime state realist. *The British Journal for the Philosophy of Science*, 71(3):933–957.

Taylor, J. R. (1966). Cluster decomposition of S-matrix elements. *Physical Review*, 142(4):1236.

Valente, G. (2013). Local disentanglement in relativistic quantum field theory. *Studies in History and Philosophy of Modern Physics*, 44(4):424–432.

Wallace, D. (2006). In defence of naiveté: The conceptual status of lagrangian quantum field theory. *Synthese*, 151(1):33–80.

Wallace, D. and Timpson, C. G. (2010). Quantum mechanics on spacetime I: Spacetime state realism. *The British Journal for the Philosophy of Science*, 61(4):697–727.

Weinberg, S. (1964). Systematic solution of multiparticle scattering problems. *Physical Review*, 133:232–256.

Weinberg, S. (1979). Phenomenological Lagrangians. *Physica A*, 96(1–2):327–340.

Weinberg, S. (1995). *The Quantum Theory of Fields*. Cambridge University Press, New York.

Weinberg, S. (1999). What is quantum field theory, and what did we think it was? In Cao, T. Y., editor, *Conceptual Foundations of Quantum Field Theory*, pages 241–51. Cambridge University Press.

Weinberg, S. (2021). On the development of effective field theory. *The European Physical Journal H*, 46(1):1–6.

Wichmann, E. H. and Crichton, J. H. (1963). Cluster decomposition properties of the S matrix. *Physical Review*, 132(6):2788–2799.

Wightman, A. S. (1963). Recent achievements of axiomatic field theory. In Salam, A., editor, *Theoretical Physics: Lectures Presented at the Seminar on Theoretical Physics Organized by the International Atomic Energy Agency from 16 July to 25 August, 1962*, pages 11–58. International Atomic Energy Agency.

Wilson, K. G. (1974). Confinement of quarks. In Korthals-Altes, C., de Rafael, E., and Stora, R., editors, *Recent Progress in Lagrangian Field Theory and Applications: proceedings of the colloquium held at Marseille, France, June 24–28, 1974*, pages 125–147. Centre National de la Recherche Scientifique.

Witten, E. (2018). Notes on some entanglement properties of quantum field theory. *arXiv preprint:1907.08126.*