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Finite-Size Scaling Theory:
Quantitative and Qualitative Approaches to Critical Phenomena

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Abstract The finite-size scaling (FSS) theory is a relatively new and important attempt to study critical phenomena; this paper aims to contribute to clarifying the philosophical significance of this theory. We maintain that, contrary to initial appearances and to some recent claims in the literature, the FSS theory cannot arbitrate the debate between the reductionists and anti-reductionists about phase transitions. Although the theory allows scientists to provide predictions for finite systems, the analysis we carry on here shows that it involves the intertwinement of both finite and infinite systems. But, we argue, the FSS theory has another virtue, as it provides quantitative predictions and explanations for finite systems close to the critical point; it thus complements in a distinctive manner the standard Renormalization Group qualitative approach relying on infinite systems.

Keywords

Finite-size scaling – Phase transitions – Critical phenomena – Renormalization group – Finite systems – Quantitative predictions – Infinite systems

1 Introduction

For more than two decades now, the investigation of the question as to whether thermodynamics reduces to statistical mechanics has focused on phase transitions. It has been argued that the explanations of phenomena such as vaporization or magnetization support an antireductionist, or emergentist view

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(e.g., Batterman 2005, 2011; Morrison 2012, 2015). Crucially, the statistical mechanical explanations of these phase changes require infinite limits to obtain the singularities and divergences appearing in the thermodynamic definitions of phase transitions. Infinite limits are needed, moreover, to explain universal behaviour (i.e., second-order phase transitions), by appealing to renormalization group (RG) methods. Thus, it seems that the statistical mechanics of finite systems is incapable to explain these familiar phenomena as described by thermodynamics.[1]

The question of the reduction of thermodynamics to statistical mechanics has been posed mainly in the context of discussions of limiting reduction (Nickles 1973). In contrast to the earlier, Nagelian concept of reduction, limiting reduction – or asymptotic reduction – has been claimed to be more adequate for the investigation of inter-theoretic relations in physics (Batterman 2005, 2016). Within this approach, an antireductionist position is supported by the existence of singular limits, viz. limits for which the behaviour of systems approaching the limit is qualitatively different from the behaviour of the systems at the limit. This is the case for the thermodynamic limits involved in obtaining second-order phase transitions in statistical mechanics. This problem also arises when we seem to face 'ineliminable' infinite idealizations, viz. mandatory idealizations that involve singular limits. While some authors defend this approach (e.g., Liu 1999 and Bangu 2009), others argue that the appeal to the thermodynamic limit does not threaten reductionism (e.g., Callender 2001, Butterfield 2011, Norton 2012, Menon and Callender 2013).[2] Moreover, it has been claimed that limiting reduction would actually not be adequate for discussing the reduction of phase transitions; so one should amend this approach (Palacios 2019), or even go back to a notion of reduction close to Nagel's approach (Butterfield 2011).

The work in this paper has been prompted by a recent critique of antireductionism relying on a specific theory of phase transitions in finite-N systems, namely the finite-size scaling theory (FSS henceforth).[3] This criticism has been advanced by Hüttemann, Kühn and Terzidis (2015), who maintain that "finite-size scaling theory makes available reductive explanations" (2015, 188). In this paper, we analyze how FSS addresses this issue and claim that, contrary to initial appearances, it does not allow us to straightforwardly arbitrate the debate between reductionism and anti-reductionism. On the one hand, the FSS theory allows scientists to make predictions on critical phenomena for finite-N

[1] As Kadanoff once said, "the philosopher might wish to note that, strictly speaking, no phase transition can ever occur in a finite system" (2009, 778).

[2] See also Ardourel (2018), who follows Menon and Callender (2013) by investigating theories that study phase transitions without the thermodynamic limit. Reutlinger (2017) defends a commonality strategy to explain critical phenomena, which has been recently discussed by Rodriguez (2021). Saatsi and Reutlinger (2018) acknowledge the explanatory indispensability of fixed points but without endorsing anti-reductionism. Finally, Franklin (2019) argues for a lower-level explanation of the scale-invariance of critical systems. For a review of these debates, see Shech (2013) and Bangu (2021).

[3] 'N' refers to the number of degrees of freedom (particles, spins, etc.) constituting the physical system of interest.

systems, which is an important and distinctive virtue of this theory. However, on the other hand, these finite-N predictions still require, in the first place, RG methods and infinite limits to obtain fixed points needed in the FSS theory. Thus the FSS theory does not dispense with infinite limits. Moreover, as we argue, in the FSS theory finite and infinite systems are twice intertwined. Not only are infinite limits required to obtain fixed points from which finite-N predictions are made, but, afterwards, extrapolations from these finite-N predictions are made in order to obtain predictions on infinite systems and thermodynamical bulk. Below we will clarify how this back-and-forth between finite and infinite systems works by distinguishing two kinds of predictions of critical phenomena: quantitative and qualitative predictions.

The aims and the uses of the FSS theory deserve to be investigated since, as we will see, this theory deals with critical phenomena in a distinctive manner. More precisely, our analysis of its content and use in scientific practice will show that the FSS theory has an important virtue – and, moreover, that it this virtue that justifies physicists' interest in proposing and developing it: the theory allows making quantitative predictions (and explanations) for finite-N systems (i.e., predictions of critical exponents and critical temperatures). This is, we stress, a paramount gain of *operational* nature that the FSS theory affords, to be distinguished from the theory's potential role in settling the *foundational* debate between reductionism and anti-reductionism. So even if our analysis of the FSS theory will show that it does not allow us to definitely decide this debate, we would like to distinguish these two perspectives here – the FSS theory as the arbiter of a foundational debate v. FSS as a tool to fulfill of a certain scientific need – since these viewpoints have not been clearly delineated so far, despite the fact that the theory has attracted some attention in the past.[4]

We shall also highlight the fact that the FSS theory is based on RG methods, but as applied to finite systems. Thus, FSS is *not* a conceptual alternative to the usual RG infinite approach, but rather a *practically useful* variant of it. By yielding numerical values for the quantities of interest in finite-N systems, the FSS theory embodies a quantitative approach to critical phenomena in finite systems. As such, it complements the qualitative RG explanations (and predictions) made for infinite systems. In a nutshell, the FSS theory extends the standard approach to critical phenomena by employing RG methods for finite-N systems.

The paper is organized as follows. First, we need to set the stage properly, and the first three brief sections below will be devoted to this task. Thus, in the immediately following section, we sketch the RG approach to second-order phase transitions. Then, we introduce the ideas of the FSS theory; since the theory is still not very familiar to philosophers, we present its basic concepts in section 3. Next (in section 4) we shall come back to one of the main philosoph-

[4] In addition to the authors mentioned above, it features in Menon and Callender's (2013) discussion, and it is examined by Butterfield and Bouatta (2012), Butterfield (2011), and Mainwood (2005). Even if these authors usually refer to "crossover theory" or "finite size crossover theory", the theoretical content turns out to be the same, which is the FSS theory.

ical motivations of this paper, and introduce the distinction underlying our arguments here, between a *qualitative* and a *quantitative* approach to predicting and explaining phase phenomena. Then, in section 5, the crux of the paper, we draw on these preparatory points and offer a detailed discussion of the FSS theory and the RG explanation of the universality of critical phenomena. More specifically, in this section we argue for three key-claims:

(i) The FSS theory affords the physicists the means to recognize differences between two large systems with different values of the number N of components – since, obviously, some predictions about their critical behaviour will be different. Hence, this theory puts the physicists in the position to fill the gap between explanations in the case of infinite and finite systems.

(ii) The FSS approach provides us with the only natural answer to the question as to why the peaks of the graphs describing finite systems are rounded. Without such a theory, all that the physicists have is a qualitative, hand-waving justification: "this is due to finite effects". Thus, we maintain, the standard RG approach to the derivation of the fixed points, albeit necessary, is not sufficient to explain (and predict) quantitatively what is going on at the critical point of a *finite* system.

(iii) The FSS theory involves a two-fold intertwinement of finite *and* infinite systems, and this fact constitutes a challenge to the reductionists willing to invoke FSS in their support.

2 The renormalization group approach to critical phenomena

To begin with, let us recall very briefly the starting point of the philosophical discussions on phase transitions and infinite systems. This is what Kadanoff calls the "extended singularity theorem" (Kadanoff 2013, sect. 2.2; see also Batterman 2017, 562). In a nutshell, the statement implies that an infinite limit is indispensable to recover the singularities of the thermodynanic functions within statistical mechanics. This claim is justified by considering the partition function $Z(N)$ of the two-dimension Ising model with N spins. Its Hamiltonian $\mathcal{H}$ is:

$$\mathcal{H} = -J \sum_{<i,j>} S_i S_j - h \sum_i S_i \tag{1}$$

where the first sum is over nearest-neighbour sites, J is the spin-spin coupling, and h is the external magnetic field. The partition function, defined as $Z(N) = \sum_{i,j} \exp^{-\mathcal{H}/kT}$, where k is the Boltzmann constant and T the temperature, is a finite sum of N exponential functions, i.e. a sum of N analytical functions. The statistical mechanics free energy, defined as $f(N) = -kT \log Z(N)$, is thus also an analytical function. However, the thermodynamic free energy is singular at the phase transition. Consequently, the sole possibility for the statistical mechanics free energy to be singular is to take the limit $N \to \infty$. This indispensability result pertains to any phase transitions. However, as we will see, in the case of second-order phase transitions, the ones in which we are interested, another infinite limit is required.

Second-order phase transitions, or continuous phase transitions, are associated with universal behavior: it is an empirically well-documented fact that different materials, having a widely different internal constitution (e.g., magnets and water), obey the same physical laws in the critical region (i.e., close to the critical temperature). These laws are power laws; their exponents are called 'critical exponents'. Critical phenomena exhibit universality in the sense that the values of the critical exponents are exactly the same for different phase transitions, such as the ferromagnetic-paramagnetic transition or the liquid-vapor transition. As Ken Wilson noted, "the correspondence of exponents does seem remarkable, however, when the values are not round numbers but fractions such as 0.63. The convergence of many systems on these values cannot be coincidental." (Wilson 1979, 174)

Thus, perhaps the main achievement of the RG methods is to allow physicists to derive such laws – and thus to explain this coincidence away. This section sketches the main ideas of this derivation, as a precursor to our account of the FSS theory (in section 3). Let us begin by considering the case of the paramagnetic-ferromagnetic phase transition. Close to the critical temperature T_c, thermodynamic quantities obey the following power laws:

$$M \sim |t|^{\beta} \qquad \chi \sim |t|^{-\gamma} \qquad C \sim |t|^{-\alpha} \tag{2}$$

where M is the magnetization, χ magnetic susceptibility, C specific heat, and $t = (T - T_C)/T_C$ the reduced temperature; α, β, and γ are the critical exponents. As noted, if one studies the liquid-vapor phase transition, one uncovers the very same critical exponents. Another important critical exponent is found in the ferromagnetic-paramagnetic transition, namely the exponent ν, associated with the correlation length ξ of the system. This quantity corresponds to the distance over which spins are still correlated. Close to the critical region, it obeys the power law $\xi \sim |t|^{-\nu}$.

Let us now very briefly introduce the RG methods in the case of the Ising model, the model typically used to describe the ferromagnetic-paramagnetic transition.[5] For that purpose, consider the Hamiltonian $\mathcal{H}$ (see eq. 1). RG methods consist in applying a series of transformations to this Hamiltonian that rescale the system by zooming out; they are called *block spin* transformations (see fig. 1). For example, after one transformation, the initial Hamiltonian $\mathcal{H}$ becomes :

$$\mathcal{H}' = -J' \sum_{<i,j>} S_i S_j - h' \sum_i S_i \tag{3}$$

where the new spins $S_{i,j}$ are effective spins which replace the former spins with a majority rule (e.g., if five spins out of the nine in a small square in the figure 1 are 'up', the effective spin is also set to 'up'). If the distance between two consecutive spins is a, the new distance between effective spins is $a' = l \times a$, with l the scaling factor. Therefore, the effective system is now composed of Nl^{-d} effective spins, with d the dimension of the system (here $d = 2$). The Hamiltonian, the spin-spin coupling J and the external magnetic

[5] See also, e.g., Batterman (2017), Butterfield and Bouatta (2012), or Cardy (1996).

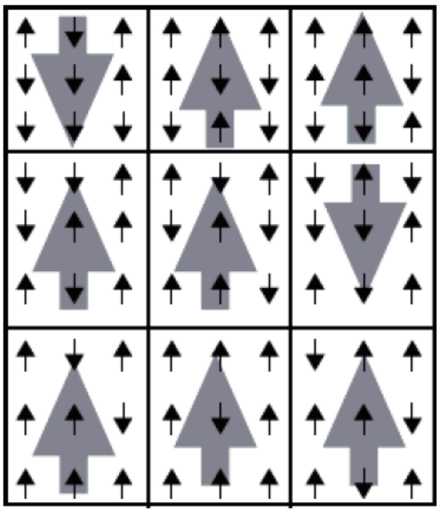

Fig. 1 Illustration of a block spin transformation. Figure extracted from (Kadanoff 2013).

field h are rescaled at each RG transformation with the scaling factor l, such as: $\mathcal{H} \to \mathcal{H}' \to \mathcal{H}'' \to ... \to \mathcal{H}^{(m)}$..., where (m) represents the m^{th} iteration of the block spin transformation. The coupling constants K, which include J and h, transform similarly:

$$K \to K' \to K'' \to ... \to K^{(m)} \quad ... \tag{4}$$

Therefore, after one transformation, the effective correlation length is: $\xi[K'] = \xi[K]/l$. Consequently, the series of the effective correlation lengths is such as :

$$\xi[K] = l\xi[K'] = l^2\xi[K''] = ... = l^m\xi[K^{(m)}] \quad ... \tag{5}$$

where the effective correlation lengths are decreasing with respect to the number of iterations.

The series of RG transformations makes use of a recursion relation R defined as $K' = R(K)$. The mathematical analysis of this recursion relation allows scientists to investigate the critical region. More precisely, the recursion relation exhibits a fixed-point K^* defined as $K^* = R(K^*)$. This means that the effective Hamiltonian and the coupling constants are unchanged by new scaling transformations R. Importantly, as we have seen above in this section, (i) the statistical mechanics phase transitions require the thermodynamic limit (recall the ?extended singularity theorem? above), and (ii) the correlation length diverges at the critical temperature (since $\xi[K] \sim |t|^{-\nu}$). Consequently, the number m of iterations of the RG tends to infinity at the fixed point.[6]

[6] We focus on nontrivial fixed points, i.e., those associated with an infinite correlation length.

This follows from the equation 5 for which $\xi[K] \to \infty$ and $\xi[K^{(m)}]$ is small and thus finite (see Palacios 2019, sect. 2.3).

In the neighborhood of the fixed point, a linearized transformation of the recursion relation R is applied, which leads to power laws for the scaling fields u_i that satisfy the following equation:

$$u'_i = l^{y_i} u_i \tag{6}$$

where y_i are the exponents of the power laws. More specifically, in the case of the Ising model, the scaling fields u_i (and their exponents y_i) are the reduced temperature t (with the exponent y_t) and the magnetic field h (with y_h). Indeed, the Hamiltonian (and the free energy f) varies with the coupling constant $K = J/kT$ (with T the temperature) and h the external magnetic field. The notion of a 'scaling field' comes from the fact that these quantities *scale* with the free energy f in the critical regime, i.e. satisfy the relation:

$$f(t', h') = l^d f(t, h) \tag{7}$$

after one iteration R, with d the dimension of the system.[7]

Now, depending on the sign of the exponents y_i, the scaling variables can either increase or vanish under the RG iterations. If $y_i < 0$, then the scaling variable decreases, and it is called *irrelevant*; if $y_i > 0$, the scaling variable increases, and it is called *relevant*. After some mathematical manipulation, it is then possible to derive the equations for *critical exponents* and the relationships between them. For example, by considering the magnetic susceptibility, and together with (2) above, we obtain that $-\gamma = (d - 2y_h)/y_t$. By using other thermodynamic quantities, e.g., magnetization and heat capacity, we obtain other scaling relations, such as: $\alpha + 2\beta + \gamma = 2$.

3 The finite-size scaling theory

We now turn to the FSS theory, which has been introduced about 50 years ago by Michael Fisher and Michael Barber, among others.[8] Within this theory, the number N of components or the finite size L of systems are explicit variables. Importantly, while divergences occur for quantities such as magnetic susceptibility in the usual RG approach, in this theory, in which L and N are finite, they become rounded peaks.

The basics of the FSS theory were originally developed without the RG framework. Today, however, it is more straightforward to introduce the theory within this framework. As Barber said,

[7] Moreover, the reason why the linearization leads to a power law comes from the semi-group property of the recursion relation R, i.e., $K'' = R(K') = R.R(K)$. For technical details, see Goldenfeld 1992, 237 and 244. See also (Wu 2021) for a recent philosophical analysis of the process of linearization in the vicinity of a fixed point.

[8] See Ferdinand and Fisher (1969), Fisher (1971), Fisher and Barber (1972), Barber (1983), Cardy (1988). This section is based mainly on Barber (1983), Goldenfeld (1992, section 9.11), Pelissetto and Vicari (2002, section 2.2.3), Cardy (1996, section 4.4). See also Privman (1990).

> [R]enormalization group techniques are well-known as powerful methods for investigating bulk critical behaviour. The same ideas can, however, also be used to study finite-size effects and to compute thermodynamic quantities of finite systems. (Barber 1983, 162)

Let us go back to the Ising model.[9] Now, instead of taking the thermodynamic limit $N \to \infty$, we recall that the number N of spins remains finite, albeit large. In that case, according to the extended singularity theorem's Kadanoff (see Section 2), the system can be close but, strictly speaking, not *at* the thermodynamic phase transition.

Nevertheless, the FSS theory will use all the tools of the usual RG approach to infinite systems in order to get information on finite systems. In particular, the FSS theory studies what happens *close* to the fixed point K^* that has been found with the usual RG approach (Section 2). Under these conditions, close to the fixed point K^*, the FSS theory will consider the same RG transformations for the Hamiltonian $\mathcal{H}$ and coupling constants K. In that case, one iteration of the previous RG transformation for the free energy (eq. 7) becomes for a N-finite system:

$$f(t', h', N'^{-1}) = l^d f(t, h, N^{-1}) \tag{8}$$

with $N' = Nl^{-d}$. There is thus an extra term N^{-1} in the free energy for a finite system close to the fixed point, which plays the same role as a scaling field u_i (i.e., t and h). More precisely, the term can be viewed as a *relevant* scaling variable with an exponent $y = d$ corresponding to the dimension of the system. With this procedure, one can derive thermodynamic quantities, such as the magnetic susceptibility $\chi(N)$ or the the heat capacity $C(N)$, which no longer diverge (see Fig. 2). These thermodynamic quantities vary with the size L of the system (for $L^d = N$). Specifically, as it can be seen from figure 2, the peaks heighten as the size L increases. However, since the size L of the system remains finite, the peaks remain rounded. Moreover, the maxima of the magnetic susceptibility and the heat capacity are *shifted* with respect to the critical temperature T_C. This shift scales like the quantity $L^{-1/\nu}$.

This behaviour for the thermodynamic quantities in finite systems obtained with the FSS theory occurs when the size L of the system is of the same order of magnitude as the correlation length ξ, i.e., when $\xi/L \sim 1$. When the correlation length ξ is much smaller than the size L of the system, i.e., when $\xi/L \ll 1$, the thermodynamic quantities behave as if the system was infinite. However, in that case, it is not possible to get significant information on how finite systems behave close to a fixed point. Indeed, as we have seen, the correlation length ξ diverges at the critical temperature (Section 2). Therefore, if ξ is much smaller than the size L of the system, the system is far from the critical regime. In order to get information on the critical region, one has to study the behaviour of a system with large correlation length, i.e., when $\xi/L \sim 1$ (which is the maximum size of ξ for a finite system). Finite-size effects appear in this region,

[9] We focus on the FSS theory within the real-space approach of RG techniques. However, the FSS theory can also be studied within a field-theoretic approach (e.g., Suzuki (1977); Brézin and Zinn-Justin (1985)).

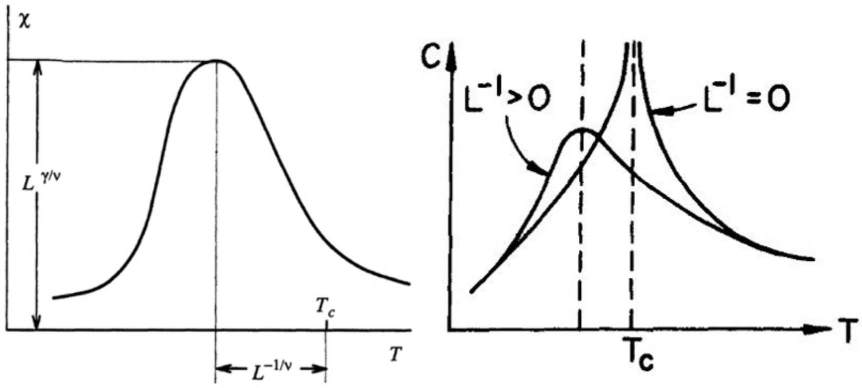

Fig. 2 Predictions of thermodynamic quantities with the FSS theory. On the left, the magnetic susceptibility χ for finite systems. It grows with $L^{\gamma/\nu}$ and its maximum is shifted to $L^{-1/\nu}$ with respect to the critical temperature T_C. On the right, the heat capacity C. In the region $L^{-1} > 0$, i.e., for finite systems, its maximum is also shifted to $L^{-1/\nu}$ with respect to T_C. It also grows with $L^{\alpha/\nu}$. The region $L^{-1} = 0$ corresponds to the case of infinite systems, with a divergence and a maximum at T_C. (Figures extracted from Cardy (1996, 74) and Goldenfeld (1992, 281), with a few modifications.)

and they lead to different behaviours when compared to infinite systems. This is so since there is a shift of the maxima of the magnetic susceptibility and of the heat capacity; moreover, the height of the peaks is finite. But note that these changes in behaviour are noticeable for *small* systems, since the shift varies with $L^{-1/\nu}$ and the maximum height varies with $L^{\gamma/\nu}$ or $L^{\alpha/\nu}$. In contrast, for *large* systems, these shifts decrease, and the maximum heights increase. If one takes the infinite limit $L \to \infty$, the shifts tend to zero and the rounded peaks turn into divergences.

As we have just seen, the FSS theory provides predictions about the physical quantities (e.g., $\chi(N)$ or $C(N)$) that link the size of the system L with the critical exponents α, β, γ, ν and the critical temperature T_c. Therefore, by investigating finite systems, one can find the critical quantities otherwise defined only in the thermodynamic limit. We shall also note that FSS makes possible to investigate finite-N systems by running numerical simulations, such as Monte Carlo simulations. It is common to hear physicists praising the FSS theory for allowing precisely this method of generating critical values. Pelissetto and Vicari (2002, 581) note:

> The FSS techniques are particularly important in numerical work. With respect to the infinite-volume methods, they do not need to satisfy the condition $\xi \ll L$. One can work with $\xi \sim L$ and thus is better able to probe the critical region. FSS Monte Carlo simulations are at present one of the most effective techniques for the determination of critical quantities.

Different results of the FSS theory can be exploited to link the size dependence of the systems to the critical quantities. In particular, one can use the *shift* of the critical temperature between finite systems and infinite systems. Let us call $T_c(L)$ the temperature associated with the maximum of the finite thermodynamic quantities, namely the "effective critical temperature". It varies as follows (Ferrenberg *al.* 2018, 3; see Fig. 2 on the left):

$$T_c - T_c(L) \sim L^{-1/\nu} \tag{9}$$

By studying $T_c(L)$ for different L, one can extract T_c and ν. The numerical values of the other critical exponents α, β, and γ are determined by investigating the amplitudes of the thermodynamic functions (Pelissetto and Vicari 2002, 583).

4 Qualitative and quantitative approaches to phase transitions

As announced at the outset, in this section we shall put the technicalities about critical phenomena to one side, and explain how our analysis here, although about a seemingly narrow issue, can be integrated into a more general discussion about the methodology of physics. To begin, let us quote Kadanoff one more time:

> When we look at a natural system, we tend to see phase transitions that look very sharp indeed, but are actually slightly rounded. However, a *conceptual understanding* of phase transitions requires that we consider the limiting, infinite-N case. (2013, 156; Our emphasis)

Not only do we agree with these remarks, but we also note that they gesture at the larger philosophical point we would like to convey in this paper. In particular, we draw attention to a certain complementarity implied by Kadanoff's remarks. He talks about a "conceptual understanding" of phase transitions, and this refers to a kind of idealized understanding of these phenomena. But, it is fair to assume, this type of understanding is to be supplemented by a more realistic understanding of them, since, after all, the systems of interest are non-idealized (are finite). This is to say that while a conceptual understanding of the transitions between phases would amount to comprehending their existence and dynamics in *qualitative* terms, one may ask whether physics has the resources to convey an understanding of them in *quantitative* terms as well.

Thus, although it is clear that a conceptual-qualitative approach to critical phenomena requires the appeal to (infinite) idealizations, the question that immediately arises is what can be said, if anything, about the real, non-idealized, finite systems. When it comes to them, one may wonder, more specifically, whether we can also obtain an understanding as to why the shapes of the graphs coming from the laboratories are rounded (that is, are neither divergent, nor displaying kinks). Also, whether it is possible to gain an understanding of where the actual numerical values characterizing the transitions come from (e.g., the critical temperatures). The overarching question then is whether we can acquire this kind of quantitative understanding – both predictive and explanatory – of the real, finite-N thermal systems in the critical regime. And, as we hope to demonstrate here, it is precisely the FSS theory that provides affirmative answers to these *practical* questions.

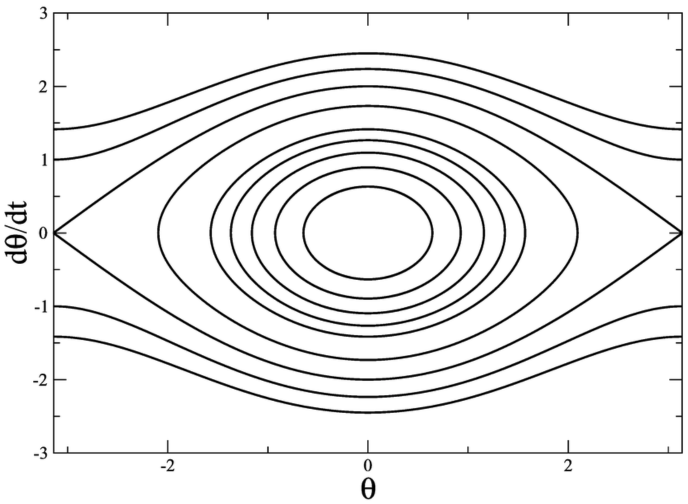

Fig. 3 Phase space portrait for the nonlinear simple pendulum. Figure extracted from Amore et al. 2007.

Now let us also point out that the kind of qualitative-quantitative complementarity signaled here has a larger methodological signification. This complementarity can be discerned in many areas of physics, not only in the domain of critical phenomena. We can see this if we consider an entirely different system, the simple pendulum (displaced by an angle θ). For a realistic, 'large'-angle pendulum, it is known that its phase space portrait *deviates* from an ellipsoidal shape – which is the shape of the phase space for a 'small'-angle pendulum, when the approximation $\sin(\theta) = \theta$ holds. See Figure 3; the smaller the angle θ, the closer the shape is to an ellipse. So, a natural question to ask here is how to account for these deviations. And, facing this query, we can provide, on the one hand, a *qualitative*, hand-waving type of answer: 'there are non-linear terms in the differential equation that describes the motion of the bob'. But, on the other hand, if we would like a *quantitative* answer, i.e., one enabling us to understand why the shape looks exactly as we see it, we have to solve the complicated differential equation $d^2\theta/dt^2 = -\frac{g}{l} sin(\theta)$.[10] In an analogous fashion, when it comes to statistical mechanical systems, if we ask why there

[10] To extract quantitative predictions from this equation for 'large' angles, we can use the Taylor expansion of $\sin(x) \approx x - x^3/3! + x^5/5! \ldots$. We can also use the analytical solution of this equation (Belendez et al. 2007, 647).

are rounded peaks in the isotherms, then, similarly, we can provide either a *qualitative*, hand-waving answer – as we recall, that 'the system is finite' – or we can indicate a *quantitative* reason, case in which we have to use the FSS.

The important point is then that the study of critical phenomena is not the only domain where this complementarity manifests itself. As we saw, it can be discerned in the study of the simple pendulum and, we presume, in many other areas in physics. We face, roughly speaking, the same situation: an unrealistic, idealized system – an infinite statistical mechanical system, and a very small angle pendulum (i.e. a harmonic oscillator), respectively – provides a kind of 'conceptual', qualitative understanding. While this type of qualitative insight is extremely important to have, we may also want to go beyond mere hand-waving and search for quantitative answers. In these two cases, these idealized systems give us shapes that do not really occur in nature (sharp corners and perfect ellipses, respectively). So, if we want to explain/predict in quantitative terms the shapes that are in fact observed, and which deviate from these 'ideal' shapes (i.e., that the peaks are rounded, and that the elliptical trajectories are distorted), then we need to appeal to the FSS, or to solve the equation of motion for a realistic large-angle pendulum, respectively.

So, to return to our main concern, recall that we aim to tackle the question about the relevance of the FSS theory: why have physicists been interested in introducing and developing it? While, as we'll see, its role in the debate on reductionism is rather unclear, the main function of the FSS theory is different: it is primarily *operational*. We call it thus since FSS allows making quantitive predictions and explanations for finite systems in the critical region (again, by applying RG methods to finite systems.)

To be able to do this, the key-move is to regard the variable N^{-1} as a new scaling field that scales with the other ones. It is this scaling property that allows one to derive the quantitative behaviour of physical quantities in finite systems, such as the magnetic susceptibility or the heat capacity. The FSS theory thus extends the RG treatment of critical phenomena, and makes possible a quantitative approach in addition to a qualitative one.

5 RG explanations and the FSS theory: intertwining finite and infinite systems

To begin, the first key-point we would like to stress is that the FSS theory is *not* an alternative, or rival theory, to the usual RG approach to critical phenomena. Indeed, it shares the same fundamental principles (i.e., it is based on scaling and invariance properties), but it aims to deal with *finite* systems. This section analyses how the FSS theory complements the usual RG approach to predict and explain critical phenomena in finite systems. We will highlight the fact that the theory appeals to infinite systems, hence it cannot serve the reductionists' philosophical agenda straightforwardly.

5.1 Quantitative predictions on criticality for finite systems

As we have seen in Section 2, it is generally claimed that the RG approach provides explanations for *infinite* systems since they are mandatory to obtain fixed points.[11] Then, the key-question becomes, what about the *finite* systems? It is noteworthy that this aspect is not ignored by Batterman (2019). After claiming that critical phenomena necessarily occur within infinite systems, he says the following about the finite systems:

> [S]ystems that are near criticality (real, large finite systems) will start off close to the critical systems and their behaviour can be understood by examining the topology of the RG flow in the neighborhood of the fixed point. So, the RG explains the behaviour of near critical, real systems. It explains what is going on in the neighborhood of the critical point [...]. (2019, 39)[12]

He stresses this idea (in a footnote): "if RG only explained the behaviour of idealized systems, it would not be a big deal. Hardly worthy a Nobel prize!" (2019, 39). The claim is then that the RG approach also explains what happens in finite and real systems, not only in infinite systems. This comes about from an investigation of the topology of the RG flow in the neighborhood of a fixed point.

This is a crucial juncture in the argument here, and it is imperative to clarify Batterman's claim: this kind of explanation (and prediction) of the behaviour of finite systems is *qualitative*. When it comes to the *quantitative* aspects of the critical-point behavior of a real, N-finite system, we stress that it is precisely the FSS theory that allows us to make such predictions and explanations. Without the resources afforded by this theory, we cannot discern any difference between two large systems with different values of the number N of components. Yet, obviously, some predictions regarding the critical behaviour of those two finite systems will be different; and, again, it is the FSS theory that explains such differences. Therefore, this theory allows one to fill the gap between explanations in the case of infinite and finite systems. This is achieved even if one requires topology and fixed points. Hence, while we agree with Batterman that physicists are not helpless when it comes to dealing with finite systems, we stress that the key-factor responsible for their quantitative success is the FSS theory, which complements the RG methods.[13]

[11] Incidentally, this is how the "paradox" of infinite idealizations arises (Shech 2013): although infinite systems are required to explain phase transitions and critical phenomena, real systems are finite systems.

[12] A similar idea appears in an earlier paper: "the RG is not just a theory of the critical point, but rather it is a theory of the critical region. And, this covers large but finite systems." (Batterman 2017, 571).

[13] In fairness to Batterman, both his (2019) and his (2017) discuss the case of the finite systems only in passing. For instance, in footnote 15 in (2017, 571), he writes: "If we want to explain the universal behavior of finite but large systems using the RG, then we need to find a fixed point and, to my knowledge, this requires an infinite system". While this is true under the condition given (by using the RG, i.e., in the standard approach), the explanation *is* possible using the FSS theory.

So, to emphasize, the FSS theory allows us to provide quantitative details regarding the behaviour of real, finite (and large) systems approaching criticality – and this is something that the usual RG approach cannot provide. This ability to deal with the real systems is a natural consequence of the finitistic approach to RG methods taken by FSS, in which the variable N^{-1} scales with the other scaling fields. This leads to finite-N predictions for physical quantities in finite systems, such as $\chi(N)$ and $C(N)$ (see Section 3). Therefore, if scientists want to show that, for instance, water and 3-d ferromagnets have the same critical exponent, and also explain what is observed experimentally, then the FSS theory plays a key-role. Nevertheless, we do not claim that *all* predictions with the FSS theory for large N finite systems are empirically distinguishable from the predictions made by using the usual RG approach, because of experimental resolution. However, for many finite systems, the FSS theory provides observable predictions that do differ from the predictions made working with infinite systems (e.g., see Lavis *et al.* 2021, 51).

Also crucially, without the FSS approach, physicists cannot explain why finite systems' peaks are rounded. What they usually give is a qualitative and, after all, vague – albeit fundamentally correct! – justification, e.g., that this is "due to finite effects". Lavis *et al.* (2021) – to our knowledge the most recent paper that investigates the FSS theory – supports this view, by making what is in effect the contrapositive of our point here, in the form of two observations; namely, that FSS (i) "explains in a quantitative way, how singularities that might occur in infinite systems are smoothed out by finite-size effects"(2021, 50), and (ii) gives "a quantitative measure of the deviations of critical phenomena, as observed in finite systems, from the behaviour expected for infinite system size" (2021, 50). In other words, the derivation of fixed points within the standard RG approach is necessary, but not sufficient, to explain (and predict) quantitatively what happens in *finite* systems.

5.2 Universality explanations and finite systems

Recall that we must explain both (i) the universality of critical phenomena and (ii) why experimental data never exhibit divergences – and that, importantly, we need a quantitative description of this latter fact. Insofar as the FSS theory is being based on the usual RG approach, while also extending it by providing finite-size predictions, the theory allows one to address the two questions.

As we pointed out, the FSS theory builds on RG methods. This means, first of all, that this theory avoids the objection that one does not need to resort to RG to explain critical phenomena in finite systems.[14] As we saw, there are indeed RG flows for finite systems within the FSS theory. Besides, if N is sufficiently large, the recursion relation R involved in the FSS theory is *identical* to the one in the infinite case. This is because the RG transformations

[14] Regarding Butterfield's work (2011), Morrison points out that "despite the explanatory power of fixed points, Butterfield (2011) has recently claimed that one needn't resort to RG in explaining phase transitions" (Morrison 2015, 110).

are local transformations. According to Cardy : "After rescaling, the coarse-grained system has an effective size L/l (with our notation). Being local, the renormalization group transformation in the finite system will be *identical* with that in the infinite one, if L/a is large" (Cardy 1988, 3. Our emphasis). Let us consider the block spin transformation once again (Section 2). At each iteration m of this RG transformation, a Hamiltonian $\mathcal{H}^{(m)}$ (defined with $\tilde{N}$ spins) is replaced by a new one $\mathcal{H}^{(m+1)}$ with $\tilde{N}l^{-d}$ spins. This transformation is *local* in the sense that it involves the relation $\mathcal{H}^{(m)} \to \mathcal{H}^{(m+1)}$. This local transformation is the same when N is finite and large, and when it is infinite (See also Barber (1983, 164)). However, the difference is that when N is finite the series of the iterations halts when the number of effective spins is zero. By contrast, with an infinite system, the block spin transformation continues until the effective Hamiltonian reaches the fixed point, which requires an infinite number of iterations (See, again, the details in section 2).

Getting back to the issue of explanation, let us further note that, roughly speaking, we take *scale invariance* to be the core explanatory ingredient of the universality of critical phenomena within the FSS theory. This is not very surprising, of course, since it already plays a crucial explanatory role within the usual RG approach of critical phenomena. More precisely put, the FSS theory is based on RG iterations with a recursion relation R applied to couplings K of Hamiltonians. The repeated application of this procedure, especially the coarse-graining step (e.g., the block spin transformation), *eliminates microscopic details*. They are integrated into an averaging rule, keeping what is common at different scales. This recursion relation rescales Hamiltonian models by keeping some properties *invariant*, viz. the partition function and the free energy (Lesne and Laguës 1998, 80; Cardy 1996, 33). This covariance ensures that the models at different scales satisfy the same physics. As Lesne and Laguës stress: "Covariance therefore expresses the essential objective of renormalisation: to exploit the inalterability of the physical reality as we change our manner of observing and describing it" (1998, 80). In that sense, the recursion relation at work in the FSS theory is an explanatory ingredient for universality. In other words, this recursion relation allows us to describe a system and a rescaled system as systems satisfying the same properties, which are *scale invariant*.

As we recall, the N^{-1} variable is a genuinely new scaling variable within the FSS theory. This means that N^{-1} scales with the relevant couplings, such as the thermal t and the magnetic h scaling variables. The N^{-1} variable is thus involved in describing universal behaviour since the N-dependence in the free energy f satisfies a *scale-invariant* law, which can be rewritten in a generic way as:

$$f(\lambda x) = \lambda f(x) \tag{10}$$

where $x = N^{-1}$, $\lambda = l^{md}$ in the case of the free energy, with l the scaling factor, m the number of applications of the recursion relation, and d the dimension. This property comes from the partition function which has an N-dependence that satisfies this scale invariance. It makes N^{-1} a *relevant* scaling variable, with a positive exponent $y_i = d$. Unlike irrelevant scaling variables that vanish the more the recursion relation is applied, the N^{-1} scaling variable remains a relevant variable.

However, as is perhaps clear at this point, these ingredients are *not* sufficient to explain the universality of behaviour displayed by critical phenomena. We stress that, without the usual RG approach, i.e., without employing the *infinite* limits, the FSS theory cannot explain why two different systems belong to the same universality class. Indeed, in order to exemplify a university class, we need to get a fixed point, which is only obtained with infinite limits for the number of spins and the number of iterations of RG transformations (Section 2). But note that the FSS theory is *not* concerned with showing that two finite systems have the same fixed points. As we have seen, the fixed points are already taken to be known within this theory. Instead, the theory is concerned with the predictions of thermodynamic quantities for finite systems *close* to the fixed point.

5.3 Back-and-forth between finite and infinite systems

This analysis of the FSS theory allows us to shed light not only on a distinctive practical aspect regarding the predictions and explanations in finite systems, but also on a conceptual-philosophical consequence overlooked in the literature. The FSS theory is involved in a kind of back-and-forth between infinite systems and finite systems, and this is especially evident when physicists use this theory to make quantitative predictions on critical phenomena and phase transitions. On the one hand, the FSS theory requires the RG methods – and thus the infinite limits in order to determine the fixed points and the topology of RG flows (for both the number N of degrees of freedom and the number m of RG transformations). On the other hand, from this qualitative knowledge of fixed points, the FSS theory predicts thermodynamical quantities for finite systems (with an explicitly finite variable such as the number N of particles or the size L of systems).

However, the FSS theory is also used to *extrapolate*, from finite systems, the values of thermodynamical quantities for infinite systems. In practice, the numerical values of critical exponents (for infinite systems) are often calculated with the FSS theory, as an extrapolation of the thermodynamic quantities in finite systems. As the physicists Ferrenberg et *al.* point out:

> At a second-order phase transition the critical behaviour of a system in the thermodynamic limit can be *extracted* from the size dependence of the free energy density. (2018, 3)

As we have seen in section 3, a common method for that purpose is to use the relation (9) between the critical temperature T_c, defined for infinite systems, and the effective critical temperature $T_c(L)$, defined for finite systems (see Fig. 4). In this figure, we can see how predictions for finite systems, with different values of the size L, converge when the size L tends to infinity.

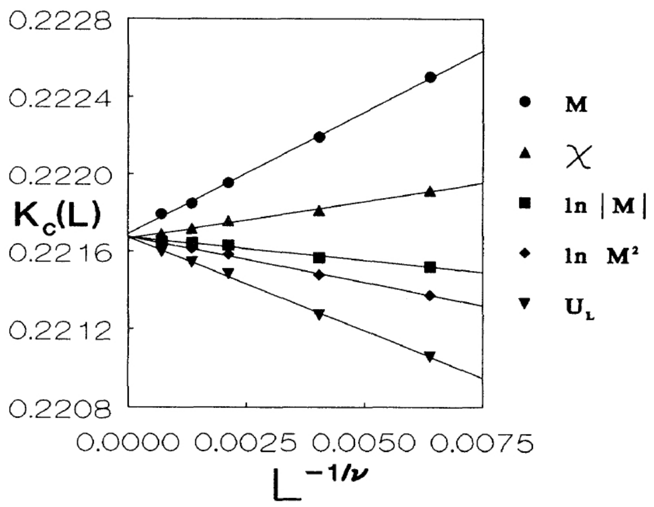

Fig. 4 We can see how the critical temperature for infinite systems is extracted from data on finite systems. Here, the effective critical temperature $T_c(L)$ is expressed here with the quantity $K_c(L)$. When $L^{-1/\nu}$ tends to 0 ($L \to \infty$), $K_C(L)$ tends to K_C, which corresponds to the critical temperature for an infinite system. (Figure extracted from Ferrenberg and Landau 1991, 5088).

Therefore, the FSS theory does not only provide quantitative predictions for *finite* systems, but also (and perhaps foremost), it provides quantitative predictions for *infinite* systems too. So, in a nutshell, infinite systems are used to obtain topological information on the RG flows; then, finite systems are used to obtain quantitative relations on thermodynamical quantities on finite systems. Finally, these quantitative relations are re-used to obtain numerical values for infinite systems. This reveals how finite and infinite systems are strongly intertwined within the FSS theory.[15] For these reasons, it should be clear that the challenge to the reductionists relying on the FSS theory is reframed: they will now need to provide a reductionist-friendly account of the topological information on the RG flows. Although initially it may have seemed that the FSS's achievements served the reductionists' philosophical agenda, our analysis shows that this is actually far from clear.

[15] We do not have in mind other simple examples in physics with such back-and-forth between infinite and finite systems – and we do not claim that there are any. Nevertheless, this feature seems to be quite distinctive to some physical phenomena, and maybe phenomena described by asymptotic theories.

6 Conclusion

The FSS theory has mostly been discussed in the literature as a means to break the tie between reductionism and anti-reductionism in favour of the former; here, however, we have suggested that this direction of research should be reviewed. Insofar as infinite limits are required, the theory features a sophisticated intertwinement between infinite and finite systems, and thus its successes do not speak in favor of reductionism. Thus, while this philosophical role assigned to the theory cannot be said to be definitely established, we highlighted another distinctive aspect of it which, we believe, should not be overlooked; specifically, we argued that it has an operational role. This role is by no means unimportant; it is actually vital *in practice*, insofar as it enables the physicists to use (operate within) the RG framework, by obtaining concrete predictions (and explanations) for finite systems. The FSS theory allows scientists to complement the RG approach to critical phenomena, by providing quantitative predictions and explanations for finite *and* infinite systems.

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