THE PREDICATE OF THE CURRENT MATHEMATICAL KNOWLEDGE INCREASES THE SCOPE OF MATHEMATICS WHAT DISTINGUISHES MATHEMATICS FROM OTHER FIELDS OF STUDY

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Abstract. Nicolas D. Goodman observed that epistemic notions increase the understanding of mathematics without changing its content. We show that the predicate of the current mathematical knowledge increases the scope of mathematics. This distinguishes mathematics from other fields of study. We present statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that refer to the current mathematical knowledge on $\mathcal{X}$. We assume that the current mathematical knowledge is a finite set of statements, which is time-dependent. For a set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, we define which elements of $\mathcal{X}$ are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. (1) *A known algorithm with no input returns an integer* n *satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. (2) *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$. (3) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$. (4) *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is infinite.* (5) $\mathcal{X}$ *is naturally defined. The infiniteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.* The set $\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains more than } 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \textit{ primes of the form } k! + 1\}$ satisfies Conditions (1)-(5) except the requirement that $\mathcal{X}$ is naturally defined. $501893 \in \mathcal{X}$. Condition (1) holds with $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains at least } 30 \textit{ primes of the form } k! + 1\}$. We present a table that shows satisfiable conjunctions of the form #(Condition 1) $\wedge$ (Condition 2) $\wedge$ #(Condition 3) $\wedge$ (Condition 4) $\wedge$ #(Condition 5), where # denotes the negation $\neg$ or the absence of any symbol. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau's conjecture implies the following unproven statement Φ: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, (((24!)!)!)!]$. We heuristically justify the statement Φ. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$. We present a new heuristic argument for the infiniteness of $\mathcal{P}_{n^2+1}$, which is not based on the statement Φ.

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1. Introduction

This is an expanded and revised version of the article [18]. The main results were presented at the 25th Conference Applications of Logic in Philosophy and the Foundations of Mathematics, see `http://applications-of-logic.uni.wroc.pl/XXV-Konferencja-Zastosowania-Logiki-w-Filozofii-i-Podstawach-Matematyki`.

Nicolas D. Goodman observed that epistemic notions increase the understanding of mathematics without changing its content, see [5]. We show that the predicate of the current mathematical knowledge, which is not studied in [5], increases the scope of mathematics. This distinguishes mathematics from other fields of study. We present statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that refer to the current mathematical knowledge on $\mathcal{X}$. We assume that the current mathematical knowledge is a finite set of statements, which is time-dependent.

2. Basic definitions

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [12], [14, p. 9]. A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs:* 1 *(one),* + *(addition),* − *(subtraction),* · *(multiplication),* ^ *(exponentiation with exponent in $\mathbb{N}$),* ! *(factorial of a non-negative integer),* (*(left parenthesis),*) *(right parenthesis).*

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is smaller than $(((((((9!)!)!)!)!)!)!)!$. The number t is an unknown element of the set of twin primes.

Definition 2. *Conditions* (1)-(5) *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
(1) *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
(2) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
(3) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.
(4) *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.*
(5) *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Condition (3) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Example 1. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies Condition* (3).

Let $[\cdot]$ denote the integer part function.

Example 2. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } \left[\frac{(((((((9!)!)!)!)!)!)!)!}{\pi}\right] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$$

does not satisfy Condition (3) *because we know an algorithm with no input that computes* $\left[\frac{(((((((9!)!)!)!)!)!)!)!}{\pi}\right]$. *The set of known elements of $\mathcal{X}$ is empty. Hence, Condition* (5) *fails for $\mathcal{X}$.*

Example 3. *(*[2], [12], [14, p. 9]*). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ *or there exists* $k \in \mathbb{N}$ *such that*

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h.

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

is decidable. This $\mathcal{X}$ *satisfies Conditions* `(1)` *and* `(3)` *and does not satisfy Conditions* `(2)`, `(4)`*, and* `(5)`. *These facts will hold forever.*

Statement 1. *Condition* `(1)` *remains unproven for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\texttt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition `(1)`, but the algorithm $\texttt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

3. Main results

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [15], [16], [20].

Statement 2. *The statement*

$$\exists n \in \mathbb{N}\ (\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n.

Statement 3. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions `(1)-(4)`. *Condition* `(5)` *fails for* $\mathcal{X}$.

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \ldots, 10^6\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. Due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^6)$, see [9]. Thus Condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, Condition `(1)` holds with $n = 10^6$. Condition `(5)` fails as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, 10^6\}$. □

Statements 4 and 6 provide stronger examples.

Conjecture 1. *(*[1, p. 443], [6]*). The are infinitely many primes of the form* $k! + 1$.

For a non-negative integer n, let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

Statement 4. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1,n] \textit{ contains more than } \rho(n) \textit{ primes of the form } k!+1\}$$

satisfies Conditions `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.* $501893 \in \mathcal{X}$. *Condition* `(1)` *holds with* $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0,501893]) = 159827$. $\mathcal{X} \cap [501894,\infty) = \{n \in \mathbb{N} : \textit{the interval } [-1,n] \textit{ contains at least } 30 \textit{ primes of the form } k!+1\}$.

Proof. For every integer $n \geqslant 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!,\infty)$, then $n+1$, $n+2$, $n+3$, $\ldots \in \mathcal{X}$. Hence, Condition `(1)` holds with $n = 11!-1$. We explicitly know 24 positive integers k such that $k!+1$ is prime, see [4]. The inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is prime}\}) > 24$ remains unproven. Since $24 < 30$, Condition `(3)` holds. The interval $[-1,11!-1]$ contains exactly three primes of the form $k!+1$: $1!+1$, $2!+1$, $3!+1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line [501893.0, 159827], which proves Condition `(1)` with $n = 501893$ and Condition `(4)` with $\mathrm{card}(\mathcal{X}) \geqslant 159827$. □

Definition 3. *Conditions* `(1a)-(5a)` *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

`(1a)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty,n]$.

`(2a)` *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

`(3a)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) < \omega$.

`(4a)` *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is finite.*

`(5a)` $\mathcal{X}$ *is naturally defined. The finiteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

Statement 5. *The set*

$$\mathcal{X} = \Big\{n \in \mathbb{N} : \textit{ the interval } [-1,n] \textit{ contains more than}$$

$$6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \textit{ squares of the form } k!+1\Big\}$$

satisfies Conditions `(1a)-(5a)` *except the requirement that* $\mathcal{X}$ *is naturally defined.* $95151 \in \mathcal{X}$. *Condition* `(1a)` *holds with* $n = 95151$. $\mathrm{card}(\mathcal{X} \cap [0,95151]) = 30311$. $\mathcal{X} \cap [95152,\infty) = \{n \in \mathbb{N} : \textit{the interval } [-1,n] \textit{ contains at least } 7 \textit{ squares of the form } k!+1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in \mathcal{X} \cap (10^6, \infty)$, then $n+1,\ n+2,\ n+3,\ \ldots \in \mathcal{X}$. Hence, Condition `(1a)` holds with $n = 10^6$. It is conjectured that $k!+1$ is a square only for $k \in \{4,5,7\}$, see [19, p. 297]. Hence, the inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is a square}\}) > 3$ remains unproven. Since $3 < 7$, Condition `(3a)` holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k!+1$: $4!+1$, $5!+1$, $7!+1$. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line [95151.0, 30311], which proves Condition `(1a)` with $n = 95151$ and Condition `(4a)` with $\mathrm{card}(\mathcal{X}) \geqslant 30311$. □

To formulate Statement 6 and its proof, we need some lemmas. For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. For a non-negative integer n, let $\theta_1(n)$ denote the largest integer divisor of 10^{10} smaller than n.

Lemma 1. *For every integer* $j > 10^{10^{10}}$, $\theta(j) = 10^{10^{10}}$. *For every integer* $j > 10^{10}$, $\theta_1(j) = 10^{10}$.

Lemma 2. *For every integer* $j \in (6553600, 7812500]$, $\theta(j) = 6553600$.

Proof. 6553600 equals $2^{18} \cdot 5^2$ and divides $10^{10^{10}}$. $7812500 < 2^{24}$. $7812500 < 5^{10}$. We need to prove that every integer $j \in (6553600, 7812500)$ does not divide $10^{10^{10}}$. It holds as the set

$$\left\{2^u \cdot 5^v : (u \in \{0,\ldots,23\}) \wedge (v \in \{0,\ldots,9\})\right\}$$

contains 6553600 and 7812500 as consecutive elements. □

Lemma 3. *The number* 6553600^2+1 *is prime.*

Proof. The following `PARI/GP` ([10]) command

```
isprime(6553600^2+1,{flag=2})
```

returns 1. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([21, p. 226]). It rigorously shows that the number 6553600^2+1 is prime. □

In the next lemmas, the execution of the command `isprime(`n`,{flag=2})` proves the primality of n. Let κ denote the function

$$\mathbb{N} \ni n \xrightarrow{\kappa} the\_exponent\_of\_2\_in\_the\_prime\_factorization\_of\_\underbrace{n+1} \in \mathbb{N}$$

Lemma 4. *The set* $\mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1$ *is prime*$\}$ *is infinite.*

Proof. Let $i = 142101504$. By the inequality $2^i \geqslant 2 + 10^{10}$ and Lemma 1, for every non-negative integer m, the number

$$\left(\theta_1\left(2^i \cdot (2m+1) - 1\right) + \kappa\left(2^i \cdot (2m+1) - 1\right)\right)^2 + 1 = \left(10^{10} + i\right)^2 + 1$$

is prime. □

Before Open Problem 1, $\mathcal{X}$ denotes the set $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1$ *is prime*$\}$.

Lemma 5. *For every* $n \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$ *and for every non-negative integer* j, $3^j \cdot (n+1) - 1 \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$.

Proof. By the inequality $3^j \cdot (n+1) - 1 \geqslant n$ and Lemma 1,

$$\theta\left(3^j \cdot (n+1) - 1\right) + \kappa\left(3^j \cdot (n+1) - 1\right) = 10^{10^{10}} + \kappa(n) = \theta(n) + \kappa(n)$$

□

Lemma 6. $\mathrm{card}(\mathcal{X}) \geqslant 629450$.

Proof. By Lemmas 2 and 3, for every even integer $j \in (6553600, 7812500]$, the number $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$ is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\mathrm{card}(\mathcal{X}) \geqslant \mathrm{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

□

Lemma 7. $10242 \in \mathcal{X}$ *and* $10242 \notin \mathcal{X}_1$.

Proof. The number $10240 = 2^{11} \cdot 5$ divides $10^{10^{10}}$. Hence, $\theta(10242) = 10240$. The number $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$ is prime. The set

$$\left\{2^u \cdot 5^v : (u \in \{0, \dots, 10\}) \wedge (v \in \{0, \dots, 10\})\right\}$$

contains 10000 and 12500 as consecutive elements. Hence, $\theta_1(10242) = 10000$. The number $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$ is composite. □

Statement 6. *The set* $\mathcal{X}$ *satisfies Conditions* `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.*

Proof. Condition `(2)` holds trivially. Let δ denote $10^{10^{10}}$. By Lemma 5, Condition `(1)` holds for $n = \delta$. Lemma 5 and the unproven statement $\mathcal{P}_{n^2+1} \cap \left[\delta^2 + 1, \infty\right) \neq \emptyset$ show Condition `(3)`. The same argument and Lemma 6 yield Condition `(4)`. By Lemma 4, the set $\mathcal{X}_1$ is infinite. Since Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, Condition `(5)` holds except the requirement that $\mathcal{X}$ is naturally defined. □

The set $\mathcal{X}$ satisfies Condition `(5)` except the requirement that $\mathcal{X}$ is naturally defined. It is true because $\mathcal{X}_1$ is infinite by Lemma 4 and Definition 1 applies only to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven. Ignoring this restriction, $\mathcal{X}$ still satisfies the same identical condition due to Lemma 7.

Proposition 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions* `(1)-(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition `(1)`. Since Conditions `(1)-(3)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

`(T)` $$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots$$

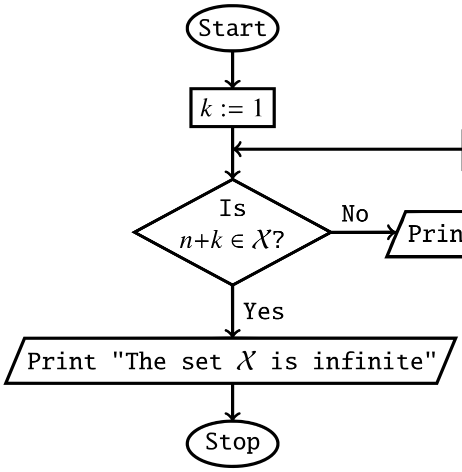

Figure 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition `(4)`. □

The physical limits of computation ([9]) disprove the assumption of Proposition 1.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* `(1)-(5)`*?*

Open Problem 1 asks about the existence of a year $t \geqslant 2023$ in which the conjunction

$$(\textit{Condition}\ 1) \wedge (\textit{Condition}\ 2) \wedge (\textit{Condition}\ 3) \wedge (\textit{Condition}\ 4) \wedge (\textit{Condition}\ 5)$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2023$ and for every $i \in \{1, 2, 3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–5 are negative.

4. SATISFIABLE CONJUNCTIONS WHICH CONSIST OF CONDITIONS (1)-(5) AND THEIR NEGATIONS

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)$$

The set $\mathcal{X} = \{0, \dots, 10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [8, p. 158] and [13, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [8, p. 159] and [13, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [7, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ 2^{2^{f(9^9)}} + 1\ is\ composite \\ \{0, \dots, 10^6\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ 2^{2^{f(9^9)}} + 1\ is\ composite \\ \{0, \dots, 10^6\} \cup \\ \{n \in \mathbb{N} : n\ is\ the\ sixth\ prime\ number\ of\ the\ form\ 2^{2^k} + 1\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1\ is\ composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition } 1) \land (\text{Condition } 2) \land \#(\text{Condition } 3) \land (\text{Condition } 4) \land \#(\text{Condition } 5)$$

where # denotes the negation ¬ or the absence of any symbol. Table 1 differs from Table 1 in [18] for three sets $\mathcal{X}$. These sets $\mathcal{X}$ have the index *new*.

	(Cond. 2) $\wedge$ (Cond. 3) $\wedge$ (Cond. 4)	(Cond. 2) $\wedge$ $\neg$(Cond. 3) $\wedge$ (Cond. 4)
(Cond. 1) $\wedge$ (Cond. 5)	Open Problem 1	Open Problem 2
(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X}_{new} = \{n \in \mathbb{N} : \textit{ the interval } [-1, n] \textit{ contains more than } 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \textit{ primes of the form } k! + 1\}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\}, \textit{ otherwise} \end{cases}$
$\neg$(Cond. 1) $\wedge$ (Cond. 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg$(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \textit{ is} \\ \textit{the sixth prime number of} \\ \textit{the form } 2^{2^k} + 1\}, \textit{ otherwise} \end{cases}$

Table 1 Five satisfiable conjunctions

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Open Problem 4. *Is there a known threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 4 asks about the existence of a year $t \geqslant 2023$ in which the implication $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, n]$ will hold for some known integer n.

Let $\mathcal{T}$ denote the set of twin primes.

Open Problem 5. *Is there a known threshold number of $\mathcal{T}$?*

Open Problem 5 asks about the existence of a year $t \geqslant 2023$ in which the implication $\mathrm{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$ will hold for some known integer n.

5. NUMBER-THEORETIC STATEMENTS Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations $\{x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \begin{aligned} x_1! &= x_1 \\ x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{aligned} \end{cases}$$

Lemma 8. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.*

Let B_n denote the following system of equations:

$$\{x_j! = x_k : \ j,k \in \{1,\dots,n\}\} \cup \{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\dots,n\}\}$$

For every positive integer n, no known system $\mathcal{S} \subseteq B_n$ with a finite number of solutions in positive integers $x_1,\dots,x_n$ has a solution $(x_1,\dots,x_n) \in (\mathbb{N} \setminus \{0\})^n$ satisfying $\max(x_1,\dots,x_n) > f(n)$. For every positive integer n and for every known system $\mathcal{S} \subseteq B_n$, if the finiteness/infiniteness of the set

$$\{(x_1,\dots,x_n) \in (\mathbb{N} \setminus \{0\})^n : \ (x_1,\dots,x_n) \textit{ solves } \mathcal{S}\}$$

is unknown, then the statement

$$\exists x_1,\dots,x_n \in \mathbb{N} \setminus \{0\} \ ((x_1,\dots,x_n) \textit{ solves } \mathcal{S}) \wedge (\max(x_1,\dots,x_n) > f(n))$$

remains unproven.

For a positive integer n, let Ψ_n denote the following statement: *if a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then each such solution $(x_1,\dots,x_n)$ satisfies $x_1,\dots,x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statement $\forall n \in \mathbb{N} \setminus \{0\} \ \Psi_n$ is dubious, see [17, p. 70].

Theorem 1. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 8 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 2. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\dots,x_n$, then for every integer $i \in \{1,\dots,n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1,\dots,x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 3. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

6. A SPECIAL CASE OF THE STATEMENT Ψ_9 APPLIES TO THE CONJECTURE THAT $\mathrm{card}(\mathcal{P}_{n^2+1}) = \omega$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 9. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 9 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

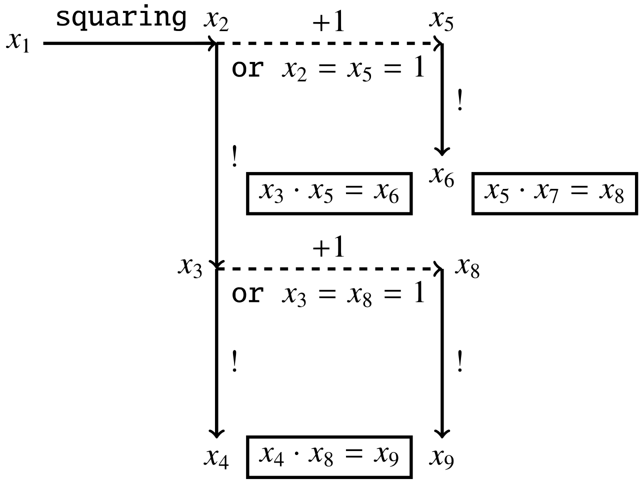

Figure 2 Construction of the system $\mathcal{A}$

Lemma 10. *(Wilson's theorem,* [3, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Lemma 11. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if x_1^2+1 is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= \left(x_1^2\right)! \\
x_4 &= \left(\left(x_1^2\right)!\right)! \\
x_5 &= x_1^2+1 \\
x_6 &= \left(x_1^2+1\right)! \\
x_7 &= \frac{\left(x_1^2\right)!+1}{x_1^2+1} \\
x_8 &= \left(x_1^2\right)!+1 \\
x_9 &= \left(\left(x_1^2\right)!+1\right)!
\end{aligned}$$

Proof. By Lemma 9, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if x_1^2+1 divides $(x_1^2)!+1$. Hence, the claim of Lemma 11 follows from Lemma 10. □

Lemma 12. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. It is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\right\} \cap (\mathbb{N} \setminus \{0\}) = \{1, 2\}$. □

Conjecture 2. *The statement* Ψ_9 *is true when is restricted to the system* $\mathcal{A}$.

Theorem 4. *Conjecture 2 proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* $x_1^2 + 1$ *is prime and greater than* $f(7)$, *then the set* $\mathcal{P}_{n^2+1}$ *is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 11, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \dots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 2 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 11 and 12, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Landau's conjecture implies the following unproven statement Φ:

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, (((24!)!)!)!]$$

Theorem 5 heuristically justifies the statement Φ. This justification does not yield the finiteness/infiniteness of $\mathcal{P}_{n^2+1}$.

Theorem 5. *Conjecture 2 implies the statement* Φ.

Proof. It follows from Theorem 4 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 6. *The statement* Φ *implies Conjecture 2.*

Proof. By Lemmas 11 and 12, if positive integers $x_1, \dots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 11 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Lemma 13. $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2((((24!)!)!)!)))))))) \approx 1.42298$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Statement 7. *Conditions* `(2)-(5)` *hold for* $\mathcal{X} = \mathcal{P}_{n^2+1}$. *The statement* Φ *implies Condition* `(1)` *for* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *and does not falsify Conditions* `(2)-(5)`.

Proof. Conditions `(2)`, `(3)`, and `(5)` hold trivially. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 2199894223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28})$, see [16]. These two facts imply Condition `(4)`. The statement Φ implies that Condition `(1)` holds for $\mathcal{X} = \mathcal{P}_{n^2+1}$ with $n = (((24!)!)!)!$. By Lemma 13, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)!$, see [9]. Hence, the statement Φ does not falsify Conditions `(2)-(5)`. □

Proving Landau's conjecture will disprove Statement 7. We do not conjecture that

$$(\text{Conditions } \texttt{(1)-(5)} \text{ hold for } \mathcal{X} = \mathcal{P}_{n^2+1}) \wedge \Phi$$

7. A NEW HEURISTIC ARGUMENT FOR THE INFINITENESS OF $\mathcal{P}_{n^2+1}$

The system $\mathcal{A}$ contains four factorials and four multiplications. Let $\mathcal{F}$ denote the family of all systems $\mathcal{S} \subseteq B_9$ which contain at most four factorials and at most four multiplications.

Among known systems $\mathcal{S} \in \mathcal{F}$, the following system C

$$\begin{cases} x_1! &= x_2 \\ x_2 \cdot x_9 &= x_1 \\ x_2 \cdot x_2 &= x_3 \\ x_3 \cdot x_3 &= x_4 \\ x_4 \cdot x_4 &= x_5 \\ x_5! &= x_6 \\ x_6! &= x_7 \\ x_7! &= x_8 \end{cases}$$

attains the greatest solution in positive integers $x_1, \dots, x_9$ and has at most finitely many solutions in $(\mathbb{N} \setminus \{0\})^9$. Only the tuples $(1, \dots, 1)$ and $(2, 2, 4, 16, 256, 256!, (256!)!, ((256!)!)!, 1)$ solve C and belong to $(\mathbb{N} \setminus \{0\})^9$.

For every known system $\mathcal{S} \in \mathcal{F}$, if the finiteness of the set

$$\{(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9 : (x_1, \dots, x_9) \textit{ solves } \mathcal{S}\}$$

is unproven and conjectured, then the statement

$$\exists x_1, \dots, x_9 \in \mathbb{N} \setminus \{0\} \; ((x_1, \dots, x_9) \textit{ solves } \mathcal{S}) \wedge (\max(x_1, \dots, x_9) > ((256!)!)!)$$

remains unproven.

Let Γ denote the statement: *if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \dots, x_9$, then each such solution $(x_1, \dots, x_9)$ satisfies $x_1, \dots, x_9 \leqslant ((256!)!)!$.* The number $46^{512} + 1$ is prime ([11]) and greater than 256!, see also [13, p. 239] for the primality of $150^{2048} + 1$. Hence, the statement Γ is equivalent to the infiniteness of $\mathcal{P}_{n^2+1}$. It heuristically justifies the infiniteness of $\mathcal{P}_{n^2+1}$ in a sophisticated way.

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