THE PREDICATE OF THE CURRENT MATHEMATICAL KNOWLEDGE SUBSTANTIALLY INCREASES THE CONSTRUCTIVE MATHEMATICS WHAT IS IMPOSSIBLE FOR ANY EMPIRICAL SCIENCE

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Abstract. This is a shortened and revised version of the article: A. Tyszka, *Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$*, Pi Mu Epsilon J. 15 (2023), no. 8, 493-504. The main results were presented at the 25th Conference Applications of Logic in Philosophy and the Foundations of Mathematics, see `http://applications-of-logic.uni.wroc.pl/XXV-Konferencja-Zastosowania-Logiki-w-Filozofii-i-Podstawach-Matematyki`. We assume that the current mathematical knowledge is a finite set of statements which is time-dependent. In every branch of mathematics, the set of all knowable truths is the set of all theorems. This set exists independently of our current scientific knowledge. Nicolas D. Goodman observed in *Synthese* that epistemic notions increase the understanding of mathematics without changing its content. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivial statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. This and the next sentence justify the article title. For any empirical science, we can identify the current knowledge with that science because truths from the empirical sciences are not necessary truths but working models of truth from a particular context.

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Key words and phrases: composite numbers of the form $2^{2^n}+1$, constructive algorithms, current mathematical knowledge, decidable sets $\mathcal{X} \subseteq \mathbb{N}$, epistemic notions, informal notions, known algorithms, known elements of $\mathbb{N}$, primes of the form n^2+1, primes of the form $n!+1$, primes of the form $2^{2^n}+1$.

1. Non-technical summary

This is a shortened and revised version of the article [15]. The main results of this article were presented at the 25th Conference Applications of Logic in Philosophy and the Foundations of Mathematics, see `http://applications-of-logic.uni.wroc.pl/XXV-Konferencja-Zastosowania-Logiki-w-Filozofii-i-Podstawach-Matematyki`. We assume that the current mathematical knowledge is a finite set of statements which is time-dependent. In every branch of mathematics, the set of all knowable truths is the set of all theorems. This set exists independently of our current scientific knowledge. Nicolas D. Goodman in [4] observed that epistemic notions increase the understanding of mathematics without changing its content. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivial statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. This and the next sentence justify the article title. For any empirical science, we can identify the current knowledge with that science because truths from the empirical sciences are not necessary truths but working models of truth from a particular context.

2. Basic definitions and examples

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [10], [12, p. 9]. A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs:* 1 *(one),* + *(addition),* − *(subtraction),* · *(multiplication),* ^ *(exponentiation with exponent in $\mathbb{N}$),* ! *(factorial of a non-negative integer),* (*(left parenthesis),*) *(right parenthesis).*

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is smaller than $((((((((9!)!)!)!)!)!)!)!)!$. The number t is an unknown element of the set of twin primes.

Definition 2. *Conditions* (1)-(5) *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
(1) *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
(2) *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
(3) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.
(4) *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.*
(5) *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Condition (3) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Example 1. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies Condition* (3).

Let $[\cdot]$ denote the integer part function.

Example 2. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } \left[\frac{((((((((9!)!)!)!)!)!)!)!)!}{\pi}\right] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$$

does not satisfy Condition (3) *because we know an algorithm with no input that computes* $\left[\frac{((((((((9!)!)!)!)!)!)!)!)!}{\pi}\right]$. *The set of known elements of $\mathcal{X}$ is empty. Hence, Condition* (5) *fails for $\mathcal{X}$.*

Example 3. *([2], [10], [12, p. 9]). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \dots\} \times \{0\})$$

No known algorithm computes the function h.

Example 4. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies Conditions `(1)` *and* `(3)` *and does not satisfy Conditions* `(2)`*,* `(4)`*, and* `(5)`*. These facts will hold forever.*

3. Main results

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [13], [14], [17].

Statement 1. *Condition* `(1)` *remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\texttt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition `(1)`, but the algorithm $\texttt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

Statement 2. *The statement*

$$\exists n \in \mathbb{N}\ (\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n.

Statement 3. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions `(1)-(4)`*. Condition* `(5)` *fails for $\mathcal{X}$.*

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \dots, 10^6\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. Due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^6)$, see [8]. Thus Condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, Condition `(1)` holds with $n = 10^6$. Condition `(5)` fails as the set of known elements of $\mathcal{X}$ equals $\{0, \dots, 10^6\}$. □

Statements 4 and 7 provide stronger examples.

Conjecture 1. *([1, p. 443], [5]). The are infinitely many primes of the form $k! + 1$.*

For a non-negative integer n, let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

Statement 4. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1,n] \textit{ contains more than } \rho(n) \textit{ primes of the form } k!+1\}$$

satisfies Conditions (1)-(5) *except the requirement that* $\mathcal{X}$ *is naturally defined.* $501893 \in \mathcal{X}$. *Condition* (1) *holds with* $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0,501893]) = 159827$. $\mathcal{X} \cap [501894,\infty) = \{n \in \mathbb{N} :$ *the interval* $[-1,n]$ *contains at least* 30 *primes of the form* $k!+1\}$.

Proof. For every integer $n \geqslant 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!,\infty)$, then $n+1,\ n+2,\ n+3,\ \ldots \in \mathcal{X}$. Hence, Condition (1) holds with $n = 11!-1$. We explicitly know 24 positive integers k such that $k!+1$ is prime, see [3]. The inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is prime}\}) > 24$ remains unproven. Since $24 < 30$, Condition (3) holds. The interval $[-1, 11!-1]$ contains exactly three primes of the form $k!+1$: $1!+1$, $2!+1$, $3!+1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line [501893.0, 159827], which proves Condition (1) with $n = 501893$ and Condition (4) with $\mathrm{card}(\mathcal{X}) \geqslant 159827$. □

Definition 3. *Conditions* (1a)-(5a) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1a) *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty,n]$.

(2a) *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

(3a) *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) < \omega$.

(4a) *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is finite.*

(5a) $\mathcal{X}$ *is naturally defined. The finiteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

Statement 5. *The set*

$$\mathcal{X} = \Big\{n \in \mathbb{N} : \textit{ the interval } [-1,n] \textit{ contains more than}$$

$$6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \textit{ squares of the form } k!+1\Big\}$$

satisfies Conditions (1a)-(5a) *except the requirement that* $\mathcal{X}$ *is naturally defined.* $95151 \in \mathcal{X}$. *Condition* (1a) *holds with* $n = 95151$. $\mathrm{card}(\mathcal{X} \cap [0,95151]) = 30311$. $\mathcal{X} \cap [95152,\infty) = \{n \in \mathbb{N} :$ *the interval* $[-1,n]$ *contains at least* 7 *squares of the form* $k!+1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in \mathcal{X} \cap (10^6, \infty)$, then $n+1,\ n+2,\ n+3,\ \ldots \in \mathcal{X}$. Hence, Condition `(1a)` holds with $n = 10^6$. It is conjectured that $k! + 1$ is a square only for $k \in \{4, 5, 7\}$, see [16, p. 297]. Hence, the inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k! + 1 \textit{ is a square}\}) > 3$ remains unproven. Since $3 < 7$, Condition `(3a)` holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k! + 1$: $4! + 1$, $5! + 1$, $7! + 1$. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line [95151.0, 30311], which proves Condition `(1a)` with $n = 95151$ and Condition `(4a)` with $\mathrm{card}(\mathcal{X}) \geqslant 30311$. □

Statement 6. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \textit{if a formal proof in ZFC is shorter than } k,$$
$$\textit{then the equality } \mathrm{card}(\mathcal{P}(n^2+1)) = \omega \textit{ is not its conclusion}\}$$

satisfies the conjunction

$$\neg(\textit{Condition } \texttt{1a}) \wedge (\textit{Condition } \texttt{2a}) \wedge (\textit{Condition } \texttt{3a}) \wedge (\textit{Condition } \texttt{4a}) \wedge (\textit{Condition } \texttt{5a})$$

To formulate Statement 7 and its proof, we need some lemmas. For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. For a non-negative integer n, let $\theta_1(n)$ denote the largest integer divisor of 10^{10} smaller than n.

Lemma 1. *For every integer $j > 10^{10^{10}}$, $\theta(j) = 10^{10^{10}}$. For every integer $j > 10^{10}$, $\theta_1(j) = 10^{10}$.*

Lemma 2. *For every integer $j \in (6553600, 7812500]$, $\theta(j) = 6553600$.*

Proof. 6553600 equals $2^{18} \cdot 5^2$ and divides $10^{10^{10}}$. $7812500 < 2^{24}$. $7812500 < 5^{10}$. We need to prove that every integer $j \in (6553600, 7812500)$ does not divide $10^{10^{10}}$. It holds as the set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 23\}) \wedge (v \in \{0, \ldots, 9\})\right\}$$

contains 6553600 and 7812500 as consecutive elements. □

Lemma 3. *The number* $6553600^2 + 1$ *is prime.*

Proof. The following `PARI/GP` ([9]) command

```
isprime(6553600^2+1,{flag=2})
```

returns 1. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([18, p. 226]). It rigorously shows that the number $6553600^2 + 1$ is prime. □

In the next lemmas, the execution of the command `isprime(`n`,{flag=2})` proves the primality of n. Let κ denote the function

$$\mathbb{N} \ni n \xrightarrow{\kappa} the\_exponent\_of\_2\_in\_the\_prime\_factorization\_of\_\underbrace{n+1} \in \mathbb{N}$$

Lemma 4. *The set* $\mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1$ *is prime*$\}$ *is infinite.*

Proof. Let $i = 142101504$. By the inequality $2^i \geqslant 2 + 10^{10}$ and Lemma 1, for every non-negative integer m, the number

$$\left(\theta_1\left(2^i \cdot (2m+1) - 1\right) + \kappa\left(2^i \cdot (2m+1) - 1\right)\right)^2 + 1 = \left(10^{10} + i\right)^2 + 1$$

is prime. □

Before Open Problem 1, $\mathcal{X}$ denotes the set $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1$ *is prime*$\}$.

Lemma 5. *For every* $n \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$ *and for every non-negative integer* j, $3^j \cdot (n+1) - 1 \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$.

Proof. By the inequality $3^j \cdot (n+1) - 1 \geqslant n$ and Lemma 1,

$$\theta\left(3^j \cdot (n+1) - 1\right) + \kappa\left(3^j \cdot (n+1) - 1\right) = 10^{10^{10}} + \kappa(n) = \theta(n) + \kappa(n)$$

□

Lemma 6. $\mathrm{card}(\mathcal{X}) \geqslant 629450$.

Proof. By Lemmas 2 and 3, for every even integer $j \in (6553600, 7812500]$, the number $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$ is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\mathrm{card}(\mathcal{X}) \geqslant \mathrm{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

□

Lemma 7. $10242 \in \mathcal{X}$ *and* $10242 \notin \mathcal{X}_1$.

Proof. The number $10240 = 2^{11} \cdot 5$ divides $10^{10^{10}}$. Hence, $\theta(10242) = 10240$. The number $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$ is prime. The set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 10\}) \wedge (v \in \{0, \ldots, 10\})\right\}$$

contains 10000 and 12500 as consecutive elements. Hence, $\theta_1(10242) = 10000$. The number $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$ is composite. □

Statement 7. *The set $\mathcal{X}$ satisfies Conditions* (1)-(5) *except the requirement that $\mathcal{X}$ is naturally defined.*

Proof. Condition (2) holds trivially. Let δ denote $10^{10^{10}}$. By Lemma 5, Condition (1) holds for $n = \delta$. Lemma 5 and the unproven statement $\mathcal{P}_{n^2+1} \cap \left[\delta^2+1, \infty\right) \neq \emptyset$ show Condition (3). The same argument and Lemma 6 yield Condition (4). By Lemma 4, the set $\mathcal{X}_1$ is infinite. Since Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, Condition (5) holds except the requirement that $\mathcal{X}$ is naturally defined. □

The set $\mathcal{X}$ satisfies Condition (5) except the requirement that $\mathcal{X}$ is naturally defined. It is true because $\mathcal{X}_1$ is infinite by Lemma 4 and Definition 1 applies only to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven. Ignoring this restriction, $\mathcal{X}$ still satisfies the same identical condition due to Lemma 7.

Proposition 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions* (1)-(4) *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots$$

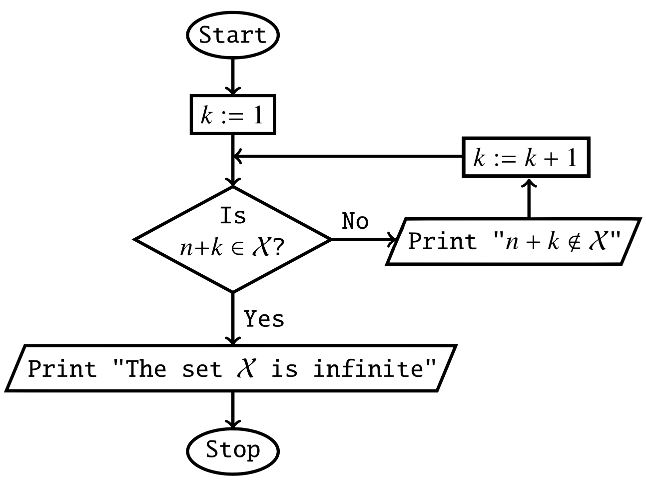

Figure 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition (4). □

The physical limits of computation ([8]) disprove the assumption of Proposition 1.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* `(1)-(5)`*?*

Open Problem 1 asks about the existence of a year $t \geqslant 2023$ in which the conjunction

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2023$ and for every $i \in \{1,2,3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–5 are negative.

4. Satisfiable conjunctions which consist of Conditions (1)-(5) and their negations

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)$$

The set $\mathcal{X} = \{0,\ldots,10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

The numbers $2^{2^k}+1$ are prime for $k \in \{0,1,2,3,4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k}+1$, see [7, p. 158] and [11, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k}+1$, see [7, p. 159] and [11, p. 74]. Most mathematicians believe that $2^{2^k}+1$ is composite for every integer $k \geqslant 5$, see [6, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if \ 2^{2^{f(9^9)}}+1 \ is \ composite \\ \{0,\ldots,10^6\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)?$$

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if \ 2^{2^{f(9^9)}}+1 \ is \ composite \\ \{0,\ldots,10^6\} \cup \\ \{n \in \mathbb{N} : n \ is \ the \ sixth \ prime \ number \ of \ the \ form \ 2^{2^k}+1\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge \neg(\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k}+1 \ \textit{is composite}\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge \#(\text{Condition } 3) \wedge (\text{Condition } 4) \wedge \#(\text{Condition } 5)$$

where # denotes the negation $\neg$ or the absence of any symbol. Table 1 differs from Table 1 in [15] for three sets $\mathcal{X}$. These sets $\mathcal{X}$ have the index *new*.

	(Cond. 2) $\wedge$ (Cond. 3) $\wedge$ (Cond. 4)	(Cond. 2) $\wedge$ $\neg$(Cond. 3) $\wedge$ (Cond. 4)
(Cond. 1) $\wedge$ (Cond. 5)	Open Problem 1	Open Problem 2
(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X}_{new} = \{n \in \mathbb{N} : \textit{ the interval } [-1, n] \textit{ contains more than } 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \textit{ primes of the form } k! + 1\}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\}, \textit{ otherwise} \end{cases}$
$\neg$(Cond. 1) $\wedge$ (Cond. 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg$(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \textit{ is the sixth prime number of the form } 2^{2^k} + 1\}, \textit{ otherwise} \end{cases}$

Table 1 Five satisfiable conjunctions

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\operatorname{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Open Problem 4. *Is there a known threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 4 asks about the existence of a year $t \geqslant 2023$ in which the implication $\operatorname{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, n]$ will hold for some known integer n.

Let $\mathcal{T}$ denote the set of twin primes.

Open Problem 5. *Is there a known threshold number of $\mathcal{T}$?*

Open Problem 5 asks about the existence of a year $t \geqslant 2023$ in which the implication $\operatorname{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$ will hold for some known integer n.

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