Maxwell gravitation without reference to equivalence classes of derivative operators

Eleanor March*

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Abstract

Drawing on recent work by Weatherall (2018), I provide an explicit presentation of Newtonian gravitation theory set on Maxwellian spacetime which avoids reference to any derivative operators or equivalence classes thereof. The resulting theory bears a close relationship to Dewar's (2018) "Maxwell gravitation"; it also sheds light on arguments by Wallace (2020) concerning the relationship between Newton-Cartan theory and Saunders's (2013) vector relationism.

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1 Introduction

In recent years, philosophers of physics have considered afresh the question of the appropriate spacetime setting for Newtonian gravitation theory. At the centre of this debate have been two apparently conflicting proposals for what one should take this geometry to be: on the one hand, Saunders's (2013) proposal that Corollary VI to the Laws of Motion in Newton's *Principia* reveals that Maxwellian spacetime is the correct setting for Newtonian physics, and on the

*Balliol College, University of Oxford. eleanor.march@balliol.ox.ac.uk

other hand, Knox's (2014) proposal that Corollary VI motivates a transition to a geometrised formulation of Newtonian gravitation, known as Newton-Cartan theory. Their claims have sparked a series of discussions of theories of Newtonian gravitation set on Maxwellian spacetime, and their relation to Newton-Cartan theory.[1]

One focus of these discussions has been on how Maxwellian spacetime – which is supposed to be equipped with a standard of rotation, but *not* a standard of absolute acceleration – should best be characterised. Earman (1989) originally defined the standard of rotation in terms of an equivalence class of derivative operators. But a number of authors have voiced concerns about this approach. For example, Weatherall (2018, 34) notes that it "makes reference to structure that one does not attribute to spacetime," Jacobs (2022) argues that it is not suitably "intrinsic" and so fails to offer a perspicuous formalism from which we can read off the theory's ontology,[2] and Wallace (2019, 2020) goes so far as to suggest that the awkwardness of standard differential-geometric presentations of Maxwellian spacetime obscures the similarity between Newton-Cartan theory and theories of Newtonian gravitation set on Maxwellian spacetime, and (more generally) shows that coordinate-free differential geometry is not an intuitive way of characterising certain spacetime structures. In response, Weatherall (2018) has offered an alternative definition of a standard of rotation, but there have as yet been no attempts to formulate Newtonian gravitation theory in terms of this object.[3]

The aim of this paper is to provide an explicit presentation of a theory which addresses the concerns raised by Weatherall, Wallace, and others – a theory of Newtonian gravitation set on Maxwellian spacetime, which is formulated in a coordinate-free way without reference to any derivative operators or equivalence classes thereof. First, I review some essential background from discussions of Maxwellian spacetime, including Dewar's (2018) "Maxwell gravitation". I then, in section 3, turn to the task of formulating Newtonian gravitation theory in terms of Weatherall's standard of rotation. Section 4 provides some prelimi-

1. See Weatherall (2016), Teh (2018), Wallace (2020), Jacobs (2023).
2. See also Dürr & Read (2019, 1094-1096), who raise similar concerns.
3. Although several authors – Weatherall included – have noted that such a formulation would be desirable (see Dürr and Read 2019; Jacobs 2022).

nary results concerning the relationship between this theory, Dewar's (2018) approach, and Newton-Cartan theory; sections 5 and 6 aim to dispel the remainder of Wallace's concerns about coordinate-free presentations of Maxwellian spacetime by showing that the arguments of his (2020) can also be made in the language of coordinate-free differential geometry. Section 7 concludes.

2 Background

Let M be a smooth four-manifold (assumed connected, Hausdorff, and paracompact). A temporal metric t_a on M is a smooth, closed, non-vanishing 1-form;[4] a spatial metric h^{ab} on M is a smooth, symmetric, rank-$(2,0)$ tensor field which admits, at each point in M, a set of four non-vanishing covectors $\overset{i}{\sigma}_a$, $i = 0, 1, 2, 3$, which form a basis for the cotangent space and satisfy $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ for $i = j = 1, 2, 3$ and 0 otherwise. A spatial and temporal metric are compatible iff $h^{an}t_n = 0$. We say that a vector field σ^a is spacelike iff $t_n\sigma^n = 0$, and timelike otherwise. Given the structure defined here, t_a induces a foliation of M into spacelike hypersurfaces, and relative to any such hypersurface, h^{ab} induces a unique spatial derivative operator D such that $D_a h^{bc} = 0$.[5] We say that h^{ab} is flat just in case for any such spacelike hypersurface, D commutes on spacelike vector fields, so that $D_{[a}D_{b]}\sigma^c = 0$ for all spacelike vector fields σ. Finally, let ∇ be a connection on M. We say that ∇ is compatible with the metrics just in case $\nabla_a t_b = 0$ and $\nabla_a h^{bc} = 0$.

With these structures in place, we can now introduce Earman's (1989) original definition of a standard of rotation. Let t_a, h^{ab} be compatible temporal and spatial metrics on M, and let ∇, ∇' be a pair of flat derivative operators on M, both compatible with the metrics. We say that ∇ and ∇' are *rotationally equivalent* just in case for any unit timelike vector field η^a on M, $\nabla^{[a}\eta^{b]} = 0 \Leftrightarrow \nabla'^{[a}\eta^{b]} = 0$. Then a standard of rotation compatible with t_a and h^{ab} is an equivalence class $[\nabla]$ of rotationally equivalent compatible flat derivative operators.

4. Here and throughout, abstract indices are written in Latin script; component indices are written in Greek script, with the exception of i, j, k, which are reserved for the spatial components of tensor fields in some coordinate basis; and the Einstein summation convention is used. Round brackets denote symmetrisation, square brackets antisymmetrisation.

5. See Weatherall (2018, 37-38) and Malament (2012, §4.1) for further details.

Within this framework, Dewar (2018) shows that one can formulate Newtonian gravitation theory as follows. Let t_a, h^{ab} be compatible temporal and spatial metrics on M, $[\nabla]$ an equivalence class of rotationally equivalent compatible flat derivative operators, and T^{ab} the Newtonian mass-momentum tensor for whichever matter fields are present. Let $\rho := t_a t_b T^{ab}$ be the scalar mass density field. Then $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a model of *Maxwell-Dewar gravitation*[6] just in case for all points $p \in M$ where $\rho \neq 0$, the following equations hold at p:

$$t_a \nabla_n T^{na} = 0 \tag{1a}$$

$$\nabla_m(\rho^{-1}\nabla_n T^{nm}) = -4\pi\rho \tag{1b}$$

$$\nabla^c(\rho^{-1}\nabla_n T^{na}) - \nabla^a(\rho^{-1}\nabla_n T^{nc}) = 0, \tag{1c}$$

where ∇ is an arbitrary member of $[\nabla]$.

Recently, however, Weatherall (2018, 34) has queried this definition of a standard of rotation, noting that it "makes reference to structure that one does not attribute to spacetime." Weatherall points to two criticisms of this approach. First, if a standard of rotation is defined as an equivalence class of derivative operators, then we must select an arbitrary member of this class to perform calculations. But some of the terms in these calculations may depend on the choice of derivative operator, and it is not clear how these should be interpreted. Secondly, one might worry that the appeal to derivative operators somehow obscures the structure of Maxwellian spacetime.

In response, Weatherall offers an alternative definition: if t_a, h^{ab} are compatible temporal and spatial metrics on M, a standard of rotation $\circlearrowleft$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, antisymmetric rank-$(2,0)$ tensor fields $\circlearrowleft^b \xi^a$ on M, such that

1. $\circlearrowleft$ commutes with addition of smooth vector fields;
2. Given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowleft^a (\alpha\xi^b) = \alpha \circlearrowleft^a \xi^b + \xi^{[b} d^{a]} \alpha$;

6. Note that Dewar (2018) calls this theory Maxwell gravitation; here, I reserve that name for presentations of the theory which do not make reference to any structure which is not definable from that of Maxwellian spacetime, such as the theory presented in section 3.

3. $\circlearrowright$ commutes with index substitution;

4. Given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = 0$ then $\circlearrowright^a \xi^b$ is spacelike in both indices; and

5. Given any smooth spacelike vector field σ^a, $\circlearrowright^a \sigma^b = D^{[a}\sigma^{b]}$.

One can then define a Maxwellian spacetime as a structure $\langle M, t_a, h^{ab}, \circlearrowright\rangle$, where $\circlearrowright$ is compatible with t_a and h^{ab}.

As Weatherall (2018, footnote 5) notes, this definition of a standard of rotation "only draws more attention to the question of whether this structure is sufficient to formulate Newtonian gravitational theory. One would like to find a coordinate-free presentation of the theory that makes use of precisely Maxwellian spacetime, as characterised here, and nothing else." But there have as yet been no attempts to formulate such a theory. It is this task to which I now turn.

3 Maxwell gravitation

I will begin by introducing some terminology. Fix a spacetime $\langle M, t_a, h^{ab}\rangle$, and let ∇ and $\circlearrowright$ be a connection and standard of rotation on M, both compatible with the metrics. In what follows, we will often want to consider connections and standards of rotation which "agree" with one another in the following sense: for any vector field η^a on M, $\nabla^{[a}\eta^{b]} = \circlearrowright^a \eta^b$. In this case, I will say that the connection and standard of rotation are *compatible*. This idea is made precise in proposition 1 of Weatherall (2018); the basic fact is that any connection determines a unique compatible standard of rotation, but a standard of rotation does not similarly determine a unique compatible connection.

We will also (sometimes) extend this idea to spacetimes. Thus a connection ∇ is compatible with a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ just in case it is compatible with the metrics and $\circlearrowright$. Finally, I will say that a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is *flat derivative operator compatible* just in case some flat derivative operator is compatible with $\langle M, t_a, h^{ab}, \circlearrowright\rangle$. As Weatherall (2018, proposition 1) proves, a spacetime is flat derivative operator compatible just in case h^{ab} is flat and there exists a unit timelike vector field ξ^a on M such that $\circlearrowleft^a \xi^b = 0$ and $\pounds_\xi h^{ab} = 0$.[7] Where there is no ambiguity over the temporal and spatial metrics in question, we will sometimes drop talk of the metrics and simply refer to $\circlearrowleft$ instead.

Finally, we need to say something about the Newtonian mass-momentum tensor T^{ab}. We have already seen that we can extract the scalar mass density field ρ from T^{ab} using the temporal metric. But in Maxwell-Dewar gravitation, we also used derivative operators to extract vector fields from T^{ab}. In what follows, we will likewise want to extract vector fields from T^{ab}, but without the use of derivative operators. To do this, we first impose the "Newtonian mass condition": whenever $T^{ab} \neq 0$, $T^{nm} t_n t_m > 0$. This captures the idea that the matter fields we are interested in are massive, in the sense that there can only be non-zero mass-momentum in spacetime regions where the mass density is strictly positive.[8] Since T^{ab} is symmetric, the Newtonian mass condition guarantees that whenever $T^{ab} \neq 0$, we can uniquely decompose T^{ab} as

$$T^{ab} = \rho \xi^a \xi^b + \sigma^{ab} \tag{2}$$

where $\xi^a = \rho^{-1} t_n T^{na}$ is a smooth unit timelike future-directed vector field (interpretable as the net four-velocity of the matter fields F), and σ^{ab} is a smooth symmetric rank-$(2,0)$ tensor field which is spacelike in both indices (interpretable as the stress tensor for F).

We are now in a position to formulate Newtonian gravitation theory in terms of Weatherall's standard of rotation. Let $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ be a Maxwellian spacetime, and let T^{ab} be the Newtonian mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab} \rangle$ is a model of *Maxwell gravitation*[9] just in case

(i) $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ is flat derivative operator compatible; and

7. Here and throughout, $\pounds$ denotes the Lie derivative.

8. For example, Weatherall (2012, 211) suggests that "[one] might take [the Newtonian mass condition] to be a benign and unsurprising characterisation of what we mean by "massive particle" in Newtonian gravitation."

9. Again, this should be distinguished from Dewar's (2018) "Maxwell gravitation", which I refer to here as Maxwell-Dewar gravitation.

(ii) For all points $p \in M$ such that $\rho \neq 0$, the following equations hold at p:

$$\pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} = 0 \tag{MG1}$$

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n (\xi^m \Delta_m \overset{i}{\lambda}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3} D_m(\rho^{-1} D_n \sigma^{nm}) \tag{MG2}$$

$$\pounds_\xi (\circlearrowleft^c \xi^a) + 2(\circlearrowleft^n \xi^{[c})\hat{h}_{nm} \pounds_\xi h^{a]m} + \circlearrowleft^c (\rho^{-1} D_n \sigma^{na}) = 0, \tag{MG3}$$

where $\hat{h}_{ab}$ is the spatial metric relative to ξ^a,[10] the $\overset{i}{\lambda}{}^a$ are three orthonormal connecting fields for ξ^a, and Δ is the "restricted derivative operator" defined in Weatherall (2018). This acts on arbitrary spacelike vector fields σ^a at a point p according to

$$\eta^n \Delta_n \sigma^a := \pounds_\eta \sigma^a + \sigma_n \circlearrowleft^n \eta^a - \frac{1}{2}\sigma_n \pounds_\eta h^{an} \tag{5}$$

where η^a is a unit timelike vector at p (the Lie derivative is taken with respect to any extension of η^a off of p). It also has the property that $\eta^n \Delta_n \sigma^a = \eta^n \nabla_n \sigma^a$ for any derivative operator ∇ compatible with $\circlearrowleft$ (Weatherall 2018, 37).

As promised, this theory does not make reference to any structure not definable from that of Maxwellian spacetime. The equations (MG) and the flat derivative operator compatibility condition require only the standard of rotation, and two kinds of derivatives:

- The Lie derivative, which is well-defined whenever M is a smooth manifold; and

- The spatial derivative operator D, which is just the unique Levi-Civita connection induced by h^{ab} on each spacelike hypersurface.

Let us now consider the dynamics of Maxwell gravitation in detail. I will begin by making two comments on the flat derivative operator compatibility condition. First, note that this is a necessary condition if Maxwell gravitation is to reproduce the empirical predictions of Galilean gravitation, since absolute rotations do have empirically detectable consequences in Galilean gravitation

10. That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = 0$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b \xi^a$.

theory (think of Newton's bucket experiment). Secondly, we can already see an immediate connection to Newton-Cartan theory here. Expressed in terms of some derivative operator compatible with $\circlearrowleft$, flat derivative operator compatibility becomes the requirement h^{ab} is flat, and that there exists a unit timelike vector field η^a on M which is twist-free ($\nabla^{[a}\eta^{b]} = 0$) and rigid ($\nabla^{(a}\eta^{b)} = 0$). But this is equivalent to the condition that $R^{ab}{}_{cd} = 0$ (Malament 2012, proposition 4.2.4), which is one of the supplemental curvature conditions in Newton-Cartan theory.

It is also helpful to express the equations (MG) in terms of some compatible derivative operator. First, consider (MG1). This becomes[11]

$$\xi^n \nabla_n \rho + \rho \nabla_n \xi^n = 0,$$

which is just the familiar continuity equation for the mass density field ρ. Likewise, (MG2) becomes

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3}\nabla_m(\rho^{-1}\nabla_n \sigma^{nm}). \tag{6}$$

The left hand side of (6) is the familiar expression for the *average radial acceleration* of the vector field ξ^a – that is, the average radial component of the relative acceleration of "neighbouring" integral curves of ξ^a. The right hand side of (6) asserts that this relative acceleration has two components – one due to the mass density field ρ, and one due to the stress tensor σ^{ab}. This latter term we can interpret as the non-gravitational component of the average radial acceleration.[12] Finally, a straightforward calculation shows that (MG3) can be written[13]

$$\xi^n \nabla_n (\omega^{ca}) = 2\omega^{n[c}\theta_n{}^{a]} - \nabla^{[c}(\rho^{-1}\nabla_n \sigma^{|n|a]})$$

11. For this it suffices to note that the expansion tensor $\theta^{ab} = \nabla^{(a}\xi^{b)} = -1/2\pounds_\xi h^{ab}$ associated with ξ^a is related to the quantity $\nabla_n \xi^n$ via $\nabla_n \xi^n = \theta = \hat{h}_{nm}\theta^{nm}$.

12. The $-4/3\pi\rho$ term is less straightforward. On the one hand, it is tempting to interpret it as the gravitational component of the average radial acceleration, as Dewar (2018, 259) does. But unlike in Galilean gravitation, there is no explicit representation of the gravitational field within the formalism of Maxwell gravitation. As such, whilst we can always attribute the $-4/3\pi\rho$ term to "the gravitational field", it is not clear that this can be the gravitational field understood in the usual sense – *pace* Dewar. For further discussion, see section 6.

13. cf. Malament (2012, proposition 4.3.6).

where $\omega^{ab} = \nabla^{[a}\xi^{b]}$ is the rotation tensor and $\theta^{ab} = \nabla^{(a}\xi^{b)}$ the expansion tensor associated with ξ^a. As such, I suggest that we interpret (MG3) as a continuity equation for the rotation tensor ω^{ab}, with a source term $-\nabla^{[a}(\rho^{-1}\nabla_n\sigma^{|n|b]})$ due to the presence of non-gravitational interactions.

4 Maxwell gravitation, Maxwell-Dewar gravitation, and Newton-Cartan theory

At this point, we have not said anything about the relation between Maxwell gravitation and Maxwell-Dewar gravitation, and it is perhaps not clear that the two are even empirically equivalent. In fact, there is a particularly close relationship between these theories, as summarised in the following two propositions:

Proposition 1. *Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a unique equivalence class of rotationally equivalent flat derivative operators $[\nabla]$ such that all the $\nabla \in [\nabla]$ are compatible with $\circlearrowleft$ and $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a model of Maxwell-Dewar gravitation.*

Proof. Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ be a model of Maxwell gravitation, and consider the class $[\nabla]$ of all flat derivative operators compatible with $\circlearrowleft$. The class $[\nabla]$ is manifestly unique in this regard, and is non-empty since $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ is flat derivative operator compatible. Flat derivative operator compatibility also guarantees that $[\nabla]$ is an equivalence class of flat derivative operators under the equivalence relation $\nabla^{[a}\eta^{b]} = 0 \Leftrightarrow \nabla'^{[a}\eta^{b]} = 0$ for all unit timelike vector fields η^a on M (Weatherall 2018, proposition 1). It follows that $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a kinematically possible model (KPM) of Maxwell-Dewar gravitation.[14]

We now show that $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a model of Maxwell-Dewar gravitation. Let ∇ be an arbitrary member of $[\nabla]$, and consider an arbitrary point $p \in M$ such that $\rho \neq 0$. A straightforward calculation shows that, at p

$$\nabla_n T^{na} = \rho\xi^n\nabla_n\xi^a + \xi^a(\pounds_\xi\rho - \frac{1}{2}\rho\hat{h}_{mn}\pounds_\xi h^{mn}) + D_n\sigma^{na}, \tag{7}$$

14. That is, a structure $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ which does not necessarily satisfy the equations (1).

so that

$$t_a \nabla_n T^{na} = \pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} \tag{8}$$
$$= 0.$$

Next, recalling (6), we can write

$$\begin{aligned} \frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n (\xi^m \Delta_m \overset{i}{\lambda}{}^r) &= \frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^r) \\ &= \frac{1}{3}\nabla_a (\xi^n \nabla_n \xi^a) \end{aligned} \tag{9}$$

where the last equality follows from proposition 3 of Dewar (2018). Then making use of (MG1) and (7), we have that

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n (\xi^m \Delta_m \overset{i}{\lambda}{}^r) = \frac{1}{3}\nabla_m (\rho^{-1} \nabla_n T^{nm}) - \frac{1}{3} D_m (\rho^{-1} D_n \sigma^{nm}),$$

so that comparison with (MG2) immediately yields that (1b) holds with respect to ∇. Moreover, some calculation shows that

$$\circlearrowright^c (\xi^n \nabla_n \xi^a) = \pounds_\xi (\circlearrowright^c \xi^a) + 2(\circlearrowright^c \xi^n)(\nabla_n \xi^a) + 2(\circlearrowright^n \xi^a)(\nabla_n \xi^c)$$

We note that, since $d_a(t_n \xi^n) = d_a(1) = 0$, it follows from the definition of $\circlearrowright$ that $\circlearrowright^a \xi^b$ is spacelike in both indices. Moreover, we have (Malament 2012, equation 4.1.42),

$$\nabla_a \xi^b = \hat{h}_{an} (\circlearrowright^n \xi^b - \frac{1}{2}\pounds_\xi h^{nb}) + t_a \xi^n \nabla_n \xi^b. \tag{10}$$

Hence,

$$\begin{aligned} \circlearrowright^c (\xi^n \nabla_n \xi^a) = \pounds_\xi (\circlearrowright^c \xi^a) &+ (\circlearrowright^c \xi^n) \hat{h}_{nm} (2 \circlearrowright^m \xi^a - \pounds_\xi h^{ma}) \\ &+ (\circlearrowright^n \xi^a) \hat{h}_{nm} (2 \circlearrowright^m \xi^c - \pounds_\xi h^{mc}) \\ = \pounds_\xi (\circlearrowright^c \xi^a) &- (\circlearrowright^c \xi^n) \hat{h}_{nm} \pounds_\xi h^{ma} - (\circlearrowright^n \xi^a) \hat{h}_{nm} \pounds_\xi h^{mc} \\ = \pounds_\xi (\circlearrowright^c \xi^a) &+ 2(\circlearrowright^n \xi^{[c}) \hat{h}_{nm} \pounds_\xi h^{a]m}. \end{aligned} \tag{11}$$

Then, again from (MG3) and (7),

$$\begin{aligned}\nabla^c(\rho^{-1}\nabla_n T^{na}) - \nabla^a(\rho^{-1}\nabla_n T^{nc}) &= \circlearrowleft^c(\rho^{-1}\nabla_n T^{na}) \\ &= \pounds_\xi(\circlearrowleft^c \xi^a) + 2(\circlearrowleft^n \xi^{[c})\hat{h}_{nm}\pounds_\xi h^{a]m} \\ &\qquad + \circlearrowleft^c(\rho^{-1}D_n\sigma^{na}) \\ &= 0.\end{aligned}$$

□

Proposition 2. *Let* $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ *be a model of Maxwell-Dewar gravitation. Then there exists a unique standard of rotation* $\circlearrowleft$ *such that all the* $\nabla \in [\nabla]$ *are compatible with* $\circlearrowleft$ *and* $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ *is a model of Maxwell gravitation.*

Proof. Let ∇ be an arbitrary member of $[\nabla]$. By proposition 1 of Weatherall (2018), there exists a unique standard of rotation $\circlearrowleft$ compatible with ∇; moreover, it follows from proposition 1 of Dewar (2018) that all the $\nabla \in [\nabla]$ determine the same standard of rotation $\circlearrowleft$.

To show that $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ is a model of Maxwell gravitation, let ∇ be an arbitrary member of $[\nabla]$, and again consider some point $p \in M$ such that $\rho \neq 0$. (MG1) follows immediately from (8); (MG2) and (MG3) result from substituting (1a) and (8) into (9) and (11), respectively. Finally, note that $\circlearrowleft$ is flat derivative operator compatible by construction. □

As such, there is a one-to-one correspondence between the models of Maxwell and Maxwell-Dewar gravitation.[15] Since this one-to-one correspondence relates models with the same mass-momentum tensor, this should at least provide reassurance that the two theories are empirically equivalent. But we can go further. Weatherall (2018, footnote 5) suggests that we might find "a version [...] of Neil Dewar's "Maxwell gravitation" expressed using only a standard of rotation."

15. Is there a similar correspondence between the KPMs of Maxwell and Maxwell-Dewar gravitation? It turns out that the answer depends on a rather technical issue, viz. whether one takes the flat derivative operator compatibility condition in Maxwell gravitation to arise at the level of kinematic or dynamical constraints. If it is a kinematic constraint, then the KPMs of the two theories will also be in one-to-one correspondence. If it is a dynamical constraint, they will not. An arbitrary standard of rotation need not be flat derivative operator compatible, whereas the standard of rotation associated with an equivalence class of flat derivative operators is flat derivative operator compatible by construction.

One might also take propositions 1 and 2 to show that Maxwell gravitation, as I have presented it here, is precisely that theory.

It is now straightforward to extend our discussion to Newton-Cartan theory. Let $\langle M, t_a, h^{ab}\rangle$ be a non-relativistic spacetime, ∇ a metric-compatible derivative operator on M, and T^{ab} the mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of *Newton-Cartan theory* just in case

$$\nabla_n T^{na} = 0 \tag{NCT1}$$

$$R_{ab} = 4\pi\rho t_a t_b \tag{NCT2}$$

$$R^{a\ c}_{\ b\ d} = R^{c\ a}_{\ d\ b} \tag{NCT3}$$

$$R^{ab}{}_{cd} = 0. \tag{NCT4}$$

The relation between Maxwell gravitation and Newton-Cartan theory is then characterised in the following two results:

Proposition 3. *Let $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be a model of Newton-Cartan theory. Then there exists a unique standard of rotation $\circlearrowright$ such that ∇ is compatible with $\circlearrowright$ and $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ is a model of Maxwell gravitation.*

Proof. This follows immediately from proposition 2 and the proof of proposition 5 of Dewar (2018). □

Proposition 4. *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a unique equivalence class of derivative operators $[\nabla]$ such that:*

- *All the $\nabla \in [\nabla]$ are compatible with $\circlearrowright$;*
- *For any two ∇, $\nabla' \in [\nabla]$, $\nabla' = (\nabla, t_b t_c \sigma^a)$, where σ^a is a spacelike and twist-free vector field which satisfies $\nabla_n \sigma^n = 0$ and $\rho\sigma^a = 0$;*[16]

16. The notation here follows Malament (2012, proposition 1.7.3): $\nabla' = (\nabla, C^a{}_{bc})$ iff for all smooth tensor fields $\alpha^{a_1\dots a_r}{}_{b_1\dots b_s}$ on M,

$$(\nabla'_n - \nabla_n)\alpha^{a_1\dots a_r}{}_{b_1\dots b_s} = \alpha^{a_1\dots a_r}{}_{mb_2\dots b_s} C^m{}_{nb_1} + \dots + \alpha^{a_1\dots a_r}{}_{b_1\dots b_{s-1}m} C^m{}_{nb_s} - \alpha^{ma_2\dots a_r}{}_{b_1\dots b_s} C^{a_1}{}_{nm} - \dots - \alpha^{a_1\dots a_{r-1}m}{}_{b_1\dots b_s} C^{a_r}{}_{nm}.$$

- *For any $\nabla \in [\nabla]$, $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

Proof. That there exists at least one derivative operator ∇ such that ∇ is compatible with $\circlearrowright$ and $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory follows immediately from proposition 1 and the proof of proposition 6 of Dewar (2018). So let ∇ be such a derivative operator, and suppose that ∇' is another derivative operator compatible with $\circlearrowright$ such that $\langle M, t_a, h^{ab}, \nabla', T^{ab}\rangle$ is a model of Newton-Cartan theory. So $\nabla_n T^{na} = \nabla'_n T^{na} = 0$. But since ∇ and ∇' are both compatible with $\circlearrowright$ and satisfy (NCT3) and (NCT4), we must have (Dewar 2018, proposition 4) that $\nabla' = (\nabla, t_b t_c \sigma^a)$, for some spacelike and twist-free vector field σ^a. Hence

$$\begin{aligned}\nabla'_n T^{na} &= \nabla_n T^{na} - (t_m t_n \sigma^n) T^{ma} - (t_m t_n \sigma^a) T^{nm} \\ &= \nabla_n T^{na} - \rho\sigma^a \\ &= \nabla_n T^{na}\end{aligned}$$

so that $\rho\sigma^a = 0$. Moreover, $R_{ab} = R'_{ab} = 4\pi\rho t_a t_b$, so that

$$\begin{aligned}R'_{ab} &= R_{ab} + \nabla_n (t_a t_b \sigma^n) \\ &= (4\pi\rho + \nabla_n \sigma^n) t_a t_b \\ &= 4\pi\rho t_a t_b\end{aligned}$$

and hence $\nabla_n \sigma^n = 0$. Finally, define $[\nabla]$ to be the class of all derivative operators such that $\nabla' = (\nabla, t_b t_c \sigma^a)$, where σ^a is a spacelike and twist-free vector field such that $\nabla_n \sigma^n = 0$ and $\rho\sigma^a = 0$. $[\nabla]$ is clearly unique in this regard. □

Corollary 4.1. *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation such that at all points $p \in M$, $\rho \neq 0$. Then there exists a unique derivative operator ∇ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

As such, the relationship between Maxwell gravitation and Newton-Cartan theory is less straightforward than that between Maxwell and Maxwell-Dewar gravitation. Whenever ρ is nowhere vanishing, each model of Maxwell gravitation is uniquely associated with a model of Newton-Cartan theory, and *vice* *versa*. But typically, a model of Maxwell gravitation does not carry enough information to fix a unique Newton-Cartan connection in regions where $\rho = 0$.

Now, in section 3, we noted that there is a close connection between the condition (NCT4) in Newton-Cartan theory, and the flat derivative operator compatibility condition in Maxwell gravitation. As a result, one might wonder if there is an analogue of propositions 3 and 4 for Künzle-Ehlers geometrised Newtonian gravitation, which drops the condition (NCT4).[17] It turns out that the answer is not quite. From any model of Künzle-Ehlers geometrised Newtonian gravitation, we can recover a unique KPM of Maxwell gravitation which satisfies the equations (MG). But if we drop flat derivative operator compatibility, then we cannot recover (NCT3) from the equations (MG), even if we still impose spatial flatness (for further details on this, see figure 1 in section 6 and the subsequent discussion). We can make some progress by dropping both (NCT3) and (NCT4). In this case, one can show that whenever h^{ab} is flat and $\rho \neq 0$, there is a one-to-one correspondence between KPMs of Newton-Cartan theory which satisfy (NCT1) and (NCT2), and KPMs of Maxwell gravitation which satisfy (MG1) and (MG2). But unlike propositions 3 and 4, we are only guaranteed that this is possible *at all* by the condition that $\rho \neq 0$. Without flat derivative operator compatibility, there is nothing to ensure that (NCT2) holds in regions where $\rho = 0$.

Nevertheless, one can still use Weatherall's standard of rotation to say something interesting about Künzle-Ehlers geometrised Newtonian gravitation. In particular, it is sometimes suggested that this theory lacks an absolute standard of rotation. For example, Knox (2011, 266) claims that "the constraints on the connection given by [metric compatibility and equations (NCT1)-(NCT3)] do not sufficiently restrict the class of connections to provide either an absolute standard of rotation or an absolute standard of acceleration." However, from the perspective of Weatherall's standard of rotation, this is not strictly true. Any Künzle-Ehlers connection is associated with a (unique) standard of rotation just as much as any Newton-Cartan connection. Rather, the difference between the two is that an arbitrary Künzle-Ehlers connection need not be ro-

17. Dewar (2016, 157) asks a similar question about the possibility of a "Maxwellian" analogue of Künzle-Ehlers geometrised Newtonian gravitation.

tationally equivalent to any flat connection.[18] And as I take it, *this* is the salient difference between Newton-Cartan theory and Künzle-Ehlers geometrised Newtonian gravitation, and is what explains why the latter theory is not empirically equivalent to standard Galilean gravitation.

This also leads us to a more general question about standards of rotation. Weatherall (2016, 88) asks "are there different, essentially inequivalent ways of characterising a standard of rotation?" The foregoing discussion suggests that the answer is yes. On one view, any derivative operator ∇ is associated with an absolute standard of rotation, since this is just the structure common to all those derivative operators which agree with ∇, for arbitrary η^a, on their determinations for $\nabla^{[a}\eta^{b]}$. This is precisely the idea which is captured by Weatherall's definition. But there are other plausible ways of defining a standard of rotation. For example, one might say that only derivative operators which commute on spacelike vector fields are associated with an absolute standard of rotation. Such a definition can be motivated by the idea that a unit timelike vector field ξ^a is irrotational just in case we can parallel transport any connecting field for ξ^a to another point on the same integral curve of ξ^a, and it remains parallel to itself. Or, one might say that only those derivative operators which admit irrotational unit timelike vector fields are associated with an absolute standard of rotation.

But this need not tell against adopting Weatherall's definition, at least as a definition of some "generalised" standard of rotation.[19] One motivation for this approach is that it avoids reference to spacetime structure which we then declare not to exist. Weatherall's definition also ensures that any derivative operator is associated with a unique standard of rotation, whereas if we adopt one of the other definitions suggested above, we are forced to say that some spacetimes have a connection but lack a standard of rotation. Since a standard of rotation is supposed to be "less structure" than a connection, this appears to be an advantage of Weatherall's definition.

There is also another advantage. If we define a standard of rotation as an

18. In particular, it is not that Künzle-Ehlers connections do not admit any irrotational unit timelike vector fields, as Knox (2011, 267) appears to suggest.

19. Indeed, a spacetime $\langle M, t_a, h^{ab}\rangle$ equipped with Weatherall's standard of rotation is essentially what Teh (2018) takes to define a "proto-Maxwell" spacetime, for exactly this reason.

equivalence classes of flat derivative operators, then this obscures the fact that there are more general notions of a standard of rotation available. Meanwhile, if we start with Weatherall's definition, we can then discuss what additional constraints, if any, might be needed.

5 Maxwell gravitation, vector relationism, and Newton-Cartan theory

To finish (and as a partial illustration of what one can do once Newtonian gravitation theory has been formulated using Weatherall's standard of rotation), I turn to Wallace's (2020) arguments concerning Saunders's (2013) vector relationism and Newton-Cartan theory. Here, Wallace claims to show that "mathematically speaking, there is no real distinction between Newton-Cartan theory [...] and vector relationism" (Wallace 2020, 24), and suggests that any differences between the two theories are partly an artefact of the awkwardness of standard differential-geometric presentations of Maxwellian spacetime (Wallace 2020, 28). As a result, Wallace adheres to a coordinate-based presentation of both theories in setting out his argument. My final aim here is to show that with Maxwell gravitation in hand, the same argument can also be made from a coordinate-free differential-geometric perspective.

Wallace's (2020) discussion of vector relationism and Newton-Cartan theory centres on the behaviour of dynamically isolated subsystems of particles embedded in a larger universe – showing that within vector relationism, such systems exhibit emergent inertial behaviour which can be idealised in terms of test particles. This forms the basis of his argument that vector relationism and Newton-Cartan theory are equivalent. When non-gravitational interactions vanish, the equations governing the relative acceleration vectors of infinitesimally separated test particles can be written to take the same form as the (coordinate-based) equation of geodesic deviation in Newton-Cartan theory, and thus, Wallace claims, may equally well be interpreted as such (Wallace 2020, §8).

Wallace is not explicit about the standard of theoretical equivalence he is working with here. But it is fairly straightforward to reconstruct from his remarks the sort of criterion he may have in mind. Having recovered the Newton-Cartan equation of geodesic deviation within vector relationism, Wallace claims of the two theories that

> both are built using Maxwellian spacetime as a background; both have dynamics that can be expressed as a set of inertial trajectories defined by the matter distribution and in turn constraining the matter distribution via a matter dynamics according to which material particles follow those trajectories except when acted on by non-gravitational forces. (Wallace 2020, 24)

Similarly, in his concluding remarks, Wallace argues that

> there is essentially no difference between Newton-Cartan theory [...] and Saunders's relational version of Newtonian dynamics: at the formal level, the latter can be reformulated as the former; at the substantive level, the inertial structure of Saunders's theory is well defined and coincides with that defined by the Newton–Cartan connection. (Wallace 2020, 28)

These comments suggest the following standard of theoretical equivalence: two theories are equivalent just in case they have the same background spacetime structure, and their central dynamical equations can be rewritten so as to take the same form.[20]

The connection to our current framework is immediate. Given (NCT4), any Newton-Cartan spacetime determines a unique flat derivative operator compatible Maxwellian spacetime. And as Malament (2012, proposition 4.3.2) shows, (NCT2) holds at a point p just in case for all geodesic reference frames ξ^a, the average radial acceleration of ξ^a at p is equal to $-4/3\pi\rho$. But this, together with (NCT1), entails (MG2). The only difference, as far as this pair of equations is concerned, is the interpretation of (MG2) – in Newton-Cartan

20. Why only the *central* dynamical equations? Wallace does not explicitly consider all the equations of Newton-Cartan theory in his comparison with vector relationism, and as we will see in section 6, not all the equations of Maxwell gravitation and Newton-Cartan theory (or vector relationism and Newton-Cartan theory, for that matter) can be written so as to appear mathematically identical.

theory, the $-4/3\pi\rho$ term is naturally understood as a manifestation of geodesic deviation in curved spacetime, whereas in Maxwell gravitation it is not.

However, in order to make the link to Wallace's argument precise, we need to say something about the relationship between Maxwell gravitation and vector relationism. I will begin by considering Wallace's (2020, 8) claim that vector relationism should also be thought of as a theory set on Maxwellian spacetime. Saunders (2013) presents vector relationism as a theory of the displacement vectors between point particles, formulated with reference to some Maxwellian coordinate system. Within this framework, the dynamics are specified by the following pair of equations:

$$\mathbf{r}_{ij} = \mathbf{X}_i - \mathbf{X}_j \tag{VR1}$$

$$\frac{d^2\mathbf{r}_{ij}}{dt^2} = \frac{1}{m_i}\sum_{k\neq i}\mathbf{F}_{ik} - \frac{1}{m_j}\sum_{k\neq j}\mathbf{F}_{jk}, \tag{VR2}$$

where $\mathbf{X}_i(t)$ denotes the position of particle i at time t with respect to such a coordinate system, m_i its mass, and the $\mathbf{F}_{ij}$ denote interparticle forces. These are taken to be antisymmetric in i and j (this is the import of Newton's third law) and functions of $\mathbf{r}_{ij}$ only. The equations (VR) are invariant under the Maxwell group of symmetries – transformations of the form

$$t \to t + \tau \tag{13a}$$

$$x^i(t) \to R^i{}_j x^j(t) + a^i(t), \tag{13b}$$

where $R^i{}_j$ is an arbitrary rotation matrix, $a^i(t)$ an arbitrary vector-valued function of time, and τ an arbitrary scalar.

To argue that Maxwellian spacetime is the appropriate setting for vector relationism, Wallace then makes tacit appeal to Earman's (1989, 46) "adequacy conditions" on the construction of spacetime theories.[21] These demand that there be a match between the spacetime and dynamical symmetries of a theory, in the following sense:

SP1: Any dynamical symmetry of T is a spacetime symmetry of T.

21. For recent discussion of the status of these conditions, see Myrvold (2019).

SP2: Any spacetime symmetry of T is a dynamical symmetry of T.

It is then straightforward to argue that the automorphism group of any flat derivative operator compatible Maxwellian spacetime $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ is, indeed, the Maxwell group. These transformations must preserve the spatial metric, so act rigidly on each spacelike hypersurface. Since the spacetime is flat derivative operator compatible, they must preserve the property that parallel transport of spacelike vectors, relative to any compatible connection, is path independent. And they must preserve the temporal metric. This is sufficient to restrict the spacetime symmetries to the Maxwell group. Hence, Wallace claims that Saunders's theory is naturally set on (flat derivative operator compatible) Maxwellian spacetime.

We can then connect vector relationism to Maxwell gravitation via the equations (MG) and (VR). First, following Wallace (2020, 11), we can decompose the forces in (VR2) into "universal" and "non-universal" components – characterised, respectively by whether the ratio q_i/m_i is constant for that force, where m_i is the inertial mass of a particle and q_i its charge. For the case of only potential forces, (VR) may then be written as

$$\frac{d^2\mathbf{X}_i}{dt^2} - \frac{d^2\mathbf{X}_j}{dt^2} = -\sum_{k\neq i} \nabla\phi(\mathbf{X}_i - \mathbf{X}_k) + \sum_{k\neq j} \nabla\phi(\mathbf{X}_j - \mathbf{X}_k) - \frac{q_i}{m_i}\sum_{k\neq i} \nabla V(\mathbf{X}_i - \mathbf{X}_k) + \frac{q_j}{m_j}\sum_{k\neq j} \nabla V(\mathbf{X}_j - \mathbf{X}_k), \quad (14)$$

where ϕ is the potential associated with the universal force, and V the potential for the non-universal force (there could be multiple such; I omit them for simplicity). Now consider the continuum limit, where point-particle trajectories are parametrised by some continuous spatial parameter $\mathbf{x}$. In this limit, (14) becomes

$$\partial_i \left(\frac{d^2\mathbf{X}(\mathbf{x}, t)}{dt^2} \right) \delta x^i = -\partial_i \int d^3\mathbf{x}' \nabla\phi(\mathbf{x} - \mathbf{x}', t)\delta x^i - \partial_i \int d^3\mathbf{x}' \tilde{\rho}(\mathbf{x}, t)\rho^{-1}(\mathbf{x}, t)\nabla V(\mathbf{x} - \mathbf{x}', t)\delta x^i,$$

where $\rho(\mathbf{x}, t)$ is the mass density, and $\tilde{\rho}(\mathbf{x}, t)$ the charge density associated with the non-universal interaction, so that

$$\partial_i \left(\frac{d^2 X^j(\mathbf{x},t)}{dt^2} \right) = -\partial_i \int d^3\mathbf{x}' (\partial^j \phi(\mathbf{x}-\mathbf{x}',t) + \tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t)). \tag{15}$$

When ϕ is the familiar gravitational potential, we have

$$\phi(\mathbf{x}-\mathbf{x}',t) = \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|},$$

so that

$$\partial_i \left(\frac{d^2 X^j(\mathbf{x},t)}{dt^2} \right) = -\partial_i \int d^3\mathbf{x}' \rho(\mathbf{x}',t)\partial^j (|\mathbf{x}-\mathbf{x}'|)^{-1} - \partial_i \int d^3\mathbf{x}' \tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t). \tag{16}$$

We have seen that the appropriate spacetime setting for vector relationism is a flat derivative operator compatible Maxwellian spacetime, $\langle M, t_a, h^{ab}, \circlearrowright \rangle$. Since we can always (if M is simply connected) find a globally defined scalar field t such that $d_a t = t_a$, we can then set up an arbitrary Maxwellian coordinate system x^μ on M as follows: we take $x^\mu = (t, x^i)$, where t is as above and the x^i are three smooth scalar fields such that the vector fields $(\partial/\partial x^i)^a$ are spacelike, orthonormal, and twist-free (with respect to $\circlearrowright$).[22]

Let x^μ be such a coordinate system, and let ∇ be the coordinate derivative operator on M canonically associated with x^μ.[23] ∇ is flat (since it is a coordinate derivative operator); it is compatible with t_a by construction, and is compatible with h^{ab} since the $(\partial/\partial x^i)^a$ are spacelike and orthonormal. Moreover, since the $(\partial/\partial x^\mu)^a$ are all twist-free with respect to $\circlearrowright$ and $\circlearrowright$ is flat derivative operator compatible, ∇ is also compatible with $\circlearrowright$.[24] Now consider a smooth unit time-like vector field ξ^a on M. The integral curves ξ of any such field can always be parametrised by their temporal length, which differs from t by at most an

22. If M is not simply connected then the same analysis goes through locally; I suppress it here for reasons of brevity.
23. That is, the unique derivative operator such that all the $\nabla_a(\partial/\partial x^\mu)^b = 0$.
24. Note that $(\partial/\partial t)^a$ is twist-free by construction, since t_a is closed.

arbitrary additive constant. Then on any such curve ξ, we have

$$\xi^a = \frac{dx^\mu(\xi(t))}{dt}\left(\frac{\partial}{\partial x^\mu}\right)^a$$

so that, since ∇ is flat

$$\xi^n \nabla_n \xi^a = \frac{d^2x^\mu(\xi(t))}{dt^2}\left(\frac{\partial}{\partial x^\mu}\right)^a.$$

Clearly, the only non-vanishing d^2x^μ/dt^2 are the d^2x^i/dt^2. Moreover, if σ^{ab} is a (symmetric) tensor field which is spacelike in both indices, then we can write

$$D_n \sigma^{na} = \partial_\mu \sigma^{\mu\nu}\left(\frac{\partial}{\partial x^\nu}\right)^a$$

where the only non-vanishing $\partial_\mu \sigma^{\mu\nu}$ are the $\partial_\mu \sigma^{\mu i}$. If we now take ξ^a to represent the four velocity field of a fluid, and σ^{ab} the stress tensor for that fluid, then these suggest the following identifications:

$$\xi^n \nabla_n \xi^m (d_m x^i) = \frac{d^2X^i(\mathbf{x}, t)}{dt^2} \tag{17a}$$

$$\rho^{-1} D_n \sigma^{nm}(d_m x^i) = \int d^3\mathbf{x}' \tilde{\rho}(\mathbf{x}, t)\rho^{-1}(\mathbf{x}, t)\partial^i V(\mathbf{x} - \mathbf{x}', t). \tag{17b}$$

Why? Take (17a). We are looking for something with which to identify the (non-zero) components of the acceleration vector field of a fluid $\xi^n \nabla_n \xi^m (d_m x^i)$ with respect to the coordinate derivative operator canonically associated with some Maxwellian coordinate system x^μ. Not only is this precisely what the $d^2X^i(\mathbf{x}, t)/dt^2$ represent, we have also seen that when ∇ is such a derivative operator, the $\xi^n \nabla_n \xi^m (d_m x^i) = d^2x^i(\xi(t))/dt^2$ take this same form. Now consider (17b). The left hand side of this equation are the (non-zero) components of a spacelike vector field which is supposed to describe the acceleration due to non-gravitational interactions – think of (the geometrised version of) Newton's second law

$$\rho \xi^n \nabla_n \xi^a = -\nabla_n \sigma^{na}. \tag{NII}$$

And this is precisely the role of the term on the right hand side. We can then write (16) as

$$\nabla_r(\xi^n\nabla_n\xi^m)(d_m x^j)\left(\frac{\partial}{\partial x^i}\right)^r = -\partial_i\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\partial^j(|\mathbf{x}-\mathbf{x}'|)^{-1} - D_r(\rho^{-1}D_n\sigma^{nm})(d_m x^j)\left(\frac{\partial}{\partial x^i}\right)^r. \quad (19)$$

Now consider the case where $i = j$. In this case, carrying out the differentiation in the right hand side of (19) gives

$$\nabla_m(\xi^n\nabla_n\xi^m) = -4\pi\rho - D_m(\rho^{-1}D_n\sigma^{nm})$$

where we have used the fact that $\xi^n\nabla_n\xi^a$ and $\rho^{-1}D_n\sigma^{na}$ are both spacelike. This immediately yields (MG2). Meanwhile, if we take $i \neq j$ in (19), then differentiating and raising indices we have

$$\nabla^r(\xi^n\nabla_n\xi^m)(d_m x^j)(d_r x^i) = \int d^3\mathbf{x}'\rho(\mathbf{x}',t)\left(3\frac{(x^j - x'^j)(x^i - x'^i)}{|\mathbf{x}-\mathbf{x}'|^5}\right) - D^r(\rho^{-1}D_n\sigma^{nm})(d_m x^j)(d_r x^i),$$

so that, since $\circlearrowleft^a$ $(\xi^n\nabla_n\xi^b)$ is spacelike in both indices,

$$\nabla^a(\xi^n\nabla_n\xi^b) - \nabla^b(\xi^n\nabla_n\xi^a) = -D^a(\rho^{-1}D_n\sigma^{nb}) + D^b(\rho^{-1}D_n\sigma^{na}),$$

which, given the continuity equation (MG1) and the fact that ∇ is flat by construction, entails (MG3) (see the proof of proposition 2). For (MG1) itself, note that in Newtonian point particle mechanics, mass is transported only by particles along their (continuous) worldlines, and is *a fortiori* locally conserved.

Conversely, it is also possible to recover (VR) from (MG). Given the identifications (17), we can use (MG) to derive expressions for $\partial_i(d^2X^i/dt^2)$ and $\partial^{[i}(d^2X^{j]}/dt^2)$ in any Maxwellian coordinate system x^μ on M. These are sufficient to specify (15) uniquely, providing that $\partial_i(d^2X^i/dt^2)$ and $\partial^{[i}(d^2X^{j]}/dt^2)$ fall off at least as $1/r^2$ at spatial infinity. If we then specialise to the case of a point-particle distribution (which justifies making the above assumptions about d^2X^i/dt^2), this gives $\phi(\mathbf{x}-\mathbf{x}',t) \to \phi(\mathbf{x}-\mathbf{x}',t)\sum_i \delta^3(\mathbf{x}'-\mathbf{X}_i(t))$ and analogously for V. Hence,

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right) = -\partial_i\sum_k \partial^j\phi(\mathbf{x}-\mathbf{X}_k,t) - \partial_i\sum_k \tilde{\rho}\rho^{-1}\partial^j V(\mathbf{x}-\mathbf{X}_k,t). \quad (20)$$

Since $\tilde{\rho}\rho^{-1} = \sum_i q_i/m_i\delta^3(\mathbf{x}-\mathbf{X}_i(t))$, (14) then follows from integrating along any path between $\mathbf{X}_i(t)$ and $\mathbf{X}_j(t)$.

This suggests a particularly close relationship between Maxwell gravitation and vector relationism. Both are set on a flat derivative operator compatible Maxwellian spacetime. Moreover, the equations of Maxwell gravitation emerge naturally in the continuum limit of vector relationism, whilst vector relationism is precisely what results from restricting Maxwell gravitation to the point particle sector. Since Wallace's arguments also involve the limit of infinitesimally-separated particles (Wallace 2020, 23), one would therefore expect them to carry over once we move from vector relationism to Maxwell gravitation.

Before turning to discuss Wallace's claims in detail, however, it is worth pausing briefly on some other immediate consequences of this relationship between Maxwell gravitation and vector relationism. One is that it suggests a particularly elegant way of interpreting the equations (MG2) and (MG3) – as encoding the radial and transverse components of the relative acceleration vector field between neighbouring fluid elements. This, in turn, provides a sense in which Maxwell gravitation, like Saunders's theory, might be thought of as one which is fundamentally concerned with relative accelerations. The other is that this lends additional support to Dewar's (2018, 268) claim that "[Maxwell-Dewar] gravitation [...] represents the natural extension of Saunders's remarks to the field-theoretic context." Dewar argues for this on the basis that Maxwell gravitation, like vector relationism, collapses the distinction between models of Newton-Cartan theory which disagree only as to the connection in regions where $\rho = 0$. However, the fact that Maxwell gravitation emerges naturally in the continuum limit of vector relationism, and *vice versa*, provides a more direct route to this conclusion.

6 Understanding Wallace from a coordinate-free perspective

To make the link to Wallace's argument, it is instructive to begin by comparing the equations of Maxwell gravitation with those of Newton-Cartan theory. In particular, let $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ be a Maxwellian spacetime. Then for any derivative operator ∇ compatible with $\langle M, t_a, h^{ab}, \circlearrowright \rangle$, the following implications hold (illustrated in figure 1).

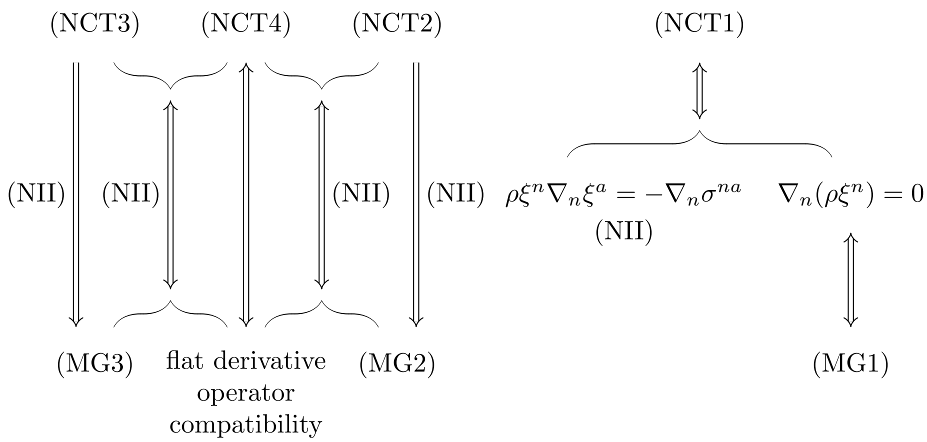

Figure 1: Relationships between the equations of Maxwell gravitation and Newton-Cartan theory. Labelled arrows are to be understood as in the scope of a conditional – so, for example, the first arrow from the left says that if (NII) holds, then (NCT3) implies (MG3)

There are several features of figure 1 worth noting. First, whilst (NCT4) is equivalent to flat derivative operator compatibility, there is no similarly sharp correspondence between (NCT3) and (MG3). (NCT3) and (NII) jointly imply (MG3), but (MG3) and (NII) do not imply (NCT3). This points to the fact that the flat derivative operator compatibility condition plays double duty in relating the two theories. From (MG3) and (NII) we can infer that $\xi^n\xi^m(R^{c\ \,a}_{\ n\ \,m} - R^{a\ \,c}_{\ m\ \,n}) = 0$; the flat derivative operator compatibility condition allows us to further infer from this that $\eta^n(R^{c\ \,a}_{\ n\ \,m} - R^{a\ \,c}_{\ m\ \,n}) = 0$ for arbitrary η^a, which yields (NCT3).[25]

25. We know from propositions 4.2.4 and 4.3.1 of Malament (2012) that a Maxwellian spacetime is flat derivative operator compatible just in case parallel transport of spacelike vectors is at least locally path-independent. The idea is to appeal to the fact that an arbitrary vector field η^a can be written as $\eta^a = \alpha\xi^a + \sigma^a$ for some scalar field α and spacelike vector field σ^a, rewrite the expression $\xi^n\eta^m(R^{c\ \,a}_{\ n\ \,m} - R^{a\ \,c}_{\ m\ \,n})$ in terms of ∇, and then use that, since

Secondly, although (NCT2) and (MG2) are not in general equivalent, they are equivalent on assumption of (NII) and flat derivative operator compatibility. Likewise, given (NII), flat derivative operator compatibility and (MG3) are equivalent to (NCT4) and (NCT3). As such, once (NII) has been fixed, we can then move freely between the remaining pairs of equations.

Now recall that for Wallace, what underwrites the claim that vector relationism and Newton-Cartan theory are equivalent is that for an idealised congruence of test particle trajectories, the equations (VR) can be rewritten so as to take the same form as the equation of geodesic deviation in Newton-Cartan theory. But we have just seen that this has an obvious analogy for Maxwell gravitation and Newton-Cartan theory: by replacing (NCT2) with the expression for the average radial acceleration (MG2), we can reformulate the two theories so that their central dynamical equations appear mathematically identical. Within Newton-Cartan theory, (MG2) encodes the relative acceleration of neighbouring fluid elements due to both spacetime curvature and non-gravitational interactions, so represents the natural generalisation of Wallace's geodesic deviation equation to non-test matter. And just as in Wallace's example, the resulting pair of equations differ at most as to whether they are interpreted as describing the influence of universal (i.e. gravitational) forces, or geodesic deviation in curved spacetime.

Moreover, once we move from vector relationism to Maxwell gravitation, the case for regarding this disagreement as merely verbal appears even stronger. After all, in vector relationism, the gravitational field is explicitly represented elsewhere in the formalism. But in Maxwell gravitation, we do not even have that. Of course, we are always free to ascribe the $-4/3\pi\rho$ term in (MG2) to "the gravitational field" – but without some further indication of what this is supposed to be, the gravitational field is simply that whereby neighbouring test particles have non-zero relative acceleration. And since this is precisely the role of the Newton-Cartan spacetime curvature, the difference between the two begins to look insubstantive. As such, we seem to have in the relationship between (MG2) and (NCT2) a coordinate-free realisation of Wallace's argument.

parallel transport of spacelike vectors is path-independent, ∇ commutes on spacelike vector fields. Repeating this argument twice gives the desired result.

However, we can also say a little more about this reasoning. Given the relationships illustrated in figure 1, not only are we free to replace (NCT2) with (MG2) in Newton-Cartan theory, we can also replace (NCT3) with (MG3), (NCT4) with the flat derivative operator compatibility condition, and rewrite (NCT1) as the conjunction of (NII) and (MG1). From this perspective, the only difference between these sets of equations is the presence of (NII) in Newton-Cartan theory, whose role is essentially to provide a (partial) gauge fixing of the connection. This provides a further sense in which Wallace's argument is strengthened when we move from vector relationism to Maxwell gravitation – *all* the equations of Newton-Cartan theory, with the exception of (NII), can be written so as to appear mathematically identical to the equations of Maxwell gravitation.[26]

All this serves to blunt the force of Wallace's (2019, 134; 2020, 28) recent claims that Maxwellian spacetime is not naturally characterised in coordinate-free differential geometric terms, and that this is partly what obscures the similarities between Maxwell gravitation and Newton-Cartan theory. Rather, we have seen that once cast in terms of Weatherall's standard of rotation, the formal similarities which Wallace discusses re-emerge from a coordinate-free perspective. As a result, one might suspect that the problem lies not with coordinate-free differential geometry *per se*, but with formulating a theory in terms of geometric objects which cannot be defined from the structure it ascribes to the world.[27]

But it does suggest an alternative moral. Both Maxwell and Maxwell-Dewar gravitation are formulated in the language of coordinate-free differential geometry. But the fact that a theory has been formulated in a coordinate-free way does not automatically mean that this is a perspicuous way of presenting that theory. When working with coordinate-free differential geometry, as ever, it is important to be attentive to this possibility.

26. Note that this also highlights why it is that Newton-Cartan theory cannot be the continuum limit of vector relationism. If one assumes that the dynamics for test particles in Newton-Cartan theory are given by the geodesic equation, then it is possible to show that in both Newton-Cartan theory and the continuum limit of vector relationism, test particles satisfy the equation of geodesic deviation. But precisely what one cannot recover in the continuum limit of vector relationism is the geodesic equation itself.

27. For an extended discussion of other possible issues relating to this in the context of the interpretation vs. motivation and reduction vs. sophistication debates, see Jacobs (2022).

7 Conclusions

I have provided an explicit presentation of a theory of Newtonian gravitation, which is formulated in a coordinate-free way, using just the structure of Maxwellian spacetime. Moreover, I hope to have shown that this theory of Newtonian gravitation is not "awkward" – it does not make reference to structure which we then declare not to exist, and its equations can be given sensible physical interpretations. This theory also makes apparent the similarities between vector relationism and Newton-Cartan theory which Wallace (2020) discusses from a coordinate-based perspective. Along the way, I have made some remarks of independent interest about Künzle-Ehlers geometrised Newtonian gravitation, vector relationism, and standards of rotation in general.

This is sufficient to address the concerns raised by Weatherall, Wallace, and others about previous attempts to formulate a theory of Newtonian gravitation set on Maxwellian spacetime. It also provides a useful basis for future work. One would like to consider what light, if any, this sheds on the question of whether Maxwell gravitation is equivalent to Newton-Cartan theory. A proper treatment of that issue will have to wait for another time.

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