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The Non-Relativistic Geometric Trinity of Gravity

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We complete a non-relativistic geometric trinity of gravity, by (a) taking the non-relativistic limit of the well-known geometric trinity of gravity, and (b) converting the curvature degrees of freedom of Newton-Cartan theory to purely non-metric degrees of freedom.

I. INTRODUCTION

It has become increasingly well-known that general relativity (GR) constitutes but one vertex in a 'geometric trinity' of gravitational theories [1]. The other two vertices of this trinity are 'teleparallel gravity' (TPG), in which the curvature degrees of freedom of GR are traded for spacetime torsion, and 'symmetric teleparallel gravity' (STGR), in which the curvature degrees of freedom of GR (and torsion degrees of freedom of TPG) are traded for spacetime non-metricity. The actions of all three theories are equivalent up to a total divergence term—in this sense, all three theories are dynamically equivalent.

In a parallel vein, it has been known since Trautman in the 1960s [2] that standard Newtonian gravity (NGT) is equivalent to a formulation of non-relativistic gravity in which, as with GR, gravitational effects are again a manifestation of spacetime curvature: this theory is known as Newton-Cartan theory (NCT), and was first developed in the 1920s by Cartan and Friedrichs: see [3, 4] for the original sources, and [5] for a recent review of non-relativistic gravity. In [6], it was shown that there is a precise sense in which NGT can be understood as the teleparallelised version of NCT, in which the gravitational potential of flat-spacetime NGT can be understood as a manifestation of the 'mass torsion' which arises once one gauges the Bargmann algebra (on which see e.g. [7]). Adding to this, it was shown recently in [8] that NGT can be secured as the non-relativistic limit of TPG using a $1/c$ expansion of the TPG action (in [6] the same result was shown using null reduction), just as NCT is by now well-known to be the non-relativistic limit of GR (on which see [5] and references therein).

These results invite the following question: *can one complete a non-relativistic geometric trinity*, by constructing a 'purely non-metric' theory equivalent to both NGT (understood as a torsionful theory) and NCT (understood as a theory with spacetime curvature)? In this article, we answer this question in the affirmative—indeed, we triangulate a non-relativistic version of Newtonian gravity (which we dub 'symmetric Newtonian gravity', shortened to SNGT) in two ways: (a) by taking the non-relativistic limit of STGR (using exactly the same $1/c$ expansion found in [8]), and (b) by proving the analogues of the Trautman geometrisation/recovery theorems (see [9, Ch. 4]) relating NCT and SNGT.

The structure of this article is as follows. In §II, we review the essential details of the geometric trinity of gravity; in §III, we construct SNGT by taking a non-relativistic limit of STGR; in §IV, we state and prove geometrisation/recovery theorems relating NCT and SNGT. We close in §V with some discussion of the upshots of this work.

II. BACKGROUND: THE GEOMETRIC TRINITY

The bulk of this section constitutes a review of the relativistic geometric trinity of gravity (§II A). In addition, we review briefly the state-of-play regarding geometric reformulations of non-relativistic gravity (§II B).

A. Relativistic Gravity

Spacetime theories are typically formulated in terms of a metric tensor $g_{\mu\nu}$ and an affine connection $\Gamma^\alpha_{\mu\nu}$. General relativity (GR) is of course the paradigmatic theory of gravity and makes use of the Levi-Civita connection, with components

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} := \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \right), \tag{1}$$

which is the unique connection that is compatible with the metric and torsion-free. The metric-compatibility condition is given by $\nabla_\alpha g_{\mu\nu} = 0$ and the torsion-free condition is given by ${\Gamma^\alpha}_{[\mu\nu]} = 0$ [10, Ch. 3]. Famously, GR describes gravity as a manifestation of spacetime curvature, as encoded in the Riemann tensor, which has components

$$R^\alpha{}_{\beta\mu\nu}(\Gamma) := \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\beta}. \tag{2}$$

Spacetime curvature measures the rotation of a vector when it is parallel transported along a closed curve.

One can alter or otherwise relax the above assumptions in order to construct spacetime theories that manifest torsion and/or non-metricity. Torsion is given by the

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antisymmetric part of the connection

$$T^{\alpha}{}_{\mu\nu} := 2\Gamma^{\alpha}{}_{[\mu\nu]} \tag{3}$$

and can be thought of as a measure of the non-closure of the parallelogram formed by two vectors being parallel transported along each other. Non-metricity is given by the non-vanishing of the covariant derivative of the metric tensor

$$Q_{\alpha\mu\nu} := \nabla_{\alpha} g_{\mu\nu} \tag{4}$$

and can be thought of as a measure of how the length of a vector changes when parallel transported.

We can thus categorize spacetimes as

1. metric (i.e., $Q_{\alpha\mu\nu}(\Gamma) = 0$)
2. torsionless (i.e., $T^{\alpha}{}_{\mu\nu}(\Gamma) = 0$)
3. flat (i.e., $R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = 0$)

As we have seen, GR is a spacetime theory that is metric and torsionless but non-flat as the Levi-Civita connection possesses curvature. In this article, we will also be concerned with two other spacetime theories: 'teleparallel gravity' (TPG) and 'symmetric teleparallel gravity' (STGR). TPG spacetimes are metric and flat but possess torsion; STGR spacetimes are torsionless and flat but possess non-metricity. Both TPG and STGR are dynamically equivalent to GR, in the sense that the actions of all three theories are equivalent up to total divergence terms; thereby, the theories are capable of modelling the same empirical phenomena, and constitute a 'geometric trinity' of gravity—see [1, 11, 12] for recent discussions.

Curvature, torsion, and non-metricity are all geometric properties of an affine connection which can, in full generality, be decomposed in the following way [13]:

$$\Gamma^{\alpha}{}_{\mu\nu} = \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{5}$$

where

$$K^{\alpha}{}_{\mu\nu} := \frac{1}{2} T^{\alpha}{}_{\mu\nu} + T_{(\mu}{}^{\alpha}{}_{\nu)} \tag{6}$$

is referred to as the 'contorsion tensor', and

$$L^{\alpha}{}_{\mu\nu} := \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)} \tag{7}$$

is referred to as the 'distortion tensor'. We can use (5) to facilitate translations between different spacetime theories with different connections (and associated different geometrical properties). For example, we can find GR's torsionful and non-metric equivalents by taking the expressions for the Riemann curvature and Ricci scalar in GR in terms of the Levi-Civita connection, and re-expressing these in terms of the 'Weitzenböck' connection of TPG or the non-metricity connection of STGR. Consider that we can express a generic Riemann curvature tensor $\hat{R}^{\alpha}{}_{\beta\mu\nu}$ as [14]:

$$\begin{aligned} \hat{R}^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} M^{\alpha}{}_{\nu\beta} - \nabla_{\nu} M^{\alpha}{}_{\mu\beta} \\ + M^{\gamma}{}_{\nu\beta} M^{\alpha}{}_{\mu\gamma} - M^{\gamma}{}_{\mu\beta} M^{\alpha}{}_{\nu\gamma}, \end{aligned} \tag{8}$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the standard Riemann tensor from the Levi-Civita connection and $M^{\alpha}{}_{\mu\nu} := K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$.

One can choose to work with TPG and the contorsion tensor ($K^{\alpha}{}_{\mu\nu} \neq 0$ and $L^{\alpha}{}_{\mu\nu} = 0$) or with STGR and the distorsion tensor ($K^{\alpha}{}_{\mu\nu} = 0$ and $L^{\alpha}{}_{\mu\nu} \neq 0$). Upon index contraction, one constructs the curvature scalar and finds:

$$-R = T + 2\nabla_{\alpha} T^{\lambda\alpha}{}_{\lambda} = Q + \nabla_{\alpha}\left(Q^{\alpha}{}_{\lambda}{}^{\lambda} - Q^{\lambda}{}_{\lambda}{}^{\alpha}\right), \tag{9}$$

where R is the scalar curvature of the Levi-Civita connection, T is the scalar torsion of the TPG connection, and Q is the non-metricity scalar of the STGR connection [1]. Importantly, this shows that the scalar expressions of curvature, torsion, and non-metricity are equivalent up to a boundary term.[1] This justifies the above claim that GR, TPG, and STGR can be formulated in terms of dynamically equivalent Lagrangian expressions.

While these particular theories are empirically equivalent to each other, there are a number of reasons why physicists are interested in investigating such alternative geometric representations. One reason has to do with the fact that these theories possess different gauge structure. In particular, TPG and STGR can be understood as gauge theories of translations [18, 19], which allows one to formulate the theories in a language more closely resembling other fundamental interactions and potentially suggests different routes towards quatisation. Another reason can be found in resolving cosmological puzzles. Despite the incredible successes of the current ΛCDM model, there are a number of unresolved issues that are the subject of heated debate, including our modeling of both early and late time expansion of the universe [20–23]. While the theories within the trinity are indeed equivalent, their geometric structures based on curvature, torsion, and non-metricity suggest different routes to modifying gravity. Indeed, the equivalence is broken when we move to modifications that consist in higher order scalar invariants of the relevant geometric quantities. That is, $f(R)$, $f(T)$, and $f(Q)$ theories are not equivalent to each other, and this has motivated exploring this theory space as possible novel realisations of dark energy, inflation, and bouncing cosmologies [24–27]

B. Non-Relativistic Gravity

So much by way of background on the relativistic geometric trinity of gravity; what is the current state-of-the-

[1] See [15–17] for some discussions concerning the role and significance of these boundary terms.

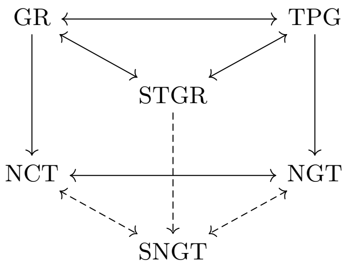

FIG. 1. The geometric trinity and its (conjectured) non-relativistic limit.

art with respect to non-relativistic physics? It has been known since the 1960s that standard, flat-spacetime Newtonian gravity (NGT) is equivalent to a curved spacetime theory known as 'Newton-Cartan theory' (NCT); this equivalence is codified in the Trautman geometrisation and recovery theorems [9, Ch. 4]. NCT (and non-relativistic gravity more generally) is still a very active field of research as it has found important applications in non-relativistic holography [28], quantum gravity [29, 30], and condensed matter systems [7, 31–33]. What constitutes much more recent knowledge is that it is possible to understand NGT as the teleparallel equivalent of NCT, in the sense that the gravitational field $G_\mu := d_\mu \varphi$ in the theory can be understood as the torsion of the mass gauge field m_μ obtained by gauging the Bargmann algebra; moreover, NGT can be obtained by taking a $1/c$ expansion of TPG [8].

This invites the following questions: (a) can one construct a purely non-metric non-relativistic theory of gravity by taking a $1/c$ expansion of STGR, and (b) can one prove Trautman-style geometrisation and recovery theorems relating this theory to NCT? In the remainder of this article, we answer in the affirmative both (a) and (b); thereby, we fill in the dotted lines in the Figure 1, and so complete for the fist time a non-relativistic geometric trinity of gravity.

III. THE NON-RELATIVISTIC LIMIT OF STGR

We begin by taking the non-relativistic limit of STGR via a $1/c$ expansion—this will give us a non-relativistic theory of gravity the degrees of freedom of which are purely non-metric. Our non-relativistic limit begins with an expansion of the relativistic objects in terms of powers of c. As there is no speed limit in non-relativistic physics, one takes $c \to \infty$, which can be thought of as 'flattening' the null cones at all spacetime points. Essentially, "in the limit the cones are all tangent to a family of hypersurfaces, each of which represents "space" at a given "time", which corresponds to the standard Newtonian picture of spacetime" [34].

More precisely, consider (using adapted coordinates at some spacetime point) the metric tensor can be written

$$g_{\mu\nu} = \mathrm{diag}\left(+1, -1/c^2, -1/c^2, -1/c^2\right) \tag{10}$$

while its inverse can be written

$$g^{\mu\nu} = \mathrm{diag}\left(+1, -c^2, -c^2, -c^2\right). \tag{11}$$

Upon taking the $c \to \infty$ limit, we have that $g_{\mu\nu} \to \mathrm{diag}\,(+1, 0, 0, 0)$ and $g^{\mu\nu}/c^2 \to \mathrm{diag}\,(0, -1, -1, -1)$. Thus, under this limit, the metric tensor and its inverse (appropriately scaled by c, *per* the above) become degenerate, splitting into the spatial metric $h^{\mu\nu}$ and temporal metric $t_{\mu\nu} = t_\mu t_\nu$ of a non-relativistic spacetime (here, we assume temporal orientability—see [9, Ch. 4]).

In the case of taking the non-relativistic limit of TPG, the prescription followed in [8] is this: (a) write the Einstein equations of GR in terms of the TPG connection (and associated torsion); (b) take the non-relativistic limit—construed in the above way—of that equation. Following the same prescription for the non-relativistic limit of STGR, we first express the Einstein equations as

$$\begin{aligned} -\overset{n}{\nabla}_\alpha L^\alpha{}_{\mu\nu} + \overset{n}{\nabla}_\mu L^\alpha{}_{\nu\alpha} - L^\alpha{}_{\mu\beta} L^\beta{}_{\alpha\nu} + L^\alpha{}_{\alpha\beta} L^\beta{}_{\mu\nu} \\ = \frac{8\pi G}{c^4}\left(T_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}\right), \end{aligned} \tag{12}$$

where the LHS is the Ricci tensor of the Levi-Civita connection expressed terms of the distorsion and the non-metric connection. Throughout this paper, we will denote the curvature based connection as $\overset{c}{\nabla}$ and the non-metricity based connection as $\overset{n}{\nabla}$. Taking the non-relativistic limit of the RHS gives [8, p. 20]:

$$\frac{8\pi G}{c^4}\left(T_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}\right) \to 4\pi G \rho t_\mu t_\nu. \tag{13}$$

In order to take the non-relativistic limit of the LHS, we now specialise to the case of 'Weylian' non-metricity, of the form $Q_{\alpha\mu\nu} = \sigma_\alpha g_{\mu\nu}$—we will remark on this further in §V. (In brief: without this specialisation, one cannot use the known behaviour of the metric tensor under the limit to fix the behaviour of the non-metricity tensor in that limit.) In the non-relativistic limit, the non-metricity and distortion tensors become

$$Q_{\alpha\mu\nu} \to \sigma_\alpha t_\mu t_\nu, \quad L^\alpha{}_{\mu\nu} \to -Q_{(\mu}{}^\alpha{}_{\nu)} = -\sigma_{(\mu}\delta^\alpha{}_{\nu)}, \tag{14}$$

where σ_α is a nowhere vanishing 1-form such that $\sigma_\beta h^{\alpha\beta} = 0$. This follows because $\frac{1}{2} Q^\alpha{}_{\mu\nu}$ vanishes in the limit, which is equivalent to $\sigma_\beta h^{\alpha\beta} t_\mu t_\nu = 0$.

Taking the non-relativistic limit of (12) then leads to:

$$-\frac{3}{2}\overset{n}{\nabla}_\mu \sigma_\nu + \frac{3}{4}\sigma_\mu \sigma_\nu = 4\pi G \rho t_\mu t_\nu. \tag{15}$$

These are the field equations of STGR in the non-relativistic limit, expressing gravity in terms of non-metricity—we take them to be the field equations of SNGT.

IV. THE NON-RELATIVISTIC TRINITY OF GRAVITY

Having taken the non-relativistic limit of STGR, we now demonstrate that the non-relativistic geometric trinity of gravity functions in an analogous manner to the relativistic geometric trinity, in the sense that it is possible to 'translate' between the geometrical structures of each vertex of the trinity via Trautman-style geometrisation/recovery theorems. As a starting point, we have standard NCT with degenerate temporal and spatial metrics. The connection $\overset{c}{\nabla}$ is metric compatible:

$$\begin{aligned}\overset{c}{\nabla}_\alpha t_\mu &= 0,\\ \overset{c}{\nabla}_\alpha h^{\mu\nu} &= 0.\end{aligned} \tag{16}$$

Furthermore, as in GR, gravity is a manifestation of spacetime curvature, so that test bodies traverse geodesics of the curved connection:

$$\xi^\lambda \overset{c}{\nabla}_\lambda \xi^\alpha = 0. \tag{17}$$

However, we also know that in the relativistic case we have an empirically equivalent theory to GR (namely, STGR) which uses a non-metric but flat connection. By analogy, then, we consider a non-metric non-relativistic connection $\overset{n}{\nabla}$:

$$\begin{aligned}\overset{n}{\nabla}_\alpha t_\mu &= \sigma_\alpha t_\mu,\\ \overset{n}{\nabla}_\alpha h^{\mu\nu} &= \sigma_\alpha h^{\mu\nu},\end{aligned} \tag{18}$$

where σ_α is a 1-form such that $h^{\mu\alpha}\sigma_\alpha = 0$. Curiously, the particular scaling of $h^{\mu\nu}$ will not much matter for our purposes, since the geometrised Poisson equation of NCT does not feature $h^{\mu\nu}$ (for further discussion on the scalings of $h^{\mu\nu}$ and t_μ, see [35]). In addition, we assume that σ_α is exact, i.e. such that $\sigma_\alpha = d_\alpha\lambda$ where λ is some scalar function and d denotes the exterior derivative. This can be motivated by considering some conformally rescaled $\bar{t}_\mu := \eta t_\mu$ compatible with $\overset{n}{\nabla}$—we then have:

$$\overset{n}{\nabla}_\nu \bar{t}_\mu = (\overset{n}{\nabla}_\nu \eta) t_\mu + \eta \overset{n}{\nabla}_\nu t_\mu = 0 \tag{19}$$

Defining

$$\sigma_\alpha := d_\alpha \ln\eta =: d_\alpha\lambda \tag{20}$$

justifies our assumption. Finally, from $h^{\mu\alpha}\sigma_\alpha = 0$, it follows that

$$\sigma_\alpha = \kappa t_\alpha \tag{21}$$

for some scalar function κ [9, p. 253]; one can also show that κ is spatially constant [35, p. 1421].

As can be verified straightforwardly, the difference tensor between $\overset{c}{\nabla}$ and $\overset{n}{\nabla}$ is

$$U^\alpha{}_{\mu\nu} := \sigma_{(\mu}\delta^\alpha{}_{\nu)}, \tag{22}$$

which can also be seen from the non-relativistic limit taken in (14). Given that our task is to construct an equivalent theory to NCT which expresses gravity using purely non-metric degrees of freedom, we move between the connections in the geodesic equation as follows:

$$\xi^\lambda \overset{c}{\nabla}_\lambda \xi^\alpha = \xi^\lambda \overset{n}{\nabla}_\lambda \xi^\alpha - U^\alpha{}_{\rho\lambda}\xi^\rho\xi^\lambda = 0. \tag{23}$$

We can now compute the Ricci from this new connection using (8):

$$\overset{n}{R}{}^\alpha{}_{\beta\mu\nu} = \overset{c}{R}{}^\alpha{}_{\beta\mu\nu} + 2\overset{c}{\nabla}_{[\nu}U^\alpha{}_{\mu]\beta} + 2U^\lambda_{\beta[\nu}U^\alpha{}_{\mu]\lambda}. \tag{24}$$

Following straightforward computations and contractions, we obtain:

$$\overset{c}{R}_{\mu\nu} = -\frac{3}{2}\overset{c}{\nabla}_\mu\sigma_\nu - \frac{3}{4}\sigma_\mu\sigma_\nu, \tag{25}$$

because of course $\overset{n}{R}{}^\alpha{}_{\mu\nu\beta} = 0$ as this spacetime is flat. Now we have

$$\overset{c}{R}_{\mu\nu} = 4\pi\rho t_\mu t_\nu, \tag{26}$$

which is the familiar dynamical equation of NCT; together with (25), this yields:

$$4\pi\rho t_\mu t_\nu = -\frac{3}{2}\overset{c}{\nabla}_\mu\sigma_\nu - \frac{3}{4}\sigma_\mu\sigma_\nu \tag{27}$$

which is, of course, equivalent to (15) because we have that

$$(\overset{n}{\nabla}_\mu - \overset{c}{\nabla}_\mu)\sigma_\nu = \sigma_\mu\sigma_\nu. \tag{28}$$

Reassuringly, therefore, one obtains the same dynamics for SNGT whether one proceeds via (a) taking a non-relativistic limit of STGR or (b) converting the connection of NCT to one manifesting pure non-metricity.

In a sense, our job is done. Indeed, what we have just presented is, in effect, a 'recovery theorem'-type result, mapping models of NCT to those of SNGT. This is correct—although it is worth noting a disanalogy with the recovery theorem relating NCT to NGT (on which see [9, Ch. 4]). In that case, one finds that the mapping one-to-many, in the sense that a given NCT model $(M, t_\mu, h^{\mu\nu}, \overset{c}{\nabla}, \rho)$ will map to an orbit of NGT models $(M, t_\mu, h^{\mu\nu}, \nabla', \varphi, \rho)$, parameterised by the 'Trautman symmetry' (see e.g. [36, p. 205]):

$$\begin{aligned}&\overset{c}{\nabla} \mapsto \nabla' = \left(\overset{c}{\nabla}, t_\mu t_\nu \nabla^\lambda\psi\right),\\ &\varphi \mapsto \varphi' = \varphi + \psi.\end{aligned} \tag{29}$$

One might, therefore, wonder whether the mapping from NCT to SNGT is likewise one-to-many. But what would the relevant gauge orbits be in this case? By analogy with the well-known 'Weyl symmetry' of Weyl geometries (see e.g. [37, §3]), which likewise manifest non-metricity of the form $Q_{\mu\nu\lambda} = \sigma_\mu g_{\nu\lambda}$, one might postulate the following Weyl-type symmetry for SNGT:

$$\begin{aligned} t_\mu &\mapsto e^{-f} t_\mu, \\ h^{\mu\nu} &\mapsto e^{-f} h^{\mu\nu}, \\ \sigma_\alpha &\mapsto \sigma_\alpha - \overset{n}{\nabla}_\alpha f. \end{aligned} \tag{30}$$

While these transformations are indeed symmetries of the kinematical conditions (18), an elementary computation confirms that they are not dynamical symmetries of (15). Thus, if there is indeed a 'gauge orbit' in the recovered theory SNGT, (30) cannot be it. In any case, one can prove a Trautman-style recovery theorem relating NCT and SNGT. We do so now, in the style found in [9, Ch. 4].

Proposition 1 *Let $(M, t_a, h^{ab}, \overset{n}{\nabla})$ be a time-orientable classical spacetime where t_μ and $h^{\mu\nu}$ are orthogonal and $\overset{n}{\nabla}$ is flat ($\overset{n}{R}{}^\alpha{}_{\mu\nu\beta} = 0$), but $\overset{n}{\nabla}$ is not compatible with the metrics t_μ and $h^{\mu\nu}$ such that $\overset{n}{\nabla}_\alpha t_\mu = \sigma_\alpha t_\mu$ and $\overset{n}{\nabla}_\alpha h^{\mu\nu} = \sigma_\alpha h^{\mu\nu}$. Let σ_μ and ρ be smooth fields on M satisfying $-\frac{3}{2}\overset{n}{\nabla}_\mu \sigma_\nu + \frac{3}{4}\sigma_\mu\sigma_\nu = 4\pi\rho t_\mu t_\nu$. Let $\overset{c}{\nabla} = (\overset{n}{\nabla}, U^\alpha{}_{\mu\nu})$, where the difference tensor $U^\alpha{}_{\mu\nu} := -\sigma_{(\mu}\delta^\alpha{}_{\nu)}$. Then all the following hold.*

(G1) $(M, t_a, h^{ab}, \overset{c}{\nabla})$ is a classical spacetime in which the connection $\overset{c}{\nabla}$ is compatible with the metrics t_μ and $h^{\mu\nu}$ and the metrics are orthogonal.

(G2) $\overset{c}{\nabla}$ is the unique derivative operator on M such that, for all timelike curves on M with four-velocity field ξ^α,

$$\xi^\lambda \overset{c}{\nabla}_\lambda \xi^\alpha = 0 \Longleftrightarrow \xi^\lambda \overset{n}{\nabla}_\lambda \xi^\alpha = U^\alpha{}_{\rho\lambda}\xi^\rho\xi^\lambda. \tag{31}$$

(G3) The curvature field $\overset{c}{R}{}^\alpha{}_{\mu\nu\beta}$ associated with $\overset{c}{\nabla}$ has the following properties.

$$\begin{aligned} \overset{c}{R}_{\mu\nu} &= 4\pi\rho t_\mu t_\nu, \\ \overset{c}{R}{}^\alpha{}_\nu{}^\mu{}_\beta &= \overset{c}{R}{}^\mu{}_\beta{}^\alpha{}_\nu, \\ \overset{c}{R}{}^{\alpha\mu}{}_{\nu\beta} &= 0, \end{aligned} \tag{32}$$

Proof: We take (G1)–(G3) in turn:

(G1) This follows from $\overset{c}{\nabla}_\alpha t_\mu = \overset{n}{\nabla}_\alpha t_\mu - \sigma_\alpha t_\mu = 0$ because $\overset{n}{\nabla}_\alpha t_\mu = \sigma_\alpha t_\mu$ Similarly, for the spatial metric.

(G2) Compatible, torsion-free non-relativistic derivative operators are parameterised by a 2-form $F_{\mu\nu}$, known as the 'Newton-Coriolis 2-form'; this object encodes the difference tensor relating any such connection (see [7, 33]). Given this, (31) picks out just one element of the 'gauge orbit' of compatible connections parameterised by $F_{\mu\nu}$ (cf. [36]), for all others will manifest geodesic deviation.

(G3) This follows from a simple computation of the Ricci tensor. Beginning with the derivation of the first curvature condition (i.e., the geometrised Poisson equation), we have

$$\overset{c}{R}{}^\alpha{}_{\beta\mu\nu} = \overset{n}{R}{}^\alpha{}_{\beta\mu\nu} + 2\overset{n}{\nabla}_{[\nu} U^\alpha{}_{\mu]\beta} + 2U^\lambda_{\beta[\nu}U^\alpha{}_{\mu]\lambda}. \tag{33}$$

from which it follows that

$$\overset{c}{R}_{\mu\nu} = -\frac{3}{2}\overset{n}{\nabla}_\mu\sigma_\nu + \frac{3}{4}\sigma_\mu\sigma_\nu, \tag{34}$$

Using $-\frac{3}{2}\overset{n}{\nabla}_\mu\sigma_\nu + \frac{3}{4}\sigma_\mu\sigma_\nu = 4\pi\rho t_\mu t_\nu$, we have of course that $\overset{c}{R}_{bc} = 4\pi\rho t_\mu t_\nu$. The other two curvature conditions can likewise be verified straightforwardly using that σ_α is closed, which follows on the assumptions (20) and (21). □

V. CONCLUSION

In this article, we have taken the non-relativistic limit of STGR, and have also converted the curvature degrees of freedom of NCT into non-metricity; thereby, we have triangulated a purely non-metric alternative theory to NCT and NGT, and so in turn have completed a non-relativistic geometric trinity for gravity—this also makes good on a question raised in [8] as to what one would obtain on taking the non-relativistic limit of STGR.

It is worth closing with two remarks. First: we should be completely explicit that we have specialised to the 'Weylian' case of non-metricity, $Q_{\mu\nu\lambda} = \sigma_\mu g_{\mu\lambda}$; this assumption was necessary in order to track the behaviour of the non-metricity tensor under the non-relativistic limit. Second, we should be explicit that we have assumed moreover that $\sigma_\alpha = d_\alpha\lambda = \kappa t_\alpha$.

There are many future prospects. To name just two, (1): recently, in [38], a novel version of NCT (so-called 'Type II NCT') has been constructed by taking a more careful and systematic $1/c$ expansion of the GR dynamics. Type II NCT has revealed several novel features of non-relativistic gravitational theories, including that non-relativistic theories can account for many of the strong gravitational effects previously believed to belong to relativistic theories and can also reproduce much of the solution space of GR [39, 40]. This raises the question: what would be the 'Type II' equivalents of NGT and SNGT? Indeed, finding such theories would have conceptual payoff, for in this article we have demonstrated equivalence of the three vertices of the non-relativistic geometric trinity only at the level of equations of motion, whereas the equivalence of the relativistic geometric trinity can—as we have seen—be demonstrated at the level of the action. However, action principles for the 'Type I' theories provably do not exist [40]; not so for 'Type II' theories (and, indeed, an action for Type II NCT is explicitly constructed in [40]); therefore, to construct a non-relativistic trinity using action principles (as in the relativistic case), one would have to construct and work with the 'Type II' theories. And (2): it has very recently been shown that there exists an 'extended' geometric trinity between (roughly) $f(R)$, $f(T)$ and $f(Q)$ theories [41]—does a similar extension of the non-relativistic geometric trinity exist?

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