Newtonian Gravitation in Maxwell Spacetime

Elliott D. Chen

July 7, 2023

[Forthcoming in *Studies in History and Philosophy of Science*]

Abstract

This paper argues for the appropriateness of Maxwell spacetime as the minimal spacetime structure in which one may formulate a theory of Newtonian gravity. I begin by presenting an intrinsic characterization of Maxwell gravitation that, eschewing covariant derivative operators, makes use only of a standard of rotation and other more primitive structures. I then revisit the question of whether Maxwell gravitation and Newton-Cartan theory are equivalent, demonstrating that previous results may be extended to all but the vacuum case since candidate geometrizations are not free to vary through purely gravitational degrees of freedom. Lastly, I consider the space of possible geometrizations of Maxwell gravitation more broadly and argue for a sense in which curvature is not entirely a matter of convention in classical spacetimes.

1 Introduction

What is the proper spacetime setting for Newtonian gravitation theory? Over the past decade, papers by Saunders (2013) and Knox (2014) have revitalized discussion of this question in the context of corollary 6 to Newton's laws of motion:

> *If bodies are moving in any way whatsoever with respect to one another and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces.* (Newton, 423)

Both contend that neo-Newtonian spacetime has superfluous structure and that one should move to a spacetime with a standard of rotation but no means of distinguishing freely-falling from inertial frames. However, Saunders has taken this to motivate a move to Maxwell spacetime, which is defined merely in terms of a standard of rotation, while Knox sees it as impetus instead for a move to Newton-Cartan theory, which is endowed with a curved derivative operator. In light of this apparent discrepancy, what is the relationship between the two theories?

Weatherall (2016) tackles this question and shows that if one takes into account a set of dynamically-allowable trajectories for some mass distribution, there corresponds to every Maxwell spacetime a unique Newton-Cartan spacetime. With this, Weatherall concludes that "Knox and Saunders do not end up in different places at all [...] Once one fully considers the effects of gravitation in Maxwell-Huygens space-time, Newton-Cartan theory is precisely the result" (Weatherall, 2016, 90).[1] This work has been followed by Dewar (2018), who gives an explicit formulation of Newtonian gravitation on Maxwell spacetime. Dewar demonstrates, moreover, that the resulting theory, "Maxwell gravitation," is equivalent to Newton-Cartan theory when one's mass density is everywhere nonvanishing. The discrepancy between the two theories, then, is merely apparent and the structural gap may be bridged by dynamical considerations.

But two difficulties remain. Part of the appeal of Maxwell spacetime is its minimalism—one has excised all superfluous structure and preserved only what is required to support Newtonian physics. Yet, in formulating Maxwell gravitation, Dewar makes use of an equivalence class of covariant derivative operators and expresses his conditions on the mass-momentum tensor in terms of arbitrary members of this class, each of which possesses structure that was meant to be expressly excluded from the theory. The motivating minimalism of Maxwell spacetime, however, ought to extend both to those structures in which it is defined, as well as

[1] Wallace (2020) has similarly argued that the difference between Maxwell spacetime and Newton-Cartan theory disappears once one considers the inertial structure that emerges in the former from the behavior of isolated subsystems.

those to which one explicitly appeals when constructing its dynamical theories.[2]

In response to this worry, Weatherall (2018) has characterized Maxwell spacetime in terms of an alternative "standard of rotation" that makes no reference to covariant derivative operators at all. The question is left open, however, as to whether this new standard of rotation is sufficient to articulate Newtonian gravity: "One would like to find a coordinate-free presentation of the theory that makes use of precisely Maxwellian spacetime, as characterized here, and nothing else—a version, say, of Neil Dewar's 'Maxwell gravitation' expressed using only a standard of rotation" (Weatherall, 2017, 83). Such a presentation would circumvent worries of mathematical impropriety plaguing Maxwell gravitation while also being the most direct characterization of the theory.

Secondly, the equivalence between Maxwell spacetime and Newton-Cartan theory relies on substantive further assumptions: one must either have knowledge of the dynamically-allowable trajectories, be content with an effective spacetime geometry of subsystems, or restrict attention to the case where nature is a plenum. This is adequate as a demonstration that there are indeed conditions under which the apparent structural deficit may be bridged. But if the two theories are truly to be seen as on a par, one would like an equivalence result supported by much more modest physical assumptions. It seems reasonable to require that the result extend to generic mass distributions, be cast in terms of information available to local observers, and preserve the ambition that Maxwell gravitation might yet be a fully cosmological theory.

In this paper, I begin by remedying the first concern and present a formulation of Newtonian gravitation that uses only the resources intrinsic to Maxwell spacetime. I then clarify the extent to which Maxwell gravitation and Newton-Cartan theory are equivalent, and show that one may relax the above assumptions and only require that the universe not be a vacuum. To make sense of this equivalence, I consider the space of possible geometrizations and argue for a sense in which curvature is not entirely a matter of convention in classical spacetimes. I take this to redeem Maxwell spacetime as an appropriate, minimal setting for a theory of Newtonian gravitation.

2 Preliminaries

I here present some preliminaries concerning the relevant background formalism of Newtonian gravitation theory. Readers familiar with the recent literature should

[2]Knox makes this same criticism in her talk from the 2017 conference "The Philosophy of Howard Stein."

feel free to skip ahead to the next section.[3]

A *Leibnizian spacetime* is a triple (M, t_a, h^{ab}) where (i) M is a smooth, simply connected, four-dimensional manifold; (ii) t_a is a "temporal metric," a smooth 1-form on M, which we require to be closed;[4] (iii) h^{ab} is a "spatial metric," a smooth symmetric field on M of signature $(0,1,1,1)$;[5] and (iv) the "orthogonality" condition $h^{ab}t_b = \mathbf{0}$ is satisfied. Conceptually, t_a induces a foliation of spacetime into hypersurfaces of constant time, with h^{ab} inducing a metric on each such hypersurface. It will be assumed that these hypersurfaces are complete relative to this induced metric and diffeomorphic to $\mathbb{R}^3$. Given a temporal metric t_a, moreover, a vector ξ^a at a point $p \in M$ is said to be "timelike" if $t_a\xi^a \neq 0$, and "spacelike" otherwise. Finally, a covariant derivative operator ∇ on M is said to be *compatible* with the spatial and temporal metrics if both $\nabla_a t_b = \mathbf{0}$ and $\nabla_a h^{bc} = \mathbf{0}$.

A *Galilean spacetime* is a structure (L, ∇), where L is a Leibnizian spacetime and ∇ is a flat covariant derivative operator compatible with t_a and h^{ab}. Models of *Galilean gravitation* are composed of a Galilean spacetime along with a mass density ρ and gravitational potential ϕ satisfying Poisson's equation $4\pi\rho = \nabla_a\nabla^a\phi$. When only gravitational forces are present, a body will obey the following equation of motion: $\xi^n\nabla_n\xi^a = -\nabla^a\phi$, where ξ^n is the body's four-velocity.[6]

A model of *Newton-Cartan theory*, or geometrized Newtonian gravitation, is a structure $(L, \tilde{\nabla}, T^{ab})$ where L is again a Leibnizian spacetime, but now $\tilde{\nabla}$ is a derivative operator compatible with L whose associated curvature tensor $\tilde{R}^a{}_{bcd}$ satisfies (i) $\tilde{R}^{ab}{}_{cd} = \mathbf{0}$ and (ii) $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$, and T^{ab} is a smooth, symmetric tensor field such that (iii) $\tilde{R}_{bc} = 4\pi\rho t_b t_c$ and (iv) $\tilde{\nabla}_n T^{na} = \mathbf{0}$. Such models may be systematically related to families of Galilean spacetimes whose covariant derivative operators $[\nabla]$ are *rotationally equivalent*—or such that, for all unit timelike vector fields ξ^a and any pair $\nabla, \nabla' \in [\nabla]$, $\nabla^{[n}\xi^{a]} = \mathbf{0}$ iff $\nabla'^{[n}\xi^{a]} = \mathbf{0}$. This relation is captured by the Trautman Geometrization Lemma and Recovery Theorem: test particles will traverse the geodesics of a Newton-Cartan connection $\tilde{\nabla}$ just in case they satisfy an equation of motion of the same form as that above for all rotationally-equivalent flat derivative operators ∇ (and appropriately chosen

[3]This is roughly in keeping with Weatherall (2016) and Dewar (2018), stemming from Earman (1989). For further details about Newtonian gravitation theory more generally, see Malament (2012).

[4]This is to guarantee that t_a will be exact, and so capable of being expressed as the gradient of a smooth scalar field t, or time function.

[5]We require, moreover, that h^{ab} is flat, i.e. $D_{[\underline{c}}D_{\underline{d}]}\mu^{\underline{a}} = \mathbf{0}$ for all spacelike fields $\mu^{\underline{a}}$. (We define the spatial derivative operator D below.)

[6]In anticipation of equations (1)-(3), one may alternatively express models of Galilean gravitation as structures $(L, \nabla, \phi^a, T^{ab})$ where L and ∇ are as above, while ϕ^a and T^{ab} represent the gravitational field and mass-momentum tensor, respectively, and satisfy the following relations: (i) $\rho\phi^a = \nabla_n T^{na}$, (ii) $-4\pi\rho = \nabla_a\phi^a$, and (iii) $\mathbf{0} = \nabla^{[c}\phi^{a]}$. See Dewar (2018) for more details.

gravitational potentials ϕ).[7]

Accordingly, one may wish to broaden attention beyond individual Galilean spacetimes and instead consider the structure that is shared by rotationally-equivalent families of Galilean spacetimes. This shared structure is naturally captured by the following definitions: a *Maxwell spacetime** is a pair $(L, [\nabla])$ consisting of a Leibnizian spacetime $L = (M, t_a, h^{ab})$, along with an equivalence class of flat covariant derivative operators $[\nabla]$ whose members are rotationally equivalent and compatible with t_a and h^{ab}.[8] A model of *Maxwell gravitation**—denoted $(L, [\nabla], T^{ab})$—consists of two parts: a Maxwell spacetime* $(L, [\nabla])$ and a smooth, symmetric tensor field T^{ab} satisfying the following equations for each $\nabla \in [\nabla]$ wherever $\rho = T^{ab}t_a t_b \neq 0$:

$$0 = t_a \nabla_n T^{na} \tag{1}$$
$$-4\pi\rho = \nabla_a(\rho^{-1}\nabla_n T^{na}) \tag{2}$$
$$\mathbf{0} = \nabla^{[c}(\rho^{-1}\nabla_n T^{|n|a]}). \tag{3}$$

The first of these equations expresses the conservation of mass, i.e. $\xi^n \nabla_n \rho + \rho \nabla_n \xi^n = 0$; the second is Poisson's equation stated in terms of the gravitational field $\phi^a = \rho^{-1}\nabla_n T^{na}$; while the third guarantees the existence of a smooth scalar field ϕ, or gravitational potential, such that $\phi^a = \nabla^a \phi$.

As mentioned before, one may argue that these definitions are problematic on the grounds that they use equivalence classes of covariant derivative operators, and so cast doubt on the suitability of Maxwell spacetime as a setting for a theory of Newtonian gravitation. I will therefore characterize these notions so as to circumvent these worries and place the theory on more secure footing. To do so, we must first introduce the alternative "standard of rotation" found in Weatherall (2018), along with a new definition of a Maxwell spacetime.

A *standard of rotation* compatible with a particular t_a and h^{ab} is a map $\circlearrowleft$ from pairs (n, ξ^a) to smooth, antisymmetric fields $\circlearrowleft^n \xi^a = \circlearrowleft^{[n} \xi^{a]}$ on M such that for any vector fields ξ^a and η^a, scalar field α, and spacelike vector field $\sigma^{\underline{a}}$: (i) $\circlearrowleft$ commutes with vector addition; (ii) $\circlearrowleft^n (\alpha\xi^a) = \alpha \circlearrowleft^n \xi^a + \xi^{[a} d^{n]}\alpha$; (iii) if $d_a(\xi^n t_n) = 0$, then $\circlearrowleft^n \xi^a$ is spacelike in both indices; (iv) $\circlearrowleft^{\underline{n}} \sigma^{\underline{a}} = D^{[\underline{n}}\sigma^{\underline{a}]}$;[9] and (v) $\circlearrowleft$ commutes with index substitution.[10]

[7]See propositions 4.2.1. and 4.2.5., respectively, of Malament (2012).

[8]Since we will define both Maxwell spacetime and models of Maxwell gravitation somewhat differently, these terms will be followed by an asterisk when referring to Dewar's definitions. With proposition 2, however, the two formulations will be shown to be equivalent.

[9]The operator d_a is the exterior derivative and D is the spatial derivative operator defined in the next subsection. Here and in what follows, an underlined contravariant index indicates that the tensor is spacelike in that index, while an underlined covariant index indicates that the tensor's action on timelike vectors is undefined in that index.

[10]For those curious about the transformation properties of $\circlearrowleft$, note that there is a natural

Relative to a standard of rotation $\circlearrowleft$, a unit timelike vector field ξ^a will be said to be *nonrotating* just in case $\circlearrowleft^n \xi^a = \mathbf{0}$. This is intended to be a direct analog of the case in Galilean spacetime where a unit timelike vector field ξ^a may be said to be nonrotating when $\nabla^{[n}\xi^{a]} = \mathbf{0}$. Accordingly, a derivative operator ∇ is said to *determine*, or *agree with*, a standard of rotation $\circlearrowleft$ just in case $\nabla^{[n}\xi^{a]} = \circlearrowleft^n \xi^a$ for all smooth vector fields ξ^a.[11]

With this in hand, we may define a *Maxwell spacetime* as a structure $(L, \circlearrowleft)$ where (i) L is a Leibnizian spacetime (M, t_a, h^{ab}), and (ii) $\circlearrowleft$ is a standard of rotation compatible with t_a and h^{ab}. It remains to be shown, however, that one may re-express the restrictions on the mass-momentum tensor T^{ab} using this new standard of rotation.

To do this, we will need to disassemble the covariant derivative into two restricted operators, such that each acts only on either spacelike or timelike vectors, respectively, in spacelike directions. We have the former from Weatherall (2018): fixing a spatial metric h^{ab} on M guarantees the existence of a unique *spatial derivative operator* D compatible with h^{ab} taking pairs $(\underline{x}, \alpha^{\underline{a}_1 \cdots \underline{a}_n}_{\underline{b}_1 \cdots \underline{b}_m})$ to smooth tensors $D_{\underline{x}} \alpha^{\underline{a}_1 \cdots \underline{a}_n}_{\underline{b}_1 \cdots \underline{b}_m}$, where $\alpha^{\underline{a}_1 \cdots \underline{a}_n}_{\underline{b}_1 \cdots \underline{b}_m}$ is a smooth tensor field with exclusively underlined indices, all distinct from $\underline{x}$.[12] The next section shows how to capture the latter of these operators, as well as how to use them to formulate a theory of Maxwell gravitation.

3 Maxwell Gravitation Anew

Since Maxwell spacetime is a setting in which one lacks the resources to speak of the absolute acceleration of bodies, there are naturally constraints on how one may take derivatives of timelike vector fields. Observe, however, that we may define

push-forward for standards of rotation: given a diffeomorphism $\varphi : M \to M$ and a smooth vector field ξ^a, $\varphi_*(\circlearrowleft)^a \xi^b = \varphi_*(\circlearrowleft^a \varphi^*(\xi^b))$. By adapting the proof of proposition 1 from Weatherall (2018), together with facts about φ_*, one may show that $\varphi_*(\circlearrowleft)$ is indeed a standard of rotation (i.e. it satisfies conditions (i), (ii), and (v)). Note, however, that it is not compatible with the original metrics t_a and h^{ab}, but is compatible instead with their push-forwards, $\varphi_*(t_a)$ and $\varphi_*(h^{ab})$ (i.e. it satisfies conditions (iii) and (iv) only with respect to these new metrics). In this way, it is natural to think of the triple $(\circlearrowleft, t_a, h^{ab})$ as together constituting a geometric object, rather than a standard of rotation by itself.

[11] Note that proposition 1.4 of Weatherall (2018) gives the condition that two derivative operators ∇ and ∇' (compatible with the same metrics) will determine the same standard of rotation $\circlearrowleft$ just in case $\nabla' = (\nabla, \sigma^{\underline{a}} t_b t_c)$ for some spacelike vector field $\sigma^{\underline{a}}$. If, further, $\circlearrowleft$ admits of any non-rotating unit timelike vector fields, this condition holds just in case ∇ and ∇' are rotationally equivalent.

[12] For the sake of completeness, the rate of change of a spacelike vector field $\sigma^{\underline{a}}$ in the direction of a timelike vector ξ at some point p is given by equation 1 of Weatherall (2018): $\xi^n \triangle_n \sigma^{\underline{a}} = \pounds_\xi \sigma^{\underline{a}} + \sigma_n \circlearrowleft^n \xi^a - \frac{1}{2}\sigma_n \pounds_\xi h^{an}$, where $\pounds_\xi$ is the Lie derivative with respect to ξ^n.

a restricted *temporal derivative operator* $\Box$ taking pairs $(\underline{a}, \xi^b)$ to smooth tensors $\Box_{\underline{a}}\xi^b$ by:

$$\Box_{\underline{a}}\xi^b = h_{\underline{an}}\,\Box^n\,\xi^b = h_{\underline{an}}(\circlearrowleft^n \xi^b - \frac{1}{2}\pounds_\xi h^{nb}),$$

where ξ^b is a smooth timelike vector field and $\pounds_\xi$ is the Lie derivative with respect to ξ^b.[13] When contracted with a spacelike vector field $\sigma^{\underline{a}}$, this has the natural interpretation of the rate of change of a timelike vector field in a spacelike direction.

Importantly, one has that $\sigma^{\underline{a}}\Box_{\underline{a}}\xi^b = \sigma^a\nabla_a\xi^b$ for any spacelike vector field $\sigma^{\underline{a}}$ and any derivative operator ∇ compatible with L whose standard of rotation agrees with $\circlearrowleft$.[14] That is, if two covariant derivatives agree on a standard of rotation, they must also agree on the rate of change of timelike vector fields in spacelike directions.[15] In this way, Maxwell spacetime proves to have enough structure for arbitrary derivatives of both scalar and spacelike vector fields to be taken, as well as derivatives of timelike vector fields if restricted to spacelike directions.

I may now present an intrinsic characterization of Maxwell gravitation. A model of *Maxwell gravitation*—denoted by a triple $(L, \circlearrowleft, T^{ab})$—consists of a Maxwell space-time $(L, \circlearrowleft)$ and a tensor field T^{ab} satisfying the following three equations wherever $\rho \neq 0$:

$$0 = \rho\,\Box_{\underline{n}}\,\xi^n + \xi^n d_n\rho \tag{1*}$$

$$-4\pi\rho = D_{\underline{a}}(\rho^{-1}D_{\underline{n}}\sigma^{\underline{na}}) + \xi^n d_n(\Box_{\underline{a}}\xi^a) + (\Box_{\underline{a}}\xi^n)(\Box_{\underline{n}}\xi^a) \tag{2*}$$

$$0 = \circlearrowleft^c(\rho^{-1}D_{\underline{n}}\sigma^{\underline{na}}) + 2(\Box^{[c}\xi^{|n|})(\Box_{\underline{n}}\xi^{a]}) + (\Box^n\xi^{[a})(\Box_{\underline{n}}\xi^{c]}) + \pounds_\xi(\circlearrowleft^c \xi^a), \tag{3*}$$

where $\xi^a = \rho^{-1}T^{ab}t_b$ and $\sigma^{ab} = T^{ab} - \rho\xi^a\xi^b$.[16] Although obscured by this manner of presentation, proposition 2 will demonstrate that these three conditions are

[13]Note that while h^{ab} is not the inverse of a metric, its action is invertible just in case it is contracted with a tensor field that is spacelike in the relevant indices. When ξ^a is a unit timelike vector field, this will be the case for both indices of $(\circlearrowleft^a \xi^b - \frac{1}{2}\pounds_\xi h^{ab})$, and for just a otherwise. Accordingly, the divergence of a unit timelike vector field ξ^a may be defined as $\Box_{\underline{n}}\xi^n = h_{\underline{na}}(\circlearrowleft^a \xi^n - \frac{1}{2}\pounds_\xi h^{an})$.

[14]The second part of proposition 1 of Weatherall (2018) guarantees that there exists at least one derivative operator with these properties.

[15]That this does not depend on one's choice of derivative operator may be shown quickly: $\sigma^{\underline{a}}\nabla'_a\xi^b = \sigma^{\underline{a}}\nabla_a\xi^b - \sigma^{\underline{a}}(\xi^z C^{\underline{b}}_{za}) = \sigma^{\underline{a}}\nabla_a\xi^b - \sigma^{\underline{a}}(\xi^z\eta^{\underline{b}}t_z t_a) = \sigma^{\underline{a}}\nabla_a\xi^b$, where $\eta^{\underline{a}}$ (whose existence is given by proposition 4 of Dewar (2018)) is the spacelike vector field satisfying $\nabla' = (\nabla, \eta^{\underline{a}}t_b t_c)$.

[16]Observe that ξ^a is a unit timelike vector: $t_a\xi^a = t_a(\rho^{-1}T^{ab}t_b) = \frac{T^{ab}t_a t_b}{T^{mn}t_m t_n} = 1$.

equivalent to those imposed on models of Maxwell gravitation*. As such, they may be interpreted in just the same way as their more transparent counterparts.

We will say that a Maxwell spacetime $(L, \circlearrowright)$ *corresponds* to a Maxwell spacetime* $(L, [\nabla])$, and vice versa, just in case they determine the same standard of rotation.[17]

Proposition 1. If $(L, \circlearrowright)$ is a Maxwell spacetime, then there exists a unique corresponding Maxwell spacetime* $(L, [\nabla])$.[18]

For ease of reading, proofs of propositions appear in an appendix.

Proposition 2. Let $(L, \circlearrowright)$ be a Maxwell spacetime and $(L, [\nabla])$ its corresponding Maxwell spacetime*. Given some T^{ab}, $(L, \circlearrowright, T^{ab})$ is a model of Maxwell gravitation if and only if $(L, [\nabla], T^{ab})$ is a model of Maxwell gravitation*.

We thus see that there is a one-to-one correspondence between models of Maxwell gravitation* and models of Maxwell gravitation. As the latter makes no reference to covariant derivative operators, one finds that the use of an equivalence class of such operators is, in fact, inessential, and so the earlier criticism of mathematical impropriety is met.

4 Equivalence and the Vacuum

There remain worries, however, about the relationship between models of Maxwell gravitation and Newton-Cartan theory. In what follows, I restrict attention to Dewar's equivalence result. This is in part for the ease of comparison owing to our shared formalism, and will be rewarded by a straightforward extension of his equivalence result. Of equal interest, however, is his accompanying claim concerning the existence of purely gravitational degrees of freedom that allegedly underdetermine geometrizations of Maxwell gravitation.[19] This particular claim is explored in this section by way of extending the equivalence, while the more general question of the space of possible geometrizations will be taken up in the final section of the

[17]We restrict attention to standards of rotation $\circlearrowright$ for which there exist some unit timelike vector field η^a such that (i) $\circlearrowright^n \eta^a = \mathbf{0}$ and (ii) $\pounds_\eta h^{ab}$. This is to ensure that there is always at least one flat covariant derivative operator compatible with our metrics and agreeing with our standard of rotation.

[18]Note that the converse of proposition follows from Proposition 1.1 of Weatherall (2018).

[19]Beyond the specific questions considered here, this discussion is hopefully of more general interest, particularly for projects considering analogs of the Weyl tensor and gravitational wave phenomena in classical spacetimes. See, for instance, Wallace (2017), Dewar and Weatherall (2018), Hansen et al. (2019), and Linnemann and Read (2021).

paper.

Dewar (2018) claims that there is a sense in which Maxwell gravitation and Newton-Cartan theory are equivalent: for each model of Maxwell gravitation there corresponds a unique model of Newton-Cartan theory provided ρ is everywhere nonvanishing.[20] But this qualification is rather strong. If the idea is that Maxwell spacetime is the weakest structure on which a theory of Newtonian gravity may be expressed, can this be sustained if it is only true for worlds in which matter fills all of space?

To understand why the equivalence is not general, Dewar considers two triples: (L, ∇, T^{na}) and $(L, (\nabla, (\nabla^a \phi) t_b t_c), T^{na})$, where $\nabla \in [\nabla]$ and $\phi = e^x e^y \sin(\sqrt{2}z)$ for a coordinate system (t, x, y, z) adapted to L. One can show in the special case where $T^{na} = \mathbf{0}$ that each structure qualifies as a distinct model of Newton-Cartan theory, and yet they both correspond to the same model of Maxwell gravitation.

Dewar gathers from this that one has uniqueness only when there is "sufficient material structure to everywhere 'probe' the spatiotemporal structure" (Dewar, 23). Accordingly, when ρ vanishes in some but not all regions, Dewar claims that there exist "distinct but 'materially identical' " models of Newton-Cartan theory corresponding to a given model of Maxwell gravitation. Matter would be present in some parts of spacetime—and so its spatiotemporal structure could there be probed—but the structure of empty regions would be left underdetermined. The thought then is that there would be many models of Newton-Cartan theory possessing the spatiotemporal structure prescribed by the behavior of matter, which, nonetheless, would differ through purely gravitational degrees of freedom. Such models would be "materially identical," in the sense of agreeing on all the motions of actual matter, and yet could be distinguished in principle by the motions of test particles in empty parts of spacetime.

With this, Dewar takes the more general equivalence of the theories to turn on whether these "unactualised dispositions may properly be considered as empirically respectable properties" (Dewar, 24). But as we will see, this is to misunderstand the nature of curvature in classical spacetimes: a model of Maxwell gravitation will continue to have a unique geometrization as long as ρ is not everywhere vanishing.

Let us begin by considering how candidate geometrizations of a model of Maxwell gravitation must be related. By proposition 4 of Dewar (2018), any two models of Newton-Cartan theory are compatible with a particular model of Maxwell gravitation—i.e. set over the same Leibnizian spacetime and agreeing on a standard of rotation—just in case their connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ are related by

$$\tilde{\nabla}' = (\tilde{\nabla}, \eta^{\underline{a}} t_b t_c), \tag{4}$$

[20] Or, alternatively, that every Maxwell spacetime is uniquely associated with a model of Newton-Cartan theory when one is given a set of dynamically allowable trajectories (Weatherall, 2016).

for some spacelike vector field $\eta^{\underline{a}}$ such that $\tilde{\nabla}^{[a}\eta^{b]} = \mathbf{0}$. But more may be said:[21]

Proposition 3. Let $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ be two rotationally-equivalent models of Newton-Cartan theory where $\tilde{\nabla}' = (\tilde{\nabla}, \eta^{\underline{a}} t_b t_c)$. Then there exists a field η such that $\eta^{\underline{a}} = \tilde{\nabla}^a \eta$ and $\tilde{\nabla}_a \tilde{\nabla}^a \eta = 0$.

We thus find that restricting attention to models of Newton-Cartan theory agreeing on a standard of rotation determines the curved derivative operator up to a spacelike field that is both twist- and divergence-free. In this way, one specifies the absolute acceleration of bodies everywhere on a spacelike hypersurface up to a multiple of $-\tilde{\nabla}^a \eta$, where this quantity is the gradient of some potential satisfying Laplace's equation. While in general there are many such potentials, one may reduce the class of qualifying solutions. Of special interest is the following:

Proposition 4. Let $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ be two rotationally-equivalent models of Newton-Cartan theory. Then, if $\tilde{\nabla}$ and $\tilde{\nabla}'$ agree with respect to any open set O, they must agree everywhere.

While a standard of acceleration is not explicitly defined for a model of Maxwell gravitation, proposition 4 shows that the curvature of candidate geometrizations is not thereby free to vary arbitrarily if ρ is anywhere nonvanishing: one cannot require two Newton-Cartan derivative operators to agree with respect to some region while allowing them to differ elsewhere. In particular, two geometrizations cannot make the same determinations regarding matter-inhabited regions and yet disagree when it comes to regions of vanishing mass density, i.e. there can be no "distinct but materially identical" models of Newton-Cartan theory. In this sense, we see that specifying the curvature of a geometrization in any one part is sufficient, in fact, to determine the curvature of the entire manifold.

Settling the question concerning materially-identical models, proposition 4 thus enables us to extend Dewar's uniqueness result to all but the vacuum case:

Corollary 4.1. Let $(L, \circlearrowleft, T^{ab})$ be a model of Maxwell gravitation such that ρ is smooth and somewhere nonvanishing. Then there is a unique Newton-Cartan connection $\tilde{\nabla}$ such that $(L, \tilde{\nabla}, T^{ab})$ is a compatible model of Newton-Cartan theory.

As it happens, this correspondence is a consequence of how we have defined models of Newton-Cartan theory: for all such models, (i) T^{ab} must be divergence-

[21] Note that the enunciation of the proposition specifies that $\tilde{\nabla}' = (\tilde{\nabla}, \eta^{\underline{a}} t_b t_c)$ merely for the sake of convenience. It holds by a direct application of the aforementioned proposition by Dewar since two rotationally-equivalent models of Newton-Cartan theory are trivially compatible with the same model of Maxwell gravitation.

free and (ii) the Ricci curvature $\tilde{R}_{bc}$ must be related to the mass density ρ by the geometrized Poisson equation. The former suffices to uniquely determine a curved derivative operator for an open set in which $\rho \neq 0$, while the latter extends this determination to the whole manifold.

To see this more perspicuously, consider how $\tilde{\nabla}_n T^{na} = \mathbf{0}$ implies

$$\rho\xi^n \tilde{\nabla}\xi^a = -\tilde{\nabla}_n \sigma^{na}.$$

By requiring one's curved derivative operator to determine that T^{na} be divergence-free, one is, in fact, purchasing an equation of motion. It is essential to note, however, that this is an equation specifying not acceleration simpliciter, but the quantity $\rho\xi^n\tilde{\nabla}\xi^a$, which as Dewar rightly cautions, vanishes trivially in empty regions of spacetime. It is for this reason that geometrizations of models of Maxwell gravitation fail to be unique when ρ vanishes everywhere. It does, however, uniquely determine the absolute acceleration of bodies in regions where ρ is positive since all rotationally-equivalent covariant derivative operators will agree with respect to the divergence of σ^{na}, being a tensor field spacelike in both indices. We thus find that models of Newton-Cartan theory have been defined such that a standard of acceleration is determined globally by fixing a local standard for some region inhabited by matter.

5 Geometrizing Maxwell Gravitation

Adopting for the moment the success condition of establishing the equivalence of the two theories, one might declare a partial result, like ours, which fails in the vacuum case, as still unsatisfactory and grounds for abandoning Maxwell spacetime altogether. But this would not be in keeping with the spirit of the project: Maxwell spacetime could never hope to be the equal of Newton-Cartan theory on the merits of its "fundamental" spatiotemporal structure alone—the former is defined so as to have strictly less structure of this kind than the latter. What a successful formulation of Maxwell gravitation would seem to show is that one need not begin with a standard of acceleration, for one may be constructed by appropriately taking into account the material structure of one's spacetime.[22]

[22]If one is willing to entertain any emergent structure at all, it becomes natural to ask if one may make do with a setting even weaker than Maxwell spacetime. With Barbour-Bertotti spacetime, for instance, even a standard of rotation is omitted from one's inventory of spacetime structure; instead, one restricts attention to models in which the total angular momentum of the universe vanishes (Barbour, 1982). Even more radically, one might turn to Leibniz gravitation, in which the only spacetime structure is metrical but one begins with the collection of all allowable trajectories for a given matter distribution (Dewar, 28). Each loss of spacetime structure, however, comes with a price that must be paid elsewhere in the theory. For Barbour-Bertotti

It is not without precedent for structure to emerge dynamically in this way, but there is something peculiar about the conditions under which this occurs in Maxwell gravitation. It was observed in the preceding section that the existence of a unique curved derivative operator compatible with a model of Maxwell gravitation is contingent upon the satisfaction of two conditions: (i) T^{ab} must be divergence-free and (ii) the Ricci curvature $\tilde{R}_{bc}$ must be related to the mass density ρ by the geometrized Poisson equation. The second of these conditions is unproblematic—again, equation (2*) encodes this relation independently of one's choice of covariant derivative operator—but the first should give one pause. Consider, for instance, how Dewar interprets this condition physically in the context of his equivalence result: "the standard of acceleration is defined as that according to which the net gravitational acceleration of the matter encoded by T^{ab} is zero" (Dewar, 20). In other words, requiring the divergence-freedom of T^{ab} is a way of constraining the total acceleration experienced by matter in one's spacetime, which is precisely the sort of physical judgment we had sought to refrain from making. Put most strongly, the ordinary approach to geometrized Newtonian gravity would seem to be incompatible with corollary 6.

Now just as with standard Newtonian gravity where one can respond by moving to Maxwell spacetime, so too one might respect corollary 6 by entertaining a wider class of geometrizations in which T^{ab} need not be divergence-free. To get a better sense of the space of possible geometrizations, consider how a Galilean spacetime comes to be uniquely associated with a model of Newton-Cartan theory. Traditionally, the Geometrization Lemma includes the following stipulation:

$$\xi^n \tilde{\nabla}_n \xi^a = \mathbf{0} \iff \xi^n \nabla_n \xi^a = -\nabla^a \phi,$$

i.e. the new curved derivative operator $\tilde{\nabla}$ must determine a timelike curve ξ^a to be a geodesic just in case an equation of motion is satisfied with respect to the original flat derivative operator ∇ and gravitational potential ϕ. The Trautman recovery theorem reveals, furthermore, that there is a whole equivalence class $[(\nabla, \phi)]$ of paired flat derivative operators and gravitational potentials that satisfy the above requirement for a given curved Newton-Cartan connection. Indeed, if (∇, ϕ) is one such pair, then any other pair (∇', ϕ'), where $\nabla' = (\nabla, t_b t_c \tilde{\nabla}^a(\phi' - \phi))$ and $\tilde{\nabla}^a \tilde{\nabla}^b(\phi' - \phi) = \mathbf{0}$, will as well. Physically speaking, each model of Galilean gravitation (L, ∇, ϕ, ρ), with (∇, ϕ) selected from this equivalence class, will possess the same

spacetime, this comes in the form of an assumption about what is physically possible, and in some formulations, a reshuffling of structure to phase space (as discussed in Belot (2000)). With Leibniz gravitation, one loses the ability to pick out models of the theory by a set of equations defined in terms of only intrinsic structures (Dewar, 28). Maxwell gravitation seems to me to strike a nice balance of having just enough spacetime structure to make do without such hefty physical assumptions.

inertial structure modulo any universal forces represented by the differences in their respective gravitational potentials.

It is this last qualification that is of principal interest for our purposes. When we prefer a curved derivative operator satisfying the biconditional above, we treat as physically significant only that curvature which arises due to mutual gravitational interactions and disregard contributions from possible universal forces. But this is not the only way of proceeding: one may equally well choose a geometrization according to which bodies are accelerated under a universal force. For take any scalar field ψ such that $\tilde{\nabla}^a\tilde{\nabla}^b\psi = \mathbf{0}$ and any pair $(\nabla, \phi) \in [(\nabla, \phi)]$. By the same reasoning as in the proof of the Geometrization Lemma, there corresponds to each ψ a unique curved derivative operator $\tilde{\nabla}' = (\nabla, -t_b t_c \nabla^a(\phi - \psi))$ such that

$$\xi^n \tilde{\nabla}'_n \xi^a = -\tilde{\nabla}'^a \psi \iff \xi^n \nabla_n \xi^a = -\nabla^a \phi,$$

and the tuple $(L, \tilde{\nabla}', T^{ab})$ meets all the requirements of a model of Newton-Cartan theory save that T^{ab} be divergence-free. A similarly altered proof of the Trautman recovery theorem shows, moreover, that any such pseudo-model will correspond to the very same equivalence class $[(\nabla, \phi)]$ as $\tilde{\nabla}$. In direct analogy with Maxwell spacetime, we find that there exists a whole family of pseudo-Newton-Cartan connections $[\tilde{\nabla}]$ agreeing on the relative motions of particles, but which disagree on their absolute acceleration by some universal force ψ.

Interestingly, however, these differences in acceleration are not reflected as differences in the curvature of these pseudo-models. For consider any $\tilde{\nabla}, \tilde{\nabla}' \in [\tilde{\nabla}]$ and let ψ be the scalar field relating them.[23] By equation 1.8.2. of Malament (2012), we have that

$$\tilde{R}'^a{}_{bcd} = \tilde{R}^a{}_{bcd} + 2\tilde{\nabla}_{[c}\tilde{\nabla}^a\psi t_{d]} t_b + 2\psi^n t_b t_{[c}\psi^a t_{d]} t_n,$$

where $\tilde{R}^a{}_{bcd}$ and $\tilde{R}'^a{}_{bcd}$ are the Riemann curvature tensors associated, respectively, with $\tilde{\nabla}$ and $\tilde{\nabla}'$. But both the second and third right-hand terms of this expression must vanish. For the second term, observe that since $\tilde{\nabla}^a\tilde{\nabla}^b\psi = \mathbf{0}$, one has

$$t_{[d}\tilde{\nabla}_{c]}\tilde{\nabla}^a\psi = t_{[d}t_{c]}\xi^n\tilde{\nabla}_n\tilde{\nabla}^a\psi = 0;$$

while the third must vanish because ψ^n is always spacelike.[24] The Riemann curvature, therefore, must be the same across all members of $[\tilde{\nabla}]$.

Moreover, since the space of vectors at a point is spanned by the timelike vectors—and rotationally-equivalent Newton-Cartan derivative operators must agree

[23]If we follow the construction procedure above for the equivalence class $[\tilde{\nabla}]$, any two members will satisfy $\tilde{\nabla}' = (\tilde{\nabla}, t_b t_c \nabla^a \psi)$ for some scalar field ψ.

[24]Recall that since $\tilde{\nabla}$ and $\tilde{\nabla}$ are rotationally equivalent, both proposition 4 of Dewar (2018) and our proposition 3 apply.

on how timelike vectors change in spacelike directions—there is a canonical bijection between standards of acceleration and Newton-Cartan derivative operators (relative to each standard of rotation). So there cannot exist some other collection $[\bar{\nabla}]$ whose members assign acceleration to bodies in the way described and yet are associated with distinct Riemann tensors. Even after broadening attention to pseudo-Newton-Cartan connections, there is no way of expressing these disagreements on the acceleration of bodies as variations in the associated curvature tensors—such disagreements may only be expressed by the addition of potentials corresponding to their respective universal forces.

We thus find that the usual approach to geometrized Newtonian gravity does indeed respect corollary 6, constraints on the divergence of T^{ab} notwithstanding. The curvature of spacetime is not altered by the introduction of these universal forces. So even if the relative motions of bodies are preserved after the addition of some uniform linear acceleration, there remains an important difference between this kind of acceleration and that produced by the mutual gravitational attraction of bodies. In fact, the foregoing sheds some light on how it is that Newton-Cartan theory and Maxwell gravitation could ever be equivalent in the first place. If one requires of a successful geometrization that all gravitational interactions be encoded purely as spacetime curvature, then there is always a privileged model of pseudo-Newton-Cartan theory at hand. For any model with non-vanishing universal forces has either not fully incorporated the gravitational interactions of matter or is entertaining the existence of some additional, but non-gravitational, force. In this sense, one is not making some further choice in adopting the usual model of Newton-Cartan theory—this simply comes with the business of geometrizing Newtonian gravity. The apparent structural gap between Maxwell spacetime and Newton-Cartan theory may then be bridged because the "additional" structure of the latter comes for free.

This reflection is of special interest for the question of the conventionality of geometry in Newtonian gravitation theory. Weatherall and Manchak (2014), for instance, have shown that given a classical spacetime (L, ∇) and an arbitrary derivative operator $\tilde{\nabla}$ (compatible with L), there exists a unique antisymmetric field G_{ab} such that $\xi^n \nabla_n \xi^a = \mathbf{0}$ just in case $\xi^n \tilde{\nabla} \xi^a = G^a_n \xi^n$. They interpret this result as follows:

> This proposition means that one is free to choose any derivative operator one likes (compatible with the fixed classical metrics) and, by postulating a universal force field, one can recover all of the allowed trajectories of either a model of standard Newtonian gravitation or a model of geometrized Newtonian gravitation. Thus, since the derivative operator determines both the collection of geodesics—i.e. non-accelerating curves—and the curvature of spacetime, there is a sense

in which both acceleration and curvature are conventional in classical spacetimes. (Weatherall and Manchak, 10)

This claim is true with respect to the acceleration of bodies in classical spacetimes.[25] There is more to say, however, with respect to the conventionality of curvature in this setting. Weatherall and Manchak, naturally, point to the derivative operators of the standard models of ordinary Newtonian gravitation theory—in which universal forces are absent—and Newton-Cartan theory as convenient but non-canonical choices. Little is said, however, about the curvature of other models—only that their curvature will be determined by the choice of derivative operator, since changes in the latter will induce corresponding changes in the former.

In the same spirit as the pseudo-models of Newton-Cartan theory considered above, one can show that there exist intermediate models in which one's gravitational potential is only absorbed as spacetime curvature to a certain degree, such that a residual potential remains over a partially-geometrized spacetime. Consider an arbitrary model of Galilean gravitation (L, ∇, ϕ, ρ). There exists a one-parameter family of curved derivative operators $[\tilde{\nabla}]$ where each member $\tilde{\nabla}$ is related to the original model by $\tilde{\nabla} = (\nabla, -t_b t_c \nabla^a(\alpha\,\phi))$ with $0 \leq \alpha \leq 1$. One may easily verify that the structure $(L, \tilde{\nabla}, \phi, \rho, \alpha)$, for any $\tilde{\nabla} \in [\tilde{\nabla}]$, will satisfy the usual requirements of models of Newton-Cartan theory, save that now the "geodesic" equation becomes

$$\xi^n \tilde{\nabla}_n \xi^a = (\alpha - 1)\tilde{\nabla}^a \phi \iff \xi^n \nabla_n \xi^a = -\nabla^a \phi$$

and the associated Ricci curvature is given by $\tilde{R}_{bc} = \alpha\, 4\pi\rho\, t_b t_c$. In this way, one has an explicit procedure for smoothly geometrizing a flat classical spacetime through intermediate models up to the standard model of Newton-Cartan theory.

This brings out Weatherall and Manchak's insight concretely: given a classical spacetime, one may equally well describe the allowed trajectories with any member of an equivalence class of derivative operators, each of which determines a distinct curvature tensor. In this sense, curvature is surely a matter of convention. But the discussion of pseudo-models of Newton-Cartan theory above teaches us that we ought to be careful when interpreting the scope of this conventionality. One may always introduce universal forces to translate between derivative operators, yet only some of these universal forces will be of the appropriate form to be represented as the curvature of a classical spacetime: those of the above or a similar form will represent the difference between intermediate degrees of geometrization of a gravitational potential, while those universal forces relating families of Trautman

[25]One sees this, if to a lesser degree, even with the existence of non-singleton equivalence classes of Trautman recoveries, which grounds a sense in which gravitational force is a gauge quantity in Newtonian gravitation theory (Malament, 2014, 278). Teh (2018) develops this gauge interpretation of the theory at greater length.

recoveries—which have vanishing spatial gradients—are only expressible as further potentials themselves. So while there is no fact of the matter about the absolute degree of curvature of a classical spacetime—or even whether the spacetime is curved at all—there is nonetheless a canonical distinction between that part of the gravitational potential of any given model that is interpretable as curvature and that which is not.[26]

6 Conclusion

Following the strategy of Dewar (2018), I have expressed a version of his "Maxwell gravitation" in a way that refrains from using any covariant derivative operators—or equivalence classes thereof. The definition is equivalent to Dewar's and yet makes use of only the structure intrinsic to Maxwell spacetime, thereby securing the theory against claims of mathematical impropriety and providing a more direct characterization of the theory.

Furthermore, I have shown that there is a unique correspondence between models of Maxwell gravitation and Newton-Cartan theory so long as one's mass distribution be somewhere nonvanishing. There cannot exist "distinct but materially identical" models of Newton-Cartan theory that vary through purely gravitational degrees of freedom—fixing the curvature of even a single region suffices for it to be determined across the entire manifold. The structural gap between the two theories may thus be bridged with the modest assumption that there simply be some matter in one's spacetime with which to probe its emergent dynamical structure.

In discussing the divergence-freedom of T^{ab} vis-à-vis corollary 6, moreover, we found that the Riemann curvature of a model of Newton-Cartan theory is unaffected by the introduction of universal forces like those relating its Trautman recoveries. This lends a certain naturalness to a conception of Maxwell gravitation that countenances both its fundamental structure and the emergent standard of acceleration given by its uniquely associated Newton-Cartan connection, and shows, additionally, a sense in which spacetime curvature is not entirely a matter of convention in classical spacetimes.

[26] It is, of course, possible to entertain more exotic geometrizations of classical spacetimes in which such terms are incorporated, but only by means of a corresponding expansion of Poisson's equation. For instance, one may introduce a "cosmological constant" term to be reflected as a curvature term of possible geometrizations. Malament (2012, 269), in demonstrating how the geometrized Poisson equation is a special case of Einstein's equation, considers the simple addition $\nabla_a \nabla^a \phi + \Lambda = 4\pi\rho$, which yields $\tilde{R}_{bc} = 4\pi\rho t_b t_c - \Lambda t_b t_c$ as the Ricci curvature of the standard model of Newton-Cartan theory. But note that this addition only carries through to the Ricci curvature because it is made to the Laplacian of the gravitational potential (or the matter distribution), and not the potential itself. For more on the relationship between the cosmological constant and the classical limit of general relativity, see Malament (1986).

Appendix: Proofs of Propositions

Proposition 1. If $(L, \circlearrowleft)$ is a Maxwell spacetime, then there exists a unique corresponding Maxwell spacetime* $(L, [\nabla])$.

Proof. Let $A = (L, \circlearrowleft)$ be a Maxwell spacetime, where $L = (M, t_a, h^{ab})$ is some Leibnizian spacetime and $\circlearrowleft$ is a standard of rotation compatible with L. Let $[\nabla]$ be the set of all flat covariant derivative operators agreeing with $\circlearrowleft$ that are compatible with t_a and h^{ab}. Part 2 of proposition 1 of Weatherall (2018) guarantees that there exists at least one such derivative operator (which may be chosen to be flat by part 3 of the same proposition), and so $[\nabla]$ is nonempty; and trivially, $[\nabla]$ is unique.

Consider the structure $A^* = (L, [\nabla])$, where L and $[\nabla]$ are as above. By construction, A^* is a Maxwell spacetime* corresponding to A since all the members of $[\nabla]$ agree with $\circlearrowleft$ (and so are rotationally equivalent to one another) and are compatible with both of L's metrics. But since $[\nabla]$ is unique (and maximal), A^* is the only Maxwell spacetime* with this property.

□

Proposition 2. Let $(L, \circlearrowleft)$ be a Maxwell spacetime and $(L, [\nabla])$ its corresponding Maxwell spacetime*. Given some T^{ab}, $(L, \circlearrowleft, T^{ab})$ is a model of Maxwell gravitation if and only if $(L, [\nabla], T^{ab})$ is a model of Maxwell gravitation*.

Proof. Let $(L, \circlearrowleft, T^{ab})$ be a model of Maxwell gravitation. Since it has been assumed that $[\nabla]$ and $\circlearrowleft$ determine the same standard of rotation, it suffices to show that equations (1)–(3) are satisfied for an arbitrary derivative operator $\nabla \in [\nabla]$ to establish that $(L, [\nabla], T^{ab})$ is a model of Maxwell gravitation*.

We begin by demonstrating that equation (1) holds. Consider the first right-hand term of (1*):

$$\begin{aligned}
\rho\, \Box_{\underline{n}}\, \xi^n &= \rho\, h_{\underline{nz}}\, \Box^z\, \xi^n \\
&= \rho\, (h_{\underline{nz}}(\circlearrowleft^z \xi^n - \frac{1}{2} \pounds_\xi h^{zn})) \\
&= \rho\, (h_{\underline{az}}(\nabla^{[z}\xi^{a]} + \nabla^{(z}\xi^{a)})) \\
&= \rho\, \nabla_a \xi^a,
\end{aligned}$$

where the last two lines follow by proposition 1.7.4 of Malament (2012), the compatibility of h^{na} with ∇, and the fact that $\nabla^z\xi^a$ is spacelike in both indices as ξ^a is a unit timelike vector. Since $\xi^n d_n \rho = \xi^n \nabla_n \rho$ and the quantity $t_a \nabla_n \xi^a + t_a \nabla_n \sigma^{\underline{na}}$ vanishes—recall, t_a is also compatible with ∇ while $\sigma^{\underline{na}}$ is spacelike in both indices by construction—equation (1*) becomes

$$\begin{aligned}0 &= \rho\nabla_n\xi^n + \xi^n\nabla_n\rho\\ &= (t_a\xi^a)\rho\nabla_n\xi^n + (t_a\xi^a)\xi^n\nabla_n\rho + t_a\rho\xi^n\nabla_n\xi^a + t_a\nabla_n\sigma^{\underline{na}}\\ &= t_a\nabla_n(\rho\xi^n\xi^a + \sigma^{\underline{na}})\\ &= t_a\nabla_n T^{na},\end{aligned}$$

which is equation (1).

We turn next to the case of equation (2). Let η^a be a smooth vector field everywhere satisfying $\eta^a t_a = 1$, and let $\hat{h}_{ab}$ be the spatial projector relative to η^a (i.e. the unique field satisfying $\hat{h}_{ab}\eta^b = \mathbf{0}$ and $\hat{h}_{ab}h^{bc} = \delta^c_a - t_a\eta^c$). One may then express the first right-hand term of (2*) as

$$\begin{aligned}D_{\underline{a}}(\rho^{-1}D_{\underline{n}}\sigma^{\underline{na}}) &= h_{\underline{az}}\nabla^z(\rho^{-1}h_{\underline{nm}}\hat{h}^{\underline{n}}{}_x\hat{h}^{\underline{a}}{}_y\nabla^m\sigma^{xy})\\ &= \nabla_a(\rho^{-1}\nabla_n\sigma^{na}).\end{aligned}$$

Moreover, by the same reasoning as in the case of $h_{\underline{nz}}\,\Box^z\,\xi^n$, one has for the remaining right-hand terms of (2*)

$$\xi^n d_n(\Box_{\underline{a}}\xi^a) = \xi^n\nabla_n(\nabla_a\xi^a) = \xi^n\nabla_a\nabla_n\xi^a$$

and

$$(\Box_{\underline{a}}\xi^n)(\Box_{\underline{n}}\xi^a) = (\nabla_a\xi^n)(\nabla_n\xi^a),$$

as ∇ is flat and $\nabla_a\xi^a$ a scalar field. These expressions, together with the observation that $\rho^{-1}\xi^a(\rho\nabla_n\xi^n + \xi^n\nabla_n\rho) = \mathbf{0}$, allow one to re-write equation (2*) as

$$\begin{aligned}-4\pi\rho &= \nabla_a(\rho^{-1}\nabla_n\sigma^{na}) + \xi^n\nabla_a\nabla_n\xi^a + (\nabla_a\xi^n)(\nabla_n\xi^a)\\ &= \nabla_a(\rho^{-1}\nabla_n\sigma^{na} + \rho^{-1}\rho\xi^n\nabla_n\xi^a + \rho^{-1}\rho\xi^a\nabla_n\xi^n + \rho^{-1}\xi^a\xi^n\nabla_n\rho)\\ &= \nabla_a(\rho^{-1}\nabla_n(\sigma^{na} + \rho\xi^n\xi^a))\\ &= \nabla_a(\rho^{-1}\nabla_n T^{na}),\end{aligned}$$

i.e. equation (2).

It remains to establish that equation (3) is satisfied. We begin by observing that the first right-hand term of (3*) is

$$\circlearrowleft^c(\rho^{-1}D_{\underline{n}}\sigma^{\underline{na}}) = \frac{1}{2}[\nabla^c(\rho^{-1}\nabla_n\sigma^{na}) - \nabla^a(\rho^{-1}\nabla_n\sigma^{nc})],$$

while, by application of proposition 1.7.4 of Malament (2012), one has

$$\begin{aligned}\pounds_\xi(\circlearrowleft^c\xi^a) = \frac{1}{2}&[\xi^n\nabla_n(\nabla^c\xi^a) - (\nabla^n\xi^a)(\nabla_n\xi^c) - (\nabla^c\xi^n)(\nabla_n\xi^a)]\\ &- \frac{1}{2}[\xi^n\nabla_n(\nabla^a\xi^c) - (\nabla^n\xi^c)(\nabla_n\xi^a) - (\nabla^a\xi^n)(\nabla_n\xi^c)]\end{aligned}$$

for the last term of (3*). The remaining right-hand terms, finally, are spacelike in all their indices, and so are just

$$2(\circlearrowright^{[c}\xi^{|n|})(\circlearrowright_{\underline{n}}\xi^{a]}) + (\circlearrowright^{n}\xi^{[a})(\circlearrowright_{\underline{n}}\xi^{c]})$$
$$= (\nabla^c\xi^n)(\nabla_n\xi^a) - (\nabla^a\xi^n)(\nabla_n\xi^c) + \frac{1}{2}(\nabla^n\xi^a)(\nabla_n\xi^c) - \frac{1}{2}(\nabla^n\xi^c)(\nabla_n\xi^a).$$

A straightforward calculation then verifies, by taking the sum of the right-hand sides of the three expressions above, that

$$\mathbf{0} = \nabla^{[c}(\rho^{-1}\nabla_n T^{|n|a]}),$$

and so equation (3) is satisfied.

This gives us the implication from left to right. The converse may be shown similarly.[27]

□

Proposition 3. Let $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ be two rotationally-equivalent models of Newton-Cartan theory where $\tilde{\nabla}' = (\tilde{\nabla}, \eta^{\underline{a}} t_b t_c)$. Then there exists a field η such that $\eta^{\underline{a}} = \tilde{\nabla}^a\eta$ and $\tilde{\nabla}_a\tilde{\nabla}^a\eta = 0$.

Proof. The existence of at least one field η such that $\tilde{\nabla}^a\eta = \eta^{\underline{a}}$ is guaranteed by proposition 4.1.6 of Malament (2012) (which holds globally for spacetimes whose spacelike slices are connected and simply connected) as $\eta^{\underline{a}}$ is twist-free by proposition 4 of Dewar (2018).

Since $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ are both models of Newton-Cartan theory, they each satisfy Poisson's equation: $\tilde{R}_{bc} = 4\pi\rho\, t_b t_c = \tilde{R}'_{bc}$. Further, by equation 1.8.2 of Malament (2012), one has that

$$\tilde{R}'^{a}{}_{bcd} = \tilde{R}^{a}{}_{bcd} + 2\tilde{\nabla}_{[c}\eta^{\underline{a}} t_{d]} t_b + 2\,\eta^{\underline{n}} t_b t_{[c}\eta^{\underline{a}} t_{d]} t_n,$$

which, through $(d \to a)$ index substitution, is

$$\tilde{R}'^{a}{}_{bca} = \tilde{R}^{a}{}_{bca} + 2\tilde{\nabla}_{[c}\eta^{\underline{a}} t_{a]} t_b + 2\,\eta^{\underline{n}} t_b t_{[c}\eta^{\underline{a}} t_{a]} t_n.$$

But, by Poisson's equation, the compatibility of t_a, and the fact that $\eta^{\underline{a}}$ is spacelike, this simplifies to

$$0 = \tilde{\nabla}_a\eta^{\underline{a}} = \tilde{\nabla}_a\tilde{\nabla}^a\eta,$$

i.e. η must be a solution to Laplace's equation.

□

[27] Note that, when proceeding in the other direction, equations (1)-(3) would be expressed in terms of an arbitrary member of $[\nabla]$ (flat by construction). Part 1 of proposition 1 of Weatherall (2018) then guarantees there will be a unique standard of rotation $\circlearrowright$ corresponding to ∇ regardless of our choice of derivative operator (they are all rotationally equivalent). The argument will then proceed by reversing the steps of the proof for the direction shown with minor alterations.

Proposition 4. Let $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ be two rotationally-equivalent models of Newton-Cartan theory. Then, if $\tilde{\nabla}$ and $\tilde{\nabla}'$ agree with respect to any open set O, they must agree everywhere.

Proof. For suppose not. Then there exist at least two rotationally-equivalent models of Newton-Cartan theory $(L, \tilde{\nabla}, T^{ab})$ and $(L, \tilde{\nabla}', T^{ab})$ such that $\tilde{\nabla} = \tilde{\nabla}'$ at all points $p \in O$, but disagree at some point y.

Proposition 4 of Dewar (2018) shows that $\tilde{\nabla}' = (\tilde{\nabla}, \eta^{\underline{a}} t_b t_c)$ for some spacelike vector field $\eta^{\underline{a}}$ such that $\tilde{\nabla}^{[a}\eta^{b]} = \mathbf{0}$, while by the same reasoning as above, there exists some field η such that $\tilde{\nabla}^a \eta = \eta^{\underline{a}}$. Any such η, however, must satisfy Laplace's equation by proposition 3.

Consider, next, the values η must assume. By supposition, $\tilde{\nabla}$ and $\tilde{\nabla}'$ agree everywhere within O, but disagree at some point y. This is to say, η must be equal to some constant s within O and yet be changing in some neighborhood of y along the direction of $\eta^{\underline{a}}$. But by theorems 1.27[28] and 1.28 of Axler, Bourdon, and Ramey (2001), if η is constant in O, then η must be constant everywhere.

□

Corollary 4.1. Let $(L, \circlearrowleft, T^{ab})$ be a model of Maxwell gravitation such that ρ is smooth and somewhere nonvanishing. Then there is a unique Newton-Cartan connection $\tilde{\nabla}$ such that $(L, \tilde{\nabla}, T^{ab})$ is a compatible model of Newton-Cartan theory.

Proof. The existence of at least one such Newton-Cartan connection $\tilde{\nabla}$ is given by the proof of proposition 6 of Dewar (2018).

Now, $\tilde{\nabla}$ must be unique. For suppose not—then there exists another Newton-Cartan connection $\tilde{\nabla}'$ agreeing with $\circlearrowleft$ such that $(L, \tilde{\nabla}', T^{ab})$ is a model of Newton-Cartan theory. Since $\tilde{\nabla}$ and $\tilde{\nabla}'$ are both Newton-Cartan connections, one has that

$$\tilde{\nabla}_n T^{na} = \mathbf{0} = \tilde{\nabla}'_n T^{na}$$

and, by the proof of proposition 6 of Dewar (2018),

$$\tilde{\nabla}'_n T^{na} = \tilde{\nabla}_n T^{na} - \rho \eta^{\underline{a}},$$

where η^a is the vector field satisfying $\tilde{\nabla}' = (\tilde{\nabla}, t_b t_c \eta^{\underline{a}})$. Thus, the quantity $\rho \eta^{\underline{a}}$ must vanish everywhere.

Since ρ is somewhere nonvanishing by supposition, there is some open set in which the vector field $\eta^{\underline{a}}$ vanishes, i.e. $\tilde{\nabla} = \tilde{\nabla}'$. By proposition 4, however, $\tilde{\nabla}$ and $\tilde{\nabla}'$ must then agree everywhere.

□

[28]The reasoning in the proof of this theorem applies just as well for functions equal to a non-zero constant.

References

[1] Axler, Sheldon; Bourdon, Paul; and Ramsey, Wade. 2001 *Harmonic Function Theory.* New York, NY: Springer-Verlag.

[2] Barbour, Julian B. 1982 "Relational Concepts of Space and Time." *The British Journal for the Philosophy of Science* 33 (3): 251–274.

[3] Belot, Gordon. 2000. "Geometry and Motion." *The British Journal for the Philosophy of Science* 51: 561–595.

[4] Dewar, Neil. 2018. "Maxwell Gravitation." *Philosophy of Science* 85 (2): 249–270.

[5] Dewar, Neil, and Weatherall, James Owen. 2018. "On Gravitational Energy in Newtonian Theories." *Foundations of Physics* 48: 558–578.

[6] Earman, John. 1989 *World Enough and Spacetime: Absolute versus Relational Theories of Space and Time.* Cambridge, MA: The MIT Press.

[7] Hansen, Dennis; Hartong, Jelle; and Obers, Neils A. 2019. "Gravity Between Newton and Einstein." *International Journal of Modern Physics D* 28 (14): 1944010.

[8] Knox, Eleanor. 2014. "Newtonian Spacetime Structure in Light of the Equivalence Principal." *The British Journal for the Philosophy of Science* 65 (4): 863–880.

[9] Knox, Eleanor. 2017. "Inertial Frames vs. Maximal Symmetries: two kinds of spacetime structure." The Philosophy of Howard Stein. Video available at https://www.youtube.com/watch?v=YgwlgyHMvL8&feature=youtu.be.

[10] Linnemann, Neils, and Read, James. 2021. "On the Status of Newtonian Gravitational Radiation." *Foundations of Physics* 51: 53.

[11] Malament, David D. 1986. "Newtonian Gravity, Limits, and the Geometry of Space." In *From Quarks to Quasars: Philosophical Problems of Modern Physics*, edited by Robert G. Colodny. Pittsburgh, PA: University of Pittsburgh Press.

[12] Malament, David B. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory.* Chicago, IL: The University of Chicago Press.

[13] Newton, Isaac. 1999. *The Principia: Mathematical Principles of Natural Philosophy*, translated by Bernard Cohen and A. Whitman. Berkeley: The University of California Press.

[14] Saunders, Simon. 2013. "Rethinking Newton's *Principia*." *Philosophy of Science* 80: 22–48.

[15] Teh, Nicholas. 2018. "Recovering Recovery: On the Relationship between Gauge Symmetry and Trautman Recovery." *Philosophy of Science* 85 (2): 201–224.

[16] Wallace, David. 2017. "More Problems for Newtonian Cosmology." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 57: 35–40.

[17] Wallace, David. 2020. "Fundamental and emergent geometry in Newtonian physics." *The British Journal of Philosophy of Science* 71 (1): 1–32.

[18] Weatherall, James Owen, and Manchak, John Byron. 2014. "Geometry of Conventionality." *Philosophy of Science* 81 (2): 233–247

[19] Weatherall, James Owen. 2016. "Maxwell-Huygens, Newton-Cartan,and Saunders-Knox spacetimes." *Philosophy of Science* 83 (1): 82–92.

[20] Weatherall, James Owen. 2018. "A Brief Comment on Maxwell(\Newton)[-Huygens] Spacetime." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 63: 34–38